DOI 10.1007/s10958-023-06660-2
Journal of Mathematical Sciences, Vol. 275, No. 1, September, 2023

THE ORBITAL STABILITY ANALYSIS OF PENDULUM OSCILLATIONS OF A HEAVY RIGID BODY WITH A FIXED POINT UNDER THE GORIACHEV–CHAPLYGIN CONDITION

B. S. Bardin *
Moscow Aviation Institute (National Research University)
4, Volokolamskoe highway, Moscow, 125993, Russia
bsbardin@yandex.ru

B. A. Maksimov
Moscow Aviation Institute (National Research University)
4, Volokolamskoe highway, Moscow, 125993, Russia
badmamaksimov1@gmail.com

We consider the motion of a heavy rigid body with a fixed point in a uniform gravitational field under the assumption that the principal moments of inertia satisfy the Goryachev–Chaplygin condition at the fixed point. We study the orbital stability problem for small pendulum oscillations of the body. We derive the equations of perturbed motion and reduce the problem to the study of the stability of the equilibrium position of a second order 2π*-periodic Hamiltonian system. We find regions of parametric resonance and perform the nonlinear analysis of orbital stability outside these regions. Bibliography*: 14 *titles.*

Periodic motions play a special role in rigid body dynamics. The study of such motions often makes it possible to draw important conclusions on motion properties of a considered mechanical system, as well as it helps to perform qualitative analysis of the phase space of the system. By this reason, the problem of orbital stability of pendulum periodic motions of a heavy rigid body with a fixed point is of considerable interest for both theoretical mechanics and its applications. Modern methods of the theory of dynamical systems, including the method of normal forms, the methods of the Kolmogorov–Arnold–Moser theory and general theory of stability allow one to obtain rigorous conclusions about the orbital stability of periodic motions of this type.

In the general case, the problem of orbital stability of pendulum periodic motions of a heavy rigid body with one fixed point contains four parameters. To reduce the number of parameters in the problem, the most interesting special cases are usually considered. The Kovalevskaya

* To whom the correspondence should be addressed.

Source Journal *International Mathematical Schools*. Vol. 5. Topics on Logic Discussed at Sirius Mathematics Center and Some Aspects of Pure and Applied Mathematics

case and the Goryachev–Chaplygin case, where the orbital stability problem has one parameter, were considered in [1]–[5], as well as the case of a dynamically symmetric body and the Bobylev–Steklov case, where the problem has two parameters, were studied in [6]–[10].

In the Kovalevskaya case, the orbital stability of pendulum periodic motions was studied on the basis of various approaches [1]–[3]. The Goryachev–Chaplygin case was studied in [4, 5] by using the method of normal forms. In both cases, an additional first integral was used for the study of stability, which allows one to obtain rigorous conditions about orbital stability or instability. The Bobylev–Steklov case was considered in [6, 10] by the method of normal forms and the Kolmogorov–Arnold–Moser theory. Rigorous conclusions on orbital stability were obtained for any values of parameters. The results were expressed in the form of a stability diagram in the parameter plane. In [7], the orbital stability was studied for a dynamically symmetric rigid body with the mass center in the equatorial plane of the ellipsoid of inertia.

In this paper, we study the orbital stability of pendulum oscillations of a heavy rigid body with a fixed point under the following assumptions. Amplitudes of pendulum oscillations are small, and the principal moments of inertia of the body for the fixed point satisfy the Goryachev–Chaplygin condition, i.e., they are in the ratio 1:4:1. Unlike the Goryachev-Chaplygin integrable case, no additional restrictions are imposed on the position of the mass center.

The paper is organized as follows. In Section 1, the mathematical formulation of the problem is stated. In Section 2, the so-called local variables are introduced and the orbital stability problem is reduced to the study of the Lyapunov stability of the periodic Hamiltonian system with one degree of freedom. In Section 3, regions of parametric resonance are found. In Section 4, the nonlinear analysis of orbital stability is performed outside the regions of parametric resonance. The main result of the paper is formulated in Theorem 4.1.

1 Statement of the Problem

We consider the motion of a rigid body of mass m around a fixed point O in a uniform gravitational field. To describe the body motion, we introduce a fixed coordinate system $OXYZ$ such that the OZ-axis is directed vertically upwards and a moving coordinate system $Oxyz$ that is rigidly connected to the body with the axes directed along the principal axes of inertia of the body for the point O. In addition, we assume that the principal moments of inertia A, B, C of the body for the fixed point O satisfy the equality $A = C = 4B$.

