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Conformable fractional derivative in commutative algebras

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Abstract. In this paper, an analog of the conformable fractional derivative is defined in an arbitrary finite-dimensional commutative associative algebra. Functions taking values in the indicated algebras and having derivatives in the sense of a conformable fractional derivative are called φ-monogenic. A relation between the concepts of φ-monogenic and monogenic functions in such algebras has been established. Two new definitions have been proposed for the fractional derivative of the functions with values in finite-dimensional commutative associative algebras.

Keywords. Conformable fractional derivative, fractional analytic functions, local fractional derivative, α-differentiable functions, φ-monogenic functions in commutative algebras.

1. Introduction

The idea of fractional derivative was first raised by L'Hospital in 1695. After introducing it, many new definitions have been formulated. The most well-known of them are the Riemann–Liouville, Caputo, Hadamard, Riesz, Grünwald–Letnikov, and Marchaud ones, as well as others (see, e.g., [1, 2] and references therein).

Recently, Khalil *et al.* [3] introduced a new definition for the fractional derivative called the conformable fractional derivative. Unlike other definitions, this new definition satisfies the formulas for the derivatives of the product and quotient of two functions and has a simpler chain rule than other definitions. In addition to the conformable fractional derivative definition, the conformable integral definition, the Rolle theorem, and the Mean value theorem for conformable fractional differentiable functions were given in the literature. In [4], Abdeljawad improved this new theory. For instance, the definitions were provided for the left and right conformable fractional derivatives and fractional integrals of higher order (i.e., of order $\alpha > 1$), the fractional power series expansion, and the fractional Laplace transform, as well as formulas for fractional integration by parts, the chain rule, and the Gronwall inequality.

In paper [5], the conformable partial derivative of order $\alpha \in (0, 1]$ with respect to several real variables and the conformable gradient vector were defined.

In work [6], two new results on homogeneous functions involving their conformable partial derivatives were obtained; namely, the homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler's Theorem.

In paper [7], a new general definition of the local fractional derivative was given, which depends on an unknown kernel. A relation was established between this new concept and ordinary differentiation. Using the corresponding formula, most of the fundamental properties of the fractional derivative can be derived directly.

In works [8–12], a theory of fractional analytic functions in the conformable sense was developed. Namely, in work [8], the fractional Cauchy-like theorem and the fractional Cauchy-like formula for fractional analytic functions were established.

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In paper [11], some interesting results in real fractional Calculus were extended to the context of the complex-valued functions of the real variable. Finally, using all obtained results, the complex conformable integral was defined and some of its most important properties were established. In work [12], the concept of fractional contour integral was also developed. In particular, some new results concerning complex fractional integration were proposed and proved, and the necessary and sufficient conditions were determined for a continuous function to have the antiderivative in the conformable sense. Finally, in work [12], some of the well-known Cauchy's integral theorems will also be the subject of the extension that we do in this paper.

Independently of the mentioned papers, in other papers, the conformable fractional derivative of order α in the complex plane was defined. An analog of Cauchy–Riemann conditions for α-differentiable functions was proposed. Moreover, two complex conformable differential equations and solutions with their Riemann surfaces were discussed.

A good many results concerning the theory of fractional differential equations and its applications, which are based on the conformable fractional derivative, were published during a short time interval; see, for example, works [13–20].

The next natural step is to generalize the concept of the conformable fractional derivative to the case of any multidimensional algebra; first of all, to the commutative and associative algebras.

2. Conformable fractional derivative and α-analytic functions

Definition 2.1. [3] *For a given function $f : [0, \infty) \to \mathbb{R}$, the conformable fractional derivative of order α is defined as follows:*

$$(T_\alpha f)(t) := \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2.1}$$

for all $t > 0$, $0 < \alpha \leq 1$. If f is α-differentiable in some $(0, b)$, $b > 0$, and $\lim\limits_{t \to 0+0} (T_\alpha f)(t)$ exists, then the corresponding definition looks like

$$(T_\alpha f)(0) := \lim_{t \to 0+0} (T_\alpha f)(t).$$

See papers [3, 4, 11, 15] for the derivative properties.

Now consider the definition of α-differentiation in the complex plane.

