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On the solution of a complicated biharmonic equation in a hydroelasticity problem

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(Presented by R. M. Taranets')

Abstract. A hydroelastic problem of free vibrations of a thin plate that horizontally separates ideal incompressible liquids of different densities in a rigid cylindrical tank with an arbitrary cross-section has been considered in the linear formulation. To solve the corresponding complicated inhomogeneous biharmonic equation, the fundamental system of the solutions of biharmonic equation (FSS) and the eigenmodes of ideal liquid oscillations in a cylindrical cavity were used. The frequency equation was obtained for arbitrary fixation of the plate contour. On the example of a clamped plate, the frequency equation was simplified by decomposing the corresponding homogeneous biharmonic equation into two harmonic equations and using Green's formula for the Laplace operator. It was shown that in this case the frequency equation does not depend on the FSS and becomes greatly simplified because the FSS depends on the unknown frequency. The resulting equation has a single form for the cases of a right circular cylinder and a rectangular channel; in particular cases, it coincides with the previously obtained equations. Research of asymmetric vibration frequencies of a plate and a membrane, as well as axisymmetric vibration frequencies of a membrane in a circular cylinder, has been carried out. An approximation formula for high frequencies and approximate conditions for the stability of the plate and membrane vibrations were obtained.

Keywords. Hydroelasticity, thin plate, membrane, ideal incompressible fluid, free surface, boundary value problem, complicated biharmonic equation, Green's formula for the Laplace operator, frequency equation, stability.

1. Introduction

Boundary-value problems of hydroelasticity belong to rather difficult problems of mathematical physics because they require a simultaneous solution of differential equations in partial derivatives from both the theories of elasticity and hydromechanics. Currently, there is a large enough number of works where various studies of the problems of hydroelasticity were carried out. Here we cite only those of them that are closest to the problem under consideration, including the articles published in recent years. In all works cited below, the hydroelastic problem is considered in the linear formulation, the liquid is considered to be heavy, ideal, and incompressible, the plate elastic and thin, and the cavity rigid.

In monograph [1], a problem of the motion of a solid body with a cavity containing a liquid and an elastic plate located on the free liquid surface was considered. The solution of the corresponding boundary-value problem of hydroelasticity was given, and a number of individual cases were analyzed. Paper [2] was devoted to the study of free axisymmetric oscillations of a two-layer liquid in a straight coaxial cylinder with elastic bases in the form of annular plates. The frequency equation was obtained, and its analysis was carried out in peculiar cases. Work [3] generalized the results of work [2] to the case of asymmetric oscillations. On the example of a homogeneous liquid with a free surface and an elastic bottom in the form of a membrane, the frequency spectrum was examined analytically and numerically.

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In paper [4], the L.M. Sretensky problem and the problem concerning the oscillations of a physical pendulum in a multilayer liquid separated by elastic plates were generalized. From the positive definiteness of the potential energy in the former (Sretensky) problem and the modified potential energy in the latter (pendulum) one, the stability conditions for the equilibrium position were obtained. More detailed research was performed for a cylindrical cavity with an arbitrary cross-section.

The problem of free axisymmetric vibrations of a circular membrane located on the free surface of a liquid in a straight circular cylindrical tank was considered in work [5]. Taking into account two terms in the series for the frequency equation, the approximate stability conditions for the common vibrations of the membrane and the liquid were obtained. An algorithm was proposed that allows exact stability conditions to be formulated. It was pointed out that the account of two terms of the series for the frequency equation provides sufficient accuracy for practice.

The problem of free planar vibrations of a plate that horizontally separates liquids of different densities located in a rigid rectangular channel with an elastic plate-like upper end was studied in paper [6]. In the absence of the upper end (the liquid surface is free), approximate stability conditions for the common vibrations of the plate and the liquid were obtained.

In work [7], a planar problem of free oscillations in a liquid occupying a rigid rectangular channel with a hard upper end and an elastic bottom end in the form of a rectangular membrane was solved. For even and odd oscillation frequencies, an approximation formula was derived and used to obtain the approximate stability conditions for the membrane and liquid oscillations. The exact stability conditions were also derived, which coincide with the hydrostatic stability conditions. In work [8], the results of work [7] were generalized to the case of an elastic bottom in the form of a clamped rectangular plate.

