DOI 10.1007/s10958-023-06647-z
Journal of Mathematical Sciences, Vol. 274, No. 6, August, 2023

ON CONTINUOUS AND BOUNDED SOLUTIONS OF THE SYSTEMS OF DIFFERENCE-FUNCTIONAL EQUATIONS WITH NUMEROUS DEVIATIONS OF THE ARGUMENT

T. O. Yeromina[1] and **O. A. Povarova**[2, 3] UDC 517.9

We establish the existence conditions for continuous solutions of a class of systems of linear functional-difference equations with numerous deviations of the argument, propose a method for the construction of these solutions, and study the structure of the set of solutions of this kind.

Consider a system of linear equations

$$x(qt) = Ax(t) + \sum_{j=1}^{k} B_j(t)x(t + \Delta_j(t)) + F(t), \tag{1}$$

where $t \in \mathbb{R}$, A, $B_j(t)$, $j = 1, \ldots, k$, are some real $(n \times n)$ matrices, q is a real constant, $F(t)$ is a real vector of dimension n, and $\Delta_j(t)\colon \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, k$. Systems of linear difference and functional-difference equations were considered, e.g., in [1–8]. We study the problem of existence of continuous solutions bounded for $t \geq T$ in the case where the following conditions are satisfied:

1) all elements of the matrices $B_j(t)$, $j = 1, \ldots, k$, and the vector $F(t)$ are functions bounded for $t \geq T$;

2) the functions $\Delta_j(t)$, $j = 1, \ldots, k$, are continuous and bounded for $t \geq T$ and, in addition, $\Delta_j(t) \geq 1$, $q \neq 0$;

3) $\sup\limits_t |B_j(t)| = b_j$, $j = 1, \ldots, k$, $\sup_t |F(t)| = M$, and

$$|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| = a < 1;$$

4) $\tilde{\Delta} = \dfrac{\sum_{l=1}^{k} b_l}{1 - a} < 1.$

The following theorem is true:

[1] I. Sikorsky Kyiv Polytechnic Institute, Ukrainian National Technical University, Peremoha Ave., 37, Kyiv, 03056, Ukraine; e-mail: ierominat@ukr.net.
[2] I. Sikorsky Kyiv Polytechnic Institute, Ukrainian National Technical University, Peremoha Ave., 37, Kyiv, 03056, Ukraine; e-mail: olena_sivak@ukr.net.
[3] Corresponding author.

Translated from Neliniini Kolyvannya, Vol. 25, No. 2-3, pp. 191–197, April–September, 2022. Original article submitted July 12, 2022.

1072-3374/23/2746–0841

Theorem 1. *Suppose that conditions 1)–4) are satisfied. Then the system of equations (1) possesses a unique solution $x(t)$ continuous and bounded for $t \geq T$ and represented in the form of a series*

$$x(t) = \sum_{i=0}^{\infty} x_i(t), \tag{2}$$

where $x_i(t)$, $i = 0, 1, \ldots$, are vector functions continuous and bounded for $t \geq T$.

Proof. Substituting (2) in (1), we obtain

$$\sum_{i=0}^{\infty} x_i(qt) = A \sum_{i=0}^{\infty} x_i(t) + \sum_{j=1}^{k} B_j(t) \sum_{i=0}^{\infty} x_i(t + \Delta_j(t)) + F(t).$$

This directly implies that if the vector functions $x_i(t)$, $i = 0, 1, \ldots$, are solutions of the following sequence of the systems of equations:

$$x_0(qt) = Ax_0(t) + F(t), \tag{3_0}$$

$$x_i(qt) = Ax_i(t) + \sum_{j=1}^{k} B_j(t) x_{i-1}(t + \Delta_j(t)), \quad i = 1, 2, \ldots, \tag{3_i}$$

then series (2) is a formal solution of the system of equations (1).

