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GENERATING SETS OF THE COMPLETE SEMIGROUP OF BINARY RELATIONS DEFINED BY SEMILATTICES OF THE CLASS $\Sigma_1(X, 6)$

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Abstract. In this paper, we study generating sets of the complete semigroup of binary relations defined by X"=semilattices of unions of the class $\Sigma_1(X, 6)$.

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1. Introduction. Let X be an arbitrary nonempty set, D be an X-semilattice of unions, which is closed with respect to the set-theoretic union of elements from D, and f be an arbitrary mapping of the set X to the set D. To each mapping f, we assign a binary relation α_f on the set X satisfying the condition

$$\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)).$$

The set of all these α_f $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an X-semilattice of unions D.

We denote by $\varnothing$ the empty binary relation or the empty subset of the set X. The condition $(x, y) \in \alpha$ will be written in the form $x\alpha y$. Further, let

$$x, y \in X, \quad Y \subseteq X, \quad \alpha \in B_X(D), \quad \breve{D} = \bigcup_{Y \in D} Y, \quad T \in D.$$

We denote by the symbols $y\alpha$, $Y\alpha$, $V(D, \alpha)$, X^*, and $V(X^*, \alpha)$ the following sets:

$$y\alpha = \{x \in X \mid y\alpha x\}, \quad Y\alpha = \bigcup_{y \in Y} y\alpha, \quad V(D, \alpha) = \{Y\alpha \mid Y \in D\},$$
$$X^* = \{Y \mid \varnothing \neq Y \subseteq X\}, \quad V(X^*, \alpha) = \{Y\alpha \mid \varnothing \neq Y \subseteq X\},$$
$$D_T = \{Z \in D \mid T \subseteq Z\}, \quad Y_T^\alpha = \{y \in X \mid y\alpha = T\}.$$

The following assertions are well known.

Theorem 1.1. *Let $D = \{\breve{D}, Z_1, Z_2, \dots, Z_{m-1}\}$ be a finite X-semilattice of unions and $C(D) = \{P_0, P_1, P_2, \dots, P_{m-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set X (the set $\varnothing$ can be repeated several times). If φ is a mapping of the semilattice D to the family of sets $C(D)$, which satisfies the conditions*

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & \dots & Z_{m-1} \\ P_0 & P_1 & P_2 & \dots & P_{m-1} \end{pmatrix}, \quad \widehat{D}_Z = D \setminus D_Z,$$

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then the following equalities hold:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \ldots \cup P_{m-1}, \quad Z_i = P_0 \cup \bigcup_{T \in \widehat{D}_{Z_i}} \varphi(T). \tag{1.1}$$

In the sequel, these equalities are said to be *formal*.

It was proved earlier that if elements of the semilattice D are represented in the form (1.1), then, among the parameters P_i $(0 < i \le m-1)$, there exist parameters that cannot be empty sets for D. Such sets P_i are called *basis sources*, whereas the sets P_j $(0 \le j \le m-1)$, which can also be empty sets, are called *completeness sources*.

It is well known that, under the mapping φ, the number of covering elements of the preimage of the basis source is always equal to one, while the number of covering elements of the preimage of the completeness source is either zero or greater than one (see [2, Chapter 11]).

Let $P_0, P_1, P_2, \ldots, P_{m-1}$ be parameters in the formal equalities and β be any binary relation from the semigroup $B_X(D)$ and

$$\overline{\beta} = \bigcup_{i=0}^{m-1} \Big(P_i \times \bigcup_{t \in P_i} t\beta\Big) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times \overline{\beta}_2(t')), \tag{1.2}$$

where $\overline{\beta}_2$ is any mapping of the set $X \setminus \breve{D}$ to the set D. Then the representation of the binary relation β of the form $\overline{\beta}$ is said to be *subquasinormal*.

If $\overline{\beta}$ is a subquasinormal representation of the binary relation β, then, for the binary relation $\overline{\beta}$, the following assertions hold:

(a) $\overline{\beta} \in B_X(D)$;

(b) $\bigcup_{i=0}^{m-1} \big(P_i \times \bigcup_{t \in P_i} t\beta\big) \subseteq \beta$ and $\beta = \overline{\beta}$ for some mapping $\overline{\beta}_2$ of the set $X \setminus \breve{D}$ in the set D;

(c) the subquasinormal representation of the binary relation β is quasinormal;

(d) if $\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & \ldots & P_{m-1} \\ P_0\overline{\beta} & P_1\overline{\beta} & \ldots & P_{m-1}\overline{\beta} \end{pmatrix}$, then $\overline{\beta}_1$ is a mapping of the family of sets $C(D)$ in the set $D \cup \{\varnothing\}$.

Note that if P_j $(0 \le j \le m-1)$ are completeness sources such that $P_j = \varnothing$, then the equality $P_j\overline{\beta} = \varnothing$ always holds. There also exist basis sources P_i $(0 \le i \le m-1)$ such that $\bigcup_{t \in P_i} t\beta = \varnothing$, i.e., $P_i\overline{\beta} = \varnothing$.

Definition 1.1. In the sequel, the elements $\overline{\beta}_1$ and $\overline{\beta}_2$ are called normal and complement mappings, respectively, for the binary relation $\overline{\beta} \in B_X(D)$.

Theorem 1.2. *Let D be a finite set and $\alpha, \beta \in B_X(D)$. Then, for any subquasinormal representation $\overline{\beta}$ of the binary relation β, the equality $\alpha \circ \beta = \alpha \circ \overline{\beta}$ holds* (*see* [1, Theorem 1.2]).

Theorem 1.3. *Let $\widetilde{B}$ be any generating set of the semigroup $B_X(D)$. If, for some α and δ from the set $\widetilde{B}$ and a subquasinormal representation $\overline{\beta} \in B_X(D)$ of the binary relation β, the inequality $\alpha \ne \delta \circ \overline{\beta}$ holds, then the relation $\alpha \ne \delta \circ \beta$ is also true* (*see* [1, Theorem 1.3]).

Definition 1.2. We say that an element α of the semigroup $B_X(D)$ is *external* if $\alpha \ne \delta \circ \beta$ for all $\delta, \beta \in B_X(D) \setminus \{\alpha\}$ (see [2, Definition 1.15.1]).

It is well known that if B is the set of all external elements of the semigroup $B_X(D)$ and B' is any generating set for the $B_X(D)$, then $B \subseteq B'$ (see [2, Lemma 1.15.1]).

Definition 1.3. A representation

$$\alpha = \bigcup_{T \in D} (Y_T^{\alpha} \times T)$$

of a binary relation α is said to be *quasinormal* if

$$\bigcup_{T \in D} Y_T^{\alpha} = X \quad \text{and} \quad Y_T^{\alpha} \cap Y_{T'}^{\alpha} = \varnothing \quad \text{for any } T, T' \in D,\ T \neq T'.$$

2. Generating sets of the semigroup of binary relations defined by semilattices of the class $\Sigma_{10}(X, 6)$, where $P = \varnothing$ and $|X \setminus \breve{D}| \geq 1$ or $X = \breve{D}$.

Definition 2.1. We denote by $\Sigma_1(X, 6)$ the class of all X-semilattices of unions whose elements are isomorphic to an X-semilattice of unions $D = \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$, which satisfies the relations

$$Z_4 \subset Z_3 \subset Z_1 \subset \breve{D}, \quad Z_5 \subset Z_3 \subset Z_1 \subset \breve{D}, \quad Z_2 \subset \breve{D}, \quad Z_5 \setminus Z_4 \neq \varnothing,$$
$$Z_4 \setminus Z_5 \neq \varnothing, \quad Z_3 \setminus Z_2 \neq \varnothing, \quad Z_2 \setminus Z_3 \neq \varnothing, \quad Z_2 \setminus Z_1 \neq \varnothing, \quad Z_1 \setminus Z_2 \neq \varnothing,$$
$$Z_5 \cup Z_4 = Z_3, \quad Z_5 \cup Z_2 = Z_4 \cup Z_2 = Z_3 \cup Z_2 = Z_1 \cup Z_2 = \breve{D}$$

(see Fig. 1).

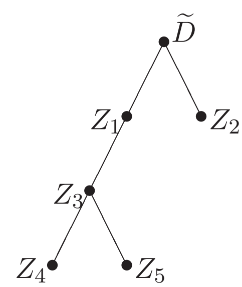

Fig. 1.

It is easy to see that $\widetilde{D} = \{Z_5, Z_4, Z_2, Z_1\}$ is an irreducible generating set for the semilattice D.

Let $C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5\}$ be a family of sets, where P_0, P_1, P_2, P_3, P_4, P_5 are pairwise disjoint subsets of the set X and

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\ P_0 & P_1 & P_2 & P_3 & P_4 & P_5 \end{pmatrix}$$

is a mapping of the semilattice D onto the family of sets $C(D)$. Then the formal equalities of the semilattice D have the following form:

$$\begin{aligned} \breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5, \\ Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5, \\ Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5, \\ Z_3 &= P_0 \cup P_2 \cup P_4 \cup P_5, \\ Z_4 &= P_0 \cup P_2 \cup P_5, \\ Z_5 &= P_0 \cup P_2 \cup P_4. \end{aligned} \tag{2.1}$$

Here the elements P_5, P_4, P_3, P_2, and P_1 are basis sources and the element P_0 is a completeness source of the semilattice D. Therefore, $|X| \geq 5$ since $|P_5| \geq 1$, $|P_4| \geq 1$, $|P_3| \geq 1$, $|P_2| \geq 1$, and $|P_1| \geq 1$

(see [2, Chapter 11]). The formal equalities of the semilattice D immediately imply that

$$\begin{gathered} P_5 = Z_4 \setminus Z_5, \quad P_4 = Z_5 \setminus Z_4, \quad P_3 = Z_1 \setminus Z_3, \\ P_2 = Z_1 \setminus Z_2, \quad P_1 = Z_2 \setminus Z_1, \quad P_0 = Z_5 \cap Z_4 \cap Z_2. \end{gathered} \tag{2.2}$$

In this paper, we study irreducible generating sets of the semigroup $B_X(D)$ defined by semilattices of the class $\Sigma_1(X, 6)$.