The fundamental system of solutions of biharmonic equation (FSS) and the eigenforms of ideal liquid oscillations were first used in hydroelastic problems by L.S. Leibenzon (in 1951) and later by L.V. Dokuchaev (in 1971) [1]. This approach was applied in works [1–8] and many others. The main difficulty of its application consists in that the FSS depends on an unknown frequency of the plate and liquid vibrations and requires the consideration of all possible FSS variants. The frequency equation was simplified for the first time in paper [9] dealing with the plane problem of elastic membrane vibrations on the free surface of an ideal liquid in a rectangular channel. In the cited work, as well as in work [10], the trigonometric functions were expanded into the simplest fractions.

This work generalizes the results of work [5] to the case of a rigid cylindrical tank with an arbitrary cross-section filled with a liquid with a free upper surface. It continues our research that was started in works [1–9] where the FSS approach was applied. As a result, analytical studies of the frequency equation turned out difficult and sometimes even impossible. In this paper, on the example of a clamped plate, the frequency equation was simplified by expanding the corresponding homogeneous biharmonic equation into two harmonic equations and using Green's formula for the Laplace operator. It was shown that in this case the frequency equation does not depend on the FSS. Since the FSS depends on an unknown frequency, this circumstance greatly simplified the equation.

2. Formulation of the problem

Consider the vibrations of an elastic thin plate that horizontally divides ideal incompressible liquids with the densities ρ_i $(i = 1.2)$ in a rigid cylindrical tank of arbitrary cross-section. The plate is characterized by a constant bending stiffness D and is prone to tensile and compressive forces in the middle surface of the intensity T. The plate contour can be fastened arbitrarily; for example, it can be clamped, supported, or free. The upper liquid of density ρ_1 fills the cylindrical tank to the depth h_1, and the lower liquid of density ρ_2 to the depth h_2. The upper liquid has a free surface. The $Oxyz$

coordinate system is so arranged that the Oxy plane coincides with the undisturbed middle surface of the plate, and the Oz axis is directed opposite to the gravitational acceleration vector $\overrightarrow{g}$ (Fig. 1). The vibrations of the plate and the liquid will be considered in the linear formulation, assuming the common vibrations of the plate and the liquid to be inseparable, that is, without cavitation, and the motions of the liquids to be potential.

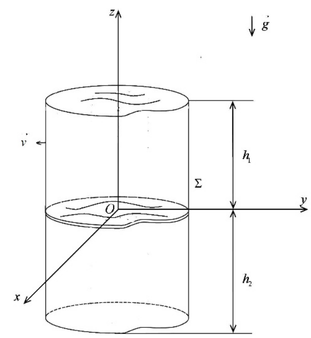

Fig. 1.

The equations of the plate and liquid vibrations look like [3–5]

$$m_0 \frac{\partial^2 W}{\partial t^2} + D\Delta_2^2 W - T\Delta_2 W + g\Delta\rho W = \rho_1 \frac{\partial \Phi_1}{\partial t} - \rho_2 \frac{\partial \Phi_2}{\partial t} + Q \text{ at } z = 0, \tag{1}$$

$$\left(\Delta_2^2 + \frac{\partial^2}{\partial z^2}\right)\Phi_i = 0 \quad (i = 1, 2)$$

with the boundary conditions

$$\left.\frac{\partial \Phi_i}{\partial \vec{\nu}}\right|_{\Sigma} = 0, \; \frac{\partial W}{\partial t} = \frac{\partial \Phi_1}{\partial z} = \frac{\partial \Phi_2}{\partial z} \text{ at } z = 0, \tag{2}$$

$$\left.(\mathcal{L}_p[W])\right|_{\gamma} = 0 \; (p = 1, 2), \tag{3}$$

$$\int_S W dS = 0; \; W, \nabla W < \infty, \tag{4}$$

$$\frac{\partial \Phi_1}{\partial z} = \frac{\partial \zeta}{\partial t}, \; \frac{\partial \Phi_1}{\partial t} + g\zeta = 0 \text{ at } z = h_1, \quad \frac{\partial \Phi_2}{\partial z} = 0 \text{ at } z = -h_2. \tag{5}$$