No conditions are imposed on the position of the mass center. Due to the dynamic symmetry of the body, the directions of the Ox– and Oz–axes can be chosen in such a way that the mass center of the body is located on the Oxy-plane. The position of the mass center is determined by the distance l to the origin and the angle α between the position vector of the mass center and the positive direction of the Ox–axis. Without loss of generality we can assume that $0 \leqslant \alpha \leqslant \pi/2$. We note that we have the Goryachev–Chaplygin case for $\alpha = 0$ and the Lagrange case for $\alpha = \pi/2$.

The position of the body in the space is specified through the Euler angles ψ, θ, φ. Then the motion equations can be written in the form of canonical equations with the Hamiltonian

$$H = \frac{(p_\theta \cos\varphi - p_\varphi \cot\theta \sin\varphi)^2}{2A} + \frac{(p_\theta \sin\varphi + p_\varphi \cot\theta \cos\varphi)^2}{2B} + \frac{{p_\varphi}^2}{2C} + mgl \sin\theta \sin(\varphi + \alpha), \quad (1.1)$$

where $p_\psi, p_\theta, p_\varphi$ are the canonically conjugated moments corresponding to the Euler angles. Th angle ψ is a cyclic coordinate, so $p_\psi = \text{const}$. In what follows, we assume that $p_\psi = 0$.

The equations of motion admit a partial solution describing the plane motion of the body, in which the Oz–axis is horizontal and the body performs plane motions about this axis. Depending on the initial conditions, in the plane motion, the body either performs periodic oscillations or rotations or asymptotically approaches an unstable equilibrium position. Since the periodic pendulum motions are unstable in the sense of Lyapunov, the orbital stability problem for such motions is of interest.

We introduce the dimensionless time $\tau = \mu t$, where $\mu^2 = mgl$. To describe the behavior of the body in the vicinity of its periodic pendulum motions, it is convenient to introduce the following coordinates and dimensionless moments:

$$q_1 = \varphi + \alpha - \frac{3\pi}{2}, \quad q_2 = \theta - \frac{\pi}{2}, \quad p_1 = \frac{p_\theta}{C\mu}, \quad p_2 = \frac{p_\varphi}{C\mu}. \tag{1.2}$$

In the new variables, the Hamiltonian of the problem takes the form

$$\begin{aligned} H = {} & 1/2\, {p_1}^2 + 1/2\, (p_2 \sin(q_1 - \alpha) - p_1 \tan q_2 \cos(q_1 - \alpha))^2 \\ & + 2\, (p_2 \cos(q_1 - \alpha) + p_1 \tan q_2 \sin(q_1 - \alpha))^2 - \cos q_2 \cos q_1. \end{aligned} \tag{1.3}$$

On pendulum motions of the body, we have $q_2 = p_2 = 0$ and the evolution of the variables q_1 and p_1 is described by the canonical system with the Hamiltonian

$$H_0 = 1/2\, {p_1}^2 - \cos q_1. \tag{1.4}$$

The type of pendulum motions depends on the value of the energy integral constant $H_0 = h$: the body performs pendulum oscillations for $|h| < 1$ or pendulum rotations about the Oz-axis for $h > 1$. As shown in [9], pendulum rotations are orbitally unstable for all $h > 1$. In what follows, we consider only pendulum oscillations when $|h| < 1$. In this case, the general solution to the canonical system with the Hamiltonian (1.4) has the form [1]

$$\begin{aligned} q_{1*}(\tau + \tau_0) &= 2 \arcsin\left(k \mathrm{sn}(\tau + \tau_0, k)\right), \\ p_{1*}(\tau + \tau_0) &= 2\, k \mathrm{cn}(\tau + \tau_0, k). \end{aligned} \tag{1.5}$$

The oscillation period is calculated by the formula

$$T = \frac{2\pi}{\omega}, \quad \omega = \frac{\pi}{2K(k)}. \tag{1.6}$$

In (1.5) and (1.6), we used usual notation for elliptic functions and integrals. The modulus of the elliptic integral is related to the energy constant by $k^2 = h/2 + 1/2$.

The goal of this paper is to analyze the orbital stability of pendulum oscillations of the body at $0 < \alpha < \pi/2$.