Definition 2.2. [8] *A complex-valued function f is called conformable fractional differentiable (or α-analytic) at a point $z \in \mathbb{C}$ if there exists the limit*

$$(T_\alpha f)(z) := \lim_{\varepsilon \to 0} \frac{f(z + \varepsilon z^{1-\alpha}) - f(z)}{\varepsilon} \tag{2.2}$$

for all z, and $0 < \alpha < 1$. The quantity $(T_\alpha f)(z)$ is called the α-derivative. If f is α-analytic in an open set U, and $\lim\limits_{z \to 0} (T_\alpha f)(z)$ exists, then the corresponding definition looks like

$$(T_\alpha f)(0) := \lim_{z \to 0} (T_\alpha f)(z).$$

Example 2.1. Let $f(z) = z^2$ and $\alpha = \frac{1}{2}$. Then

$$T_{1/2}(z^2) = \lim_{\varepsilon \to 0} \frac{\left(z + \varepsilon z^{1-1/2}\right)^2 - z^2}{\varepsilon} = 2z^{3/2}.$$

It it obvious that $T_{1/2}(z^2)$ is holomorphic outside some cut connecting the point 0 and ∞.

Remark 2.1. If a function $f(z)$ is holomorphic on $\mathbb{C}$, then, generally speaking, the conformable fractional derivative $T_\alpha f(z)$ is not a holomorphic function on $\mathbb{C}$ (but it is holomorphic outside some cut of the complex plane).

The following theorem can be found in [8].

Theorem 2.1. *Let $\alpha \in (0,1]$ and f, g be α-analytic at a point z_0. Then*

1. $T_\alpha\left(c_1 f(z) + c_2 g(z)\right) = c_1 T_\alpha f(z) + c_2 T_\alpha g(z)$ *for all* $c_1, c_2 \in \mathbb{C}$*;*
2. $T_\alpha(z^c) = c z^{c-\alpha}$ *for all* $c \in \mathbb{C}$*;*
3. $T_\alpha(\mu) = 0$ *for all constant functions* $f(z) = \mu$*;*
4. $T_\alpha\left(f(z)g(z)\right) = f(z) T_\alpha g(z) + g(z) T_\alpha f(z)$*;*
5. $T_\alpha\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)T_\alpha f(z) - f(z) T_\alpha g(z)}{g^2(z)}$*;*
6. *if, in addition, f is analytic, then* $T_\alpha f(z)|_{z=z_0} = z_0^{1-\alpha} f'(z_0)$*.*

The complex conformable fractional derivatives of some complex-valued functions are as follows:

$$T_\alpha\left(e^{cz}\right) = c z^{1-\alpha} e^{cz}, \quad c \in \mathbb{C};$$

$$T_\alpha(\sin cz) = c z^{1-\alpha} \cos cz, \quad c \in \mathbb{C};$$

$$T_\alpha(\cos cz) = -c z^{1-\alpha} \sin cz, \quad c \in \mathbb{C};$$

$$T_\alpha\left(\frac{1}{\alpha} z^\alpha\right) = 1.$$

For more results on α-analytic functions in the sense of conformable fractional derivative, see works [8, 9, 11, 12].

3. Monogenic functions in commutative associative algebras

Let $\mathbb{A}$ be an arbitrary n-dimensional ($1 \le n < \infty$) commutative associative algebra with a unity quantity over the field of complex numbers $\mathbb{C}$. E. Cartan [21, p. 33] proved that in $\mathbb{A}$ there exists a basis $\{I_k\}_{k=1}^n$ such that the first m basis vectors $I_1, I_2, \ldots, I_m$ are idempotents and the other vectors $I_{m+1}, I_{m+2}, \ldots, I_n$ are nilpotents. The element $1 = I_1 + I_2 + \cdots + I_m$ is the unit of $\mathbb{A}$.

In the algebra $\mathbb{A}$, we consider the vectors $e_1, e_2, \ldots, e_d$, $2 \le d \le 2n$. Let these vectors have the following expansion in the basis of the algebra:

$$e_j = \sum_{r=1}^{n} a_{jr} I_r\,, \quad a_{jr} \in \mathbb{C}, \quad j = 1, 2, \ldots, d. \tag{3.1}$$

Throughout this paper, we assume that at least one of the vectors $e_1, e_2, \ldots, e_d$ is invertible.