Definition 2.2. We denote by $\mathfrak{A}_5$, $\mathfrak{A}_4$, $\mathfrak{A}_3$, $\mathfrak{A}_2$, and $\mathfrak{A}_1$ the following sets:

$$\begin{aligned}
\mathfrak{A}_5 &= \Big\{ \{Z_5, Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_4, Z_3, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, Z_1, \breve{D}\} \Big\}, \\
\mathfrak{A}_4 &= \Big\{ \{Z_5, Z_4, Z_3, Z_1\},\ \{Z_5, Z_4, Z_3, \breve{D}\},\ \{Z_5, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, \breve{D}\}, \\
&\qquad \{Z_4, Z_2, Z_1, \breve{D}\},\ \{Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_1, \breve{D}\} \Big\}, \\
\mathfrak{A}_3 &= \Big\{ \{Z_5, Z_4, Z_3\},\ \{Z_5, Z_2, \breve{D}\},\ \{Z_4, Z_2, \breve{D}\},\ \{Z_2, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_1\},\ \{Z_4, Z_3, Z_1\}, \\
&\qquad \{Z_5, Z_3, \breve{D}\},\ \{Z_5, Z_1, \breve{D}\},\ \{Z_4, Z_3, \breve{D}\},\ \{Z_4, Z_1, \breve{D}\},\ \{Z_3, Z_2, \breve{D}\},\ \{Z_3, Z_1, \breve{D}\} \Big\}; \\
\mathfrak{A}_2 &= \Big\{ \{Z_5, Z_3\},\ \{Z_5, Z_1\},\ \{Z_4, Z_3\},\ \{Z_4, Z_1\},\ \{Z_3, Z_1\},\ \{Z_5, \breve{D}\}, \\
&\qquad \{Z_4, \breve{D}\},\ \{Z_3, \breve{D}\},\ \{Z_2, \breve{D}\},\ \{Z_1, \breve{D}\} \Big\}, \\
\mathfrak{A}_1 &= \Big\{ \{Z_5\},\ \{Z_4\},\ \{Z_3\},\ \{Z_2\},\ \{Z_1\},\ \{\breve{D}\} \Big\}.
\end{aligned}$$

Definition 2.3. We denote by $\Sigma_{1.0}(X, 6)$ all semilattices $D = \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ of the class $\Sigma_1(X, 6)$ such that $Z_5 \cap Z_4 \cap Z_2 \neq \varnothing$. The last inequality and the formal equalities (2.1) of the semilattice D imply that

$$Z_5 \cap Z_4 \cap Z_2 = P_0 \neq \varnothing,$$

i.e., $|X| \geq 6$, since

$$P_5 \neq \varnothing, \quad P_4 \neq \varnothing, \quad P_3 \neq \varnothing, \quad P_2 \neq \varnothing, \quad P_1 \neq \varnothing, \quad P_0 \neq \varnothing.$$

In this case, we assume that $D \in \Sigma_{1.0}(X, 6)$.

Lemma 2.1. *Let $D \in \Sigma_{1.0}(X, 6)$ and $\alpha = \delta \circ \beta$ for some $\alpha, \delta, \beta \in B_X(D)$. Then the following assertions hold.*

(a) *Let $T, T' \in \{Z_5, Z_4, Z_2\}$, $T \neq T'$. If $T, T' \in V(D, \alpha)$, then α is an external element of the semigroup $B_X(D)$.*
(b) *If $T \in \{Z_3, Z_1\}$ and $T, Z_2 \in V(D, \alpha)$, then α is an external element of the semigroup $B_X(D)$.*

Proof. Let $Z_0 = \breve{D}$ and $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_X(D) \setminus \{\alpha\}$. If the quasinormal representation of the binary relation δ has the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times Z_0),$$

then

$$\begin{aligned} \alpha = \delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) \\ \cup (Y_3^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times Z_0\beta). \end{aligned} \tag{2.3}$$

From the formal equalities (2.1) of the semilattice D we obtain that

$$\begin{aligned} Z_0\beta &= P_0\beta \cup P_1\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta \cup P_5\beta, \\ Z_1\beta &= P_0\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta \cup P_5\beta, \\ Z_2\beta &= P_0\beta \cup P_1\beta \cup P_3\beta \cup P_4\beta \cup P_5\beta, \\ Z_3\beta &= P_0\beta \cup P_2\beta \cup P_4\beta \cup P_5\beta, \\ Z_4\beta &= P_0\beta \cup P_2\beta \cup P_5\beta, \\ Z_5\beta &= P_0\beta \cup P_2\beta \cup P_4\beta, \end{aligned} \tag{2.4}$$

where $P_i\beta \neq \varnothing$ for any $P_i \neq \varnothing$ $(i = 0, 1, 2, 3, 4, 5)$ and $\beta \in B_X(D)$. Indeed, by the assumption, $P_i \neq \varnothing$ for any $i = 0, 1, 2, 3, 4, 5$ and $\beta \neq \varnothing$ since $\varnothing \notin D$. Let $y \in P_i$ for some $y \in X$. Then $y \in \check{D}$, $\beta = \alpha_f$ for some $f : X \to D$, and

$$\alpha_f = \bigcup_{x \in X} \big(\{x\} \times f(x)\big) \supseteq \{y\} \times f(y),$$

i.e., there exists an element $z \in f(y)$ such that $y\alpha_f z$ and $y\beta z$. Thus, by the definition of the set $P_i\beta$, we obtain that $z \in P_i\beta$ since $y \in P_i$, $y\beta z$. Finally, we see that $P_i\beta \neq \varnothing$, i.e., $P_i\beta \in D$ for any $i = 0, 1, 2, 3, 4, 5$.

Now let $Z_i\beta = T$ and $Z_j\beta = T'$ for some $0 \leq i \neq j \leq 5$ and $T \neq T'$, $T, T' \in \{Z_5, Z_4, Z_2\}$. Then it follows from Eq. (2.4) that $T = P_0\beta = T'$ since T and T' are minimal elements of the semilattice D. The equality $T = T'$ contradicts the inequality $T \neq T'$. Item (a) of Lemma 2.1 is proved.

(b) Let $Z_i\beta = Z_2$ and $Z_j\beta = T$ $(T \in \{Z_3, Z_1\})$ for some $0 \leq i \neq j \leq 5$. If $0 \leq i \leq 5$. Then from the formal equalities of a semilattice D, we obtain that

$$\begin{aligned} &Z_0\beta = P_0\beta \cup P_1\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta \cup P_5\beta = P_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = P_5\beta = Z_2, \\ &Z_1\beta = P_0\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta \cup P_5\beta = P_0\beta = P_2\beta = P_3\beta = P_4\beta = P_5\beta = Z_2, \\ &Z_2\beta = P_0\beta \cup P_1\beta \cup P_3\beta \cup P_4\beta \cup P_5\beta = P_0\beta = P_1\beta = P_3\beta = P_4\beta = P_5\beta = Z_2, \\ &Z_3\beta = P_0\beta \cup P_2\beta \cup P_4\beta \cup P_5\beta = P_0\beta = P_2\beta = P_4\beta = P_5\beta = Z_2, \\ &Z_4\beta = P_0\beta \cup P_2\beta \cup P_5\beta = P_0\beta = P_2\beta = P_5\beta = Z_2, \\ &Z_5\beta = P_0\beta \cup P_2\beta \cup P_4\beta = P_0\beta = P_2\beta = P_4\beta = Z_2, \end{aligned}$$

since Z_2 is a minimal element of the semilattice D. Now, let $i \neq j$.

(1) If $Z_0\beta = P_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = P_5\beta = Z_2$ and $j = 1, 2, 3, 4, 5,$ then

$$T = Z_1\beta = Z_2\beta = Z_3\beta = Z_4\beta = Z_5\beta = Z_2,$$

which contradicts the inequality $T \neq Z_2$.

(2) If $Z_1\beta = P_0\beta = P_2\beta = P_3\beta = P_4\beta = P_5\beta = Z_2$ and $j = 0, 2, 3, 4, 5,$ then

$$T = Z_0\beta = Z_2\beta = Z_2 \cup P_1\beta, \quad T = Z_3\beta = Z_4\beta = Z_5\beta = Z_2,$$

where $P_1\beta \in D$. The last equalities are impossible since $T \neq Z_2 \cup Z$ for any $Z \in D$ and $T \neq Z_2$ by the definition of the semilattice D.

(3) If $Z_2\beta = P_0\beta = P_1\beta = P_3\beta = P_4\beta = P_5\beta = Z_2$ and $j = 0, 1, 3, 4, 5,$ then

$$T = Z_0\beta = Z_1\beta = Z_3\beta = Z_4\beta = Z_5\beta = Z_2 \cup P_2\beta,$$

where $P_2\beta \in D$. The last equalities are impossible since $T \neq Z_2 \cup Z$ for any $Z \in D$ by the definition of the semilattice D.

(4) If $Z_3\beta = P_0\beta = P_2\beta = P_4\beta = P_5\beta = Z_2$ and $j = 0, 1, 2, 4, 5,$ then

$$T = Z_0\beta = Z_2\beta = Z_2 \cup P_1\beta \cup P_3\beta,$$

$$T = Z_1\beta = Z_2 \cup P_3\beta, \quad T = Z_4\beta = Z_5\beta = Z_2,$$

where $P_1\beta, P_3\beta \in D$. The last equalities are impossible since $T \neq Z_2 \cup Z \cup Z'$, $T \neq Z_2 \cup Z$, and $T \neq Z_2$ for any $Z, Z' \in D$ by the definition of the semilattice D.

(5) If $Z_4\beta = P_0\beta = P_2\beta = P_5\beta$ and $j = 0, 1, 2, 3, 5$, then

$$T = Z_0\beta = Z_2\beta = Z_2 \cup P_1\beta \cup P_3\beta \cup P_4\beta,$$
$$T = Z_1\beta = Z_2 \cup P_3\beta \cup P_4\beta, \quad T = Z_3\beta = Z_5\beta = Z_2 \cup P_4\beta,$$

where $P_1\beta, P_3\beta, P_4\beta \in D$. The last equalities are impossible since $T \neq Z_2 \cup Z \cup Z' \cup Z''$, $T \neq Z_2 \cup Z \cup Z'$, and $T \neq Z_2 \cup Z$ for any $Z, Z', Z'' \in D$ by the definition of the semilattice D.