2 Local Variables and Isoenergetic Reduction

Following the method developed in [10], we introduce local coordinates in a neighborhood of the unperturbed periodic motion according to the formulas

$$\begin{aligned} q_1 &= q_{1*}(\xi) + \frac{\sin q_{1*}(\xi)}{V^2(\xi)}\eta - \frac{\sin q_{1*}(\xi)}{2V^4(\xi)}\eta^2 + O(\eta^3), \\ p_1 &= p_{1*}(\xi) + \frac{p_{1*}(\xi)}{V^2(\xi)}\eta - \frac{p_{1*}(\xi) \cos q_{1*}(\xi)}{2V^4(\xi)}\eta^2 + O(\eta^3), \end{aligned} \tag{2.1}$$

where $V^2(\xi) = {p_{1*}}^2(\xi) + \sin^2 q_{1*}(\xi)$ and the functions $q_{1*}(\xi)$, $p_{1*}(\xi)$ are defined by (1.5).

In the new variables, the Hamiltonian function periodically depends on ξ and is an analytic function of the variable η. Let us perform one more canonical change of variables according to the formula

$$\xi = \frac{w}{\omega}, \quad \eta = \omega r \tag{2.2}$$

and expand the Hamiltonian function in the neighborhood of $\eta = q_2 = p_2 = 0$

$$\Gamma = \Gamma_2 + \Gamma_4 + \ldots, \tag{2.3}$$

where

$$\begin{aligned} \Gamma_2 &= \omega r + \Phi_{20}(q_2, p_2, w), \\ \Gamma_4 &= \omega^2 \chi(w) r^2 + \omega r \Psi_{20}(q_2, p_2, w) + \Phi_{40}(q_2, p_2, w); \end{aligned} \tag{2.4}$$

here,

$$\Phi_{20}(q_2, p_2, w) = \sum_{i+j=2} \varphi_{ij} {q_2}^i {p_2}^j,$$

$$\Phi_{40}(q_2, p_2, w) = \sum_{i+j=4} \varphi_{ij} {q_2}^i {p_2}^j,$$

$$\Psi_{20}(q_2, p_2, w) = \sum_{i+j=2} \psi_{ij} {q_2}^i {p_2}^j.$$

The coefficients in (2.4) are 2π-periodic functions of the variable w and have the following explicit form:

$$\begin{aligned}
\chi &= -\frac{(\cos q_{1*} - 1)(\cos^2 q_{1*} + {p_{1*}}^2 - 1)}{2(\cos^2 q_{1*} - {p_{1*}}^2 - 1)^2}, \\
\psi_{20} &= \frac{6\,\cos(2\,q_{1*} - 2\,\alpha){p_{1*}}^2 - 3\,{p_{1*}}^2\cos(q_{1*} - 2\,\alpha)}{-4\,{p_{1*}}^2 + 2\,\cos 2\,q_{1*} - 2} \\
&\quad + \frac{3\,{p_{1*}}^2\cos(3\,q_{1*} - 2\,\alpha) - 10\,{p_{1*}}^2 - \cos(2\,q_{1*}) + 1}{-4\,{p_{1*}}^2 + 2\,\cos 2\,q_{1*} - 2}, \\
\psi_{11} &= \frac{3\,p_{1*}\sin(2\,q_{1*} - 2\,\alpha) + 3\,p_{1*}\sin(3\,q_{1*} - 2\,\alpha)}{2\,{p_{1*}}^2 - \cos 2\,q_{1*} + 1} - \frac{3p_{1*}\sin(q_{1*} - 2\,\alpha)}{2\,{p_{1*}}^2 - \cos 2\,q_{1*} + 1}, \\
\psi_{02} &= \frac{3\,\cos(q_{1*} - 2\,\alpha)3\,\cos(3\,q_{1*} - 2\,\alpha)}{-4\,{p_{1*}}^2 + 2\,\cos 2\,q_{1*} - 2}, \\
\varphi_{40} &= -1/2\,\cos(2\,q_{1*} - 2\,\alpha){p_{1*}}^2 - 1/24\,\cos q_{1*} + 5/6\,{p_{1*}}^2, \\
\varphi_{31} &= 1/2\,p_{1*}\sin(2\,q_{1*} - 2\,\alpha), \\
\varphi_{20} &= -3/4\,\cos(2\,q_{1*} - 2\,\alpha){p_{1*}}^2 + 1/2\,\cos q_{1*} + 5/4\,{p_{1*}}^2, \\
\varphi_{11} &= 3/2\,p_{1*}\sin(2\,q_{1*} - 2\,\alpha), \\
\varphi_{02} &= 3/4\,\cos(2\,q_{1*} - 2\,\alpha) + 5/4.
\end{aligned} \tag{2.5}$$