For the element $\zeta = x_1 e_1 + x_2 e_2 + \cdots + x_d e_d$, where $x_1, x_2, \ldots, x_d \in \mathbb{R}$, the complex numbers

$$\xi_u := x_1 a_{1u} + x_2 a_{2u} + \cdots + x_d a_{du}, \qquad u = 1, 2, \ldots, m,$$

form the spectrum of the point ζ.

Consider in the algebra $\mathbb{A}$ a linear span

$$E_d := \{\zeta = x_1 e_1 + x_2 e_2 + \cdots + x_d e_d : \; x_1, x_2, \ldots, x_d \in \mathbb{R}\}$$

generated by the vectors $e_1, e_2, \ldots, e_d$ of $\mathbb{A}$.

The following assumption is essential: for each fixed $u \in \{1, 2, \ldots, m\}$ at least one of the numbers $a_{1u}, a_{2u}, \ldots, a_{du}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

We identify a domain S in the space $\mathbb{R}^d$ with the domain

$$S := \{\zeta = x_1 e_1 + x_2 e_2 + \cdots + x_d e_d : (x_1, x_2, \ldots, x_d) \in S\} \text{ in } E_d \subset \mathbb{A}.$$

Definition 3.1. [22] *We will call a continuous function $\Phi : \Omega \to \mathbb{A}$ monogenic in a domain $\Omega \subset E_d$ if Φ is differentiable in the sense of Gâteaux at every point of this domain, that is, if for each $\zeta \in \Omega$ there exists an element $\Phi'_G(\zeta) \in \mathbb{A}$ such that the equality*

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h \Phi'_G(\zeta) \quad \forall\, h \in E_d \tag{3.2}$$

Consider the expansion of the function $\Phi : \Omega \to \mathbb{A}$ in the basis $\{I_k\}_{k=1}^n$,

$$\Phi(\zeta) = \sum_{k=1}^{n} U_k(x_1, x_2, \ldots, x_d)\, I_k\,. \tag{3.3}$$

If the functions $U_k : \Omega \to \mathbb{C}$ are $\mathbb{R}$-differentiable in the domain Ω, that is, for an arbitrary $(x_1, x_2, \ldots, x_d) \in \Omega$,

$$U_k\,(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_d + \Delta x_d) - U_k(x_1, x_2, \ldots, x_d)$$
$$= \sum_{j=1}^{d} \frac{\partial U_k}{\partial x_j} \Delta x_j + o\left(\sqrt{\sum_{j=1}^{d} (\Delta x_j)^2}\right), \qquad \sum_{j=1}^{d} (\Delta x_j)^2 \to 0\,,$$

the function Φ is monogenic in the domain Ω if and only if the following analogs of the Cauchy–Riemann conditions are fulfilled at each point of the domain Ω:

$$\frac{\partial \Phi}{\partial x_j} e_1 = \frac{\partial \Phi}{\partial x_1} e_j \qquad \text{for all} \quad j = 2, 3, \ldots, d.$$

Note that the expansion of the resolvent has the form [23]

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^{m} \frac{1}{t - \xi_u} I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{(t - \xi_{u_s})^r} I_s \tag{3.4}$$
$$\forall\, t \in \mathbb{C} : \; t \neq \xi_u\,, \quad u = 1, 2, \ldots, m,$$

where the coefficients $Q_{r,s}$ are determined by the following recurrence relationships:

$$Q_{2,s} = \xi_s\,, \quad Q_{r,s} = \sum_{q=r+m-2}^{s-1} Q_{r-1,q}\, B_{q,\,s}\,, \quad r = 3, 4, \ldots, s - m + 1,$$

$$B_{q,s} := \sum_{p=m+1}^{s-1} \xi_p \Upsilon_{q,s}^{p}\,, \;\; p = m + 2, m + 3, \ldots, n.$$

Here the structure constants $\Upsilon_{r,p}^s \in \mathbb{C}$ are defined by the equality $I_r I_s = \sum_p \Upsilon_{r,p}^s I_p$, and the natural numbers u_s by the following rule:

for any natural $m+1 \leq s \leq n$, there exist a unique natural $1 \leq u_s \leq m$ such that for all natural $1 \leq r \leq m$,

$$I_r I_s = \begin{cases} 0 & \text{if } r \neq u_s\,, \\ I_s & \text{if } r = u_s\,. \end{cases}$$

