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DETERMINATION OF THE STATIC THERMOELASTIC STATE OF LAYERED THERMOSENSITIVE PLATE, CYLINDER, AND SPHERE

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We propose a method for the determination of static thermoelastic states of multilayer bodies of canonical shapes with regard for the thermal radiation, convective heat exchange, and arbitrary temperature dependences of the physical and mechanical characteristics of the material under the action of surface and volume heat sources. For the solution of the corresponding heat-conduction and thermoelasticity problems with piecewise constant characteristics, we use the Kirchhoff transformation, Newton iterative method, generalized functions, and Green functions. The results of numerical investigations are presented.

Keywords: thermosensitive plate, cylinder, sphere, thermal radiation, thermoelastic state, Kirchhoff transformation, Newton iterative method, Green functions, generalized functions.

It is known that the presence of temperature-dependent physical and mechanical characteristics in mathematical models used to describe the thermoelastic states of structural elements operating under the conditions of significant temperature variations leads to the necessity of solution of nonlinear problems of heat conduction and thermoelasticity with coordinate-dependent coefficients. Among the methods used for their solution, an important place is occupied by numerical-analytic methods including, in particular, the methods aimed at the determination of temperature fields based on the Kirchhoff transformation. Most frequently, these methods are used in the problems with linear [1–3, 5, 6, 8–10, 15, 16], quadratic [3, 4, 9], or cubic [9, 13] thermal conductivity coefficients. In [7, 15], for the solution of the corresponding problems, it was proposed to approximate physical and mechanical characteristics by piecewise constant functions of temperature.

In [13], without using the Kirchhoff transformation, the problems of heat conduction with heat fluxes and temperature given on the surfaces of the first and last layers, respectively, were reduced to the solution of systems of integral Volterra equations of the second kind for any temperature dependences of the thermal conductivity coefficients.

In the present paper, we develop the approaches proposed in [4–6, 13, 15] and generalize them to the problems of evaluation of static thermoelastic states of multilayer bodies of canonical shapes with heat sources under linear and nonlinear boundary conditions for any character of temperature dependences of the physical and mechanical characteristics of the layers whose thicknesses may be noticeably different.

In order to determine temperature fields, it is necessary to solve the problems of heat conduction with boundary conditions of the first kind on one of the boundaries and of the second kind on the opposite boundary for any dependence of the thermal conductivity coefficients on temperature. These problems are solved with the help of the Kirchhoff transformation, generalized functions, Green functions, and the Newton iterative method.

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With the help of the solutions of these problems, the original problems are reduced to a single nonlinear algebraic equation.

To determine the thermoelastic state of the plate, we use the solution of the problem of thermoelasticity in stresses with boundary conditions satisfied on the cylindrical surface in the integral form.

The distributions of displacements, strains, and stresses in isotropic cylinders and spheres are described by relations represented in the unified form. We obtain them from the analytic solutions of the systems of integro-algebraic equations (in displacements) of the problems of thermoelasticity for piecewise inhomogeneous anisotropic cylinders [11] and spheres [12]. The indicated solutions are found by the method of successive approximations in which we restrict ourselves to the first approximation. As the zero-order approximation, we take the exact solutions of the problems of heat conduction for the corresponding multilayer bodies with constant moduli of elasticity and Poisson's ratios and given temperature dependences of the coefficients of linear thermal expansion.

1. Statement and Solution of the Problems of Heat Conduction

Assume that the bounding surfaces of the multilayer plate $(k=0)$, cylinder $(k=1)$, and sphere $(k=2)$ occupy the domains $\tilde{x}_0 \le \tilde{x} \le \tilde{x}_n$, $i=1,2,\ldots,n$, where n is the number of layers, and are either kept at temperatures t_c^- and t_c^+ or heated by convective heat exchange with media kept at temperatures t_{c0} and t_{cn} and by heat fluxes with densities q_0 and q_n, respectively. The layers are in perfect contact and contain internal heat sources with density $w_0^{(i)} w_t^{(i)}(\tilde{x})$ and heat fluxes with density q_j specified on the interfaces $\tilde{x}=\tilde{x}_j$, $j=1,2,\ldots,n-1$. At the same time, the heat fluxes of self-radiation are removed from the bounding surfaces according to the Stefan–Boltzmann law. We determine stationary temperature fields in these bodies with regard for the temperature dependences of the thermal conductivity coefficients $\lambda_t^{(i)}(T_i) = \lambda_0^{(i)} \Lambda_i(T_i)$ of the materials of layers, the heat-transfer coefficients $\alpha_0(T_1) = \tilde{\alpha}_0 \alpha_0^*(T_1)$ and $\alpha_n(T_n) = \tilde{\alpha}_n \alpha_n^*(T_n)$, and the emissivity factors of the surfaces $\varepsilon_0(T_1) = \tilde{\varepsilon}_0 \varepsilon_0^*(T_1)$ and $\varepsilon_n(T_n) = \tilde{\varepsilon}_n \varepsilon_n^*(T_n)$. Here, the factors multiplying the functions have the dimensionalities of the corresponding quantities.

Under the accepted assumptions, the mathematical model includes the following relations represented in the dimensionless form:

– *the heat-conduction equations:*

$$\frac{1}{x^k}\frac{d}{dx}\left[x^k \bar{\Lambda}_i(\bar{T}_i)\frac{\lambda_0^{(i)}}{\lambda_0^{(1)}}\frac{d\bar{T}_i}{dx}\right] = -\mathrm{Po}_i\, \bar{w}_t^{(i)}(x), \qquad k=0,1,2, \quad i=1,2,\ldots,n, \tag{1}$$

– *the conditions of contact between the layers:*

$$\bar{\Lambda}_{j+1}(\bar{T}_{j+1})\frac{\lambda_0^{(j+1)}}{\lambda_0^{(1)}}\frac{d\bar{T}_{j+1}}{dx} - \bar{\Lambda}_j(\bar{T}_j)\frac{\lambda_0^{(j)}}{\lambda_0^{(1)}}\frac{d\bar{T}_j}{dx} = -\mathrm{Ki}_j,$$

$$\bar{T}_{j+1} = \bar{T}_j, \qquad x=x_j, \quad j=1,2,\ldots,n-1, \tag{2}$$

– *the boundary conditions for* $x = x_0$:

$$\bar{T}_1\big|_{x=x_0} = \bar{t}_c^{\,-} \tag{3}$$

or

$$\left(\bar{\Lambda}_1(\bar{T}_1)\frac{d\bar{T}_1}{dx} - \mathrm{Bi}_0\,\bar{\alpha}_0(\bar{T}_1)(\bar{T}_1 - \bar{t}_{c0}) - \mathrm{Sk}_0\,\bar{\varepsilon}_0(\bar{T}_1)\bar{T}_1^{\,4}\right)\Bigg|_{x=x_0} = -\mathrm{Ki}_0, \tag{4}$$

– *the boundary conditions for* $x = x_n$:

$$\bar{T}_n\big|_{x=x_n} = \bar{t}_c^{\,+} \tag{5}$$

or

$$\left(\bar{\Lambda}_n(\bar{T}_n)\frac{d\bar{T}_n}{dx} + \mathrm{Bi}_n\,\bar{\alpha}_n(\bar{T}_n)(\bar{T}_n - \bar{t}_{cn}) + \mathrm{Sk}_n\,\bar{\varepsilon}_n(\bar{T}_n)\bar{T}_n^{\,4}\right)\Bigg|_{x=x_n} = \mathrm{Ki}_n, \tag{6}$$

and also some other versions of boundary conditions that can be obtained by combining the values $\mathrm{Bi}_p = 0$ and $\mathrm{Sk}_p = 0$, $p = 0, n$.

Here,

$$\bar{T}_i = \frac{T_i}{T_s}, \quad x = \frac{\tilde{x}}{\ell}, \quad x_j = \frac{\tilde{x}_j}{\ell}, \quad j = 0, 1, \ldots, n,$$

$$\mathrm{Po}_i = \frac{\ell^2 w_0^{(i)}}{\lambda_0^{(1)} T_s}, \quad \mathrm{Bi}_0 = \frac{\ell\tilde{\alpha}_0}{\lambda_0^{(1)}}, \quad \mathrm{Bi}_n = \frac{\ell\tilde{\alpha}_n}{\lambda_0^{(n)}},$$

$$\mathrm{Sk}_0 = \frac{\ell\tilde{\varepsilon}_0\sigma_0}{\lambda_0^{(1)}}T_s^3, \quad \mathrm{Sk}_n = \frac{\ell\tilde{\varepsilon}_n\sigma_0}{\lambda_0^{(n)}}T_s^3, \quad \mathrm{Ki}_0 = \frac{\ell q_0}{\lambda_0^{(1)} T_s}, \quad \mathrm{Ki}_n = \frac{\ell q_n}{\lambda_0^{(n)} T_s},$$

$$\left[\bar{\Lambda}_i(\bar{T}), \bar{\alpha}_p(\bar{T}), \bar{\varepsilon}_p(\bar{T})\right] = \left[\Lambda_i(T), \alpha_p^*(T), \varepsilon_p^*(T)\right]\Big|_{T=T_s\bar{T}}, \quad p = 0, n, \quad \bar{w}_t^{(i)}(x) = w_t^{(i)}(x\ell), \; i = 1, 2, \ldots, n,$$

$$\mathrm{Ki}_j = \frac{\ell q_j}{\lambda_0^{(1)} T_s}, \quad j = 1, 2, \ldots, n-1, \quad \bar{t}_{cp} = \frac{t_{cp}}{T_s}, \quad \bar{t}_c^{\,\pm} = \frac{t_c^{\pm}}{T_s},$$

σ_0 is the Stefan–Boltzmann constant, T_s is the typical temperature of the problem, and ℓ is a parameter with dimensionality of length.

The temperature fields are determined depending on the version of boundary conditions with the help of analytic solutions of the following two problems:

Problem 1. We determine the temperature distributions

$$\bar{T}_i = \bar{T}_i^{(k)}\left(x, \mathrm{Ki}_0^{*(k)}, T_{cn}^{(k)}\right), \qquad k = 0, 1, 2, \quad i = 1, 2, \ldots, n, \tag{7}$$

satisfying Eqs. (1), the contact conditions (2), and the boundary conditions

$$\left.\left[\bar{\Lambda}_1(\bar{T}_1^{(k)})\frac{d\bar{T}_1^{(k)}}{dx}\right]\right|_{x=x_0} = -\mathrm{Ki}_0^{*(k)}, \tag{8}$$

$$\left.\bar{T}_n^{(k)}\right|_{x=x_n} = T_{cn}^{(k)}, \tag{9}$$

where $\mathrm{Ki}_0^{*(k)}$ and $T_{cn}^{(k)}$ are unknown values of the dimensionless heat flux and temperature, respectively, on the surfaces $x = x_0$ and $x = x_n$.