Let us consider the motion at the isoenergetic level $\Gamma = 0$ corresponding to an unperturbed periodic motion. The evolution of the variables q_2, p_2 at the level $\Gamma = 0$ can be described by using the reduced canonical system (the Whittaker equation)

$$\frac{dq_2}{dw} = \frac{\partial K}{\partial p_2}, \quad \frac{dp_2}{dw} = -\frac{\partial K}{\partial q_2}, \tag{2.6}$$

where w plays the role of a new independent variable. The evolution of the variable r is determined by the relation $r = -K(q_2, p_2, w)$ obtained by solving the equation $\Gamma = 0$ with respect to r. For small r, q_2, p_2 the Hamiltonian K can be represented as a power series in q_2, p_2. Taking into account (2.3) and (2.4), we obtain the following expansion of the Hamiltonian function K in a series in q_2, p_2

$$K = K_2 + K_4 + \ldots, \tag{2.7}$$

where

$$\begin{aligned} K_2 &= \omega^{-1}\Phi_{20}(q_2, p_2, w), \\ K_4 &= \omega^{-1}[\chi(w)\Phi_{20}^2(q_2, p_2, w) - \Psi_{20}(q_2, p_2, w)\Phi_{20}(q_2, p_2, w) + \Phi_{40}(q_2, p_2, w)]. \end{aligned} \tag{2.8}$$

The problem of orbital stability of periodic motions of a rigid body is equivalent to the problem of stability of the equilibrium position of the system (2.6).

3 Regions of Parametric Resonance

For small oscillation amplitudes, i.e., $0 < k \ll 1$, we can analytically describe the boundaries of instability regions. For this purpose we consider the linear canonical system with the Hamiltonian K_2

$$\begin{aligned} \frac{dq_2}{dw} &= \varphi_{11} q_2 + 2\,\varphi_{02} p_2, \\ \frac{dp_2}{dw} &= -2\,\varphi_{20} q_2 - \varphi_{11} p_2. \end{aligned} \tag{3.1}$$

The series expansions in powers of k of the coefficients of the system (3.1) have the form

$$\begin{aligned} \varphi_{11} &= -3\cos w \sin 2\alpha\, k + 6k^2 \sin 2\,w \cos 2\alpha + O(k^3), \\ \varphi_{02} &= \frac{1}{2} + \frac{3}{2}\cos^2\alpha + 3\,k \sin w \sin 2\alpha \cos\alpha \\ &\quad + \frac{1}{16}(\cos 2\alpha(48\cos 2w - 45) + 5)k^2 + O(k^3), \\ \varphi_{20} &= \frac{1}{2} + \frac{1}{8}(-12(\cos 2w + 1)\cos 2\alpha + 24\cos 2w + 17)k^2 + O(k^3). \end{aligned} \tag{3.2}$$

For $k = 0$ the linear system is autonomous and describes harmonic oscillations with frequency

$$\Omega_0 = \sqrt{1 + 3\cos^2\alpha}. \tag{3.3}$$

If $\Omega_0 \neq N/2$, $N \in \mathbb{Z}$, then for sufficiently small k the stability takes place in the linear approximation. If $\Omega_0 \approx N/2$, then for $k \ll 1$ the so-called parametric resonance phenomenon is possible, which leads to instability. In the problem, the parametric resonance phenomenon takes place in the following two cases: $\Omega_0 \approx 2$ and $\Omega_0 \approx 3/2$. The corresponding regions of parametric resonance in the parameter plane (α, h) emanate from the points $\alpha_* = 0$ and $\alpha_{**} = \arccos\left(\sqrt{5/12}\right)$ of the straight line $h = -1$.