(6) If $Z_5\beta = P_0\beta = P_2\beta = P_4\beta$ and $j = 0, 1, 2, 3, 4$, then

$$T = Z_0\beta = Z_2\beta = Z_2 \cup P_1\beta \cup P_3\beta \cup P_5\beta,$$
$$T = Z_1\beta = Z_2 \cup P_3\beta \cup P_5\beta, \quad T = Z_3\beta = Z_4\beta = Z_2 \cup P_5\beta,$$

where $P_1\beta, P_3\beta, P_5\beta \in D$. The last equalities are impossible since $T \neq Z_2 \cup Z \cup Z' \cup Z''$, $T \neq Z_2 \cup Z \cup Z'$, and $T \neq Z_2 \cup Z$ for any $Z, Z', Z'' \in D$ by the definition of the semilattice D.

Item (b) of Lemma 2.1 is proved. □

Let $D \in \Sigma_{1.0}(X, 6)$. We denote by $\mathfrak{A}_0$, $B(\mathfrak{A}_0)$, and B_0 the following sets:

$$\mathfrak{A}_0 = \Big\{\{Z_5, Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_4, Z_3, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_5, Z_4, Z_3, Z_1\},\ \{Z_5, Z_4, Z_3, \breve{D}\},\ \{Z_5, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, \breve{D}\},$$
$$\{Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_5, Z_4, Z_3\},\ \{Z_5, Z_2, \breve{D}\},\ \{Z_4, Z_2, \breve{D}\},\ \{Z_3, Z_2, \breve{D}\},\ \{Z_2, Z_1, \breve{D}\}\Big\},$$
$$B(\mathfrak{A}_0) = \Big\{\alpha \in B_X(D) \mid V(X^*, \alpha) \in \mathfrak{A}_0\Big\},$$
$$B_0 = \Big\{\alpha \in B_X(D) \mid\ V(X^*, \alpha) = D\Big\}.$$

Lemma 2.1 implies that the sets B_0 and $B(\mathfrak{A}_0)$ are external elements for the semigroup $B_X(D)$.

Lemma 2.2. *Let $D \in \Sigma_{1.0}(X, 6)$. Then the following assertions hold.*

(a) *If a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where Y_5^α, Y_3^α, $Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set B_0.

(b) *If a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where Y_4^α, Y_3^α, $Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set B_0.

Proof. (a) Let quasinormal representations of binary relations δ and β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = (Z_5 \times Z_5) \cup ((Z_4 \setminus Z_5) \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_2) \cup \big(((Z_2 \cap Z_1) \setminus Z_3) \times Z_1\big) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_5^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$,

$$Z_5 \cup (Z_4 \setminus Z_5) \cup (Z_2 \setminus Z_1) \cup ((Z_2 \cap Z_1) \setminus Z_3) \cup (X \setminus \breve{D})$$
$$= (P_0 \cup P_2 \cup P_4) \cup P_5 \cup P_1 \cup P_3 \cup (X \setminus \breve{D}) = \breve{D} \cup (X \setminus \breve{D}) = X$$

(see Eqs. (2.1) and (2.2)). Then $\delta, \beta \in B_0$ and

$$Z_5\beta = Z_5, \quad Z_4\beta = Z_5 \cup Z_4 = Z_3,$$
$$Z_2\beta = (P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5)\beta = \breve{D},$$
$$Z_1\beta = (P_0 \cup P_2 \cup P_3 \cup P_4)\beta = Z_1, \quad \breve{D}\beta = \breve{D},$$
$$\alpha = \delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta) =$$
$$= (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_3) \cup (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}) =$$
$$= (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_3) \cup (Y_1^\delta \times Z_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha,$$

if $Y_5^\delta = Y_5^\alpha$, $Y_4^\delta = Y_3^\alpha$, $Y_1^\delta = Y_1^\alpha$, and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (a) of Lemma 2.2 is proved.

(b) Let quasinormal representations of binary relations δ and β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = ((Z_4 \setminus Z_5) \times Z_5) \cup (Z_5 \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_2) \cup \big(((Z_2 \cap Z_1) \setminus Z_3) \times Z_1\big) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_5^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$,

$$(Z_4 \setminus Z_5) \cup Z_5 \cup (Z_2 \setminus Z_1) \cup ((Z_2 \cap Z_1) \setminus Z_3) \cup (X \setminus \breve{D})$$
$$= P_5 \cup (P_0 \cup P_2 \cup P_4) \cup P_1 \cup P_3 \cup (X \setminus \breve{D}) = \breve{D} \cup (X \setminus \breve{D}) = X$$

(see Eqs. (2.1) and (2.2)). Then $\delta, \beta \in B_0$ and

$$Z_5\beta = Z_4, \quad Z_4\beta = Z_4 \cup Z_5 = Z_3, \quad Z_2\beta = (P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5)\beta = \breve{D},$$
$$Z_1\beta = (P_0 \cup P_2 \cup P_3 \cup P_4)\beta = Z_1, \quad \breve{D}\beta = \breve{D},$$
$$\alpha = \delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)$$
$$= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times Z_3) \cup (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}) =$$
$$= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times Z_3) \cup (Y_1^\delta \times Z_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha,$$

if $Y_5^\delta = Y_4^\alpha$, $Y_4^\delta = Y_3^\alpha$, $Y_1^\delta = Y_1^\alpha$, and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (b) of Lemma 2.2 is proved. □

Lemma 2.3. *Let $D \in \Sigma_{1.0}(X, 6)$. Then the following assertions hold*:

(a) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_5^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(b) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(c) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(d) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(e) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(f) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(g) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$.

Proof. (a) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1),$$
$$\beta = (Z_5 \times Z_5) \cup ((Z_4 \setminus Z_5) \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_3) \cup \big((X \setminus (\breve{D} \setminus (Z_1 \setminus Z_3))) \times Z_1\big),$$

where $Y_5^\delta, Y_4^\delta, Y_1^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$,

$$\begin{aligned} &Z_5 \cup (Z_4 \setminus Z_5) \cup (Z_2 \setminus Z_1) \cap \big(X \setminus (\breve{D} \setminus (Z_1 \setminus Z_3))\big) \\ &\qquad = (P_0 \cup P_2 \cup P_4) \cup P_5 \cup P_1 \cup (X \setminus (\breve{D} \setminus P_3)) = (\breve{D} \setminus P_3) \cup (X \setminus (\breve{D} \setminus P_3)) = X \end{aligned}$$

and $Z_5\beta = Z_5$, $Z_4\beta = Z_5 \cup Z_4 = Z_3$, $Z_3\beta = Z_5 \cup Z_4 = Z_3$, $Z_1\beta = Z_5 \cup Z_4 \cup Z_1 = Z_1$,

$$\begin{aligned} \delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) \cup (Y_1^\delta \times Z_1\beta) \\ &= (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1) \\ &= (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_3) \cup (Y_1^\delta \times Z_1) = \alpha, \end{aligned}$$

if $Y_5^\delta = Y_5^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_3^\alpha$ and $Y_1^\delta = Y_1^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (a) of Lemma 2.3 is proved.

(b) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1),$$
$$\beta = ((Z_4 \setminus Z_5) \times Z_5) \cup (Z_5 \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_3) \cup \big((X \setminus (\breve{D} \setminus (Z_1 \setminus Z_3))) \times Z_1\big),$$

where $Y_5^\delta, Y_4^\delta, Y_1^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_5\beta = Z_4$, $Z_4\beta = Z_5 \cup Z_4 = Z_3$, $Z_3\beta = Z_5 \cup Z_4 = Z_3$, $Z_1\beta = Z_5 \cup Z_4 \cup Z_1 = Z_1$,

$$\begin{aligned} \delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) \cup (Y_1^\delta \times Z_1\beta) \\ &= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1) \\ &= (Y_5^\delta \times Z_4) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_3) \cup (Y_1^\delta \times Z_1) = \alpha, \end{aligned}$$

if $Y_5^\delta = Y_4^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_3^\alpha$, and $Y_1^\delta = Y_1^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (b) of Lemma 2.3 is proved.

(c) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = (Z_5 \times Z_5) \cup ((Z_4 \setminus Z_5) \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_3) \cup \big((X \setminus (\breve{D} \setminus (Z_1 \setminus Z_3))) \times \breve{D}\big),$$

where $Y_5^\delta, Y_4^\delta, Y_0^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$,

$$Z_5 \cup (Z_4 \setminus Z_5) \cup (Z_2 \setminus Z_1) \cap \big(X \setminus (\breve{D} \setminus (Z_1 \setminus Z_3))\big)$$
$$= (P_0 \cup P_2 \cup P_4) \cup P_5 \cup P_1 \cup (X \setminus (\breve{D} \setminus P_3)) = (\breve{D} \setminus P_3) \cup (X \setminus (\breve{D} \setminus P_3)) = X,$$

and $Z_5\beta = Z_5$, $Z_4\beta = Z_5 \cup Z_4 = Z_3$, $Z_3\beta = Z_5 \cup Z_4 = Z_3$, $\breve{D}\beta = Z_5 \cup Z_4 \cup Z_3 \cup \breve{D} = \breve{D}$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) \cup (Y_0^\delta \times \breve{D}\beta)$$
$$= (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) \cup (Y_0^\delta \times \breve{D})$$
$$= (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_3) \cup (Y_0^\delta \times \breve{D}) = \alpha$$

if $Y_5^\delta = Y_5^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_3^\alpha$, and $Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (c) of Lemma 2.3 is proved.

(d) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = ((Z_4 \setminus Z_5) \times Z_5) \cup (Z_5 \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_3) \cup \big(\big(X \setminus (\breve{D} \setminus (Z_1 \setminus Z_3))\big) \times \breve{D}\big),$$

where $Y_5^\delta, Y_4^\delta, Y_0^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_5\beta = Z_4$, $Z_4\beta = Z_5 \cup Z_4 = Z_3$, $Z_3\beta = Z_5 \cup Z_4 = Z_3$, $\breve{D}\beta = Z_5 \cup Z_4 \cup Z_3 \cup \breve{D} = \breve{D}$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) \cup (Y_0^\delta \times \breve{D}\beta) =$$
$$= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) \cup (Y_0^\delta \times \breve{D})$$
$$= (Y_5^\delta \times Z_4) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_3) \cup (Y_0^\delta \times \breve{D}) = \alpha$$

if $Y_5^\delta = Y_4^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_3^\alpha$, and $Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (d) of Lemma 2.3 is proved.