By using the Kirchhoff transformation

$$\theta_i^{(k)} = \int\limits_{\bar{T}_*}^{\bar{T}_i^{(k)}} \bar{\Lambda}_i(\zeta)\, d\zeta, \tag{10}$$

we reduce the problem of finding these temperature distributions to the solution of the problems for the Kirchhoff variables containing $\mathrm{Ki}_0^{*(k)}$ and $T_{cn}^{(k)}$ as parameters:

$$\frac{1}{x^k}\frac{d}{dx}\left(x^k\,\frac{\lambda_0^{(i)}}{\lambda_0^{(1)}}\frac{d\theta_i^{(k)}(x)}{dx}\right) = -\mathrm{Po}_i\,\bar{w}_t^{(i)}(x), \qquad k = 0, 1, 2, \quad i = 1, 2, \ldots, n, \tag{11}$$

$$\left.\left(\frac{\lambda_0^{(j+1)}}{\lambda_0^{(1)}}\frac{d\theta_{j+1}^{(k)}}{dx} - \frac{\lambda_0^{(j)}}{\lambda_0^{(1)}}\frac{d\theta_j^{(k)}}{dx}\right)\right|_{x=x_j} = -\mathrm{Ki}_j, \tag{12}$$

$$\left.\left(\theta_{j+1}^{(k)} - \theta_j^{(k)}\right)\right|_{x=x_j} = F_{j+1}^{(k)}, \qquad x = x_j, \quad j = 1, 2, \ldots, n-1, \tag{13}$$

$$\left.\frac{d\theta_1^{(k)}}{dx}\right|_{x=x_0} = -\mathrm{Ki}_0^{*(k)}, \qquad \left.\theta^{(k)}\right|_{x=x_n} = \theta_{cn}^{(k)}, \tag{14}$$

where

$$F_{j+1}^{(k)} = \int\limits_{\overline{T}_*}^{\overline{T}_{j+1}^{(k)}\left(x_j,\,\mathrm{Ki}_0^{*(k)},\,T_{cn}^{(k)}\right)} \left[\overline{\Lambda}_{j+1}(\zeta) - \overline{\Lambda}_j(\zeta)\right] d\zeta, \tag{15}$$

$$\theta_{cn}^{(k)} = \int\limits_{\overline{T}_*}^{T_{cn}^{(k)}} \overline{\Lambda}_n(\zeta)\, d\zeta\,, \qquad \overline{T}_* = \frac{T_*}{T_s}\,,$$

and T_* is the lower bound of the temperature range of variations of thermophysical characteristics.

Note that only contact conditions (13) are nonlinear in problems (11)–(14).

Further, we introduce the following functions:

$$\begin{aligned} \theta^{(k)}(x) &= \theta_1^{(k)}(x) + \sum_{i=1}^{n-1}\left[\theta_{i+1}^{(k)}(x) - \theta_i^{(k)}(x)\right] S(x-x_i), \\ \lambda_0(x) &= \lambda_0^{(1)}(x) + \sum_{i=1}^{n-1}\left[\lambda_0^{(i+1)}(x) - \lambda_0^{(i)}(x)\right] S(x-x_i), \\ \mathrm{Po}(x) &= \mathrm{Po}_1(x) + \sum_{i=1}^{n-1}\left[\mathrm{Po}_{i+1}(x) - \mathrm{Po}_i(x)\right] S(x-x_i), \\ \overline{w}_t(x) &= \overline{w}_t^{(1)}(x) + \sum_{i=1}^{n-1}\left[\overline{w}_t^{(i+1)}(x) - \overline{w}_t^{(i)}(x)\right] S(x-x_i), \end{aligned} \tag{16}$$

where $S(\zeta)$ is the Heaviside function and, for any value of k, $k = 0, 1, 2$, consider the following equation with generalized derivatives equivalent to the system of equations (11) and the contact conditions (12) and (13):

$$\frac{d}{dx}\left[x^k \frac{\lambda_0(x)}{\lambda_0^{(1)}} \frac{d\theta^{(k)}}{dx}\right] = -\sum_{j=1}^{n-1} x_j^k\, \mathrm{Ki}_j\, \delta(x-x_j)$$

$$+ \sum_{j=1}^{n-1} F_{j+1}^{(k)} \frac{d}{dx}\left[\frac{\lambda_0(x)}{\lambda_0^{(1)}}\, x^k \delta(x-x_j)\right] - x^k\, \mathrm{Po}(x)\, \overline{w}_t(x)\,. \tag{17}$$

Here, $\delta(\zeta)$ is the Dirac delta-function. Problems (17) and (14) are solved by using the Green functions

$$G^{(k)}(x,\rho) = \left[f^{(k)}(x_n) - f^{(k)}(x)\right] S(x-\rho) + \left[f^{(k)}(x_n) - f^{(k)}(\rho)\right] S(\rho - x), \tag{18}$$

which are solutions of the following problems:

$$\frac{d}{dx}\left[x^k \frac{\lambda_0(x)}{\lambda_0^{(1)}} \frac{dG^{(k)}(x,\rho)}{dx}\right] = -\delta(x-\rho), \tag{19}$$

$$\left.\frac{dG^{(k)}(x,\rho)}{dx}\right|_{x=x_0} = 0, \quad G^{(k)}(x,\rho)\big|_{x=x_n} = 0. \tag{20}$$

Here,

$$f^{(0)}(x) = x + \sum_{j=1}^{n-1} H_j(x-x_j)S(x-x_j),$$

$$f^{(1)}(x) = \ln x + \sum_{j=1}^{n-1} H_j \ln\frac{x}{x_j} S(x-x_j),$$

$$f^{(2)}(x) = -\frac{1}{x} - \sum_{j=1}^{n-1} H_j \left(\frac{1}{x}-\frac{1}{x_j}\right) S(x-x_j),$$

$$H_j = \frac{\lambda_0^{(1)}}{\lambda_0^{(j+1)}} - \frac{\lambda_0^{(1)}}{\lambda_0^{(j)}}.$$

Multiplying both sides of Eq. (17) by the function $G^{(k)}(x,\rho)$, after necessary transformations, we get

$$\frac{d}{dx}\left[x^k G^{(k)}(x,\rho)\frac{\lambda_0(x)}{\lambda_0^{(1)}}\frac{d\theta^{(k)}}{dx}\right] - \frac{d}{dx}\left[x^k \frac{dG^{(k)}(x,\rho)}{dx}\frac{\lambda_0(x)}{\lambda_0^{(1)}}\theta^{(k)}\right]$$

$$+ \frac{d}{dx}\left[x^k \frac{dG^{(k)}(x,\rho)}{dx}\frac{\lambda_0(x)}{\lambda_0^{(1)}}\right]\theta^{(k)}$$

$$= -\sum_{j=1}^{n-1} x_j^k \,\mathrm{Ki}_j\, G^{(k)}(x,\rho)\delta(x-x_j)$$

$$+ \sum_{j=1}^{n-1} F_{j+1}^{(k)} \frac{d}{dx}\left[\frac{\lambda_0(x)}{\lambda_0^{(1)}} x^k G^{(k)}(x,\rho)\delta(x-x_j)\right]$$

$$- \sum_{j=1}^{n-1} F_{j+1}^{(k)} \frac{dG^{(k)}(x,\rho)}{dx}\frac{\lambda_0(x)}{\lambda_0^{(1)}} x^k \delta(x-x_j)$$

$$- x^k G^{(k)}(x,\rho)\,\mathrm{Po}(x)\bar{w}_t(x). \tag{21}$$

We now integrate Eq. (21) from x_0 to x_n and take into account (19), the boundary conditions (12), (14), and (20), and the identity

$$x^k \frac{\lambda_0(x)}{\lambda_0^{(1)}} \frac{dG^{(k)}(x,\rho)}{dx} = -S(x-\rho).$$

Interchanging the arguments x and ρ in the obtained relation, we get

$$\theta^{(k)}(x) = \theta_{cn}^{(k)} + x_0^k \, \mathrm{Ki}_0^{*(k)} \left[f^{(k)}(x_n) - f^{(k)}(x) \right]$$

$$+ \sum_{j=1}^{n-1} x_j^k \, \mathrm{Ki}_j \, G^{(k)}(x,\rho)\Big|_{\rho=x_j} - \sum_{j=1}^{n-1} F_{j+1}^{(k)} S(x_j - x) + W^{(k)}(x), \tag{22}$$

where

$$W^{(k)}(x) = \int_{x_0}^{x_n} \rho^k G^{(k)}(x,\rho) \, \mathrm{Po}(\rho) \bar{w}_t(\rho) \, d\rho$$

$$= f^{(k)}(x_n) \int_{x_0}^{x_n} \rho^k \, \mathrm{Po}(\rho) \bar{w}_t(\rho) \, d\rho$$

$$- f^{(k)}(x) \int_{x_0}^{x} \rho^k \, \mathrm{Po}(\rho) \bar{w}_t(\rho) \, d\rho - \int_{x}^{x_n} \rho^k \, \mathrm{Po}(\rho) \bar{w}_t(\rho) f^{(k)}(\rho) \, d\rho.$$

In each layer $n, \; n-1, \ldots, 1,$ the Kirchhoff variables are determined by using the following formulas derived from (22):

$$\begin{aligned}
\theta_n^{(k)}(x) &= \theta_{cn}^{(k)} + x_0^k \, \mathrm{Ki}_0^{*(k)} \left[f_n^{(k)}(x_n) - f_n^{(k)}(x) \right] \\
&\quad + \sum_{j=1}^{n-1} x_j^k \, \mathrm{Ki}_j \, G_i^{(k)}(x, x_j) + W_n^{(k)}(x), \\
\theta_i^{(k)}(x) &= \theta_{cn}^{(k)} + x_0^k \, \mathrm{Ki}_0^{*(k)} \left[f_n^{(k)}(x_n) - f_i^{(k)}(x) \right] \\
&\quad + \sum_{j=1}^{n-1} x_j^k \, \mathrm{Ki}_j \, G_i^{(k)}(x, x_j) - \sum_{j=i}^{n-1} F_{j+1}^{(k)} + W_i^{(k)}(x), \quad i = n-1, n-2, \ldots, 1,
\end{aligned} \tag{23}$$

where

$$f_i^{(0)}(x) = \frac{\lambda_0^{(1)}}{\lambda_0^{(i)}} x - \sum_{j=1}^{i-1} H_j x_j, \qquad f_i^{(1)}(x) = \frac{\lambda_0^{(1)}}{\lambda_0^{(i)}} \ln x - \sum_{j=1}^{i-1} H_j \ln x_j,$$

$$f_i^{(2)}(x) = -\frac{\lambda_0^{(1)}}{\lambda_0^{(i)}} \frac{1}{x} + \sum_{j=1}^{i-1} H_j \frac{1}{x_j},$$

$$G_i^{(k)}(x, x_j) = [f_j^{(k)}(x_j) - f_i^{(k)}(x)] S(x - x_j) + f_n^{(k)}(x_n) - f_j^{(k)}(x_j),$$

$$W_i^{(k)}(x) = f_n^{(k)}(x_n) \sum_{m=1}^{n} \int_{x_{m-1}}^{x_m} \rho^k \,\mathrm{Po}_m\, w_t^{(m)}(\rho)\, d\rho$$

$$- f_i^{(k)}(x) \left[\sum_{m=1}^{i-1} \int_{x_{m-1}}^{x_m} \rho^k \,\mathrm{Po}_m\, w_t^{(m)}(\rho)\, d\rho + \int_{x_{i-1}}^{x} \rho^k \,\mathrm{Po}_i\, w_t^{(i)}(\rho)\, d\rho \right]$$

$$- \left[\int_{x}^{x_i} \rho^k \,\mathrm{Po}_i\, w_t^{(i)}(\rho)\, d\rho + \sum_{m=i}^{n} \int_{x_m}^{x_m} \rho^k \,\mathrm{Po}_m\, w_t^{(m)}(\rho)\, d\rho \right].$$