It follows from relationships (3.4) that the points $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ corresponding to the noninvertible elements $\zeta = \sum_{j=1}^{d} x_j\, e_j$ form the set

$$L_u: \quad \begin{cases} x_1 \operatorname{Re} a_{1u} + x_2 \operatorname{Re} a_{2u} + \cdots + x_d \operatorname{Re} a_{du} = 0, \\ x_1 \operatorname{Im} a_{1u} + x_2 \operatorname{Im} a_{2u} + \cdots + x_d \operatorname{Im} a_{du} = 0, \end{cases} \qquad u = 1, 2, \ldots, m,$$

in the d-dimensional space $\mathbb{R}^d$.

We say that a domain $\Omega \subset E_d$ is *convex with respect to the set of directions* L_u if Ω contains the segment $\{\zeta_1 + \alpha(\zeta_2 - \zeta_1) : \alpha \in [0,1]\}$ for all $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in L_u$.

Denote

$$D_u := \{\xi_u = x_1 a_{1u} + x_2 a_{2u} + \cdots + x_d a_{du} \in \mathbb{C} : \; \zeta \in \Omega\}, \quad u = 1, 2, \ldots, m.$$

In the following theorem, we present a constructive description of monogenic functions with values in the algebra $\mathbb{A}$ via holomorphic functions of the complex variable.

Theorem 3.1. [23, 24] *Let a domain $\Omega \subset E_d$ be convex with respect to the set of directions L_u, $u = 1, 2, \ldots, m$, and let for all $u = 1, 2, \ldots, m$, at least one of the numbers $a_{1u}, a_{2u}, \ldots, a_{du}$ belong to $\mathbb{C} \setminus \mathbb{R}$. Then every monogenic function $\Phi : \Omega \to \mathbb{A}$ can be represented in the form*

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1}\, dt +$$

$$+ \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int\limits_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1}\, dt, \tag{3.5}$$

where F_u and G_s are certain holomorphic functions in the domains D_u and D_{u_s}, respectively, and Γ_q is a closed Jordan rectifiable curve in D_q that surrounds the point ξ_q and does not contain points ξ_ℓ, $\ell, q = 1, 2, \ldots, m$, $\ell \neq q$.

From representation (3.5), it follows that under the conditions of Theorem 3.1, each function Φ monogenic in the domain Ω is differentiable in a strong sense, in particular, in the sense of Lorch [25].

Definition 3.2. [25] *A function $\Phi : \Omega \to \mathbb{A}$ given in a domain $\Omega \subset E_d$ is called differentiable in the sense of Lorch at a point $\zeta \in \Omega$ if there exists an element $\Phi'_L(\zeta) \in \mathbb{A}$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E_d$ with $\|h\| < \delta$ the following inequality is fulfilled:*

$$\left\|\Phi(\zeta + h) - \Phi(\zeta) - h\Phi'_L(\zeta)\right\| \leq \|h\|\, \varepsilon\,. \tag{3.6}$$

The element $\Phi'_L(\zeta)$ is called the Lorch derivative of the function Φ at the point ζ.

The representation of the monogenic function Φ in form (3.5) is unique. It was proved in [23] (for the case $\mathbb{R}^3$, see [24]) that for every monogenic function $\Phi : \Omega \to \mathbb{A}$ in an arbitrary domain Ω, the r-th Gâteaux derivatives Φ_G^r are monogenic functions in Ω for all r.

Remark 3.1. Under the conditions of Theorem 3.1, a monogenic function $\Phi : \Omega \to \mathbb{A}$ is differentiable in the sense of Lorch in Ω.

Consider examples of representation (3.5) in some low-dimensional commutative algebras.