Here $m_0 = \rho_0 \delta_0$; $W(x, y, t)$, ρ_0, and δ_0 are the normal deflection, the density, and the thickness of the plate, respectively; $\Delta\rho = \rho_2 - \rho_1$; $\Phi_i(x, y, z, t)$ is the velocity potential in the i-th liquid ($i = 1, 2$); Δ_2 is the two-dimensional Laplace operator; $\vec{\nu}$ is the unit vector of the external normal to the side surface Σ; γ is the contour of the region S; $z = \zeta(x, y, t)$ is the free surface profile of the upper liquid; $\mathcal{L}_1$ and $\mathcal{L}_2$ are differential operators of the boundary conditions describing the fixation of the plate at the contour γ (for example, in the case of a clamped plate, the operator $\mathcal{L}_1$ is a unit one, and $\mathcal{L}_2 = \frac{\partial}{\partial \vec{\nu}}$).

3. Solution procedure

Let us express the functions $\Phi_i(x, y, t)$ and $\zeta(x, y, t)$ in the form of generalized Fourier series of the eigenfunctions $\psi_n(x, y)$:

$$\Phi_i(x, y, t) = \sum_{n=1}^{\infty} \left[A_{in}(t) e^{k_n z} + B_{in}(t) e^{-k_n z}\right] \psi_n(x, y), \tag{6}$$

$$\zeta(x,y,t)=\sum_{n=1}^{\infty}\zeta_n(t)\,\psi_n(x,y)\,, \tag{7}$$

where ψ_n and k_n are the eigenfunctions and the corresponding eigenvalues, respectively, of ideal liquid oscillations in a cylindrical cavity of arbitrary cross-section. They are determined from the following boundary-value problem:

$$\Delta_2\psi_n + k_n^2\psi_n = 0 \text{ at } S, \quad \left.\frac{\partial\psi_n}{\partial\nu}\right|_{\gamma} = 0.$$

This boundary-value problem is known to be equivalent to Fredholm's integral equation of the 2nd kind with a symmetric kernel, which implies the existence of a countable set of eigenvalues k_n and the corresponding eigenfunctions ψ_n. The eigenfunctions ψ_n are linearly independent, orthogonal in the region S, and after summing up with an arbitrary constant, they become complete in S [11].

The infinite series in (6)–(7), as well as their second derivatives with respect to x, y, and t, are assumed to converge absolutely and uniformly in the domain S within a finite time interval [11].

Substituting series (6) and (7) into equations (2) and (5), and using the orthogonality of the ψ_n functions, we obtain

$$A_{2n}=\frac{\dot{W}_n e^{\kappa_{2n}}}{2k_n\sinh\kappa_{2n}}, \quad B_{2n}=\frac{\dot{W}_n e^{-\kappa_{2n}}}{2k_n\sinh\kappa_{2n}}, . \tag{8}$$

where

$$W_n=\frac{1}{N_n^2}\int_S W\psi_n dS, \quad N_n^2=\int_S \psi_n^2 dS, \quad \kappa_{in}=h_i k_n. \tag{9}$$

Taking relationships (6)–(9) into account, equation (1) looks like

$$m_0\frac{\partial^2 W}{\partial t^2}+D\Delta_2^2 W - T\Delta_2 W + g\Delta\rho W=\sum_{n=1}^{\infty}\frac{b_n\ddot{\zeta}_n - a_n\ddot{W}_n}{k_n}\psi_n + Q, \tag{10}$$

where $a_n=\rho_1\coth\kappa_{1n}+\rho_2\coth\kappa_{2n}$ and $b_n=\rho_1/\sinh\kappa_{1n}$. From the second relationship in (5) and relationships (9), we have

$$\ddot{\zeta}_n+\sigma_n^2\zeta_n-\frac{\ddot{W}_n}{\cosh\kappa_{1n}}=0. \tag{11}$$

Here $\sigma_n^2 = gk_n\tanh\kappa_{1n}$ is the squared vibration frequency of the free surface of upper liquid if the plate is absolutely rigid ($W\equiv 0$).

Thus, the common vibrations of the plate and the liquid with the free surface can be found from the system of integro-differential equations (9)–(11), boundary conditions (3), the first condition of liquid incompressibility (4), and the initial conditions.