By using (23), on the interfaces $x = x_{n-1}$, $x = x_{n-2}, \ldots,$ $x = x_1$, we obtain

$$\theta_n^{(k)}\Big|_{x=x_{n-1}} = \theta_{cn}^{(k)} + x_0^k \,\mathrm{Ki}_0^* \left[f_n^{(k)}(x_n) - f_n^{(k)}(x_{n-1}) \right]$$

$$+ \sum_{j=1}^{n-1} x_j^k \,\mathrm{Ki}_j\, G_n^{(k)}(x_{n-1}, x_j) + W_n^{(k)}(x_{n-1}),$$

$$\theta_{n-1}^{(k)}\Big|_{x=x_{n-2}} = \theta_{cn}^{(k)} + x_0^k \,\mathrm{Ki}_0^* \left[f_n^{(k)}(x_n) - f_{n-1}^{(k)}(x_{n-2}) \right]$$

$$+ \sum_{j=1}^{n-1} x_j^k \,\mathrm{Ki}_j\, G_{n-1}^{(k)}(x_{n-2}, x_j) - F_n^{(k)} + W_{n-1}^{(k)}(x_{n-2}), \tag{24}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\theta_2\big|_{x=x_1} = \theta_c + x_0^k \,\mathrm{Ki}_0^* \left[f^{(k)}(x_n) - f_2^{(k)}(x_1) \right]$$

$$+ \sum_{j=1}^{n-1} x_j^k \,\mathrm{Ki}_j\, G_2^{(k)}(x_1, x_j) - \sum_{j=1}^{n-1} F_{j+1}^{(k)} + W_2^{(k)}(x_1).$$

For known Kirchhoff variables, we determine the temperature distributions as follows:

$$\overline{T}_i^{(k)}\left(x, \mathrm{Ki}_0^{*(k)}, T_{cn}^{(k)}\right) = \overline{\vartheta}_i^{(k)}(x) + \overline{T}_* ; \tag{25}$$

in layers starting from $i = n$, by using equality (15) and the roots of the algebraic equations, we find

$$\Psi_i\left(\overline{\vartheta}_i^{(k)}(x)\right) - \theta_i^{(k)}(x) = 0, \quad i = n, n-1, \ldots, 1, \tag{26}$$

where

$$\Psi_i\left(\overline{\vartheta}_i^{(k)}(x)\right) = \int\limits_0^{\overline{\vartheta}_i^{(k)}(x)} \tilde{\Lambda}_i(\eta)\, d\eta, \qquad \tilde{\Lambda}_i\left(\overline{\vartheta}_i^{(k)}\right) = \overline{\Lambda}_i\left(\overline{T}_i^{(k)}\right).$$

We determine the solutions of Eqs. (26) for fixed values of x by the Newton iterative method. According to this method, the formula for the $(m+1)$th approximation takes the form

$$\overline{\vartheta}_{i,m+1}^{(k)}(x) = \overline{\vartheta}_{i,m}^{(k)}(x) - \frac{\Psi_i\left(\overline{\vartheta}_{i,m}^{(k)}(x)\right) - \theta_i^{(k)}(x)}{\tilde{\Lambda}_i\left(\overline{\vartheta}_{i,m}^{(k)}(x)\right)}. \tag{27}$$

To determine the values of temperature for the other versions of boundary conditions, it is necessary to substitute the values of $\mathrm{Ki}_0^{*(k)}$ and $T_{cn}^{(k)}$ determined from these conditions in relations (7). In particular, for the boundary conditions (4) and (6), we obtain

$$\mathrm{Ki}_0^{*(k)} = \varphi_n\left(T_{cn}^{(k)}\right), \tag{28}$$

where

$$\varphi_n\left(T_{cn}^{(k)}\right) = -\sum_{j=1}^{n-1} \mathrm{Ki}_j \left(\frac{x_j}{x_0}\right)^k - \frac{1}{x_0^k} \sum_{i=1}^{n} \mathrm{Po}_i \int\limits_{x_{i-1}}^{x_i} \rho^k \overline{w}_t^{(i)}(\rho)\, d\rho$$

$$+ \left(\frac{x_n}{x_0}\right)^k \frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} \left[\mathrm{Bi}_n\, \overline{\alpha}_n\left(T_{cn}^{(k)}\right)\left(T_{cn}^{(k)} - \overline{t}_{cn}\right) + \mathrm{Sk}_n\, \overline{\varepsilon}_n\left(T_{cn}^{(k)}\right)\left(T_{cn}^{(k)}\right)^4 - \mathrm{Ki}_n\right],$$

and $T_{cn}^{(k)}$ is the root of the equation

$$\Big\{\mathrm{Ki}_0^{*(k)} + \mathrm{Bi}_0\, \overline{\alpha}_0(\overline{T}_1)\big|_{\overline{T}_1 = \overline{T}_1^{(k)}(x_0, \mathrm{Ki}_0^{*(k)}, T_{cn}^{(k)})} \left[\overline{T}_1^{(k)}(x_0, \mathrm{Ki}_0^{*(k)}, T_{cn}^{(k)}) - \overline{t}_{c0}\right]$$

$$+ \ \mathrm{Sk}_0 \ \overline{\varepsilon}_0(\overline{T}_1)\Big|_{\overline{T}_1=\overline{T}_1^{(k)}\left(x_0,\,\mathrm{Ki}_0^{*(k)},\,T_{cn}^{(k)}\right)}$$

$$\times \left[\overline{T}_1^{(k)}\left(x_0, \mathrm{Ki}_0^{*(k)}, T_{cn}^{(k)}\right)\right]^4\Bigg\}\Bigg|_{\mathrm{Ki}_0^{*(k)}=\varphi_n(T_{cn}^{(k)})} - \mathrm{Ki}_0 \ = \ 0. \qquad (29)$$

We determine the root $T_{cn}^{(k)}$ by the bisection method. Moreover, for each approximation $T_{cn}^{(k)}$, we compute the values of $\mathrm{Ki}_0^{*(k)}$ by using relation (28) and determine $\overline{T}_1^{(k)}\left(x_0, \mathrm{Ki}_0^{*(k)}, T_{cn}^{(k)}\right)$ with regard for (15), (24), (25), (27), and (23) with $i=1$.

Problem 2. In the cases where either the temperature is given on the surface $x=x_0$ and condition (6) is given on the surface $x=x_n$ or condition (4) is given on the surface $x=x_0$ and the heat flux $\mathrm{Ki}_n^{*(k)}$ is given on the surface $x=x_n$, the problem of finding temperature fields is significantly simplified if, instead of (7), we determine the temperature distributions

$$\overline{T}_i^{*} \ = \ \overline{T}_i^{*(k)}\left(x, \mathrm{Ki}_n^{*(k)}, T_{c0}^{(k)}\right), \qquad k=0,1,2, \quad i=1,2,\ldots,n, \qquad (30)$$

satisfying Eqs. (1), the contact conditions (2), and the boundary conditions

$$\overline{T}_1^{*(k)}\Big|_{x=x_0} \ = \ T_{c0}^{*(k)},$$

$$\left[\overline{\Lambda}_n\left(\overline{T}_n^{*(k)}\right)\frac{d\overline{T}_n^{*(k)}}{dx}\right]\Bigg|_{x=x_n} \ = \ \mathrm{Ki}_n^{*(k)} \qquad (31)$$

different from (8) and (9).

We obtain the temperature distributions $\overline{T}_i^{*(k)}$ (30) by using the same scheme (7) of finding the distributions as in ***Problem 1***.

The Kirchhoff variables

$$\theta_i^{*(k)} \ = \ \int_{\overline{T}_*}^{\overline{T}_i^{*(k)}} \overline{\Lambda}_i(\zeta)\, d\zeta \qquad (32)$$

are determined from the equation

$$\frac{d}{dx}\left[x^k\,\frac{\lambda_0(x)}{\lambda_0^{(1)}}\,\frac{d\theta^{*(k)}}{dx}\right] \ = \ -\sum_{j=1}^{n-1} x_j^k\ \mathrm{Ki}_j\ \delta(x-x_j)$$

$$+\sum_{j=1}^{n-1} F_j^{*(k)}\,\frac{d}{dx}\left[\frac{\lambda_0(x)}{\lambda_0^{(1)}}\,x^k\delta(x-x_j)\right] - x^k\ \mathrm{Po}(x)\overline{w}_t(x) \qquad (33)$$

and the boundary conditions

$$\theta^{*(k)}\Big|_{x=x_0} = \theta_{c0}^{*(k)}, \qquad \frac{d\theta^{*(k)}}{dx}\Bigg|_{x=x_n} = \mathrm{Ki}_n^{*(k)}. \tag{34}$$

Here,

$$F_j^{*(k)} = \int\limits_{\bar{T}_*}^{\bar{T}_j^{*(k)}\left(x,\,\mathrm{Ki}_n^{*(k)},\,T_{c0}^{(k)}\right)} \left[\bar{\Lambda}_{j+1}(\zeta)-\bar{\Lambda}_j(\zeta)\right]d\zeta, \qquad \theta_{c0}^{*(k)} = \int\limits_{\bar{T}_*}^{T_{c0}^{*(k)}} \bar{\Lambda}_1(\zeta)\,d\zeta. \tag{35}$$

The corresponding Green functions, which are solutions of the problems

$$\frac{d}{dx}\left[x^k\,\frac{\lambda_0(x)}{\lambda_0^{(1)}}\,\frac{dG^{(k)}(x,\rho)}{dx}\right] = -\delta(x-\rho), \tag{36}$$

$$G^{(k)}(x,\rho)\Big|_{x=x_0} = 0, \qquad \frac{dG^{(k)}(x,\rho)}{dx}\Bigg|_{x=x_n} = 0, \tag{37}$$

have the following form:

$$G^{(k)}(x,\rho) = \left[f^{(k)}(x)-f^{(k)}(x_0)\right]S(\rho-x) + \left[f^{(k)}(\rho)-f^{(k)}(x_0)\right]S(x-\rho). \tag{38}$$

After transformations similar to that used in the previous problem but with regard for Eq. (36), the identity

$$x^k\,\frac{\lambda_0(x)}{\lambda_0^{(1)}}\,\frac{dG^{(k)}(x,\rho)}{dx} = S(\rho-x),$$

and the boundary conditions (34) and (37), we obtain

$$\theta^{*(k)}(x) = x_n^k\,\mathrm{Ki}_n^{*(k)}\left[f^{(k)}(x)-f^{(k)}(x_0)\right]\frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} + \theta_{c0}^{(k)} + \sum_{j=1}^{n-1} x_j^k\,\mathrm{Ki}_j\,G^{(k)}(x,\rho)\Big|_{\rho=x_j} + \sum_{j=1}^{n-1} F_j^{*(k)} S(x-x_j) + W^{(k)}(x), \tag{39}$$

where

$$W^{(k)}(x) = \int_{x_0}^{x_n} \rho^k G^{(k)}(x,\rho)\,\mathrm{Po}(\rho)\bar{w}_t(\rho)\,d\rho$$

$$= \left[f^{(k)}(x) - f^{(k)}(x_0)\right]\int_{x_0}^{x_n} \rho^k\,\mathrm{Po}(\rho)\bar{w}_t(\rho)\,d\rho$$

$$+ \int_{x_0}^{x}\left[f^{(k)}(\rho) - f^{(k)}(x)\right]\rho^k\,\mathrm{Po}(\rho)\bar{w}_t(\rho)\,d\rho.$$