Example 3.1. In the n-dimensional semi-simple algebra $\mathbb{A}_n$ with the multiplication table

$\cdot$	I_1	I_2	$\ldots$	I_n
I_1	I_1	0	$\ldots$	0
I_2	0	I_2	$\ldots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
I_n	0	0	$\ldots$	I_n

,

representation (3.5) of a monogenic function $\Phi(\zeta)$ has the form [26]

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + \ldots + F_n(\xi_n)I_n\,,$$

where $\zeta = \xi_1\, I_1 + \xi_2\, I_2 + \cdots \xi_n\, I_n$. In particular, in the algebra of bicomplex numbers (or commutative Segre's quaternions) $\mathbb{BC} = \{\zeta = \xi_1\, I_1 + \xi_2\, I_2 : \xi_1, \xi_2 \in \mathbb{C}\}$, the monogenic function has the form [27]

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2. \tag{3.7}$$

Example 3.2. In the biharmonic algebra $\mathbb{B}$ with the basis $\{1, \rho\}$, $\rho^2 = 0$, representation (3.5) of a monogenic function $\Phi(\zeta)$ has the form [28]

$$\Phi(\zeta) = F(\xi_1) + \Big[\xi_2 F'(\xi_1) + F_0(\xi_1)\Big]\rho, \tag{3.8}$$

where $\zeta = \xi_1 + \xi_2\, \rho, \quad \xi_1, \xi_2 \in \mathbb{C}$.

Example 3.3. In the 3-dimensional algebra $\mathbb{A}_3$ with the two-dimensional radical and the multiplication table

$\cdot$	1	ρ	ρ^2
1	1	ρ	ρ^2
ρ	ρ	ρ^2	0
ρ^2	ρ^2	0	0

,

representation (3.5) of a monogenic function $\Phi(\zeta)$ has the form [29]

$$\Phi(\zeta) = F(\xi_1) + \Big[\xi_2 F'(\xi) + F_1(\xi_1)\Big]\rho +$$

$$+\Big[\xi_3 F'(\xi_1) + \frac{\xi_1^2}{2} F''(\xi_1) + \xi_2 F_1'(\xi_1) + F_2(\xi_1)\Big]\rho^2, \tag{3.9}$$

where $\zeta = \xi_1 + \xi_2\, \rho + \xi_3\, \rho^2, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

Example 3.4. In the 3-dimensional algebra $\mathbb{A}_2$ with the one-dimensional radical and the multiplication table

$\cdot$	I_1	I_2	ρ
I_1	I_1	0	0
I_2	0	I_2	ρ
ρ	0	ρ	0

,

representation (3.5) of a monogenic function $\Phi(\zeta)$ has the form [29]

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + \Big[\xi_3 F_2'(\xi_2) + F_0(\xi_2)\Big]\rho,$$

where $\zeta = \xi_1\, I_1 + \xi_2\, I_2 + \xi_3\, \rho, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

In paper [30], analogs of the Cauchy integral theorem, the Cauchy integral formula, and the Morera theorem for a curvilinear integral were obtained for the monogenic function given in a domain of a special real subspace E_d, $2 \le d \le 2n$, of an arbitrary finite-dimensional commutative associative algebra $\mathbb{A}$. This result for the subspace E_3 was proved in work [31]. In paper [32], we proved an analog of the Cauchy integral theorem for the surface integral of hyperholomorphic functions given in a domain of three-dimensional space and taking the values in the algebra $\mathbb{A}$. In paper [33], the correspondence between a monogenic function in the algebra $\mathbb{A}$ and a finite set of monogenic functions in a special commutative associative algebra was obtained. In work [34], a relationship was proposed between monogenic functions taking values in an n-dimensional commutative associative algebra and monogenic functions taking values in a special $(n+1)$-dimensional algebra. Finally, in work [35], the previous results were applied to the solution of linear PDEs. Using the monogenic functions given in certain sequences of commutative associative algebras with the increasing dimensions of these algebras, we substantiated a recurrence procedure for constructing infinite-dimensional families of solutions for any partial differential equation with constant coefficients in the form of components of the mentioned monogenic functions.

4. φ-monogenic functions in finite-dimensional commutative associative algebras

Consider the definition of φ-monogenic functions in an arbitrary n-dimensional $(1 \le n < \infty)$ commutative associative algebra $\mathbb{A}$ with unity over the field of complex number $\mathbb{C}$.