(e) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = ((Z_2 \cap Z_1) \times Z_5) \cup ((Z_2 \setminus Z_1) \times Z_2) \cup ((Z_1 \setminus Z_2) \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_5^\delta, Y_2^\delta, Y_1^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$,

$$(Z_2 \cap Z_1) \cup (Z_2 \setminus Z_1) \cup (Z_1 \setminus Z_2) \cap (X \setminus \breve{D})$$
$$= (P_0 \cup P_3 \cup P_4 \cup P_5) \cup P_1 \cup P_2 \cup (X \setminus \breve{D}) = \breve{D} \cup (X \setminus \breve{D}) = X$$

and $Z_5\beta = Z_5$, $Z_2\beta = Z_5 \cup Z_2 \cup Z_1 = \breve{D}$, $Z_1\beta = Z_5 \cup Z_1 = Z_1$, $\breve{D}\beta = \breve{D}$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)$$
$$= (Y_5^\delta \times Z_5) \cup (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})$$
$$= (Y_5^\delta \times Z_5) \cup (Y_1^\delta \times Z_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha$$

if $Y_5^\delta = Y_5^\alpha$, $Y_1^\delta = Y_1^\alpha$, and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (e) of Lemma 2.3 is proved.

(f) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = ((Z_2 \cap Z_1) \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_2) \cup ((Z_1 \setminus Z_2) \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_5^\delta, Y_2^\delta, Y_1^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_5\beta = Z_4$, $Z_2\beta = Z_5 \cup Z_2 \cup Z_1 = \breve{D}$, $Z_1\beta = Z_5 \cup Z_1 = Z_1$, $\breve{D}\beta = \breve{D}$,

$$\begin{aligned}\delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)\\ &= (Y_5^\delta \times Z_4) \cup (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})\\ &= (Y_5^\delta \times Z_4) \cup (Y_1^\delta \times Z_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha\end{aligned}$$

if $Y_5^\delta = Y_4^\alpha$, $Y_1^\delta = Y_1^\alpha$ and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (f) of Lemma 2.3 is proved.

(g) Let quasinormal representations of binary relations δ, β have the form

$$\begin{aligned}\delta &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),\\ \beta &= (Z_3 \times Z_3) \cup ((Z_2 \setminus Z_1) \times Z_2) \cup ((Z_1 \setminus Z_3) \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D}),\end{aligned}$$

where $Y_3^\delta, Y_2^\delta, Y_1^\delta \in \{\varnothing\}$. Then, by the definition of the set $\mathfrak{A}_0$, we have that $\delta, \beta \in B(\mathfrak{A}_0)$,

$$Z_3 \cup (Z_2 \setminus Z_1) \cup (Z_1 \setminus Z_3) \cap (X \setminus \breve{D}) = (P_0 \cup P_2 \cup P_4 \cup P_5) \cup P_1 \cup P_3 \cup (X \setminus \breve{D}) = \breve{D} \cup (X \setminus \breve{D}) = X$$

and $Z_3\beta = Z_3$, $Z_2\beta = Z_3 \cup Z_2 \cup Z_1 = \breve{D}$, $Z_1\beta = Z_3 \cup Z_1 = Z_1$, $\breve{D}\beta = \breve{D}$,

$$\begin{aligned}\delta \circ \beta &= (Y_3^\delta \times Z_3\beta) \cup (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)\\ &= (Y_3^\delta \times Z_3) \cup (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D})\\ &= (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha,\end{aligned}$$

if $Y_3^\delta = Y_3^\alpha$, $Y_1^\delta = Y_1^\alpha$, and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (g) of Lemma 2.3 is proved. □

Lemma 2.4. *Let $D \in \Sigma_{1.0}(X, 6)$. Then the following assertions hold*:

(a) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3),$$

where $Y_5^\alpha, Y_3^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(b) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_1^\alpha \times Z_1),$$

where $Y_5^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(c) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3),$$

where $Y_4^\alpha, Y_3^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(d) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_1^\alpha \times Z_1),$$

where $Y_4^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(e) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(f) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$;

(g) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(h) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(i) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(j) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$.

Proof. (a) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = (Z_5 \times Z_5) \cup ((Z_1 \setminus Z_5) \times Z_4) \cup \big((X \setminus (\breve{D} \setminus (Z_2 \setminus Z_1))) \times Z_3\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then we have $\delta, \beta \in B(\mathfrak{A}_0)$,

$$Z_5 \cup (Z_1 \setminus Z_5) \cup \big(X \setminus (\breve{D} \setminus (Z_2 \setminus Z_1))\big) = (P_0 \cup P_2 \cup P_4) \cup (P_3 \cup P_5) \cup (X \setminus (\breve{D} \setminus P_1)) = (\breve{D} \setminus P_1) \cup (X \setminus (\breve{D} \setminus P_1)) = X$$

and $Z_5\beta = Z_5$, $Z_4\beta = Z_5 \cup Z_4 = Z_3$, $Z_3\beta = Z_5 \cup Z_4 = Z_3$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) = (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_3) = \alpha$$

if $Y_5^\delta = Y_5^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_3^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (a) of Lemma 2.4 is proved.

(b) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = (Z_5 \times Z_5) \cup ((Z_2 \setminus Z_1) \times Z_3) \cup \big((X \setminus (\breve{D} \setminus (Z_1 \setminus Z_5))) \times Z_1\big),$$

where $Y_4^\delta, Y_1^\delta \in \{\varnothing\}$. Then item (a) of Lemma 2.3 implies that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$,

$$Z_5 \cup (Z_2 \setminus Z_1) \cup \big(X \setminus (\breve{D} \setminus (Z_1 \setminus Z_5))\big) = (P_0 \cup P_2 \cup P_4) \cup P_1 \cup \big(X \setminus (\breve{D} \cup (P_3 \cup P_5))\big) = (\breve{D} \cup (P_3 \cup P_5)) \cup (X \setminus (\breve{D} \cup (P_3 \cup P_5))) = X,$$

and $Z_5\beta = Z_5$, $Z_4\beta = Z_5 \cup Z_1 = Z_1$, $Z_3\beta = Z_5 \cup Z_1 = Z_1$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_1) \cup (Y_3^\delta \times Z_1) = (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_1) = \alpha$$

if $Y_5^\delta = Y_5^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_1^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (b) of Lemma 2.4 is proved.

(c) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = ((Z_1 \setminus Z_5) \times Z_5) \cup (Z_5 \times Z_4) \cup \big((X \setminus (\breve{D} \setminus (Z_2 \setminus Z_1))) \times Z_3\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_5\beta = Z_4$, $Z_4\beta = Z_5 \cup Z_4 = Z_3$, $Z_3\beta = Z_5 \cup Z_4 = Z_3$,

$$\begin{aligned}\delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta)\\ &= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) = (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_3) = \alpha\end{aligned}$$

if $Y_5^\delta = Y_4^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_3^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (c) of Lemma 2.4 is proved.

(d) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = (Z_5 \times Z_4) \cup ((Z_2 \setminus Z_1) \times Z_3) \cup \big((X \setminus (\breve{D} \setminus (Z_1 \setminus Z_5))) \times Z_1\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then item (b) of Lemma 2.3 implies that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$, and $Z_5\beta = Z_4$, $Z_4\beta = Z_5 \cup Z_1 = Z_1$, $Z_3\beta = Z_5 \cup Z_1 = Z_1$,

$$\begin{aligned}\delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta)\\ &= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times Z_1) \cup (Y_3^\delta \times Z_1) = (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_1) = \alpha\end{aligned}$$

if $Y_5^\delta = Y_4^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_1^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (d) of Lemma 2.4 is proved.

(e) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = ((Z_2 \setminus Z_1) \times Z_4) \cup (Z_5 \times Z_3) \cup \big((X \setminus (\breve{D} \setminus (Z_1 \setminus Z_5))) \times Z_1\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then item (b) of Lemma 2.3 implies that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$, and $Z_5\beta = Z_3$, $Z_4\beta = Z_5 \cup Z_1 = Z_1$, $Z_3\beta = Z_5 \cup Z_1 = Z_1$,

$$\begin{aligned}\delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta)\\ &= (Y_5^\delta \times Z_3) \cup (Y_4^\delta \times Z_1) \cup (Y_3^\delta \times Z_1) = (Y_5^\delta \times Z_3) \cup ((Y_4^\delta \cup Y_3^\delta) \times Z_1) = \alpha\end{aligned}$$

if $Y_5^\delta = Y_3^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_1^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (e) of Lemma 2.4 is proved.

(f) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = ((P_0 \cup P_2 \cup P_4) \times Z_5) \cup (P_1 \times Z_1) \cup \big((X \setminus (\breve{D} \setminus (P_3 \cup P_5))) \times \breve{D}\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then item (e) of Lemma 2.3 implies that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$, and $Z_5\beta = Z_5$, $Z_4\beta = Z_5 \cup \breve{D} = \breve{D}$, $Z_3\beta = Z_5 \cup \breve{D} = \breve{D}$,

$$\begin{aligned}\delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta)\\ &= (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times \breve{D}) \cup (Y_3^\delta \times \breve{D}) = (Y_5^\delta \times Z_5) \cup ((Y_4^\delta \cup Y_3^\delta) \times \breve{D}) = \alpha\end{aligned}$$

if $Y_5^\delta = Y_5^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (f) of Lemma 2.4 is proved.

(g) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = ((P_0 \cup P_2 \cup P_4) \times Z_4) \cup (P_1 \times Z_1) \cup \big((X \setminus (\breve{D} \setminus (P_3 \cup P_5))) \times \breve{D}\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then item (f) of Lemma 2.3 implies that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$, and $Z_5\beta = Z_4$, $Z_4\beta = Z_5 \cup \breve{D} = \breve{D}$, $Z_3\beta = Z_5 \cup \breve{D} = \breve{D}$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta)$$
$$= (Y_5^\delta \times Z_4) \cup (Y_4^\delta \times \breve{D}) \cup (Y_3^\delta \times \breve{D}) = (Y_5^\delta \times Z_4) \cup ((Y_4^\delta \cup Y_3^\delta) \times \breve{D}) = \alpha$$

if $Y_5^\delta = Y_4^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (g) of Lemma 2.4 is proved.