To find the Kirchhoff variables in each layer $1, 2, \ldots, n$, we derive the following formulas from equality (39):

$$\theta_1^{*(k)}(x) = x_n^k\,\mathrm{Ki}_n^{*(k)}\left[f_1^{(k)}(x) - f_1^{(k)}(x_0)\right]\frac{\lambda_0^{(n)}}{\lambda_0^{(1)}}$$

$$+ \theta_{c0}^{*(k)} + \sum_{j=1}^{n-1} x_j^k\,\mathrm{Ki}_j\,G_1^{(k)}(x,x_j) + W_1^{(k)}(x),$$

$$\theta_i^{*(k)}(x) = x_n^k\,\mathrm{Ki}_n^{*(k)}\left[f_i^{(k)}(x) - f_1^{(k)}(x_0)\right]\frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} + \theta_{c0}^{*(k)}$$

$$+ \sum_{j=1}^{n-1} x_j^k\,\mathrm{Ki}_j\,G_i^{(k)}(x,x_j) + \sum_{j=1}^{i-1} F_j^{*(k)} + W_i^{(k)}(x), \quad i = 2, \ldots, n, \tag{40}$$

where

$$G_i^{(k)}(x,x_j) = \left[f_j^{(k)}(x_j) - f_i^{(k)}(x)\right]S(x - x_j) + f_i^{(k)}(x) - f_1^{(k)}(x_0),$$

$$W_i^{(k)}(x) = \left[f_i^{(k)}(x) - f_1^{(k)}(x_0)\right]\sum_{m=1}^{n}\int_{x_{m-1}}^{x_m} \rho^k\,\mathrm{Po}_m w_t^{(m)}(\rho)\,d\rho$$

$$+ \left[\sum_{m=1}^{i-1}\int_{x_{m-1}}^{x_m} f_m^{(k)}(\rho)\rho^k\,\mathrm{Po}_m w_t^{(m)}(\rho)\,d\rho\right.$$

$$\left. + \int_{x_{i-1}}^{x} f_i^{(k)}(\rho)\rho^k\,\mathrm{Po}_i w_t^{(i)}(\rho)\,d\rho\right]$$

$$- f_i^{(k)}(x)\left[\sum_{m=1}^{i-1}\int_{x_{m-1}}^{x_m}\rho^k\,\mathrm{Po}_m\,w_t^{(m)}(\rho)\,d\rho + \int_{x_{i-1}}^{x}\rho^k\,\mathrm{Po}_i\,w_t^{(i)}(\rho)\,d\rho\right].$$

To determine the Kirchhoff variables on the interfaces $x = x_1$, $x = x_2, \ldots,$ $x = x_{n-1}$, we derive the following relations from (40):

$$\theta_1^{*(k)}(x_1) = x_n^k\,\mathrm{Ki}_n^{*(k)}\left[f_1^{(k)}(x_1) - f_1^{(k)}(x_0)\right]\frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} + \theta_{c0}^{*(k)}$$

$$+ \sum_{j=1}^{n-1} x_j^k\,\mathrm{Ki}_j\,G_1^{(k)}(x_1, x_j) + W_1^{(k)}(x_1),$$

$$\theta_2^{*(k)}(x_2) = x_n^k\,\mathrm{Ki}_n^{*(k)}\left[f_2^{(k)}(x_2) - f_1^{(k)}(x_0)\right]\frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} + \theta_{c0}^{*(k)}$$

$$+ \sum_{j=1}^{n-1} x_j^k\,\mathrm{Ki}_j\,G_2^{(k)}(x_2, x_j) + F_1^{*(k)} + W_2^{(k)}(x_2), \tag{41}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\theta_{n-1}^{*(k)}(x_{n-1}) = x_n^k\,\mathrm{Ki}_n^{*(k)}\left[f_{n-1}^{(k)}(x_{n-1}) - f_1^{(k)}(x_0)\right]\frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} + \theta_{c0}^{*(k)}$$

$$+ \sum_{j=1}^{n-1} x_j^k\,\mathrm{Ki}_j\,G_{n-1}^{(k)}(x_{n-1}, x_j) + \sum_{j=1}^{n-2} F_j^{*(k)} + W_{n-1}^{(k)}(x_{n-1}).$$

The temperature distributions $\bar{T}_i^{*(k)}\left(x, \mathrm{Ki}_n^{*(k)}, T_{c0}^{(k)}\right) = \bar{\vartheta}_i^{*(k)}(x) + \bar{T}_*$ in the layers starting from $i = 1$ are found by using (35) and the roots of the algebraic equations

$$\Psi_i\left(\bar{\vartheta}_i^{*(k)}(x)\right) - \theta_i^{*(k)}(x) = 0, \quad i = 1, 2, \ldots, n.$$

As in the previous problem, we apply the Newton iterative method to find the roots of this equation for fixed values of x. As a result, for the $(m+1)$ th approximation, we get

$$\bar{\vartheta}_{i,m+1}^{*(k)}(x) = \bar{\vartheta}_{i,m}^{*(k)}(x) - \frac{\Psi\left(\bar{\vartheta}_{i,m}^{*(k)}(x)\right) - \theta_i^{*(k)}(x)}{\tilde{\Lambda}_i\left(\bar{\vartheta}_{i,m}^{*(k)}(x)\right)}. \tag{42}$$

To compute temperatures, e.g., for the boundary conditions (3) and (6), it is necessary to substitute in (30) solely the values of $\mathrm{Ki}_n^{*(k)}$ determined from the equation

$$\mathrm{Ki}_n^{*(k)} + \left[\mathrm{Bi}_n\,\bar{\alpha}_n(\bar{T}_n)(\bar{T}_n - \bar{t}_{cn}) + \mathrm{Sk}_n\,\bar{\varepsilon}_n(\bar{T}_n)\bar{T}_n^4\right]\Big|_{\bar{T}_n=\bar{T}_n^{*(k)}\left(x_n,\,\mathrm{Ki}_n^{*(k)},\,\bar{t}_c^-\right)} = \mathrm{Ki}_n\,.$$

The root $\mathrm{Ki}_n^{*(k)}$ of this equation is determined by the bisection method. Moreover, the temperature $\bar{T}_n^{*(k)}\left(x_n, \mathrm{Ki}_n^{*(k)}, \bar{t}_c^-\right)$ for each approximation $\mathrm{Ki}_n^{*(k)}$ is found by using relations (35), (41), (43), and (40) with $i = n$.

2. Determination of the Thermoelastic State

Assume that the studied bodies are free of force loads, the elasticity moduli $E_p\left(T_p^{(k)}\right)$, Poisson's ratios $\nu_p\left(T_p^{(k)}\right)$, and the coefficients of linear thermal expansion $\alpha_{tp}\left(T_p^{(k)}\right)$ of their components are functions of temperature, and the thermal strains are given by the formulas

$$\Phi_{rp}^{(k)}(x) = \int\limits_{T_*}^{T_p^{(k)}(x)} \alpha_{tp}(\zeta)\,d\zeta\,.$$

The temperature stresses acting in a circular ***plate*** are given by the relations obtained in [15]

$$\sigma_p^{(0)} = \sigma_{rp}^{(0)} = \sigma_{\varphi p}^{(0)} = E_p^{(0)}(z)\left[C_1 + zC_2 - \Phi_{rp}^{(0)}(z)\right], \quad p = 1, 2, \ldots, n, \tag{43}$$

where

$$C_1 = \frac{d_1 a_{22} - d_2 a_{12}}{d_3}\,, \qquad C_2 = \frac{d_2 a_{11} - d_1 a_{12}}{d_3}\,,$$

$$a_{11} = \sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} E_i^{(0)}(z)\,dz\,, \qquad a_{12} = \sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} zE_i^{(0)}(z)\,dz,$$

$$a_{22} = \sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} z^2 E_i^{(0)}(z)\,dz\,, \qquad d_3 = a_{11}a_{22} - a_{12}^2,$$

$$d_1 = \sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} E_i^{(0)}(z)\,\Phi_{ri}^{(0)}(z)\,dz\,, \qquad d_2 = \sum_{i=1}^{n} \int\limits_{x_{i-1}}^{x_i} zE_i^{(0)}(z)\,\Phi_{ri}^{(0)}(z)\,dz,$$

$$E_p^{(0)}(z) = \frac{E_p\left(T_p^{(0)}\right)}{1-\nu_p\left(T_p^{(0)}\right)}.$$

To determine the temperature stresses formed in a ***cylinder*** (with fastened faces) and in a ***sphere***, we use the following relations:

$$\sigma_{rp}^{(k)} = c_p^{(k)}(\rho)\left[\frac{du_p^{(k)}(\rho)}{d\rho} + k\nu_p^{*(k)}(\rho)\frac{u_p^{(k)}(\rho)}{\rho}\right] - c_p^{*(k)}(\rho)\Phi_{rp}^{(k)}(\rho),$$

$$\sigma_{\varphi p}^{(k)} = c_p^{(k)}(\rho)\left[\nu_p^{*(k)}(\rho)\frac{du_p^{(k)}(\rho)}{d\rho} + \left(\frac{1}{1-\nu_p^{(k)}(\rho)}\right)^{k-1}\frac{u_p^{(k)}(\rho)}{\rho}\right] - c_i^{*(k)}(\rho)\Phi_{rp}^{(k)}(\rho), \tag{44}$$

where the displacements normalized by ℓ and the corresponding strains are determined by the formulas established in [11, 12]:

$$u_p^{(k)}(\rho) = \frac{u_{tp}^{(k)}(\rho)}{c_p^{(k)}(\rho)} + \frac{L_{1p}^{(k)}\varphi_{2p}^{(k)}(\rho) + L_{2p}^{(k)}\varphi_{1p}^{(k)}(\rho)}{4q_k^2 Q_k c_p^{(k)}(\rho)} + \frac{V_p^{(1k)}(\rho) + V_p^{(2k)}(\rho)}{2q_k c_p^{(k)}(\rho)} + \sum_{i=1}^{n-1}\frac{g_{up}^{(ik)}(\rho)}{c_p^{(k)}(\rho)}\bar{u}_i^{(k)}(r_i), \tag{45}$$

$$\frac{du_p^{(k)}(\rho)}{d\rho} = \frac{\varepsilon_{tp}^{(k)}(\rho)}{c_p^{(k)}(\rho)} + \frac{L_{1p}^{(k)}\varphi_{4p}^{(k)}(\rho) + L_{2p}^{(k)}\varphi_{3p}^{(k)}(\rho)}{4q_k^2 Q_k c_p^{(k)}(\rho)} - \frac{kV_p^{(1k)}(\rho) - V_p^{(2k)}(\rho)}{2q_k\rho c_p^{(k)}(\rho)} + \sum_{i=1}^{n-1}\frac{g_{\varepsilon p}^{(ik)}(\rho)}{\rho c_p^{(k)}(\rho)}\bar{u}_i^{(k)}(r_i), \tag{46}$$