4. Boundary-value problem of common eigenmodes of the plate and the liquid

To find the eigenfrequencies of the common vibrations of the plate and the liquid, let us put

$$W(x,y,t)=w(x,y)\,e^{i\omega t}, \;\; \zeta(x,y,t)=\tilde{\zeta}(x,y)\,e^{i\omega t}, \;\; Q=C_0e^{i\omega t}, \tag{12}$$

where ω is the vibration frequency. Substituting (12) into (9)–(11) and boundary conditions (3) and (4), we obtain

$$\Delta_2^2 w - 2P\Delta_2 w + q = \frac{\omega^2}{D}\sum_{n=1}^{\infty}\frac{a_n^* w_n}{k_n}\psi_n + C \tag{13}$$

$$w_n = \frac{1}{N_n^2} \int_S w \psi_n dS, \tag{14}$$

$$\int_S w ds = 0 \text{ and } w, \nabla w < \infty, \tag{15}$$

$$\left(\mathcal{L}_p [w]\right)\big|_\gamma = 0, \ \ (p = 1, 2). \tag{16}$$

Here $P = T/2D$, $q = \left(g\Delta\rho - m_0\omega^2\right)/D$, $a_n^* = a_n - \tilde{b}_n$, $C = C_0/D$, and

$$\tilde{b}_n = \frac{\omega^2 b_n}{\left(\omega^2 - \sigma_n^2\right)\cosh\kappa_{1n}} = \frac{2\omega^2\rho_1}{\left(\omega^2 - \sigma_n^2\right)\sinh 2\kappa_{1n}}.$$

Therefore, the eigenfrequencies and the forms of the common vibrations of the plate and the liquid are determined from the system of integro-differential equations (13)–(15) and boundary conditions (16).

The solution of equation (13) is sought in the form

$$w = \sum_{k=1}^{4} A_k^0 w_k^0 + \sum_{n=1}^{\infty} \tilde{C}_n \psi_n + w_0, \tag{17}$$

where w_k^0 $(k = \overline{1,4})$ is the fundamental system of solutions of the homogeneous variant of equation (13),

$$\Delta_2^2 w_k^0 - 2P\Delta_2 w_k^0 + q w_k^0 = 0, \tag{18}$$

and the constants $\tilde{C}_n$ and w_0 will be expressed in terms of unknown constants A_k^0 that will be found from the plate fixation conditions.

It should be noted that due to the second (restriction) condition in (15), the index k should change from 1 to 2 in the case of simply connected region S. If the region S is doubly connected, then $k = \overline{1,4}$. Hence, below we adopt the index $k = 1.2$.

Substituting (17) into equation (13) and using the relationship $\Delta_2\psi_n = -k_n^2\psi_n$, $\Delta_2^2\psi_n = k_n^4\psi_n$, let us express the constant $\tilde{C}_n$ in terms of w_n,

$$\tilde{C}_n = \omega^2 a_n^* w_n / k_n d_n, \tag{19}$$

where $d_n = \left(Dk_n^2 + T\right)k_n^2 + g\Delta\rho - m_0\omega^2$. Substituting (17) into (14) and taking (19) into account, we obtain the following expressions for w_0 and w_n:

$$w_0 = -\sum_{k=1}^{2} \tilde{w}_k^0 A_k^0, \quad w_n = \frac{k_n d_n}{k_n d_n - \omega^2 a_n^*} \sum_{k=1}^{2} A_k^0 E_{kn}^0. \tag{20}$$

Here

$$\tilde{w}_k^0 = \frac{1}{mes(S)} \int_S w_k^0 dS,$$

$$E_{kn}^0 = \frac{1}{N_n^2} \int_S w_k^0 \psi_n dS. \tag{21}$$

Taking (15), (19), and (20) into account, the final expression for the plate bending profile w reads

$$w = \sum_{k=1}^{2} \left(w_k^0 - \tilde{w}_k^0 - \omega^2 \sum_{n=1}^{\infty} \frac{a_n^* E_{kn}^0}{\omega^2 a_n^* - k_n d_n} \psi_n \right) A_k^0. \tag{22}$$

Formula (22) includes unknown constants A_k^0. From the boundary conditions of the plate fixation (16), we have two linear homogeneous equations for A_k^0:

$$\sum_{k=1}^{2}\left(\mathcal{L}_{pk}^{0}-\omega^{2}\sum_{n=1}^{\infty}\alpha_{n}E_{kn}^{0}\mathcal{L}_{pn}\right)A_{k}^{0}=0\ (p=1,2). \tag{23}$$