(h) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = ((P_0 \cup P_2 \cup P_4) \times Z_3) \cup (P_1 \times Z_1) \cup \big((X \setminus (\breve{D} \setminus (P_3 \cup P_5))) \times \breve{D}\big),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then item (g) of Lemma 2.3 implies that β is generated by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$, and $Z_5\beta = Z_3$, $Z_4\beta = Z_5 \cup \breve{D} = \breve{D}$, $Z_3\beta = Z_5 \cup \breve{D} = \breve{D}$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta)$$
$$= (Y_5^\delta \times Z_3) \cup (Y_4^\delta \times \breve{D}) \cup (Y_3^\delta \times \breve{D}) = (Y_5^\delta \times Z_3) \cup ((Y_4^\delta \cup Y_3^\delta) \times \breve{D}) = \alpha$$

if $Y_5^\delta = Y_3^\alpha$, $Y_4^\delta \cup Y_3^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_4^\delta \cup Y_3^\delta| \geq 1$ ($|Y_3^\delta| \geq 0$ by the assumption). Item (h) of Lemma 2.4 is proved.

(i) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = ((P_0 \cup P_1 \cup P_3 \cup P_4 \cup P_5) \times Z_2) \cup (P_2 \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_2^\delta, Y_1^\delta \in \{\varnothing\}$. Then we have $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_2\beta = Z_2$, $Z_1\beta = Z_2 \cup Z_1 = \breve{D}$, $\breve{D}\beta = \breve{D}$,

$$\delta \circ \beta = (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)$$
$$= (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times \breve{D}) \cup (Y_0^\delta \times \breve{D}) = (Y_2^\delta \times Z_2) \cup ((Y_1^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha$$

if $Y_2^\delta = Y_2^\alpha$, $Y_1^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_1^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (i) of Lemma 2.4 is proved.

(j) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = \big((P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5) \times Z_1\big) \cup (P_1 \times Z_2) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_2^\delta, Y_1^\delta \in \{\varnothing\}$. Then we have $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_1\beta = Z_1$, $Z_2\beta = Z_1 \cup Z_2 = \breve{D}$, $\breve{D}\beta = \breve{D}$,

$$\delta \circ \beta = (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)$$
$$= (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}) = (Y_1^\delta \times Z_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times \breve{D}) = \alpha$$

if $Y_1^\delta = Y_1^\alpha$, $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. The last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by the assumption). Item (j) of Lemma 2.4 is proved. □

Lemma 2.5. *Let $D \in \Sigma_{1.0}(X, 6)$. Then the following assertions hold.*

(a) *Let $T' \in \{Z_3, Z_1, \breve{D}\}$ and $\alpha = X \times T'$, then α is generated by elements of the elements of the set $B(\mathfrak{A}_0)$.*

(b) *If $|X \setminus \breve{D}| \geq 1$ and $T \in \{Z_5, Z_4, Z_2\}$, then the binary relation $\alpha = X \times T$ is generated by elements of the elements of the set $B(\mathfrak{A}_0)$.*

(c) *If $X = \breve{D}$ and $T \in \{Z_5, Z_4, Z_2\}$, then the binary relation $\alpha = X \times T$ is an external element for the semigroup $B_X(D)$.*

Proof. Item (a).

(1) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_2^\delta \times Z_2) \cup (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}),$$
$$\beta = (Z_5 \times Z_2) \cup ((\breve{D} \setminus Z_5) \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_2^\delta, Y_1^\delta \in \{\varnothing\}$. Then we have $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_2\beta = Z_2 \cup Z_1 = \breve{D}$, $Z_1\beta = Z_2 \cup Z_1 = \breve{D}$, $\breve{D}\beta = \breve{D}$,

$$\delta \circ \beta = (Y_2^\delta \times Z_2\beta) \cup (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta)$$
$$= (Y_2^\delta \times \breve{D}) \cup (Y_1^\delta \times \breve{D}) \cup (Y_0^\delta \times \breve{D}) = (Y_2^\delta \times Z_2) = X \times \overline{D} = \alpha,$$

(2) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1),$$
$$\beta = (Z_5 \times Z_3) \cup ((\breve{D} \setminus Z_5) \times Z_1) \cup ((X \setminus \breve{D}) \times \breve{D}),$$

where $Y_3^\delta, Y_1^\delta \in \{\varnothing\}$. Then, by item (e) of Lemma 2.4 and by item (g) of Lemma 2.3, binary relations δ and β are generated by elements of the set $B(\mathfrak{A}_0)$ and $Z_3\beta = Z_3 \cup Z_1 = Z_1$, $Z_1\beta = Z_3 \cup Z_1 = Z_1$,

$$\delta \circ \beta = (Y_3^\delta \times Z_3\beta) \cup (Y_1^\delta \times Z_1\beta) = (Y_3^\delta \times Z_1) \cup (Y_1^\delta \times Z_1) = X \times Z_1 = \alpha,$$

(3) Let quasinormal representations of binary relations δ, β have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3),$$
$$\beta = ((Z_5 \cap Z_4) \times Z_5) \cup ((\breve{D} \setminus (Z_5 \cap Z_4)) \times Z_4) \cup ((X \setminus \breve{D}) \times Z_3),$$

where $Y_5^\delta, Y_4^\delta \in \{\varnothing\}$. Then we have $\delta, \beta \in B(\mathfrak{A}_0)$ and $Z_5\beta = Z_5 \cup Z_4 = Z_3$, $Z_4\beta = Z_4 \cup Z_5 = Z_3$, $Z_3\beta = Z_3$,

$$\delta \circ \beta = (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) = (Y_5^\delta \times Z_3) \cup (Y_4^\delta \times Z_3) \cup (Y_3^\delta \times Z_3) = X \times Z_3 = \alpha,$$

Item (a) of Lemma 2.5 is proved.

(b) Let a quasinormal representation of a binary relation δ have the form

$$\delta = (Y_5^\delta \times Z_5) \cup (Y_4^\delta \times Z_4) \cup (Y_3^\delta \times Z_3) \cup (Y_1^\delta \times Z_1),$$

where $Y_5^\delta, Y_4^\delta, Y_1^\delta \notin \{\varnothing\}$, then $\delta \in B(\mathfrak{A}_0) \setminus \{\alpha\}$. Let a quasinormal representation of a binary relation β have the form $\beta = (\breve{D} \times T) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times f(t'))$, where f is any mapping of the set $X \setminus \breve{D}$ to the set $\{Z_5, Z_4, Z_2\} \setminus \{T\}$. It is easy to see that $\beta \neq \alpha$ and two elements of the set $\{Z_4, Z_{3,}, Z_2\}$ belong to the semilattice $V(D, \beta)$, i.e., $\delta \in B(\mathfrak{A}_0) \setminus \{\alpha\}$. In this case, we have $Z_5\beta = Z_4\beta = Z_3\beta = Z_1\beta = T$,

$$\begin{aligned}\delta \circ \beta &= (Y_5^\delta \times Z_5\beta) \cup (Y_4^\delta \times Z_4\beta) \cup (Y_3^\delta \times Z_3\beta) \cup (Y_1^\delta \times Z_1\beta)\\ &= (Y_5^\delta \times T) \cup (Y_4^\delta \times T) \cup (Y_3^\delta \times T) \cup (Y_1^\delta \times T) = (Y_5^\delta \cup Y_4^\delta \cup Y_3^\delta \cup Y_1^\delta) \times T = X \times T = \alpha,\end{aligned}$$

since the representation of the binary relation δ is quasinormal. Thus, element α is generated by elements of the set $B(\mathfrak{A}_0)$. Item (b) of Lemma 2.5 is proved.

(c) Let $X = \breve{D}$, $\alpha = X \times T$ for some $T \in \{Z_5, Z_4, Z_2\}$ and $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_X(D) \setminus \{\alpha\}$. Then, from Eqs. (2.3) and (2.4), we obtain that

$$\begin{aligned}Z_5\beta = Z_4\beta = Z_3\beta = Z_2\beta = Z_1\beta = \breve{D}\beta = T,\\ P_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = P_5\beta = T,\end{aligned}$$

since T is a minimal element of the semilattice D.

Now, let a subquasinormal representation $\overline{\beta}$ of a binary relation β have the form

$$\overline{\beta} = \big((P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5) \times T\big) \cup \bigcup_{t' \in X \setminus \breve{D}} \big(\{t'\} \times \overline{\beta}_2(t')\big),$$

where $\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 & P_5 \\ T & T & T & T & T & T \end{pmatrix}$ is a normal mapping. But the complement mapping $\overline{\beta}_2$ is empty since $X \setminus \breve{D} = \varnothing$, i.e., in this case, the subquasinormal representation $\overline{\beta}$ of a binary relation β is uniquely defined. Therefore, we have $\beta = \overline{\beta} = X \times T = \alpha$, which contradicts the condition that $\beta \notin B_X(D) \setminus \alpha$.

Therefore, if $X = \breve{D}$ and $\alpha = X \times T$, for some $T \in \{Z_5, Z_4, Z_2\}$, then α is an external element of the semigroup $B_X(D)$. Item (c) of Lemma 2.5 is proved. □

Theorem 2.6. *Let* $D \in \Sigma_{1.0}(X, 6)$ *and*

$$\begin{aligned}\mathfrak{A}_0 = \Big\{&\{Z_5, Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_4, Z_3, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, Z_1, \breve{D}\},\\ &\{Z_5, Z_4, Z_3, Z_1\},\ \{Z_5, Z_4, Z_3, \breve{D}\},\ \{Z_5, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, \breve{D}\},\\ &\{Z_4, Z_2, Z_1, \breve{D}\},\ \{Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_5, Z_4, Z_3\},\ \{Z_5, Z_2, \breve{D}\},\\ &\{Z_4, Z_2, \breve{D}\},\ \{Z_3, Z_2, \breve{D}\},\ \{Z_2, Z_1, \breve{D}\}\Big\},\end{aligned}$$

$$B(\mathfrak{A}_0) = \Big\{\alpha \in B_X(D) \mid V(X^*, \alpha) \in \mathfrak{A}_0\Big\}, \qquad B_0 = \Big\{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\Big\}.$$

Then the following assertions hold:

(a) *if* $|X \setminus \breve{D}| \geq 1$, *then* $S_0 = B_0 \cup B(\mathfrak{A}_0)$ *is an irreducible generating set for the semigroup* $B_X(D)$;

(b) *if* $X = \breve{D}$, *then* $S_1 = B_0 \cup B(\mathfrak{A}_0) \cup \{X \times Z_5, X \times Z_4, X \times Z_2\}$ *is an irreducible generating set for the semigroup* $B_X(D)$.