where

$$c_p^{(k)}(\rho) = \frac{E_p^{(k)}(\rho)\left[1-\nu_p^{(k)}(\rho)\right]}{\left[1+\nu_p^{(k)}(\rho)\right]\left[1-2\nu_p^{(k)}(\rho)\right]}, \qquad c_p^{*(k)}(\rho) = \frac{E_p^{(k)}(\rho)}{1-2\nu_p^{(k)}(\rho)},$$

$$E_p^{(k)}(\rho) = E_p\left(\bar{T}_p^{(k)}(\rho)\right), \qquad \nu_p^{(k)}(\rho) = \nu_p\left(\bar{T}_p^{(k)}(\rho)\right), \qquad \nu_p^{*(k)}(\rho) = \frac{\nu_p^{(k)}(\rho)}{1-\nu_p^{(k)}(\rho)},$$

$$u_{tp}^{(k)}(\rho) = \frac{R_{1p}^{(k)}\varphi_{2p}^{(k)}(\rho) - R_{2p}^{(k)}\varphi_{1p}^{(k)}(\rho)}{(k+1)Q_k} + \frac{1}{\rho^k}J_{tp}^{(k)}(\rho),$$

$$\varepsilon_{tp}^{(k)}(\rho) = \frac{R_{1p}^{(k)}\varphi_{4p}^{(k)}(\rho) - R_{2p}^{(k)}\varphi_{3p}^{(k)}(\rho)}{(k+1)Q_k} - \frac{k}{\rho^{k+1}} J_{tp}^{(k)}(\rho) + c_p^*(\rho)\,\Phi_{rp}^{(k)}(\rho),$$

$$R_{1p}^{(k)} = \sum_{j=1}^{p-1} M_{1j}^{(k)+} J_{tj}^{(k)}(r_j), \qquad R_{2p}^{(k)} = \sum_{j=p}^{n} M_{2j}^{(k)-} J_{tj}^{(k)}(r_j)\frac{c_{0p}^{(k)}}{r_j^{2q_k} c_{0j}^{(k)}},$$

$$\varphi_{1p}^{(k)}(\rho) = M_{1p}^{(k)+}\rho + M_{1p}^{(k)-}\frac{r_{p-1}^{k+1}}{\rho^k}, \qquad \varphi_{3p}^{(k)}(\rho) = M_{1s}^{(k)+} - kM_{1p}^{(k)-}\left(\frac{r_{p-1}}{\rho}\right)^{k+1},$$

$$\varphi_{2p}^{(k)}(\rho) = M_{2p}^{(k)+}\frac{1}{\rho^k} - M_{2p}^{(k)-}\frac{\rho}{r_p^{k+1}}, \qquad \varphi_{4p}^{(k)}(\rho) = -kM_{2p}^{(k)+}\frac{1}{\rho^{k+1}} - M_{2p}^{(k)-}\frac{1}{r_p^{k+1}},$$

$$J_{tp}^{(k)}(\rho) = \int_{r_{p-1}}^{\rho} r^k c_p^{*(k)}(r)\,\Phi_{rp}^{(k)}(r)\,dr, \qquad p = 1, 2, \ldots, n,$$

$$Q_k = \Phi_n^{(k1)}(r_n) + \beta_n^{(k)}\Phi_n^{(k0)}(r_n), \qquad M_{11}^{(k)\pm} = 2q_k\left(q_k \mp \beta_1^{(k)}\right),$$

$$M_{1p}^{(k)\pm} = \Phi_{p-1}^{(k0)}(r_{p-1})\left(q_k \pm K_p^{(k)}\right) \pm \Phi_{p-1}^{(k1)}(r_{p-1})K_{cp}^{(k)},$$

$$\Phi_1^{(km)}(r) = q_k^m\left[q_k - \beta_1^{(k)} + (-1)^m(q_k + \beta_1^{(k)})\left(\frac{r_0}{r}\right)^{2q_k}\right],$$

$$\Phi_p^{(km)}(r) = \Phi_{p-1}^{(k0)}(r_{p-1})f_{p1}^{(km)}(r) + \Phi_{p-1}^{(k1)}(r_{p-1})f_{p2}^{(km)}(r),$$

$$f_{p1}^{(km)}(r) = \frac{1}{2q_k^{1-m}}\left[q_k + K_p^{(k)} + (-1)^m\left(q_k - K_p^{(k)}\right)\left(\frac{r_{p-1}}{r}\right)^{2q_k}\right],$$

$$f_{p2}^{(km)}(r) = \frac{K_{cp}^{(k)}}{2q_k^{1-m}}\left[1 - (-1)^m\left(\frac{r_{p-1}}{r}\right)^{2q_k}\right], \qquad K_p^{(k)} = K_{cp}^{(k)}\beta_{p-1}^{(k)} - \beta_p^{(k)},$$

$$m = 0, 1, \quad p = 2, 3, \ldots, n, \quad q_1 = 1, \quad q_2 = 3/2,$$

$$M_{2p}^{(k)\pm} = \kappa_{np}^{(k2)} + \beta_n^{(k)}\kappa_{np}^{(k1)} \pm q_k\left(\kappa_{np}^{(k4)} + \beta_n^{(k)}\kappa_{np}^{(k3)}\right),$$

$$K_{cp}^{(k)} = \frac{c_{0,p-1}^{(k)}}{c_{0p}^{(k)}}, \qquad \beta_p^{(1)} = \nu_{0p}^{*(k)}, \qquad \beta_p^{(2)} = \frac{5\nu_{0p}^{(2)} - 1}{2\left(1 - \nu_{0p}^{(2)}\right)},$$

$$c_{0p}^{(k)} = \frac{E_{0p}^{(k)}\left(1-\nu_{0p}^{(k)}\right)}{\left(1+\nu_{0p}^{(k)}\right)\left(1-2\nu_{0p}^{(k)}\right)}, \qquad \nu_{0p}^{*(k)} = \frac{\nu_{0p}^{(k)}}{1-\nu_{0p}^{(k)}}, \qquad p = 1, 2, \ldots, n,$$

$$\kappa_{ii}^{(k1)} = \kappa_{ii}^{(k4)} = 1, \qquad \kappa_{ii}^{(k2)} = \kappa_{ii}^{(k3)} = 0,$$

$$\kappa_{n,i}^{(k1)} = f_{n1}^{(k0)}(r_n)\kappa_{n-1,i}^{(k1)} + f_{n2}^{(k0)}(r_n)\kappa_{n-1,i}^{(k2)},$$

$$\kappa_{n,i}^{(k2)} = f_{n1}^{(k1)}(r_n)\kappa_{n-1,i}^{(k1)} + f_{n2}^{(k1)}(r_n)\kappa_{n-1,i}^{(k2)},$$

$$\kappa_{n,i}^{(k3)} = f_{n1}^{(k0)}(r_n)\kappa_{n-1,i}^{(k3)} + f_{n2}^{(k0)}(r_n)\kappa_{n-1,i}^{(k4)},$$

$$\kappa_{n,i}^{(k4)} = f_{n1}^{(k1)}(r_n)\kappa_{n-1,i}^{(k3)} + f_{n2}^{(k1)}(r_n)\kappa_{n-1,i}^{(k4)}, \qquad i = n, n-1, \ldots, 1,$$

$$L_{1p}^{(k)} = H_{1p}^{(k)} + 4kq_k^2\gamma_1^{(k)} r_0^k \bar{u}_1^{(k)}(r_0), \qquad L_{2p}^{(k)} = H_{2p}^{(k)} - 2kq_k\gamma_n^{(k)}\bar{u}_n^{(k)}(r_n)\frac{c_{0p}^{(k)}}{r_n c_{0n}^{(k)}},$$

$$\gamma_1^{(k)} = \left[\nu_1^{*(k)}(r_0) - \nu_{01}^{*(k)}\right]c_1^{(k)}(r_0), \qquad \gamma_n^{(k)} = \left[\nu_n^{*(k)}(r_n) - \nu_{0n}^{*(k)}\right]c_n^{(k)}(r_n),$$

$$H_{1p}^{(k)} = \sum_{j=1}^{p-1} r_j^k M_{1j}^{(k)+} V_j^{(1k)}(r_j) + \sum_{j=1}^{p} r_{j-1}^k M_{1j}^{(k)-} V_j^{(2k)}(r_{j-1}),$$

$$H_{2p}^{(k)} = \sum_{j=p+1}^{n} M_{2j}^{(k)+} V_j^{(2k)}(r_{j-1})\frac{c_{0p}^{(k)}}{r_{j-1}c_{0j}^{(k)}} - \sum_{j=p}^{n} M_{2j}^{(k)-} V_j^{(1k)}(r_j)\frac{c_{0p}^{(k)}}{r_j c_{0j}^{(k)}},$$

$$g_{up}^{(ik)}(\rho) = \frac{2q_k r_i^k K_{11}^{(ik)} g_{2up}^{(ki)}(\rho) - g_{1up}^{(ki)}(\rho)}{4q_k^2 Q_k},$$

$$g_{\varepsilon p}^{(ik)}(\rho) = \frac{2q_k r_i^k K_{11}^{(ik)} g_{2\varepsilon p}^{(ki)}(\rho) - g_{1\varepsilon p}^{(ki)}(\rho)}{4q_k^2 Q_k},$$

$$g_{1up}^{(ik)}(\rho) = \begin{cases} b_{2p}^{(ik)}\varphi_{1p}^{(k)}(\rho)\dfrac{c_{0p}}{c_{0i}}, & p \le i, \\ b_{1p}^{(ik)}\varphi_{2p}^{(k)}(\rho), & p > i, \end{cases}$$

$$g_{2up}^{(ik)}(\rho) = \begin{cases} -r_{i+1}^{-k-1} M_{2,i+1}^{(k)-} \varphi_{1p}^{(k)}(\rho) \dfrac{c_{0p}}{c_{0,i+1}}, & p < i+1, \\ M_{1,i+1}^{(k)+} \varphi_{2p}^{(k)}(\rho), & p \ge i+1, \end{cases}$$

$$g_{1\varepsilon p}^{(ik)}(\rho) = \begin{cases} b_{2p}^{(ik)} \varphi_{3p}^{(k)}(\rho) \dfrac{c_{0p}}{c_{0i}}, & p \le i, \\ b_{1p}^{(ik)} \varphi_{4p}^{(k)}(\rho), & p > i, \end{cases}$$

$$g_{2\varepsilon p}^{(ik)}(\rho) = \begin{cases} -r_{i+1}^{-k-1} M_{2,i+1}^{(k)-} \varphi_{3p}^{(k)}(\rho) \dfrac{c_{0p}}{c_{0,i+1}}, & p < i+1, \\ M_{1,i+1}^{(k)+} \varphi_{4p}^{(k)}(\rho), & p \ge i+1, \end{cases}$$

$$b_{2p}^{(ik)} = \frac{1}{r_i}\Big[m_{0i}^{(k)} m_{2i}^{(1k)} - \big(1 - K_{0c}^{(k,i+1)}\big) c_i^{(k)}(r_i) m_{2i}^{(2k)}\Big], \qquad K_{0c}^{(k,i+1)} = \frac{c_{0i}^{(k)}}{c_{0,i+1}^{(k)}},$$