Definition 4.1. *Let us fix a continuous function $\varphi : \Omega \to \mathbb{A}$ such that all its values are invertible in $\Omega \subseteq \mathbb{A}$. We call a continuous function $\Phi : \Omega \to \mathbb{A}$ φ-monogenic in the domain $\Omega \subseteq \mathbb{A}$ if there exists an element $\Phi'_\varphi(\zeta) \in \mathbb{A}$ such that for all $h \in \mathbb{A}$ the equality*

$$\lim_{\varepsilon \to 0+0} \frac{\Phi\left(\zeta + \varepsilon h \varphi(\zeta)\right) - \Phi(\zeta)}{\varepsilon} = h\, \Phi'_\varphi(\zeta) \tag{4.1}$$

holds. The element $\Phi'_\varphi(\zeta)$ is called the φ-derivative of the function Φ at a point ζ.

Remark 4.1. If $\varphi(\zeta) = \zeta^{1-\alpha}$, then the φ-derivative coincides with the α-derivative.

Example 4.1. For the function $\Phi(\zeta) = \zeta^2$, we have

$$\lim_{\varepsilon \to 0+0} \frac{\Big(\zeta + \varepsilon h \varphi(\zeta)\Big)^2 - \zeta^2}{\varepsilon} = \lim_{\varepsilon \to 0+0} \big(2h\zeta\varphi(\zeta) + \varepsilon h^2 \varphi^2(\zeta)\big) = h \cdot 2\zeta\varphi(\zeta).$$

Thus, $\left(\zeta^2\right)'_\varphi = 2\zeta\varphi(\zeta)$.

A real-valued analog of the next theorem was proved in paper [7].

Theorem 4.1. *A function $\Phi:\Omega\to\mathbb{A}$ is φ-monogenic at a point $\zeta\in\Omega$ if and only if Φ is monogenic at ζ. In thit case, we have the relationship*

$$\Phi'_{\varphi}(\zeta)=\varphi(\zeta)\Phi'_{G}(\zeta). \tag{4.2}$$

Proof. Sufficiency. Let us fix a point ζ. Let the function $\Phi:\Omega\to\mathbb{A}$ be monogenic at ζ. It means that there exists an element $\Phi'_G(\zeta)$ of the algebra $\mathbb{A}$ such that for each $h\in\mathbb{A}$ equality (3.2) holds. Since ζ is fixed, then $\varphi(\zeta)$ is an element of $\mathbb{A}$. Since equality (3.2) is true for each vector $h\in\mathbb{A}$, then it is true for the vector $h\cdot\varphi(\zeta)\in\mathbb{A}$, i.e., from (3.2), we have

$$\lim_{\varepsilon\to 0+0}\frac{\Phi(\zeta+\varepsilon h\varphi(\zeta))-\Phi(\zeta)}{\varepsilon}=h\varphi(\zeta)\Phi'_G(\zeta). \tag{4.3}$$

Thus, by virtue of relationship (4.1), the function $\Phi:\Omega\to\mathbb{A}$ is φ-monogenic at the point ζ, and equality (4.2) if obeyed

Necessity. Since the function $\Phi:\Omega\to\mathbb{A}$ is φ-monogenic at the point $\zeta\in\Omega$, then equality (4.1) is true for every direction $h\in\mathbb{A}$. Taking into account the invertibility of φ, we conclude that equality (4.1) is also true for the direction $h\cdot\big(\varphi(\zeta)\big)^{-1}\in\mathbb{A}$. Therefore, from (3.2), we have

$$\lim_{\varepsilon\to 0+0}\frac{\Phi(\zeta+\varepsilon h)-\Phi(\zeta)}{\varepsilon}=h\big(\varphi(\zeta)\big)^{-1}\Phi'_{\varphi}(\zeta). \tag{4.4}$$

Thus, the function $\Phi:\Omega\to\mathbb{A}$ is monogenic at the point ζ, and $\Phi'_G(\zeta)=\big(\varphi(\zeta)\big)^{-1}\Phi'_{\varphi}(\zeta)$. □

Thus, we have, for example, $(e^{\zeta})'_{\varphi}=\varphi(\zeta)\,e^{\zeta}$, $(\sin\zeta)'_{\varphi}=\varphi(\zeta)\,\cos\zeta$ *etc.*

In view of Remark 3.1, we have the following statement.