Here

$$\mathcal{L}_{pk}^{0}=\left.\left(\mathcal{L}_{p}\left[w_{k}^{0}-\tilde{w}_{k}^{0}\right]\right)\right|_{\gamma},\ \ \mathcal{L}_{pn}=\left.\left(\mathcal{L}_{p}\left[\psi_{n}\right]\right)\right|_{\gamma},\ \ \alpha_{n}=a_{n}^{*}/\left(\omega^{2}a_{n}^{*}-k_{n}d_{n}\right).$$

From the zero value of the determinant of the homogeneous system (23), the frequency equation for the common eigenmodes of the plate and the liquid looks like

$$\left|\left\|C_{qk}\right\|_{q,k=1}^{2}\right|=0, \tag{24}$$

where

$$C_{pk}=\mathcal{L}_{pk}^{0}-\omega^{2}\sum_{n=1}^{\infty}\alpha_{n}E_{kn}^{0}\mathcal{L}_{pn}\quad(\ p=1,2;\ k=1,2).$$

Using the series expansion of the functions w_k^0 in the complete and orthogonal system of functions ψ_n, the condition $\int_S\psi_n dS=0$, and notation (21), equation (24) can be rewritten in the form

$$\left|\left\|C_{qk}\right\|_{q,k=1}^{2}\right|=0, \tag{25}$$

where

$$C_{11}=\sum_{n=1}^{\infty}\beta_{n}E_{1n}^{0}\mathcal{L}_{1n},\ \ C_{12}=\sum_{n=1}^{\infty}\beta_{n}E_{2n}^{0}\mathcal{L}_{1n},$$

$$C_{21}=\mathcal{L}_{21}^{0}-\omega^{2}\sum_{n=1}^{\infty}\alpha_{n}E_{1n}^{0}\mathcal{L}_{2n},\quad C_{22}=\mathcal{L}_{22}^{0}-\omega^{2}\sum_{n=1}^{\infty}\alpha_{n}E_{2n}^{0}\mathcal{L}_{2n}.$$

Here $\beta_n=k_nd_n/\left(\omega^2a_n^*-k_nd_n\right)$.

Thus, knowing the functions w_k^0, the eigenfunctions ψ_n, and the eigenvalues k_n, we can find the eigenfrequencies ω^2 from equation (25), and the eigenforms of the common vibrations for the plate and the liquid from formulas (22) and (23). This approach was used in many works (see, for example, [1–8]). However, its application is difficult because the values of the functions w_k^0 and the main coefficients E_{kn}^0 depend on the signs of the quantities P and q; in turn, the quantity q depends on the unknown frequency ω^2. Therefore let us simplify the frequency equation (25).

The biharmonic equation (18) can be represented in the form [12]

$$\left(\Delta_{2}w_{k}^{0}-p_{1}^{2}w_{k}^{0}\right)\left(\Delta_{2}w_{k}^{0}-p_{2}^{2}w_{k}^{0}\right)=0, \tag{26}$$

where $p_{1,2}^2=P\pm\sqrt{P^2-q}$. Since $\Delta_2\psi_n=-k_n^2\psi_n$ and $\Delta_2w_k^0-p_k^2w_k^0=0$ $(k=1,2)$, then

$$\int_{S}w_{k}^{0}\psi_{n}dS=-\frac{1}{k_{n}^{2}}\int_{S}w_{k}^{0}\Delta_{2}\psi_{n}dS=\frac{1}{p_{k}^{2}}\int_{S}\psi_{n}\Delta_{2}w_{k}^{0}dS. \tag{27}$$

From Green's formulas for the Laplace operator and the condition $\left.\frac{\partial\psi_n}{\partial\vec{\nu}}\right|_\gamma=0$, we have

$$\int_{S}\left(w_{k}^{0}\Delta_{2}\psi_{n}-\psi_{n}\Delta_{2}w_{k}^{0}\right)dS=\oint_{\gamma}\left(w_{k}^{0}\frac{\partial\psi_{n}}{\partial\vec{\nu}}-\psi_{n}\frac{\partial w_{k}^{0}}{\partial\vec{\nu}}\right)d\gamma=-\oint_{\gamma}\left(\psi_{n}\frac{\partial w_{k}^{0}}{\partial\vec{\nu}}\right)d\gamma, \tag{28}$$

or, is we take formula (27) into account,

$$\int_S \left(w_k^0 \Delta_2 \psi_n - \psi_n \Delta_2 w_k^0\right) dS = -\left(k_n^2 + p_k^2\right) \int_S \psi_n w_k^0 dS. \tag{29}$$