For $k \ll 1$ the boundaries of the regions of parametric resonance can be obtained in the form of the convergent series in powers of the small parameter k

$$\alpha = \alpha_0 + k\alpha_1 + \ldots, \tag{3.4}$$

where on must put $\alpha_0 = \alpha_*$ in the case of resonance $\Omega_0 \approx 2$ and $\alpha_0 = \alpha_{**}$ in the case of resonance $\Omega_0 \approx 3/2$. To obtain the coefficients of the above series, we use the technique developed in [11]. In accordance with this technique, using a linear canonical change of variables $(q_2, p_2) \to (X, Y)$, we reduce the Hamiltonian K_2 to the simplest so-called normal form. At the above resonances, the normal form of the Hamiltonian reads [11]

$$K_2 = k_{20}X^2 + k_{11}XY + k_{02}Y^2, \tag{3.5}$$

where k_{20}, k_{11}, k_{02} depend analytically on k and α. To construct the above linear change of variables, as well as find the coefficients k_{20}, k_{11}, k_{02}, we use the Depri–Hori method [12], which allows us to calculate expressions for these coefficients in the form of series in powers of k up to an arbitrarily high degree. A simple analysis of the canonical system with the Hamiltonian (3.2) shows that the system is stable if ${k_{11}}^2 < 4\,k_{20}k_{02}$; otherwise, i.e., if ${k_{11}}^2 > 4\,k_{20}k_{02}$, the system is unstable. Thus, the boundaries of the parametric resonance regions are determined by the equation

$${k_{11}}^2 = 4\,k_{20}k_{02}. \tag{3.6}$$

Substituting (3.7) into (3.6) and equating the terms with equal powers of k on the right-hand side of (3.5), we can obtain equations for the coefficients of the series (3.4).

A calculation shows that, in the case of resonance $\Omega_0 \approx 2$,

$$\begin{aligned} k_{20} &= -\frac{{\alpha_1}^2}{4}k^2 - \frac{9\alpha_2\alpha_1}{2}k^3 + O(k^4), \\ k_{11} &= O(k^4), \\ k_{02} &= -\frac{{\alpha_1}^2}{4}k^2 - \frac{9\alpha_2\alpha_1}{2}k^3 + O(k^4). \end{aligned} \tag{3.7}$$

In this case, the boundaries of the parametric resonance regions are determined by

$$\alpha = 0, \tag{3.8}$$

$$\alpha = \frac{\sqrt{3}}{2}k^2 + O(k^3). \tag{3.9}$$

On the first boundary (3.8), i.e., at $\alpha = 0$, the Goryachev–Chaplygin case takes place. This case was completely studied in [4] and is not considered in this paper. On the second boundary (3.9), the normalized Hamiltonian reads

$$K_2 = -\frac{9\,X^2}{32}k^4 + O(k^5). \tag{3.10}$$

Further calculations show that, in the resonance case $\Omega_0 \approx 3/2$,

$$\begin{aligned} k_{20} &= -\frac{\alpha_1\sqrt{35}k}{12} + \left(\frac{63}{64} - \frac{\alpha_2\sqrt{35}}{6}\right)k^2 - \frac{\alpha_3\sqrt{35}}{2}k^3 + O(k^4), \\ k_{11} &= -\frac{315\,\sqrt{35}}{128}k^3 + O(k^4), \end{aligned} \tag{3.11}$$

$$k_{02} = -\frac{\alpha_1\sqrt{35}k}{12} + \Big(\frac{63}{64} - \frac{\alpha_2\sqrt{35}}{6}\Big)k^2 - \frac{\alpha_3\sqrt{35}}{2}k^3 + O(k^4).$$

In this case, on the right boundary we have

$$\alpha = \arccos\left(\frac{\sqrt{15}}{6}\right) + \frac{27\sqrt{35}}{160}k^2 + \frac{315}{128}k^3 + O(k^4) \tag{3.12}$$

and the Hamiltonian reads

$$K_2 = -\frac{105\sqrt{35}(X+Y)^2}{512}k^3 + O(k^5). \tag{3.13}$$

whereas the equation of the left boundary and the corresponding normalized Hamiltonian are determined by

$$\alpha = \arccos\left(\frac{\sqrt{15}}{6}\right) + \frac{27\sqrt{35}}{160}k^2 - \frac{315}{128}k^3 + O(k^4), \tag{3.14}$$

$$K_2 = \frac{105\sqrt{35}(X-Y)^2}{512}k^3 + O(k^5). \tag{3.15}$$

Inside the parametric resonance regions, the pendulum oscillations are orbitally unstable.

4 Nonlinear Analysis

Outside the parametric resonance regions, the linear system is stable, i.e., the pendulum oscillations are orbitally stable in the linear approximation. However, this fact does not imply the orbital stability of the original nonlinear system. To obtain a rigorous conclusion about the orbital stability for the parameter values outside the regions of parametric resonance and at their boundaries, a nonlinear analysis is required.