Proof. Let $D \in \Sigma_{1.0}(X, 5)$ and $|X \setminus \breve{D}| \geq 1$. First, we prove that each element of the semigroup $B_X(D)$ is generated by elements of the set S_0. Indeed, let α be an arbitrary element of the semigroup $B_X(D)$. Then a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_2^\alpha \cup Y_1^\alpha \cup Y_0^\alpha = X$ and $Y_i^\alpha \cap Y_j^\alpha = \varnothing$ $(0 \leq i \neq j \leq 4)$. For $|V(X^*, \alpha)|$, we consider the following cases:

(1) $|V(X^*, \alpha)| = 6$. Then $\alpha \in B_0$ and $B_0 \subset S_0$ by the definition of the set S_0.

(2) $|V(X^*,\alpha)|=5$. Then

$$\mathfrak{A}_5=\big\{\{Z_5,Z_4,Z_3,Z_2,\breve{D}\},\{Z_5,Z_4,Z_3,Z_1,\breve{D}\},\{Z_5,Z_3,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,Z_1,\breve{D}\}\big\}\subset\mathfrak{A}_0$$

i.e., $\alpha\in B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

(3) $|V(X^*,\alpha)|=4$. Then we have

$$\mathfrak{A}_4=\big\{\{Z_5,Z_4,Z_3,Z_1\},\{Z_5,Z_4,Z_3,\breve{D}\},\{Z_5,Z_3,Z_2,\breve{D}\},\\ \{Z_5,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\\ \{Z_3,Z_2,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1,\breve{D}\}\big\}.$$

By the definition of the set $\mathfrak{A}_0$, we have

$$V(X^*,\alpha)\in\big\{\{Z_5,Z_4,Z_3,Z_1\},\{Z_5,Z_4,Z_3,\breve{D}\},\{Z_5,Z_3,Z_2,\breve{D}\},\\ \{Z_5,Z_2,Z_1,\breve{D}\},\{Z_4,Z_3,Z_2,\breve{D}\},\{Z_4,Z_2,Z_1,\breve{D}\},\{Z_3,Z_2,Z_1,\breve{D}\}\big\},$$

i.e., in this case, $\alpha\in B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

If $V(X^*,\alpha)\in\{\{Z_5,Z_3,Z_1,\breve{D}\},\{Z_4,Z_3,Z_1,\breve{D}\}\}$, then items (a) and (b) of Lemma 2.2 imply that the element α is generated by elements of $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

(4) $|V(X^*,\alpha)|=3$. Then we have

$$V(X^*,\alpha)\in\mathfrak{A}_3=\big\{\{Z_5,Z_4,Z_3\},\{Z_5,Z_2,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\\ \{Z_2,Z_1,\breve{D}\},\{Z_5,Z_3,Z_1\},\{Z_4,Z_3,Z_1\},\{Z_5,Z_3,\breve{D}\},\\ \{Z_5,Z_1,\breve{D}\},\{Z_4,Z_3,\breve{D}\}\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_2,\breve{D}\},\{Z_3,Z_1,\breve{D}\}\big\}.$$

By the definition of the set $\mathfrak{A}_0$, we have

$$\big\{\{Z_5,Z_4,Z_3\},\{Z_5,Z_2,\breve{D}\},\{Z_4,Z_2,\breve{D}\},\{Z_3,Z_2,\breve{D}\},\{Z_2,Z_1,\breve{D}\}\big\}\subset\mathfrak{A}_0,$$

i.e., in this case, $\alpha\in B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

If

$$V(X^*,\alpha)\in\big\{\{Z_5,Z_3,Z_1\},\{Z_4,Z_3,Z_1\}\{Z_5,Z_3,\breve{D}\},\\ \{Z_5,Z_1,\breve{D}\},\{Z_4,Z_3,\breve{D}\},\{Z_4,Z_1,\breve{D}\},\{Z_3,Z_1,\breve{D}\}\big\},$$

then items (a)–(g) of Lemma 2.3 imply that the element α is generated by elements of $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

(5) $|V(X^*,\alpha)|=2$. Then we have

$$V(X^*,\alpha)\in\mathfrak{A}_2=\big\{\{Z_5,Z_3\},\{Z_5,Z_1\},\{Z_4,Z_3\},\{Z_4,Z_1\},\\ \{Z_3,Z_1\},\{Z_5,\breve{D}\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\}\big\}.$$

Then items (a)–(j) of Lemma 2.4 imply that the element α is generated by elements of $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

(6) $|V(X^*,\alpha)|=1$. Then we have

$$V(X^*,\alpha)\in\mathfrak{A}_1=\big\{\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\big\}.$$

If $V(X^*,\alpha)\in\{\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\}$, then item (a) of Lemma 2.5 implies that the element α is generated by elements of $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by the definition of the set S_0.

Thus, we have that S_0 is a generating set for the semigroup $B_X(D)$.

If $|X\setminus\breve{D}|\geq 1$, then the set S_0 is an irreducible generating set for the semigroup $B_X(D)$ since S_0 is a set of external elements of the semigroup $B_X(D)$. Item (a) of Theorem 2.6 is proved.

(b) Now, let $D \in \Sigma_{1.0}(X,6)$ and $X = \breve{D}$. First, we prove that each element of the semigroup $B_X(D)$ is generated by elements of the set S_1. The cases (1), (2), (3), (4), and (5) are proved similarly to the cases (1), (2), (3), (4), and (5) given above. We consider the case, where

$$V(X^*,\alpha) \in \mathfrak{A}_1 = \big\{\{Z_5\},\{Z_4\},\{Z_3\},\{Z_2\},\{Z_1\},\{\breve{D}\}\big\}.$$

If $V(X^*,\alpha) \in \{\{Z_3\},\{Z_1\},\{\breve{D}\}\}$, then item (a) of Lemma 2.5 implies that the element α is generated by elements of $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0) \subset S_0$ by the definition of the set S_1.

If $V(X^*,\alpha) \in \{\{Z_5\},\{Z_4\},\{Z_2\}\}$, then $\alpha \in S_1$ by the definition of the set S_1.

Thus, we have that S_1 is a generating set for the semigroup $B_X(D)$.

If $X = \breve{D}$, then the set S_1 is an irreducible generating set for the semigroup $B_X(D)$ since S_1 is a set of external elements of the semigroup $B_X(D)$. Item (b) of Theorem 2.6 is proved. □

Theorem 2.7. *Let* $D = \{Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_{1.0}(X,6)$ *and*

$$\mathfrak{A}_0 = \Big\{\{Z_5,Z_4,Z_3,Z_2,\breve{D}\},\ \{Z_5,Z_4,Z_3,Z_1,\breve{D}\},\ \{Z_5,Z_3,Z_2,Z_1,\breve{D}\},\ \{Z_4,Z_3,Z_2,Z_1,\breve{D}\},$$
$$\{Z_5,Z_4,Z_3,Z_1\},\ \{Z_5,Z_4,Z_3,\breve{D}\},\ \{Z_5,Z_3,Z_2,\breve{D}\},\ \{Z_5,Z_2,Z_1,\breve{D}\},\ \{Z_4,Z_3,Z_2,\breve{D}\},\ \{Z_4,Z_2,Z_1,\breve{D}\},$$
$$\{Z_3,Z_2,Z_1,\breve{D}\},\ \{Z_5,Z_4,Z_3\},\ \{Z_5,Z_2,\breve{D}\},\ \{Z_4,Z_2,\breve{D}\},\ \{Z_3,Z_2,\breve{D}\},\{Z_2,Z_1,\breve{D}\}\Big\},$$

$$B(\mathfrak{A}_0) = \Big\{\alpha \in B_X(D) \ \Big|\ V(X^*,\alpha) \in \mathfrak{A}_0\Big\}, \quad B_0 = \Big\{\alpha \in B_X(D) \ \Big|\ V(X^*,\alpha) = D\Big\}.$$

Then the following assertions hold:

(a) *if* $|X \setminus \breve{D}| \geq 1$, *then the number* $|S_0|$ *of elements of the set*

$$S_0 = B_0 \cup B(\mathfrak{A}_0)$$

is equal to

$$|S_0| = 6^n - 4^n - 2\cdot 3^n + 2\cdot 2^n;$$

(b) *if* $X = \breve{D}$, *then the number* $|S_1|$ *of elements of the set*

$$S_1 = B_0 \cup B(\mathfrak{A}_0) \cup \big\{X\times Z_4, X\times Z_3, X\times Z_2\big\}$$

is equal to

$$|S_1| = 6^n - 4^n - 2\cdot 3^n + 2\cdot 2^n + 3.$$

Proof. Let the number of elements of the set X be equal to n, i.e., $|X| = n$. Let $S_n = \{\varphi_1, \varphi_2, \dots, \varphi_{n!}\}$ be the group of all one-to-one mappings of the set $M = \{1,2,\dots,n\}$ to itself and let $\varphi_{i_1}, \varphi_{i_2}, \dots, \varphi_{i_m}$ $(m \leq n)$ be arbitrary elements of the group S_n, $Y_{\varphi_1}, Y_{\varphi_2}, \dots, Y_{\varphi_m}$ be an arbitrary partitioning of the set X. We denote by k_n^m the number of elements of the set $\{Y_{\varphi_1}, Y_{\varphi_2}, \dots, Y_{\varphi_m}\}$. It is well known that

$$k_n^m = \sum_{i=1}^{m} \frac{(-1)^{m+i}}{(i-1)!\cdot(m-i)!}\cdot i^{n-1}.$$

If $m = 2,3,4,5,6,$ then we have

$$k_n^2 = 2^{n-1} - 1, \quad k_n^3 = \frac{1}{2}\cdot 3^{n-1} - 2^{n-1} + \frac{1}{2},$$
$$k_n^4 = \frac{1}{6}\cdot 4^{n-1} - \frac{1}{2}\cdot 3^{n-1} + \frac{1}{2}\cdot 2^{n-1} - \frac{1}{6},$$
$$k_n^5 = \frac{1}{24}\cdot 5^{n-1} - \frac{1}{6}\cdot 4^{n-1} + \frac{1}{4}\cdot 3^{n-1} - \frac{1}{6}\cdot 2^{n-1} + \frac{1}{24},$$
$$k_n^6 = \frac{1}{120}\cdot 6^{n-1} - \frac{1}{24}\cdot 5^{n-1} + \frac{1}{12}\cdot 4^{n-1} - \frac{1}{12}\cdot 3^{n-1} + \frac{1}{24}\cdot 2^{n-1} - \frac{1}{120}.$$