From formulas (21), (28), and (29), we obtain

$$E_{kn}^0 = \frac{1}{N_n^2} \int_S w_k^0 \psi_n dS = \frac{1}{N_n^2 \left(k_n^2 + p_k^2\right)} D_{kn}, \tag{30}$$

where $D_{kn} = \oint_\gamma \psi_n \frac{\partial w_k^0}{\partial \vec{\nu}} d\gamma$.

Let us further simplify the frequency equation (25) on the example of a clamped contour. As was already noted, in this case, the operator $\mathcal{L}_1$ is a unit one and $\mathcal{L}_2 = \frac{\partial}{\partial \vec{\nu}}$. Accordingly, $\mathcal{L}_{1n} = \psi_n|_\gamma = B_n$, $\mathcal{L}_{2n} = 0$, and the coefficients of equation (25) take the form

$$C_{1k} = \sum_{n=1}^{\infty} \beta_n E_{kn}^0 B_n\,, \quad C_{2k} = C_k = \left.\frac{\partial w_k^0}{\partial \vec{\nu}}\right|_\gamma . \tag{31}$$

Taking relationship (31) into account, equation (25) reads

$$\sum_{n=1}^{\infty} \frac{\beta_n B_n}{N_n^2} \left(\frac{C_2 D_{1n}}{k_n^2 + p_1^2} - \frac{C_1 D_{2n}}{k_n^2 + p_2^2}\right) = 0. \tag{32}$$

An analytical solution of the frequency equation (32) can be obtained if we put

$$B_n = \text{const} \quad \text{and} \quad C_k = \text{const}. \tag{33}$$

This case can be realized, for example, for a circular or a coaxial cylindrical cavity, as well as for a cavity in the form of a rectangular channel.

Provided that conditions (33) are satisfied, we have $D_{kn} = \gamma C_k B_n$, and equation (32) can be written as follows:

$$\gamma C_1 C_2 \sqrt{P^2 - q} \sum_{n=1}^{\infty} \frac{\beta_n B_n^2}{N_n^2} \frac{1}{\left(k_n^2 + p_1^2\right)\left(k_n^2 + p_2^2\right)} = 0. \tag{34}$$

Since $\left(k_n^2 + p_1^2\right)\left(k_n^2 + p_2^2\right) = d_n/D$, then in the case $C_k \neq 0$ and $P^2 \neq q$, the frequency equation (34) takes the form

$$\sum_{n=1}^{\infty} \frac{k_n}{\omega^2 a_n^* - k_n d_n} \left(\frac{B_n}{N_n}\right)^2 = 0. \tag{35}$$

One can see that the simplified frequency equation (35) depends only on the eigenvalues k_n and eigenfunctions ψ_n of vibrations of an ideal liquid in a cylindrical tank, but it does not depend on the FSS of the biharmonic equation (18).

In the case of a rectangular channel with the width $2a$, we obtain $\psi_n(x, y) = \psi_n(x) = \cos k_n (x + a)$, $k_n = \pi n/2a$, $B_n = (-1)^n$, and $N_n^2 = a$. Then, equations (35) coincide with the relevant equation from work [6].

Equation (35) includes a number of particular cases. For instance, if the plate degenerates into a membrane, we should put $D = 0$ in this equation; in the absence of upper liquid, we should put $\rho_1 = 0$.

If the cylindrical cavity is filled completely (the upper liquid has no free surface), we should put $\zeta \equiv 0$ in the first boundary condition (5) and should not consider the second condition (the constant pressure at the free surface). We should put $\zeta_n = 0$ in equation (10) and should not consider equation (11). In this case, we adopt $a_n^* = a_n$ $\left(\tilde{b}_n = 0\right)$ in equation (13), as well as in other equations and relationships.