Corollary 4.1. *Under the conditions of Theorem 3.1, a function $\Phi:\Omega\to\mathbb{A}$ is φ-monogenic at a point $\zeta\in\Omega$ if and only if Φ is differentiable in the sense of Lorch at ζ. In this case, we have the relationship*

$$\Phi'_{\varphi}(\zeta)=\varphi(\zeta)\Phi'_{L}(\zeta)=\varphi(\zeta)\Phi'_{G}(\zeta).$$

Remark 4.2. From equality (4.2) follows the relationship

$$e_j\Phi'_{\varphi}(\zeta)=\varphi(\zeta)\frac{\partial\Phi}{\partial x_j}\,,\qquad j=1,2,\ldots,d.$$

In additional, if e_s is invertible for some $s\in\{1,2,\ldots,d\}$, then

$$\Phi'_{\varphi}(\zeta)=\varphi(\zeta)e_s^{-1}\frac{\partial\Phi}{\partial x_s}\,.$$

From Remark 4.2 follows the next properties.

Proposition 4.1. *If the function Φ is φ-monogenic and ψ-monogenic, and at least one of the vectors e_s, $s\in\{1,2,\ldots,d\}$, is invertible, then the following equalities are true:*

1. $\Phi'_{\varphi}+\Phi'_{\psi}=\Phi'_{\varphi+\psi}$;
2. $\Phi'_{\varphi\cdot\psi}=\varphi\Phi'_{\psi}=\psi\Phi'_{\varphi}$.

5. Alternative approaches to defining fractional differentiations in commutative associative algebras

Suppose that e_1 is invertible, and the function Φ of the variable $\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d$, where $x_1, x_2, \ldots, x_d \in \mathbb{R}$, is monogenic. For any $\alpha \in \mathbb{R}$, we define the power function ζ^α in the algebra $\mathbb{A}$ as follows:

$$\zeta^\alpha := \exp(\alpha \ln \zeta),$$

where $\ln\zeta$ are defined in paper [25, p. 422]. Then, for natural n, we have the equalities

$$\Phi'_G(\zeta) = \frac{\partial \Phi}{\partial x_1} e_1^{-1}, \qquad \Phi''_G(\zeta) = \frac{\partial^2 \Phi}{\partial x_1^2} e_1^{-2}, \ldots$$

$$\Phi_G^{(n)}(\zeta) = \frac{\partial^n \Phi}{\partial x_1^n} e_1^{-n}, \quad \text{where} \quad e_1^{-n} := \left(e_1^{-1}\right)^n .$$

The following definition is natural.

Definition 5.1. *Let $\alpha \in \mathbb{R}$. The derivative of order α of the function Φ at a point ζ is called the product*

$$\Phi^{(\alpha)}(\zeta) := \frac{\partial^\alpha \Phi}{\partial x_1^\alpha} \cdot e_1^{-\alpha}, \tag{5.1}$$

where the real fractional partial derivative $\frac{\partial^\alpha \Phi}{\partial x_1^\alpha}$ defined in some sense exists at the point x_1.

Note that in relationship (5.1), the real fractional partial derivative $\frac{\partial^\alpha \Phi}{\partial x_1^\alpha}$ is not defined. Considering different meanings of the real derivative $\frac{\partial^\alpha \Phi}{\partial x_1^\alpha}$, we get different meanings for the derivative $\Phi^{(\alpha)}$.

The following definition is based on Cauchy's idea in using the integral representation. We use integral representation (3.5).

Remark 5.1. Let $\alpha \in \mathbb{R}$. The derivative of order α of the function Φ at a point ζ is called the product

$$\Phi^\alpha(\zeta) = \sum_{u=1}^{m} I_u \frac{\Gamma(\alpha+1)}{2\pi i} \int\limits_{\Gamma_u} F_u(t) \left((te_1 - \zeta)^{-1}\right)^{\alpha+1} dt +$$

$$+ \sum_{s=m+1}^{n} I_s \frac{\Gamma(\alpha+1)}{2\pi i} \int\limits_{\Gamma_{u_s}} G_s(t) \left((te_1 - \zeta)^{-1}\right)^{\alpha+1} dt, \tag{5.2}$$

where $\Gamma(\alpha+1)$ is Euler's function. In this case, the integrand must be correctly defined.

Definitions (4.1), (5.1), and (5.2) are of different nature. Therefore, the issue of the relations between them is not simple and requires further research.

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