By direct substitution in (3_0), we can show that the series

$$x_0(t) = \sum_{j=0}^{\infty} A^j F\big(q^{-(j+1)} t\big), \tag{4_0}$$

is a formal solution of the system of equations (4_0). Moreover, by virtue of conditions 1)–4), series (4_0) is uniformly convergent for all $t \in \mathbb{R}$ and satisfies the condition

$$|x_0(t)| \leq \sum_{j=0}^{\infty} \left|A^j\right| \left|F\big(q^{-(j+1)} t\big)\right| \leq M \sum_{j=0}^{\infty} a^j \leq \frac{M}{1-a} = M'.$$

We now successively consider the systems of equations (3_i), $i = 1, 2, \ldots$. This enables us to show that the series

$$x_i(t) = \sum_{j=0}^{\infty} A^j \left(\sum_{l=1}^{k} B_l\big(q^{-(j+1)} t\big) x_{i-1}\left(q^{-(j+1)} t + \Delta_j\big(q^{-(j+1)} t\big)\right) \right), \quad i = 1, 2, \ldots, \tag{4_i}$$

is uniformly convergent for $t \in \mathbb{R}$ to certain continuous vector functions $x_i(t)$, $i = 1, 2, \ldots$, which are solutions of the corresponding systems (3_i), $i = 1, 2, \ldots$, and satisfy the conditions

$$|x_i(t)| \leq M' \tilde{\Delta}^i, \quad i = 1, 2, \ldots. \tag{5_i}$$

Indeed, in view of (4_1) and conditions 1)–4), we obtain

$$|x_1(t)| \le \sum_{j=0}^{\infty} |A|^j \left(\sum_{l=1}^{k} \left| B_l\left(q^{-(j+1)}t\right)\right| \left| x_0\left(q^{-(j+1)}t + \Delta_j\left(q^{-(j+1)}t\right)\right)\right|\right)$$

$$\le M' \sum_{j=0}^{\infty} a^j \sum_{l=1}^{k} b_l \le M' \frac{\sum_{l=1}^{k} b_l}{1-a} = M'\tilde{\Delta}.$$

Hence, estimate (5_1) is true. We proceed by induction and assume that estimate (5_i) holds for some $i \ge 1$ and prove it for $i+1$. Indeed, by using (4_{i+1}), (5_i), and the conditions of the theorem, we get

$$|x_{i+1}(t)| \le \sum_{j=0}^{\infty} |A|^j \left(\sum_{l=1}^{k} \left| B_l\left(q^{-(j+1)}t\right)\right| \left| x_{i-1}\left(q^{-(j+1)}t + \Delta_j\left(q^{-(j+1)}t\right)\right)\right|\right)$$

$$\le \sum_{j=0}^{\infty} a^j \left(\sum_{l=1}^{k} b_l \right) M'\tilde{\Delta}^i \le M' \frac{\sum_{l=1}^{k} b_l}{1-a} \tilde{\Delta}^i = M'\tilde{\Delta}^{i+1}.$$

Thus, estimates (5_i) are true for all $i \ge 1$.

Hence, series (4_i), $i = 0, 1, \ldots$, are uniformly convergent for all $t \ge T > 0$ to certain continuous vector functions $x_i(t)$, $i = 0, 1, \ldots$, satisfying estimates (5_i), $i = 0, 1, \ldots$. Thus, it directly follows from (5_i), $i = 0, 1, \ldots$, that series (2) uniformly converges for all $t \in \mathbb{R}$ to a certain continuous vector function $x(t)$, which is a solution of the system of equations (1).

We now assume that system (1) has one more solution $y(t)$ such that $y(t) \ne x(t)$. Since

$$y(qt) \equiv Ay(t) + \sum_{j=1}^{k} B_j(t) y(t + \Delta_j(t)) + F(t),$$

by using the conditions of Theorem 1, we obtain

$$|x(qt) - y(qt)| \le |A|\,|x(t) - y(t)| + \sum_{j=1}^{k} |B_j(t)|\,|x(t + \Delta_j(t)) - y(t + \Delta_j(t))|$$

$$\le \left(a + \sum_{j=1}^{k} b_j \right) \|x(t) - y(t)\|,$$

where

$$\|x(t) - y(t)\| = \max_t |x(t) - y(t)|.$$

This yields the relation

$$\|x(t) - y(t)\| \leq \left(a + \sum_{j=1}^{k} b_j\right) \|x(t) - y(t)\|.$$

However, according to the conditions of the theorem, this may be true only for $y(t) \equiv x(t)$. The obtained contradiction completes the proof.