$$b_{1p}^{(ik)} = r_i^k \Big[m_{0i}^{(k)} m_{1i}^{(1k)} + \big(1 - K_{0c}^{(k,i+1)}\big) c_i^{(k)}(r_i) m_{1i}^{(2k)}\Big],$$

$$m_{0i}^{(k)} = k\Big[K_{11}^{(ik)} - K_{12}^{(ik)} + \big(\nu_{0,i+1}^{*(k)} - \nu_{0,i}^{*(k)} K_{0c}^{(k,i+1)}\big) c_i^{(k)}(r_i)\Big],$$

$$K_{11}^{(ik)} = c_{i+1}^{(k)}(r_i) - c_i^{(k)}(r_i), \qquad K_{12}^{(ik)} = \nu_{i+1}^{*(k)}(r_i) c_{i+1}^{(k)}(r_i) - \nu_i^{*(k)}(r_i) c_i^{(k)}(r_i),$$

$$m_{1i}^{(1k)} = M_{1i}^{(k)+} + M_{1i}^{(k)-}\left(\frac{r_{i-1}}{r_i}\right)^{k+1}, \qquad m_{1i}^{(2k)} = M_{1i}^{(k)+} - k M_{1i}^{(k)-}\left(\frac{r_{i-1}}{r_i}\right)^{k+1},$$

$$m_{2i}^{(1k)} = M_{2i}^{(k)+} - M_{2i}^{(k)-}, \qquad m_{2i}^{(2k)} = k M_{2i}^{(k)+} + M_{2i}^{(k)-},$$

$$V_p^{(1k)}(\rho) = c_p^{(1k)}(\rho)\,\bar{u}_p^{(k)}(\rho) - c_p^{(1k)}(r_{p-1})\,\bar{u}_p^{(k)}(r_{p-1})\left(\frac{r_{p-1}}{\rho}\right)^k - \frac{1}{\rho^k}\int_{r_{p-1}}^{\rho} U_p^{(1k)}(r) r^k\,dr,$$

$$V_p^{(2k)}(\rho) = -k c_p^{(2k)}(r_p)\,\bar{u}_p^{(k)}(r_p)\frac{\rho}{r_p} + k c_p^{(2k)}(\rho)\,\bar{u}_p^{(k)}(\rho) - \rho\int_{\rho}^{r_p} U_p^{(2k)}(r)\frac{1}{r}\,dr,$$

$$U_p^{(1k)}(r) = c_p^{(1k)}(r)\left[\frac{d\bar{u}_p^{(k)}(r)}{dr} + k\,\frac{\bar{u}_p^{(k)}(r)}{r}\right],$$

$$U_p^{(2k)}(r) = kc_p^{(2k)}(r)\left[-\frac{d\bar{u}_p^{(k)}(r)}{dr} + \frac{\bar{u}_p^{(k)}(r)}{r}\right],$$

$$c_p^{(1k)}(r) = c_p^{*(k)}(r)\left(\frac{1}{1+\nu_p^{(k)}(r)}\right)^{2-k}, \qquad c_p^{(2k)}(r) = \frac{E_p^{(k)}(r)}{1+\nu_p^{(k)}(r)}.$$

Here,

$$\bar{u}_p^{(k)}(r) \quad \text{and} \quad \frac{d\bar{u}_p^{(k)}(r)}{dr}$$

are the displacements and strains obtained for constant moduli of elasticity and Poisson's ratios for which it follows from (45) and (46) that

$$\bar{u}_p^{(k)}(\rho) = \frac{\bar{R}_{1p}^{(k)}\varphi_{2p}^{(k)}(\rho) - \bar{R}_{2p}^{(k)}\varphi_{1p}^{(k)}(\rho)}{(k+1)Q_k} + \frac{k_{0p}^{*(k)}}{\rho^k} J_{tp}^{(0k)}(\rho),$$

$$\frac{d\bar{u}_p^{(k)}(\rho)}{d\rho} = \frac{\bar{R}_{1p}^{(k)}\varphi_{4p}^{(k)}(\rho) - \bar{R}_{2p}^{(k)}\varphi_{3p}^{(k)}(\rho)}{(k+1)Q_k} + k_{0p}^{*(k)}\left[-\frac{k}{\rho^{k+1}} J_{tp}^{(0k)}(\rho) + \Phi_{rp}^{(k)}(\rho)\right],$$

$$\bar{R}_{1p}^{(k)} = \sum_{j=1}^{p-1} M_{1j}^{(k)+} k_{0j}^{*(k)} J_{tj}^{(0k)}(r_j)\frac{c_{0j}^{(k)}}{c_{0p}^{(k)}}, \qquad \bar{R}_{2p}^{(k)} = \sum_{j=p}^{n} M_{2j}^{(k)-} J_{tj}^{(0k)}(r_j)\frac{k_{0j}^{*(k)}}{r_j^{k+1}},$$

$$J_{tp}^{(0k)}(\rho) = \int_{r_{p-1}}^{\rho} r^k \Phi_{rp}^{(k)}(r)\,dr, \qquad k_{0p}^{*(k)} = \frac{1+\nu_{0p}^{(k)}}{1-\nu_{0p}^{(k)}}.$$

If the accuracy of evaluation of displacements, strains, and stresses is lower than the given accuracy, then we repeat the procedure of their determination by using the same formulas (44)–(46) but with a greater number of layers. Additional layers are obtained by splitting the p th domain into n_p parts each of which has the elasticity moduli, Poisson's ratios, and the coefficients of linear thermal expansion of this domain.

3. Examples of Numerical Analysis

1. We consider ***three-layer plates*** with different temperature dependences of the thermal conductivity coefficient of the third layer. We choose VK6 and VK15 alloys as the materials of the first and second layers, respectively. The materials of the third layer are VK10 alloy (*plate* ***1***), Ti–6Al–4V alloy (*plate* ***2***), ZrO_2 ceramics (*plate* ***3***), Si_3N_4 ceramics (*plate* ***4***), and SUS304 metal (*plate* ***5***). We study the temperature fields described by solution (7) for $k = 0$, $n = 3$, $T_* = 273°K$, $T_s = 1273°K$, $\mathrm{Ki}_0^{*(k)} = 0.3283$, $T_{cn}^{(0)} = 0.2301$, $\tilde{x}_0 = 0$,

$\tilde{x}_1 = \frac{1}{3}\tilde{x}_3$, $\tilde{x}_2 = \frac{2}{3}\tilde{x}_3$, and $\tilde{x}_3 = 0.01$ m, with the following thermal conductivity coefficients:

– for VK6, VK15, and VK10 [14]:

$$\lambda_t^{(1)}(T) = 58.618\left[1 + a_0 e^{-0.4(\vartheta/100-5.2)^2} - b_0 e^{-0.275(\vartheta/100-2.2\vartheta)^2}\right] \text{ [W/(m}\cdot\text{K)]},$$

where $a_0 = 0.214454$, $b_0 = 0.4285714$, and $\vartheta = T - T_*$,

$$\lambda_t^{(2)}(T) = 71.09\big(1 - 2.8984\cdot 10^{-3}\vartheta + 6.1466\cdot 10^{-6}\vartheta^2$$

$$- 6.1789\cdot 10^{-9}\vartheta^3 + 2.2844\cdot 10^{-12}\vartheta^4\big) \text{ [W/(m}\cdot\text{K)]},$$

$$\lambda_t^{(3)}(T) = 54.057\big(1 - 7.4218\cdot 10^{-4}\vartheta - 7.7696\cdot 10^{-5}\vartheta^2$$

$$+ 1.718\cdot 10^{-8}\vartheta^3 - 8.5704\cdot 10^{-12}\vartheta^4\big) \text{ [W/(m}\cdot\text{K)]};$$

– for Ti-6Al-4V and ZrO_2 [18]:

$$\lambda_t^{(3)}(T) = 5.741(1 + 2.961\cdot 10^{-3}\vartheta) \text{ [W/(m}\cdot\text{K)]},$$

$$\lambda_t^{(3)}(T) = 1.776\big(1 + 1.539\cdot 10^{-4}\vartheta + 6.5316\cdot 10^{-8}\vartheta^2\big) \text{ [W/(m}\cdot\text{K)]};$$

– for Si_3N_4 and SUS304 [17]:

$$\lambda_t^{(3)}(T) = 10.394\big(1 - 9.9187\cdot 10^{-4}\vartheta$$

$$+ 6.365\cdot 10^{-7}\vartheta^2 - 1.04\cdot 10^{-10}\vartheta^3\big) \text{ [W/(m}\cdot\text{K)]},$$

$$\lambda_t^{(3)}(T) = 12.244\big(1 - 3.558\cdot 10^{-4}\vartheta$$

$$+ 1.884\cdot 10^{-6}\vartheta^2 - 9.072\cdot 10^{-10}\vartheta^3\big) \text{ [W/(m}\cdot\text{K)]}.$$

For plates in which the temperature dependence of the thermal conductivity coefficient of the third layer is approximated by polynomials of at most third degree, i.e.,

$$\tilde{\Lambda}(\bar{\vartheta}) = 1 + \bar{\beta}_1\bar{\vartheta} + \bar{\beta}_2\bar{\vartheta}^2 + \bar{\beta}_3\bar{\vartheta}^3, \quad \bar{\vartheta} = \bar{T} - \bar{T}_*,$$

the temperature (for the sake of comparison) is also determined by using the exact solutions of equations corresponding to Eq. (26). In particular, for $\beta_3 \neq 0$, substituting $\bar{\vartheta} = y - \bar{\beta}_2/(3\bar{\beta}_3)$, we reduce these equations to the following equation with respect to a new variable:

$$y^4 + my^2 + py + q = 0, \tag{47}$$

where

$$m = 2\left[\frac{\bar{\beta}_1}{\bar{\beta}_3} - \frac{1}{3}\left(\frac{\bar{\beta}_1}{\bar{\beta}_3}\right)^2\right], \qquad p = \frac{4}{3}\frac{\bar{\beta}_2}{\bar{\beta}_3}\left[\frac{2}{9}\left(\frac{\bar{\beta}_2}{\bar{\beta}_3}\right)^2 - \frac{\bar{\beta}_1}{\bar{\beta}_3}\right] + \frac{4}{\bar{\beta}_3},$$

$$q = \frac{1}{9}\left(\frac{\bar{\beta}_2}{\bar{\beta}_3}\right)^2\left[\frac{2\bar{\beta}_1}{\bar{\beta}_3} - \frac{1}{3}\left(\frac{\bar{\beta}_2}{\bar{\beta}_3}\right)^2\right] - \frac{4}{\bar{\beta}_3}\left(\frac{\bar{\beta}_2}{3\bar{\beta}_3} + \theta_3(x)\right).$$

The roots of Eq. (47) are found among the roots of the equations [13]:

$$y^2 \pm \sqrt{\chi}\, y + \frac{m + \chi + s}{2} = 0,$$

$$y^2 \pm \sqrt{\chi}\, y + \frac{m + \chi - s}{2} = 0,$$

where

$$s = \sqrt{(m + \chi)^2 - 4q}$$

and χ is a nonnegative root of the cubic equation

$$\chi^3 + 2m\chi^2 + (m^2 - 4q)\chi - p^2 = 0,$$

given by the Cardano formula

$$\chi = \sqrt[3]{-\frac{\tilde{q}}{2} + \sqrt{\tilde{Q}}} + \sqrt[3]{-\frac{\tilde{q}}{2} - \sqrt{\tilde{Q}}} - \frac{2}{3}m. \tag{48}$$