Let us first study the question about the orbital stability for the values of parameters outside the regions of parametric resonance and outside the boundaries of these regions. Using a linear canonical change of variables

$$\begin{aligned} q_2 &= a_{11}(w)x + a_{12}(w)y, \\ p_2 &= a_{21}(w)x + a_{22}(w)y, \end{aligned} \tag{4.1}$$

we can reduce the Hamiltonian K_2 to the normal form

$$K_2 = \frac{1}{2}\,\Omega\,(x^2+y^2), \tag{4.2}$$

where the coefficients a_{11}, a_{21}, a_{12}, a_{22} are 2π-periodic functions of the variable w and are analytic in k. The quantity Ω is an analytic function of k and α. To find a_{11}, a_{21}, a_{12}, a_{22}, Ω, we use the Depri-Hori method. A calculation shows that

$$\begin{aligned} \Omega &= \Omega_0 + \Omega_1 k^2 + O(k^3), \\ \Omega_1 &= \frac{16\,\Omega_0{}^6 - 48\,\Omega_0{}^4 + (36\sin^2 2\alpha - 69)\Omega_0{}^2 + 72\sin^2 2\alpha + 20}{4\Omega_0{}^2(1-4\,\Omega_0{}^2)}. \end{aligned} \tag{4.3}$$

We recall that Ω_0 is defined by (3.3).

For the further research it is convenient to change the scale of variables and introduce the canonical polar coordinates ρ, ϑ by

$$
\begin{aligned}
x &= k^3\sqrt{2\rho}\sin\vartheta,\\
y &= k^3\sqrt{2\rho}\cos\vartheta.
\end{aligned}
\tag{4.4}
$$

In these variables, the Hamiltonian K the complete nonlinear system takes the form

$$K = \Omega\,\rho + G_4\rho^2 + G_6\rho^3 + O(\rho^4). \tag{4.5}$$

We omit expressions for $G_4(\vartheta, w)$ and $G_6(\vartheta, w)$ because they are too cumbersome. Rigorous conclusions about the stability of the trivial solution to a system with the Hamiltonian (4.5) can be obtained on the basis of the Kolmogorov–Arnold–Moser theory. For this purpose, we normalize the Hamiltonian K to terms of order ρ^2.

If there are no resonances of the third and fourth order in the system, that is, $\Omega \neq n/3$ and $\Omega \neq n/4, n \in \mathbb{Z}$ respectively, then the Hamiltonian function (4.5) by a canonical change of variables $\rho, \vartheta \longrightarrow R, \varphi$ that is close-to-identity and analytic in k, can be reduced to the following normal form (see, for example, [13])

$$\Phi = \Omega\,R + c_2R^2 + \widetilde{G_6}R^3 + O(R^4), \tag{4.6}$$

where

$$c_2 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} G_4(\vartheta, w)\ \mathrm{d}\,\vartheta\ \mathrm{d}\,w. \tag{4.7}$$

A calculation shows that

$$c_2 = c_{20}k^6 + O(k^8), \tag{4.8}$$

where

$$c_{20} = -\frac{126\cos^6\alpha + 93\cos^4\alpha - 34\cos^2\alpha - 25}{8(4\cos^2\alpha + 1)(1 + 3\cos^2\alpha)}. \tag{4.9}$$

By the Arnold-Moser theorem, the equilibrium position of the system (2.6) is stable at $c_2 \neq 0$. In the interval $0 < \alpha < \pi/2$, the equation $c_2 = 0$ has a unique analytic solution that analytically depends on α. For sufficiently small k this solution is given by the asymptotic formula

$$\alpha^* = \alpha_0^* + O(k^2),$$

where α_0^* is a solution to the equation $c_{20} = 0$ and the calculation have shown that $\alpha_0^* \approx 0.7665103122$. Thus, outside the parametric resonance regions, at $\alpha \neq \alpha^*$ and in the absence of resonances up to the fourth order, the equilibrium position of the system (2.6) is stable for small k.

Now, let us put $\alpha = \alpha^*$ when the so-called case of degeneracy takes place and solving the stability problem requires an additional analysis including terms of degree R^3. As above, using a canonical change of variables $R, \varphi \longrightarrow \widetilde{R}, \widetilde{\varphi}$ that is close to identical and analytic in k, we can normalize the Hamiltonian up to the six order terms inclusively. In the new variables the Hamiltonian reads

$$\widetilde{\Phi} = \Omega\,\widetilde{R} + c_3\widetilde{R}^3 + O(\widetilde{R^4}), \quad c_3 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \widetilde{G_6}(\vartheta, w)\ \mathrm{d}\,\vartheta\ \mathrm{d}\,w. \tag{4.10}$$

A calculation shows that

$$c_3 = -\frac{(1+3\cos^2\alpha_0^*)^{3/2}}{12288} + O(k^2) \tag{4.11}$$

Since the quantity c_3 is nonzero for sufficiently small k, it follows that, by the Arnold–Moser theorem, the equilibrium point of the system (2.6) is stable.