In (1), we perform a one-to-one change of variables

$$x(t) = y(t) + \gamma(t),$$

where $\gamma(t)$ is the above-constructed continuous solution of system (1) bounded for $t \geq T$. As a result, we reduce the investigation of the system of equations (1) to the analysis of the following system of equations:

$$y(qt) = Ay(t) + \sum_{j=1}^{k} B_j(t) y(t + \Delta_j(t)).$$

Under the conditions of Theorem 1, this system of equations possesses a unique solution $y \equiv 0$ continuous for $t \in \mathbb{R}$. However, under certain additional conditions, it has infinitely many solutions continuous for $t \geq T > 0$. For the sake of simplicity, we prove this fact in the case where $\Delta_j(t) \equiv j$, $j = 1, \ldots, k$, and the matrix A has the form $A = \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where $0 < \lambda_i < 1$, $i = 1, \ldots, n$.

Hence, we now consider the system of equations

$$y(qt) = \Lambda y(t) + \sum_{j=1}^{k} B_j(t) y(t + j) \tag{6}$$

and prove the following theorem:

Theorem 2. *Suppose that the conditions of Theorem 1 and the conditions*

(i) $0 < \lambda_i < 1$, $i = 1, \ldots, n$, $q > 1$;

(ii) $\bar{\Delta} = \dfrac{\sum_{l=1}^{k} b_j}{1 - \lambda^*} < 1$, *where* $b_j = \sup_t |B_j(t)|$, $j = 1, \ldots, k$, $\lambda^* = \max\{\lambda_i, i = 1, \ldots, n\}$,

are satisfied. Then the system of equations (6) has a family of solutions

$$y(t) = y\left(t, \omega\left(\frac{\ln t}{\ln q}\right)\right)$$

continuous for $t \geq T > 0$ *(*T *is a sufficiently large constant), which depends on an arbitrary continuous* 1*-periodic vector function* $\omega(\tau)$.

Proof. We show that the system of equations (6) has continuous solutions in the form of a series

$$y(t) = \sum_{i=0}^{\infty} y_i(t), \tag{7}$$

where $y_i(t)$, $i = 0, 1, \ldots,$ are certain continuous vector functions. Indeed, substituting (7) in (6), we obtain

$$\sum_{i=0}^{\infty} y_i(qt) = \Lambda \sum_{i=0}^{\infty} y_i(t) + \sum_{j=1}^{k} B_j(t) \sum_{i=0}^{\infty} y_i(t+j).$$

This directly implies that if the functions $y_i(t)$, $i = 0, 1, \ldots,$ are solutions of the following sequence of systems of equations:

$$y_0(qt) = \Lambda y_0(t), \tag{8_0}$$

$$y_i(qt) = \Lambda y_i(t) + \sum_{j=1}^{k} B_j(t) y_{i-1}(t+j), \quad i = 1, 2, \ldots, \tag{8_i}$$

then series (7) is a formal solution of the system of equations (6).

The system of equations (8_0) has a set of solutions continuous for $t \geq T > 0$ of the form

$$y_0(t) = t^{\nu} \omega \left(\frac{\ln t}{\ln q} \right), \tag{9_0}$$

where $\omega(\tau) = \big(\omega_1(\tau), \omega_2(\tau), \ldots, \omega_n(\tau)\big)$, $\omega_i(\tau)$, $i = 1, \ldots, n$, are arbitrary continuous 1-periodic functions and

$$t^{\nu} = \operatorname{diag} \left(t^{\frac{\ln \lambda_1}{\ln q}}, t^{\frac{\ln \lambda_2}{\ln q}}, \ldots, t^{\frac{\ln \lambda_n}{\ln q}} \right).$$