Here,

$$\tilde{Q} = \left(\frac{\tilde{p}}{3}\right)^3 + \left(\frac{\tilde{q}}{2}\right)^2, \qquad \tilde{p} = -\frac{1}{3}m^2 - 4q, \qquad \tilde{q} = -\frac{2}{27}m^3 + \frac{8}{3}qm - p^2.$$

If $\bar{\beta}_3 = 0$ (quadratic dependence), then the required root of the corresponding cubic equation is given by the following Cardano formula:

$$\bar{\vartheta} = \sqrt[3]{-\frac{q^*}{2}+\sqrt{Q^*}} + \sqrt[3]{-\frac{q^*}{2}-\sqrt{Q^*}} - \frac{\bar{\beta}_1}{2\bar{\beta}_2}, \tag{49}$$

where

$$Q^* = \left(\frac{p^*}{3}\right)^3 + \left(\frac{q^*}{2}\right)^2, \quad p^* = \frac{3}{\bar{\beta}_2} - \frac{3}{4}\left(\frac{\bar{\beta}_1}{\bar{\beta}_2}\right)^2, \quad q^* = \frac{1}{4}\left(\frac{\bar{\beta}_1}{\bar{\beta}_2}\right)^3 - \frac{3}{2}\frac{\bar{\beta}_1}{\bar{\beta}_2} - \frac{3}{\bar{\beta}_2}\theta_3(x).$$

If $\bar{\beta}_2^{(i)} = 0$ and $\bar{\beta}_3^{(i)} = 0$ (linear dependence), then

$$\bar{\vartheta} = \frac{\sqrt{1+2\bar{\beta}_1\theta_3(x)}-1}{\bar{\beta}_1}. \tag{50}$$

The comparative analysis of temperatures in the third layer computed by using the Newton method with 30 iterations and the exact solutions (48)–(50) shows that they differ by at most last two significant digits. For the Cardano solutions (48), (49), for each $x = x_{3j}$, it is necessary to take into account the signs of the corresponding radicands. In view of this fact, for ***plate 5***, the expressions for the roots of Eq. (48) obtained for

$$x_{3j} = x_2 + (j-1)\frac{x_3 - x_2}{4}$$

with $j = 1, 2$ and $j = 3, 4, 5$ are different. Note that, in ***plate 4***, where the thermal conductivity coefficient is also cubic function of temperature, the roots of Eq. (48) for all five values of x_{3j} are given by the same formula.

We also perform the comparison with temperatures computed according to the corresponding distribution of the temperature field described by the solution of the recurrence system of Volterra integral equations [13]:

$$\bar{\vartheta}_3(x) = \bar{\vartheta}_{cn} + \mathrm{Ki}_0\frac{\lambda_0^{(1)}}{\lambda_0^{(3)}}\int_x^{x_3}\frac{1}{\tilde{\Lambda}_3(\bar{\vartheta}_3(\zeta))}d\zeta,$$

$$\bar{\vartheta}_2(x) = \bar{\vartheta}_3(x_2) + \mathrm{Ki}_0\frac{\lambda_0^{(1)}}{\lambda_0^{(2)}}\int_x^{x_2}\frac{1}{\tilde{\Lambda}_2(\bar{\vartheta}_2(\zeta))}d\zeta, \tag{51}$$

$$\bar{\vartheta}_1(x) = \bar{\vartheta}_2(x_1) + \mathrm{Ki}_0\int_x^{x_1}\frac{1}{\tilde{\Lambda}_1(\bar{\vartheta}_1(\zeta))}d\zeta,$$

obtained without using the Kirchhoff transformation. The solution of the system of equations (51) is found by the method of successive approximations. The integrals are computed by using the trapezoid rule with splitting of each layer into 1600 parts. The number of iterations does not exceed 15.

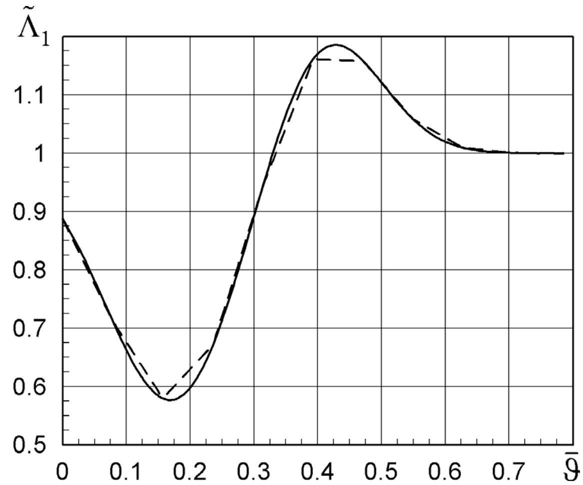

Fig. 1

In addition, in order to demonstrate the applicability of the proposed method in the case of specifying discrete (tabular) temperature dependences of the thermal conductivity coefficients, we approximate the functions $\tilde{\Lambda}_1(\bar{\vartheta})$ by linear splines with uniformly located 11 and 21 nodes (in Fig. 1, the solid line corresponds to the given function $\tilde{\Lambda}_1(\bar{\vartheta})$, while the dashed line is plotted for a spline with 11 nodes):

$$\tilde{\Lambda}_1(\bar{\vartheta}) \approx s_1\bar{\vartheta} + h_1 + \sum_{p=1}^{N_s-1}\left[(s_{p+1}-s_p)\bar{\vartheta}+h_{p+1}-h_p\right]S(\bar{\vartheta}-\bar{\vartheta}_p),$$

where

$$s_j = \frac{\tilde{\Lambda}_1(\bar{\vartheta}_j)-\tilde{\Lambda}_1(\bar{\vartheta}_{j-1})}{\bar{\vartheta}_j-\bar{\vartheta}_{j-1}}, \qquad h_j = \tilde{\Lambda}_1(\bar{\vartheta}_{j-1}) - s_j\bar{\vartheta}_{j-1},$$

and $\bar{\vartheta}_j$ are temperatures from the range $[\bar{T}_*, 1]$, $N_s = 10, 20$.

For this dependence and its approximations in Eq. (26), we, respectively, find

$$\Psi_1(\bar{\vartheta}) = \bar{\vartheta} + a_0 d_1\left[\operatorname{erf}\left(\bar{\vartheta}\sqrt{a_1}+\frac{b_1}{2\sqrt{a_1}}\right) - \operatorname{erf}\left(\frac{b_1}{2\sqrt{a_1}}\right)\right]$$

$$- b_0 d_2\left[\operatorname{erf}\left(\bar{\vartheta}\sqrt{a_2}+\frac{b_2}{2\sqrt{a_2}}\right) - \operatorname{erf}\left(\frac{b_2}{2\sqrt{a_2}}\right)\right],$$

$$d_j = \frac{1}{2}\sqrt{\frac{\pi}{a_j}}\exp\left(c_j-\frac{b_j^2}{4a_j}\right), \qquad j=1,2, \tag{52}$$

$$\Psi_1(\bar{\vartheta}) \approx \frac{s_1}{2}\bar{\vartheta}^2 + h_1\bar{\vartheta} + \sum_{p=1}^{N_s-1}\left[\frac{s_{p+1}-s_p}{2}\bar{\vartheta}^2\right.$$

$$\left. + (h_{p+1}-h_p)\bar{\vartheta} + \frac{s_{p+1}-s_p}{2}\bar{\vartheta}_p^2\right] S(\bar{\vartheta}-\bar{\vartheta}_p). \qquad (53)$$

The values of temperature on the surfaces $x=0$, $x_1/4$, $x_1/2$, $3x_1/4$, and x_1 of the first layer of *plate **1*** are presented in Table 1.

Table 1

x	Proposed method			By using (51)
	by using (52)	by using (53), $N_s=10$	by using (53), $N_s=20$	
0	0.68095934	0.68010031	0.68074456	0.68095938
$x_1/4$	0.64201194	0.64123680	0.64181598	0.64201198
$x_1/2$	0.60336321	0.60270337	0.60320513	0.60336325
$3x_1/4$	0.56538833	0.56500614	0.56528355	0.56538837
x_1	0.52856933	0.52827185	0.52850974	0.52856937

The data presented in Table 1 illustrate high accuracy of the evaluation of temperature by using the proposed method. Approximating the temperature dependence by a linear spline with a twice thicker grid, we correct the values of temperature in the 3rd – 5th decimal digits.

II. We study the influence of temperature dependence of the thermal conductivity coefficients and thermal radiation on the thermal and thermoelastic states of thermosensitive ***four-layer plate, cylinder,*** and ***sphere*** with $q_0 = 2.0\cdot 10^6$ W/m^2, $q_j = 0$, $j=1,2,\dots,n$, $w_0^{(i)}=0$, $\tilde{\alpha}_0=0$, $\alpha_n^*(T_n)=1$, $\varepsilon_0^*(T_1)=1$, $\tilde{\varepsilon}_n=0$, $t_{cn}=T^*$, $T^*=273$°K, $T_s=1373$°K, and $\mathrm{Bi}_n=3$.

As materials of layers, we take the ZnO_2 ceramics (*layer **1***), Ti-6Al-4V alloy (*layer **2***), Si_3N_4 ceramics (*layer **3***), and SUS304 metal (*layer **4***).

The geometric parameters are as follows: $x_0=0$, $\tilde{x}_1=0.32$ mm, $\tilde{x}_2=2.32$ mm, $\tilde{x}_3=3.32$ mm, and $\tilde{x}_4=4.32$ mm *for the plate* and $x_0=10$ mm, $\tilde{x}_1=10.32$ mm, $\tilde{x}_2=12.32$ mm, $\tilde{x}_3=13.32$ mm, and $\tilde{x}_4=14.32$ mm *for the cylinder and the sphere*.