The resonance cases require a special analysis. The resonance case of the first and second order take place at the boundaries of the regions of parametric resonance and will be considered below. Since the Hamilton function does not contain terms of order $\rho^{3/2}$, the third order resonance do not appear in the system. However, the fourth order resonance can occur in the case $\Omega = n/4$, $n \in \mathbb{Z}$. In our problem, two such cases can happen: $\Omega = 5/4$ and $\Omega = 7/4$. In the case of a fourth order resonance, the Hamiltonian (4.5) can be reduced to the form

$$\begin{aligned}
&\Phi = \Omega\, R + (c_2 + a_4\cos(nw - 4\,\varphi) - b_4\sin(nw - 4\,\varphi))R^2 + O(R^3),\\
&a_4 = \frac{1}{2\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} G_4(\vartheta, w)\cos(4\,\vartheta - nw)\ \mathrm{d}\,\vartheta\ \mathrm{d}\,w,\\
&b_4 = \frac{1}{2\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} G_4(\vartheta, w)\sin(4\,\vartheta - nw)\ \mathrm{d}\,\vartheta\ \mathrm{d}\,w.
\end{aligned} \tag{4.12}$$

In the case $\Omega = 5/4$, the value of the parameter α is respectively determined by the following resonance relation which can be easily obtained by using the expression (4.3):

$$\alpha = \arccos\left(\frac{\sqrt{3}}{4}\right) + \frac{3\sqrt{39}}{7}\,k^2 + O(k^3)\,. \tag{4.13}$$

In the resonance case $\Omega = 5/4$, the coefficients of the normal form read (4.12)

$$\begin{aligned}
&c_2 = \frac{55859}{11200}k + O(k^3),\\
&a_4 = O(k^3), \quad b_4 = O(k^3).
\end{aligned} \tag{4.14}$$

In the resonance case $\Omega = 7/4$, the value of the parameter α and the coefficients of the normal form (4.12) are determined by the asymptotic formulas

$$\begin{aligned}
&\alpha = \arccos\left(\frac{\sqrt{11}}{4}\right) + \frac{3\,\sqrt{55}}{55}k^2 + O(k^3),\\
&c_2 = -\frac{4987}{3136}k + O(k^3),\\
&a_4 = O(k^3), \quad b_4 = O(k^3).
\end{aligned} \tag{4.15}$$

A sufficient condition for the stability of an equilibrium position at a fourth order resonance has the form

$$\sqrt{{a_4}^2 + {b_4}^2} < |c_2|. \tag{4.16}$$

In both resonance cases, the condition (4.16) is obviously satisfied, which means the Lyapunov stability of the system (2.6) and, consequently, the orbital stability of pendulum oscillations.

To study the orbital stability at the boundaries of parametric resonance regions, we use the same research technique. The resonances of the first ($\Omega = 2$) and second ($2\Omega = 3$) order take place at the boundaries of the parametric resonance regions. In the following cases, making a linear change of variables $X, Y \to u, v$, we reduce the Hamiltonian to the form

$$K = \frac{\delta}{2}u^2 + K_4(u,v) + O_6, \quad K_4 = \sum_{i+j=4} s_{ij}u^i v^j. \tag{4.17}$$

The coefficients s_{ij} of the form K_4 are 2π-periodic functions of the variable w. The coefficient δ is determined in the process of constructing the specific linear change of variables and takes the value $\delta = 1$ or $\delta = -1$. Further, by a nonlinear close-to-identity change of variables $u, v \to x, y$, we can normalize the Hamiltonian K up to the fourth degree terms. In this case, the normal form of the Hamiltonian is written as

$$\begin{gathered} K = \frac{\delta}{2}x^2 + c_4 y^4 + O_6, \\ c_4 = \frac{1}{2\pi}\int_0^{2\pi} s_{04}dw. \end{gathered} \tag{4.18}$$

By the Ivanov–Sokol'skij theorem [14], the equilibrium position of the system with the Hamiltonian (4.18) is stable if

$$\delta c_4 > 0 \tag{4.19}$$

and it is unstable if the inequality (4.19) holds with the opposite sign.