Successively considering the systems of equations (8_i), $i = 1, 2, \ldots,$ we can show that they have formal solutions in the form of series

$$y_i(t) = \sum_{j=0}^{\infty} \Lambda^j \sum_{l=1}^{k} B_l\big(q^{-(j+1)} t\big) y_{i-1}\big(q^{-(j+1)} t + l + 1\big), \quad i = 1, 2, \ldots. \tag{9_i}$$

We now prove that, under the conditions of the theorem, series (9_i), $i = 1, 2, \ldots,$ uniformly converge to certain continuous vector functions $y_i(t)$, $i = 1, 2, \ldots,$ satisfying the estimates

$$|y_i(t)| \leq \tilde{M} \bar{\Delta}^i, \quad i = 1, 2, \ldots. \tag{10}$$

Indeed, since

$$|y_0(t)| \leq |t^{\upsilon}| \, |\omega(\tau)| \leq t^{\frac{\ln \lambda_*}{\ln q}} \tilde{M},$$

where $\tilde{M} = \max_{\tau} |\omega(\tau)|$ and $\lambda_* = \min\{\lambda_i, i = 1, \dots, n\}$, in view of ($8_i$) and

$$\frac{\ln \lambda_*}{\ln q} < 0,$$

we obtain

$$|y_1(t)| \le \sum_{j=0}^{\infty} |\Lambda|^j \left(\sum_{l=1}^{k} \left| B_l\big(q^{-(j+1)}t\big) \right| \left| y_0\big(q^{-(j+1)}t + l + 1\big) \right| \right)$$

$$\le \tilde{M} \sum_{j=0}^{\infty} |\Lambda|^j \left(\sum_{l=1}^{k} |b_l| \left(\frac{t}{q^{j+1}} + l + 1 \right)^{\frac{\ln \lambda_*}{\ln q}} \right)$$

$$\le \tilde{M} \sum_{j=0}^{\infty} \left(\lambda^*\right)^j \sum_{l=1}^{k} |b_l| \le \tilde{M} \frac{\sum_{l=1}^{k} |b_l|}{1 - \lambda^*} \le \tilde{M} \bar{\Delta}.$$

Thus, estimate (10) holds for $i = 1$. We assume that it is true for some $i \ge 1$ and prove it for $i + 1$. In view of (9_{i+1}), (10), $i = 1, 2, \dots,$ we find

$$|y_{i+1}(t)| \le \sum_{j=0}^{\infty} |\Lambda|^j \left(\sum_{l=1}^{k} \left| B_l\big(q^{-(j+1)}t\big) \right| \left| y_i\big(q^{-(j+1)}t + l + 1\big) \right| \right)$$

$$\le \sum_{j=0}^{\infty} \left|\lambda^*\right|^j \left(\tilde{M} \sum_{l=1}^{k} |b_l| \right) \bar{\Delta}^i \le \tilde{M} \frac{\sum_{l=1}^{k} |b_l|}{1 - \lambda^*} \bar{\Delta}^i = \tilde{M} \bar{\Delta}^{i+1}.$$

Thus, estimates (10) are true for all $i \ge 1$ and series (9_i), $i = 1, 2, \dots,$ uniformly converge to certain continuous vector functions $y_i(t)$, $i = 1, 2, \dots$. Hence, we have proved that series (7) is uniformly convergent for all $t \ge T > 0$ to a certain continuous function $y(t)$, which is a solution of the system of equations (6) and satisfies the condition

$$|y(t)| \le \sum_{i=0}^{\infty} |y_i(t)| \le \tilde{M} \sum_{i=0}^{\infty} \bar{\Delta}^i \le \frac{\tilde{M}}{1 - \bar{\Delta}}.$$

Theorem 2 is proved.

Theorem 3. *Suppose that the conditions of Theorem 1 and the conditions*

(i) $\lambda_i > 1$, $i = 1, \dots, n$, $0 < q < 1$;

(ii) $\bar{\Delta} = \dfrac{\sum_{l=1}^{k} b_j}{\lambda_* - 1} < 1$, *where* $b_j = \sup_t |B_j(t)|$, $j = 1, \dots, k$, $\lambda_* = \min\{\lambda_i, i = 1, \dots, n\}$,

are satisfied. Then the system of equations (6) has a family of solutions

$$y(t) = y\left(t, \omega\left(\frac{\ln t}{\ln q}\right)\right)$$

continuous for $t \geq T > 0$ (T is a certain sufficiently large constant), which depends on an arbitrary continuous 1-periodic vector function $\omega(\tau)$.