Let Y_{φ_1}, Y_{φ_2} be any two elements of the partitioning of the set X and

$$\overline{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2),$$

where $T_1, T_2 \in D$ and $T_1 \neq T_2$. Then number of different binary relations $\overline{\beta}$ of the semigroup $B_X(D)$ is equal to

$$2 \cdot k_n^2 = 2^n - 2. \tag{2.5}$$

Let Y_{φ_1}, Y_{φ_2}, and Y_{φ_3} be any three-element partitioning of the set X and $\overline{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2) \cup (Y_{\varphi_3} \times T_3)$, where T_1, T_2, and T_3 are pairwise different elements of a given semilattice D. Then the number of distinct binary relations $\overline{\beta}$ of the semigroup $B_X(D)$ is equal to

$$6 \cdot k_n^3 = 3^n - 3 \cdot 2^n + 3. \tag{2.6}$$

Let Y_{φ_1}, Y_{φ_2}, Y_{φ_3}, and Y_{φ_4} be any four-element partitioning of the set X and

$$\overline{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2) \cup (Y_{\varphi_3} \times T_3) \cup (Y_{\varphi_4} \times T_4),$$

where T_1, T_2, T_3, and T_4 are pairwise different elements of a given semilattice D. Then the number of distinct binary relations $\overline{\beta}$ of the semigroup $B_X(D)$ is equal to

$$24 \cdot k_n^4 = 4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4. \tag{2.7}$$

Let Y_{φ_1}, Y_{φ_2}, Y_{φ_3}, Y_{φ_4}, and Y_{φ_5} be any four-element partitioning of the set X and

$$\overline{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2) \cup (Y_{\varphi_3} \times T_3) \cup (Y_{\varphi_4} \times T_4) \cup (Y_{\varphi_5} \times T_5),$$

where T_1, T_2, T_3, T_4, and T_5 are pairwise different elements of a given semilattice D. Then the number of distinct binary relations $\overline{\beta}$ of the semigroup $B_X(D)$ is equal to

$$120 \cdot k_n^5 = 5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5. \tag{2.8}$$

If Y_{φ_1}, Y_{φ_2}, Y_{φ_3}, Y_{φ_4}, Y_{φ_5}, and Y_{φ_6} is any six-element partitioning of the set X and

$$\overline{\beta} = (Y_{\varphi_1} \times T_1) \cup (Y_{\varphi_2} \times T_2) \cup (Y_{\varphi_3} \times T_3) \cup (Y_{\varphi_4} \times T_4) \cup (Y_{\varphi_5} \times T_5) \cup (Y_{\varphi_6} \times T_6),$$

where T_1, T_2, T_3, T_4, T_5, and T_6 are pairwise different elements of a given semilattice D. Then the number of distinct binary relations $\overline{\beta}$ of the semigroup $B_X(D)$ is equal to

$$720 \cdot k_n^6 = 6^n - 6 \cdot 5^n + 15 \cdot 4^n - 20 \cdot 3^n + 15 \cdot 2^n - 6. \tag{2.9}$$

If $\alpha \in B_0$, then a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and the system Y_5^α, Y_4^α, Y_2^α, Y_1^α, or the system Y_5^δ, Y_4^δ, Y_3^δ, Y_2^δ, Y_1^δ, or the system Y_5^δ, Y_4^δ, Y_2^δ, Y_1^δ, Y_0^δ, or the system Y_5^δ, Y_4^δ, Y_3^δ, Y_2^δ, Y_1^δ, Y_0^δ is a partitioning of the set X.

Let the system Y_5^δ, Y_4^δ, Y_2^δ, or the system Y_5^α, Y_4^α, Y_2^α, Y_1^α, or the system Y_5^δ, Y_4^δ, Y_3^δ, Y_2^δ, Y_1^δ or the system Y_5^δ, Y_4^δ, Y_2^δ, Y_1^δ, Y_0^δ, or the system Y_5^α, Y_4^α, Y_3^α, Y_2^α, Y_1^α, Y_0^α be a partitioning of the set X. This and Eqs. (2.6)–(2.9) imply that

$$\begin{aligned} |B_0| = (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4) + 2 \cdot (5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5) \\ + (6^n - 6 \cdot 5^n + 15 \cdot 4^n - 20 \cdot 3^n + 15 \cdot 2^n - 6) = 6^n - 4 \cdot 5^n + 6 \cdot 4^n - 4 \cdot 3^n + 2^n. \end{aligned}$$

Now, we find the number of elements of the set $B(\mathfrak{A}_0)$. We have the following.

(1) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_4^\alpha, Y_2^\alpha \notin \{\varnothing\}$, then the partitioning of the set X has following form: Y_5^δ, Y_4^δ, Y_3^δ, Y_2^δ, Y_0^δ, or Y_5^δ, Y_4^δ, Y_2^δ, Y_0^δ, or Y_5^α, Y_4^α, Y_3^α, Y_2^α, or Y_5^α, Y_4^α, Y_2^α and, from Eqs. (2.6)–(2.8), the number of binary relations α is equal to

$$m_1 = (5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5) + 2\cdot(4^n - 4\cdot 3^n + 6\cdot 2^n - 4) + (3^n - 3\cdot 2^n + 3) = 5^n - 3\cdot 4^n + 3\cdot 3^n - 2^n.$$

(2) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_4^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_4^δ, Y_3^δ, Y_1^δ, Y_0^δ, or Y_5^δ, Y_4^δ, Y_1^δ, Y_0^δ, or Y_5^δ, Y_4^δ, Y_3^δ, Y_0^δ or Y_5^α, Y_4^α, Y_1^α, and, from Eqs. (2.6)–(2.8), the number of binary relations α is equal to

$$m_2 = (5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5) + 2\cdot(4^n - 4\cdot 3^n + 6\cdot 2^n - 4) + (3^n - 3\cdot 2^n + 3) = 5^n - 3\cdot 4^n + 3\cdot 3^n - 2^n.$$

(3) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\delta, Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_3^δ, Y_2^δ, Y_1^δ, Y_0^δ, or Y_5^δ, Y_3^δ, Y_2^δ, Y_1^δ, and, from Eqs. (2.7) and (2.8), the number of binary relations α is equal to

$$m_3 = (5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5) + (4^n - 4\cdot 3^n + 6\cdot 2^n - 4) = 5^n - 4\cdot 4^n + 6\cdot 3^n - 4\cdot 2^n + 1.$$

(4) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\delta, Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_4^δ, Y_3^δ, Y_2^δ, Y_1^δ, Y_0^δ, or Y_4^δ, Y_3^δ, Y_2^δ, Y_1^δ, and, from Eqs. (2.7) and (2.8), the number of binary relations α is equal to

$$m_4 = (5^n - 5\cdot 4^n + 10\cdot 3^n - 10\cdot 2^n + 5) + (4^n - 4\cdot 3^n + 6\cdot 2^n - 4) = 5^n - 4\cdot 4^n + 6\cdot 3^n - 4\cdot 2^n + 1.$$

(5) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_5^\delta, Y_4^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_4^δ, Y_3^δ, Y_1^δ, or Y_5^δ, Y_4^δ, Y_1^δ, and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_5 = (4^n - 4\cdot 3^n + 6\cdot 2^n - 4) + (3^n - 3\cdot 2^n + 3) = 4^n - 3\cdot 3^n + 3\cdot 2^n - 1.$$

(6) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\delta, Y_4^\delta, Y_0^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_4^δ, Y_3^δ, Y_0^δ, or Y_5^δ, Y_4^δ, Y_0^δ, and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_7 = (4^n - 4\cdot 3^n + 6\cdot 2^n - 4) + (3^n - 3\cdot 2^n + 3) = 4^n - 3\cdot 3^n + 3\cdot 2^n - 1.$$

(7) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\delta, Y_3^\delta, Y_2^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_3^δ, Y_2^δ, Y_0^δ, or Y_5^δ, Y_3^δ, Y_2^δ and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_8 = (4^n - 4\cdot 3^n + 6\cdot 2^n - 4) + (3^n - 3\cdot 2^n + 3) = 4^n - 3\cdot 3^n + 3\cdot 2^n - 1.$$

(8) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_2^δ, Y_1^δ, Y_0^δ, or Y_5^δ, Y_2^δ, Y_1^δ, and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_9 = (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4) + (3^n - 3 \cdot 2^n + 3) = 4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1.$$

(9) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\delta, Y_3^\delta, Y_2^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_4^δ, Y_3^δ, Y_2^δ, Y_0^δ, or Y_4^δ, Y_3^δ, Y_2^δ, and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_9 = (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4) + (3^n - 3 \cdot 2^n + 3) = 4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1.$$

(10) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_4^δ, Y_2^δ, Y_1^δ, Y_0^δ, or Y_4^δ, Y_2^δ, Y_1^δ, and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_{10} = (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4) + (3^n - 3 \cdot 2^n + 3) = 4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1.$$

(11) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_3^\delta, Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_3^δ, Y_2^δ, Y_1^δ, Y_0^δ, or Y_3^δ, Y_2^δ, Y_1^δ, and, from Eqs. (2.6) and (2.7), the number of binary relations α is equal to

$$m_{11} = (4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4) + (3^n - 3 \cdot 2^n + 3) = 4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1.$$