The temperature dependences of the thermal conductivity coefficients have been presented earlier, whereas the remaining physical and mechanical characteristics are given by the following relations:

$$E_1(T) = \left(132 - 50.3\cdot 10^{-3} T - 8.1\cdot 10^{-6} T^2\right) \text{[GPa]}, \qquad \nu_1(T) = 0.333,$$

$$\alpha_{t1}(T) = 9.087\cdot 10^{-6}\left(1 - 13.168\cdot 10^{-4}\vartheta + 13.976\cdot 10^{-7}\vartheta^2\right) \text{[K}^{-1}\text{]},$$

$$E_2(T) = (122.7 - 0.056T) \text{[GPa]}, \qquad \nu_2(T) = 0.2888 + 32\cdot 10^{-6} T,$$

$$\alpha_{t2}(T) = 8.747\cdot 10^{-6}\left(1 + 46.771\cdot 10^{-5}\vartheta - 3.075\cdot 10^{-7}\vartheta^2\right) \text{[K}^{-1}\text{]},$$

$$E_3(T) = \left(348.43 - 106.96801\cdot 10^{-3} T\right.$$

$$\left. + \; 752.6088\cdot 10^{-7} T^2 - 311.7054\cdot 10^{-10} T^3\right) \text{[GPa]}, \qquad \nu_3(T) = 0.24,$$

$$\alpha_{t3}(T) = 7.33\cdot 10^{-6}\left(1 + 72.859\cdot 10^{-5}\vartheta\right) \text{[K}^{-1}\text{]},$$

$$E_4(T) = 201.04\left(1 + 3.079\cdot 10^{-4} T - 6.534\cdot 10^{-7} T^2\right) \text{[GPa]},$$

$$\nu_4(T) = 0.3263 - 65.26\cdot 10^{-9} T + 12.39\cdot 10^{-8} T^2,$$

$$\alpha_{t4}(T) = 15.05\cdot 10^{-6}\left(1 + 66.238\cdot 10^{-5}\vartheta\right) \text{[K}^{-1}\text{]}.$$

The values of $\text{Ki}_0^{*(k)}$ are given by relation (28) with

$$\varphi_n\left(T_{cn}^{(k)}\right) = \left(\frac{x_n}{x_0}\right)^k \frac{\lambda_0^{(n)}}{\lambda_0^{(1)}} \text{Bi}_n\left(T_{cn}^{(k)} - \bar{t}_{cn}\right).$$

The values of $T_{cn}^{(k)}$, to within 10^{-8}, were determined by using the bisection method from the equation

$$\left\{\text{Ki}_0^{*(k)} + \text{Sk}_0\left[\bar{T}_1^{(k)}(x_0, \text{Ki}_0^{*(k)}, T_{cn}^{(k)})\right]^4\right\}\Big|_{\text{Ki}_0^{*(k)} = \varphi_n(T_{cn}^{(k)})} = \text{Ki}_0.$$

Note that the values of the temperature computed for $\text{Sk}_0 = 0$ and constant vales of the thermal conductivity coefficients on the basis of the exact solution of the corresponding problems:

$$\bar{T}_i^{(k)}(x) = \text{Ki}_0\left(\frac{x_0}{x_n}\right)^k \left\{\frac{\lambda_0^{(1)}}{\lambda_0^{(n)}\,\text{Bi}_n} + x_n^k\left[f_n^{(k)}(x_n) - f_i^{(k)}(x)\right]\right\} + \bar{t}_{cn}$$

have the same accuracy.

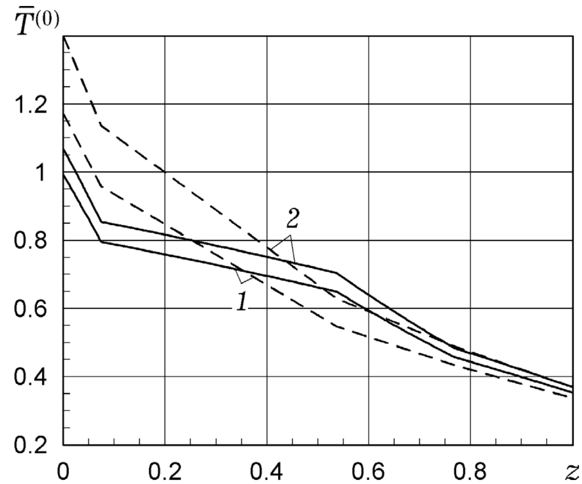

Fig. 2

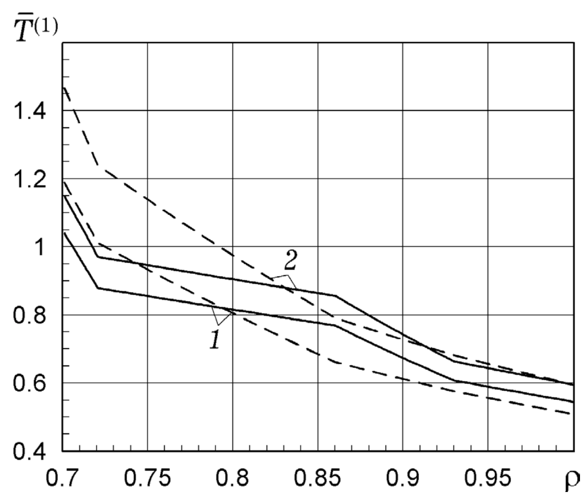

Fig. 3

The results of numerical analyses are presented in the form of the plots in Figs. 2–9. The solid lines correspond to the temperature-dependent thermal conductivity coefficients, while the dashed lines correspond to the case of constant thermal conductivity coefficients whose values are obtained from the temperature-dependent coefficients at $\bar{T} = \bar{T}_*$. Curves ***1*** are plotted with regard for the presence of thermal radiation $(\mathrm{Sk}_0 = 1.1833)$, while curves ***2*** are plotted in the absence of radiation $(\mathrm{Sk}_0 = 0)$. The distributions of temperature over the thickness of the plate (Fig. 2), cylinder (Fig. 3), and sphere (Fig. 4) demonstrate, in particular, that, in the case where the temperature dependences of the thermal conductivity coefficients and thermal radiation are neglected, the maximum temperatures become $1.2 \div 1.5$ times higher. Moreover, the radial stresses in the second layers of the cylinder (Fig. 5) and the sphere (Fig. 6) may differ by a factor of two. Furthermore, the qualitative character of the behavior of radial and circular stresses in the plate (Fig. 7) and circular stresses in the cylinder (Fig. 8) and in the sphere (Fig. 9) in the first layers differs from the behaviors obtained with regard for the above-mentioned factors.

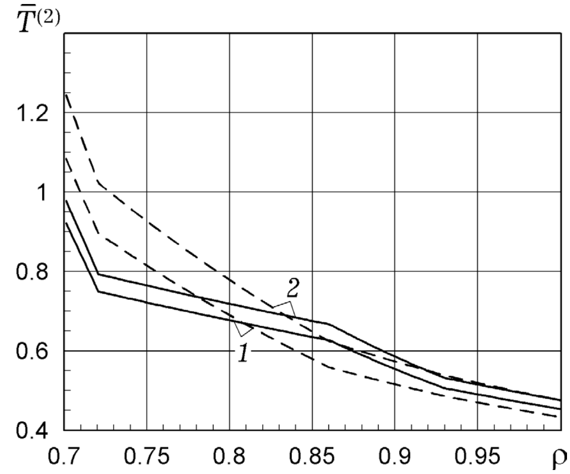

Fig. 4

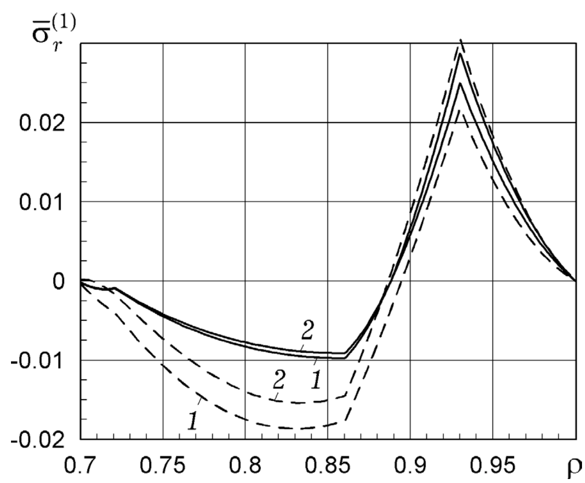

Fig. 5

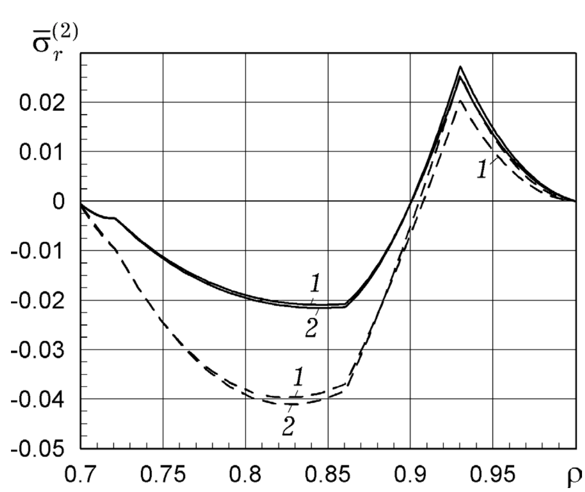

Fig. 6

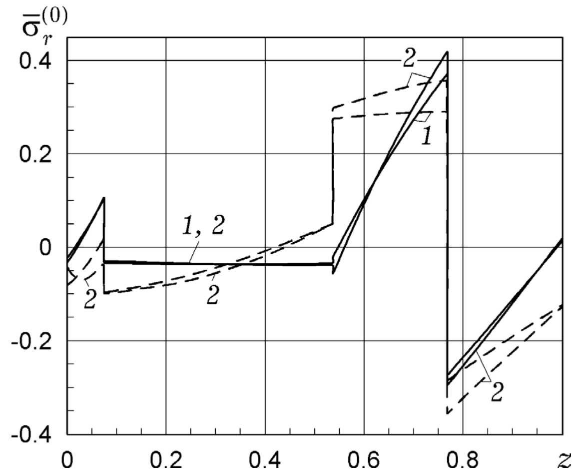

Fig. 7

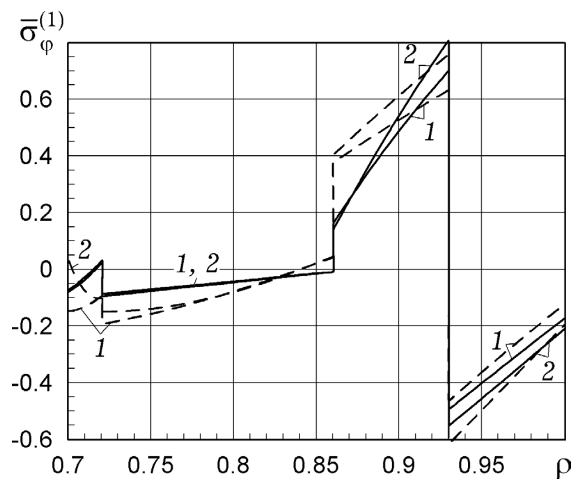

Fig. 8

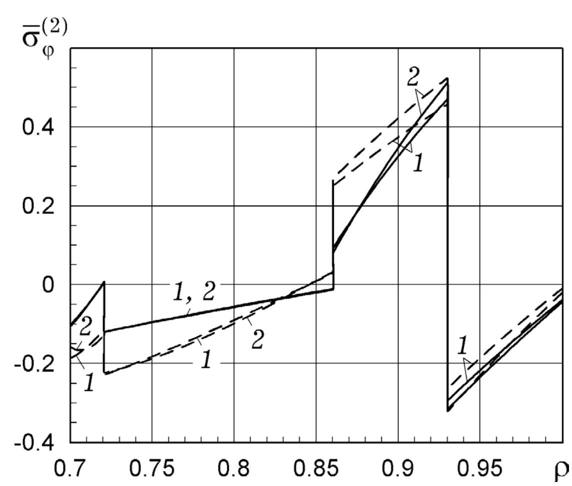

Fig. 9

Conclusions

The results of numerical investigations illustrate the efficiency of the proposed method for the determination of static thermoelastic states of multilayer bodies of canonical shapes and its applicability to the solution of static problems of thermoelasticity with temperature-dependent thermal conductivity coefficients given by discrete values. We also establish the existence of significant qualitative and quantitative difference between the behaviors of temperature stresses in separated layers computed with and without taking into account the temperature dependences of the thermal conductivity coefficients and thermal radiation.

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