Proof. We now show that the system of equations (6) has continuous solutions in the form of a series

$$y(t) = \sum_{i=0}^{\infty} y_i(t), \tag{11}$$

where $y_i(t)$, $i = 0, 1, \ldots$, are continuous vector functions. Indeed, substituting (11) in (6), we obtain

$$\sum_{i=0}^{\infty} y_i(qt) = \Lambda \sum_{i=0}^{\infty} y_i(t) + \sum_{j=1}^{k} B_j(t) \sum_{i=0}^{\infty} y_i(t+j).$$

This directly implies that if the functions $y_i(t)$, $i = 0, 1, \ldots$, are solutions of the sequence of systems of equations

$$y_0(qt) = \Lambda y_0(t), \tag{12_0}$$

$$y_i(qt) = \Lambda y_i(t) + \sum_{j=1}^{k} B_j(t) y_{i-1}(t+j), \quad i = 1, 2, \ldots, \tag{12_i}$$

then series (11) is a formal solution of the system of equations (6).

The system of equations (12_0) has a set of solutions continuous for $t \geq T > 0$ of the form

$$y_0(t) = t^{\nu} \omega\left(\frac{\ln t}{\ln q}\right), \tag{13_0}$$

where $\omega(\tau) = (\omega_1(\tau), \omega_2(\tau), \ldots, \omega_n(\tau))$, $\omega_i(\tau)$, $i = 1, \ldots, n$, are arbitrary continuous 1-periodic functions, and

$$t^{\nu} = \operatorname{diag}\left(t^{\frac{\ln \lambda_1}{\ln q}}, t^{\frac{\ln \lambda_2}{\ln q}}, \ldots, t^{\frac{\ln \lambda_n}{\ln q}}\right).$$

We successively consider the systems of equations (12_i), $i = 1, 2, \ldots$, and show that they have formal solutions in the form of the following series:

$$y_i(t) = -\sum_{j=0}^{\infty} \Lambda^{-(j+1)} \sum_{l=1}^{k} B_l(q^j t) y_{i-1}(q^j t + l + 1), \quad i = 1, 2, \ldots. \tag{13_i}$$

We now show that, under the conditions of the theorem, series (13_i), $i = 1, 2, \ldots$, uniformly converge to continuous vector functions $y_i(t)$, $i = 1, 2, \ldots$, satisfying the estimates

$$|y_i(t)| \leq \tilde{M}\,\bar{\Delta}^i, \quad i = 1, 2, \ldots. \tag{14}$$

Indeed, since

$$|y_0(t)| \leq |t^v|\,|\omega(\tau)| \leq t^{\frac{\ln\lambda_*}{\ln q}}\,\tilde{M} \leq \frac{\tilde{M}}{t^{\left|\frac{\ln\lambda_*}{\ln q}\right|}},$$

where $\tilde{M} = \max_\tau |\omega(\tau)|$ and $\lambda_* = \min\{\lambda_i, i = 1, \ldots, n\}$, in view of ($12_1$) and

$$\frac{\ln\lambda_*}{\ln q} < 0,$$

we obtain

$$|y_1(t)| \leq \sum_{j=0}^{\infty} \left|\Lambda^{-1}\right|^{j+1} \left(\sum_{l=1}^{k} \left|B_l(q^j t)\right| \left|y_0(q^j t + l + 1)\right|\right)$$

$$\leq \tilde{M} \sum_{j=0}^{\infty} \left(\frac{1}{\lambda_*}\right)^{j+1} \left(\sum_{l=1}^{k} |b_l| \frac{1}{\left(q^j t + l + 1\right)^{\left|\frac{\ln\lambda_*}{\ln q}\right|}}\right)$$

$$\leq \frac{\tilde{M}\sum_{l=1}^{k}|b_l|}{\lambda_*} \sum_{j=0}^{\infty} \left(\frac{1}{\lambda_*}\right)^{j} \leq \frac{\tilde{M}\sum_{l=1}^{k}|b_l|}{\lambda_*} \frac{1}{1 - \dfrac{1}{\lambda_*}}$$

$$\leq \tilde{M} \frac{\sum_{l=1}^{k}|b_l|}{\lambda_* - 1} \leq \tilde{M}\,\bar{\Delta}.$$