(12) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3),$$

where $Y_5^\delta, Y_4^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_4^δ, Y_3^δ, or Y_5^δ, Y_4^δ, and, from Eqs. (2.5) and (2.6), the number of binary relations α is equal to

$$m_{12} = (3^n - 3 \cdot 2^n + 3) + (2^n - 2) = 3^n - 2 \cdot 2^n + 1.$$

(13) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\delta, Y_2^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_5^δ, Y_2^δ, Y_0^δ, or Y_5^δ, Y_2^δ, and, from Eqs. (2.5) and (2.6), the number of binary relations α is equal to

$$m_{13} = (3^n - 3 \cdot 2^n + 3) + (2^n - 2) = 3^n - 2 \cdot 2^n + 1.$$

(14) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\delta, Y_2^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_4^δ, Y_2^δ, Y_0^δ, or Y_4^δ, Y_2^δ, and, from Eqs. (2.5) and (2.6), the number of binary relations α is equal to

$$m_{14} = (3^n - 3 \cdot 2^n + 3) + (2^n - 2) = 3^n - 2 \cdot 2^n + 1.$$

(15) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_3^\delta, Y_2^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_3^δ, Y_2^δ, Y_0^δ, or Y_3^δ, Y_2^δ, and, from Eqs. (2.5) and (2.6), the number of binary relations α is equal to

$$m_{15} = (3^n - 3 \cdot 2^n + 3) + (2^n - 2) = 3^n - 2 \cdot 2^n + 1.$$

(16) If a quasinormal representation of a binary relation α has the form

$$\alpha = (Y_2^\alpha \times Z_2) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_2^\delta, Y_1^\delta \notin \{\varnothing\}$, then partitionings of the set X have the following forms: Y_2^δ, Y_1^δ, Y_0^δ, or Y_2^δ, Y_1^δ, and, from Eqs. (2.5) and (2.6), the number of binary relations α is equal to

$$m_{16} = (3^n - 3 \cdot 2^n + 3) + (2^n - 2) = 3^n - 2 \cdot 2^n + 1.$$

This implies that

$$\begin{aligned} m = |B(\mathfrak{A}_0)| = \sum_{i=1}^{16} m_i &= 2 \cdot (5^n - 3 \cdot 4^n + 3 \cdot 3^n - 2^n) + 2 \cdot (5^n - 4 \cdot 4^n + 6 \cdot 3^n - 4 \cdot 2^n + 1) \\ &\quad + 7 \cdot (4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1) + 5 \cdot (3^n - 2 \cdot 2^n + 1) \\ &= (2 \cdot 5^n - 6 \cdot 4^n + 6 \cdot 3^n - 2 \cdot 2^n) + (2 \cdot 5^n - 8 \cdot 4^n + 12 \cdot 3^n - 8 \cdot 2^n + 2) \\ &\quad + (7 \cdot 4^n - 21 \cdot 3^n + 21 \cdot 2^n - 7) + (5 \cdot 3^n - 10 \cdot 2^n + 5) = 4 \cdot 5^n - 7 \cdot 4^n + 2 \cdot 3^n + 2^n. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |S_0| &= |B_0 \cup B(\mathfrak{A}_0)| = (6^n - 4 \cdot 5^n + 6 \cdot 4^n - 4 \cdot 3^n + 2^n) + (4 \cdot 5^n - 7 \cdot 4^n + 2 \cdot 3^n + 2^n) \\ &= 6^n - 4^n - 2 \cdot 3^n + 2 \cdot 2^n, \\ |S_1| &= \left|B_0 \cup B(\mathfrak{A}_0) \cup \{X \times Z_5, X \times Z_4, X \times Z_2\}\right| = 6^n - 4^n - 2 \cdot 3^n + 2 \cdot 2^n + 3 \end{aligned}$$

since

$$B_0 \cap B(\mathfrak{A}_0) = B_0 \cap \{X \times Z_5, X \times Z_4, X \times Z_2\} = B(\mathfrak{A}_0) \cap \{X \times Z_5, X \times Z_4, X \times Z_2\} = \varnothing.$$

Theorem 2.7 is proved. □

3. Generating sets of the semigroup of binary relations defined by semilattices of the class $\Sigma_{11}(X, 6)$, where $P = \varnothing$ and $|X \setminus \breve{D}| \geq 1$ or $X = \breve{D}$.

Definition 3.1. We denote by $\Sigma_{1.1}(X, 6)$ all semilattices $D = \{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ of the class $\Sigma_1(X, 6)$, for which $Z_5 \cap Z_4 \cap Z_2 = \varnothing$. This inequality and the formal equalities (2.1) of a semilattice D imply that $Z_5 \cap Z_4 \cap Z_2 = P_0 = \varnothing$, i.e., $|X| \geq 5$ since $P_5 \neq \varnothing$, $P_4 \neq \varnothing$, $P_3 \neq \varnothing$, $P_2 \neq \varnothing$, $P_1 \neq \varnothing$.

In this section, we study irreducible generating sets of the semigroup $B_X(D)$ defined by semilattices of the class $\Sigma_{1.1}(X, 6)$.

The following Lemmas 3.1, 3.3, 3.4, and 3.5 and Theorems 3.6 and 3.7 can be proved similarly to Lemmas 2.1, 2.3, 2.4, and 2.5 and Theorems 2.6 and 2.7.

Lemma 3.1. *Let $D \in \Sigma_{1.1}(X, 6)$ and $\alpha = \delta \circ \beta$ for some $\alpha, \delta, \beta \in B_X(D)$. Then the following assertions hold:*

(a) *let $T, T' \in \{Z_5, Z_4, Z_2\}$, $T \neq T'$; if $T, T' \in V(D, \alpha)$, then α is an external element of the semigroup $B_X(D)$;*
(b) *if $T \in \{Z_3, Z_1\}$ and $T, Z_2 \in V(D, \alpha)$, then α is an external element of the semigroup $B_X(D)$.*

Lemma 3.2. *Let $D \in \Sigma_{1.1}(X, 6)$. Then the following assertions hold:*

(a) *if a quasinormal representation of a binary relation α has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, then α is generated by elements of the elements of the set B_0;

(b) *if a quasinormal representation of a binary relation* α *has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* B_0;

Lemma 3.3. *Let* $D \in \Sigma_{1.1}(X, 6)$. *Then the following assertions hold*:

(a) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_5^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(b) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(c) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(d) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(e) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(f) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(g) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$.

Lemma 3.4. *Let* $D \in \Sigma_{1.1}(X, 6)$. *Then the following assertions hold*:

(a) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_3^\alpha \times Z_3),$$

where $Y_5^\alpha, Y_3^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(b) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_1^\alpha \times Z_1),$$

where $Y_5^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(c) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3),$$

where $Y_4^\alpha, Y_3^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(d) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_1^\alpha \times Z_1),$$

where $Y_4^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(e) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1),$$

where $Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(f) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_5^\alpha \times Z_5) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_5^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(g) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_4^\alpha \times Z_4) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_4^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(h) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_3^\alpha \times Z_3) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(i) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_2^\alpha \times Z_2) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(j) *if a quasinormal representation of a binary relation has the form*

$$\alpha = (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D}),$$

where $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$.

Lemma 3.5. *Let* $D \in \Sigma_{1.1}(X, 6)$. *Then the following assertions hold*:

(a) *let* $T' \in \{Z_3, Z_1, \breve{D}\}$ *and* $\alpha = X \times T'$, *then* α *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(b) *if* $|X \setminus \breve{D}| \geq 1$ *and* $T \in \{Z_5, Z_4, Z_2\}$, *then the binary relation* $\alpha = X \times T$ *is generated by elements of the elements of the set* $B(\mathfrak{A}_0)$;

(c) *if* $X = \breve{D}$ *and* $T \in \{Z_5, Z_4, Z_2\}$, *then the binary relation* $\alpha = X \times T$ *is an external element for the semigroup* $B_X(D)$.

Theorem 3.6. *Let* $D \in \Sigma_{1.1}(X, 6)$ *and*

$$\mathfrak{A}_0 = \Big\{\{Z_5, Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_4, Z_3, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_5, Z_4, Z_3, Z_1\},\ \{Z_5, Z_4, Z_3, \breve{D}\},\ \{Z_5, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_5, Z_4, Z_3\},\ \{Z_5, Z_2, \breve{D}\},\ \{Z_4, Z_2, \breve{D}\},\ \{Z_3, Z_2, \breve{D}\},\ \{Z_2, Z_1, \breve{D}\}\Big\},$$

$$B(\mathfrak{A}_0) = \Big\{\alpha \in B_X(D) \ \Big|\ V(X^*, \alpha) \in \mathfrak{A}_0\Big\}, \quad B_0 = \Big\{\alpha \in B_X(D) \ \Big|\ V(X^*, \alpha) = D\Big\}.$$

Then the following assertions hold:

(a) *if* $|X \setminus \breve{D}| \geq 1$, *then*

$$S_0 = B_0 \cup B(\mathfrak{A}_0)$$

is an irreducible generating set for the semigroup $B_X(D)$;

(b) *if* $X = \breve{D}$, *then*

$$S_1 = B_0 \cup B(\mathfrak{A}_0) \cup \big\{X \times Z_5, X \times Z_4, X \times Z_2\big\}$$

is an irreducible generating set for the semigroup $B_X(D)$.

Theorem 3.7. *Let* $D = \{Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_{1.1}(X, 6)$ *and*

$$\mathfrak{A}_0 = \Big\{\{Z_5, Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_4, Z_3, Z_1, \breve{D}\},\ \{Z_5, Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, Z_1, \breve{D}\},$$
$$\{Z_5, Z_4, Z_3, Z_1\},\ \{Z_5, Z_4, Z_3, \breve{D}\},\ \{Z_5, Z_3, Z_2, \breve{D}\},\ \{Z_5, Z_2, Z_1, \breve{D}\},\ \{Z_4, Z_3, Z_2, \breve{D}\},\ \{Z_4, Z_2, Z_1, \breve{D}\},$$
$$\{Z_3, Z_2, Z_1, \breve{D}\},\ \{Z_5, Z_4, Z_3\},\ \{Z_5, Z_2, \breve{D}\},\ \{Z_4, Z_2, \breve{D}\},\ \{Z_3, Z_2, \breve{D}\},\ \{Z_2, Z_1, \breve{D}\}\Big\},$$

$$B(\mathfrak{A}_0) = \Big\{\alpha \in B_X(D) \mid V(X^*, \alpha) \in \mathfrak{A}_0\Big\}, \quad B_0 = \Big\{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\Big\}.$$

Then the following assertions hold:

(a) *if* $|X \setminus \breve{D}| \geq 1$, *then the number* $|S_0|$ *of elements of the set*

$$S_0 = B_0 \cup B(\mathfrak{A}_0)$$

is equal to

$$|S_0| = 6^n - 4^n - 2 \cdot 3^n + 2 \cdot 2^n;$$

(b) *if* $X = \breve{D}$, *then the number* $|S_1|$ *of elements of the set*

$$S_1 = B_0 \cup B(\mathfrak{A}_0) \cup \big\{X \times Z_4, X \times Z_3, X \times Z_2\big\}$$

is equal to

$$|S_1| = 6^n - 4^n - 2 \cdot 3^n + 2 \cdot 2^n + 3.$$

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