A calculation shows that, on the boundaries (3.9) and (3.14), the coefficients δ, c_4 have the form

$$c_4 = -\frac{1}{4}k^4 + O(k^5), \quad \delta = -1, \tag{4.20}$$

and

$$c_4 = \frac{445}{6144}k^4 + O(k^5)\,, \quad \delta = 1 \tag{4.21}$$

respectively. Since (4.19) is obviously satisfied, the oscillations with small amplitudes are orbitally stable on these boundaries. On the boundary (3.12), we have

$$c_4 = \frac{445}{6144}k^4 + O(k^5), \quad \delta = -1. \tag{4.22}$$

Here, the inequality (4.19) holds with the opposite sign and the oscillations with small amplitudes are orbitally unstable.

Thus, we have established the following result.

Theorem 4.1. *At sufficiently small amplitudes, the pendulum oscillations of a heavy rigid body with a fixed point, whose principal moments of inertia are related such that* $1:4:1$ *are orbitally unstable in narrow regions of parametric resonance. The boundaries of these regions are analytically described by Equations* (3.9), (3.12), (3.14). *The pendulum oscillations with small amplitudes are orbitally stable outside these regions and on the boundaries* (3.9), (3.14) *and are orbitally unstable on the boundary* (3.12).

Declarations

Funding This work was supported by the grant of the Russian Science Foundation (project No. 19-11-00116) at the Moscow Aviation Institute (National Research University).

Data availability This manuscript has no associated data.

Ethical Conduct Not applicable.

Conflicts of interest The authors declare that there is no conflict of interest.

References

1. A. P. Markeev, "The stability of the plane motions of a rigid body in the Kovalevskaya case," *J. Appl. Math. Mech.* **65**, No. 1, 47–54 (2001).

2. A. Z. Bryum, "Orbital stability analysis using first integrals," *J. Appl. Math. Mech.* **53**, No. 6, 689-695 (1989).

3. A.Z. Bryum and A. Ya. Savchenko, "On the orbital stability of a periodic solution of the equations of motion of a Kovalevskaya gyroscope," *J. Appl. Math. Mech.* **50**, No. 6 , 748-753 (1986).

4. B. S. Bardin, "Stability problem for pendulum-type motions of a rigid body in the Goryachev–Chaplygin case," *Mech. Solids* **42**, No. 2, 177–183 (2007).

5. A. P. Markeev, "Pendulum-like motions of a solid in the Goryachev–Chaplygin case," *J. Appl. Math. Mech.* **68**, No. 2, 249-258 (2004).

6. B. S. Bardin, T. V. Rudenko, and A. A. Savin, "On the orbital stability of planar periodic motions of a rigid body in the Bobylev–Steklov case," *Regul. Chaotic Dyn.* **17**, No. 6, 533–546 (2012).

7. B. S. Bardin and A. A. Savin, "On the orbital stability of pendulum-like oscillations and rotations of a symmetric rigid body with a fixed point," *Regul. Chaotic Dyn.* **17**, No. 3-4, 243–257 (2012).

8. B. S. Bardin and A. A. Savin, " The stability of the plane periodic motions of a symmetrical rigid body with a fixed point," *J. Appl. Math. Mech.* **77**, No. 6, 578-587 (2013).

9. H. M. Yehia and E. G. El-Hadidy, "On the orbital stability of pendulum-like vibrations of a rigid body carrying a rotor," *Regul. Chaotic Dyn.* **18**, No. 5, 539–552 (2013).

10. B. S. Bardin, "On a method of introducing local coordinates in the problem of the orbital stability of planar periodic motions of a rigid body," *Rus. J. Nonlinear Dyn.* **16**, No. 4, 581–594 (2020).

11. A. P. Markeev, *Linear Hamiltonian Systems and Some Problems of Stability of the Satellite Center of Mass* [in Russian], Institute of Computer Research Press, Izhevsk (2009).

12. G. E. O. Giacaglia, *Perturbation Methods in Non-Linear Systems*, Springer, New York etc. (1972).

13. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York (1989).

14. A. P. Ivanov and A. G. Sokol'skij, "On the stability of a non-autonomous Hamiltonian system in parametric resonance of dominant mode," *J. Appl. Math. Mech.*, **44**, No. 6, 687-691 (1980).

Submitted on March 28, 2023