Thus, estimate (14) holds for $i = 1$. We assume that it is true for some $i \geq 1$ and prove it for $i + 1$. Indeed, according to (13_{i+1}) and (14), $i = 1, 2, \ldots,$ we get

$$|y_{i+1}(t)| \leq \sum_{j=0}^{\infty} \left|\Lambda^{-1}\right|^{j+1} \left(\sum_{l=1}^{k} \left|B_l(q^j t)\right| \left|y_i(q^j t + l + 1)\right|\right)$$

$$\leq \tilde{M} \sum_{j=0}^{\infty} \left(\frac{1}{\lambda_*}\right)^{j+1} \left(\sum_{l=1}^{k} |b_l|\Delta^i\right)$$

$$\leq \frac{\tilde{M}}{\lambda_*} \sum_{j=0}^{\infty} \left(\frac{1}{\lambda_*}\right)^{j} \left(\sum_{l=1}^{k} |b_l|\Delta^i\right)$$

$$\leq \frac{\tilde{M}\sum_{l=1}^{k}|b_l|}{\lambda_*}\frac{\Delta^i}{1-\dfrac{1}{\lambda_*}}$$

$$\leq \tilde{M}\frac{\sum_{l=1}^{k}|b_l|}{\lambda_*-1}\Delta^i \leq \tilde{M}\,\bar{\Delta}^{i+1}.$$

Hence, estimates (14) are true for all $i \geq 1$ and series (13_i), $i = 1, 2, \ldots$, uniformly converge to certain continuous vector functions $y_i(t)$, $i = 1, 2, \ldots$. This proves that series (11) is uniformly convergent for all $t \geq T > 0$ to a certain continuous function $y(t)$, which is a solution of the system of equations (6) and satisfies the condition

$$|y(t)| \leq \sum_{i=0}^{\infty}|y_i(t)| \leq \tilde{M}\sum_{i=0}^{\infty}\bar{\Delta}^i \leq \frac{\tilde{M}}{1-\bar{\Delta}}.$$

Theorem 3 is proved.

REFERENCES

1. G. D. Birkhoff, "General theory of linear difference equations," *Trans. Amer. Math. Soc.,* **12**, No. 2, 243–284 (1911).
2. R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods, and Applications,* 2nd ed., Marcel Dekker, New York (2000).
3. W. J. Trjitzinsky, "Analytic theory of linear q-difference equations," *Acta Math.,* **61**, No. 1, 1–38 (1933); DOI: 10.1007/BF02547785.
4. D. I. Martynyuk, *Lectures on the Qualitative Theory of Difference Equations* [in Russian], Naukova Dumka, Kiev (1972).
5. A. A. Mirolyubov and M. A. Soldatov, *Linear Inhomogeneous Difference Equations* [in Russian], Nauka, Moscow (1986).
6. G. P. Pelyukh, "On the theory of systems of linear difference equations with continuous argument," *Dokl. Akad. Nauk,* **73**, No. 2, 269–272 (2006).
7. H. P. Pelyukh and O. A. Sivak, "On the structure of the set of continuous solutions of functional difference equations with linearly transformed argument," *Nelin. Kolyv.,* **13**, No. 1, 75–95 (2010); ***English translation:*** *Nonlin. Oscillat.,* **13**, No. 1, 85–107 (2010).
8. T. O. Er'omina, "Investigation of the structure of the set of continuous solutions for systems of linear functional-difference equations," *Nelin. Kolyv.,* **17**, No. 3, 341–350 (2014); ***English translation:*** *J. Math. Sci.,* **212**, No. 3, 264–274 (2016).