Hence, if the cylindrical cavity is filled completely, the frequency equation (35) transforms into the equation

$$\sum_{n=1}^{\infty} \frac{k_n}{\omega^2 a_n - k_n d_n} \left(\frac{B_n}{N_n}\right)^2 = 0. \tag{36}$$

Consider axisymmetric plate and liquid vibrations in a circular cylinder. In this case, the eigenfunctions $\psi_n(x, y)$ have the form $\psi_n(x, y) = \psi_n(r) = J_0(k_n r)$, where J_0 is the Bessel function of the first kind, and the eigenvalues k_n are determined from the equation $J_1(\xi_n) = 0$, where $\xi_n = k_n a$ and a is the radius of the cylinder. The other quantities in equation (36) are $B_n = J_0(\xi_n)$ and $N_n = a\sqrt{\pi} J_0(\xi_n)$. So the frequency equations (35) and (36) look like

$$\sum_{n=1}^{\infty} \frac{k_n}{\omega^2 a_n^* - k_n d_n} = 0, \tag{37}$$

$$\sum_{n=1}^{\infty} \frac{k_n}{\omega^2 a_n - k_n d_n} = 0. \tag{38}$$

It follows from works [1, 13] that axisymmetric vibrations of a clamped circular plate and an incompressible liquid in a limited volume are impossible. Therefore, equations (37)–(38) can be used only for the membrane case ($D = 0$); in this case, the coefficient $d_n = T k_n^2 + g\Delta\rho - m_0\omega^2$.

In the absence of upper liquid ($\rho_1 = 0$), equation (37) coincides with the relevant equation from work [5] and generalizes the latter to the case of a liquid with a free surface on the membrane surface.

In the case of non-axisymmetric plate and liquid vibrations (spatial vibrations) in a circular cylinder, the index n, which corresponds to radial harmonics, has to be appended by the index m corresponding to circular harmonics. In this case, the eigenfunctions $\psi_n(x, y) = \psi_{nm}(r, \theta)$ are represented in the form

$$\psi_{nm}(r, \theta) = J_m(k_{nm} r) \left\{ \begin{array}{l} \cos m\theta \\ \sin m\theta \end{array} \right. ,$$

and the eigenvalues k_{nm} are determined from the equation $J_m'(\xi_{nm}) = 0$, where $\xi_{nm} = k_{nm} a$, and J_m is the Bessel function of the first kind of the m-th order. This representation of eigenfunctions makes it possible to separate the radial and circular harmonics in the main differential and integral equations, and write down the frequency equation (35) in the form

$$\sum_{n=1}^{\infty} \frac{k_{nm}}{\omega^2 a_{nm}^* - k_{nm} d_{nm}} \left(\frac{B_{nm}}{N_{nm}}\right)^2 = 0. \tag{39}$$

Here

$$a_{nm} = \rho_1 \coth \kappa_{1nm} + \rho_2 \coth \kappa_{2nm}, \quad \kappa_{inm} = h_i k_{nm}, \quad a_{nm}^* = a_{nm} - \tilde{b}_{nm},$$

$$\tilde{b}_{nm} = \frac{2\omega^2 \rho_1}{(\omega^2 - \sigma_{nm}^2) \sinh 2\kappa_{1nm}}, \quad d_{nm} = \left(D k_{nm}^2 + T\right) k_{nm}^2 + g\Delta\rho - m_0\omega^2,$$

$$B_{nm} = J_m(\xi_{nm}), \; N_{nm}^2 = \frac{\pi a^2 (1 + \delta_{m0})}{2\xi_{nm}^2} J_m^2(\xi_{nm}) \left(\xi_{nm}^2 - m^2\right). \tag{40}$$

Taking (40) into account, equation (39) reads

$$\sum_{n=1}^{\infty} \frac{\xi_{nm}^3}{(\xi_{nm}^2 - m^2)(a\omega^2 a_{nm}^* - \xi_{nm} d_{nm})} = 0. \tag{41}$$

If the cylindrical cavity is filled completely (the upper liquid has no free surface), we should put $a_{nm}^* = a_{nm}$ in the frequency equation (41), and it transforms into the equation

$$\sum_{n=1}^{\infty} \frac{\xi_{nm}^3}{(\xi_{nm}^2 - m^2)(a\omega^2 a_{nm} - \xi_{nm} d_{nm})} = 0. \tag{42}$$

It was mentioned above that axisymmetric vibrations are impossible for the plate and the liquid. Therefore, the obtained equations (41)–(42) are valid for the membrane ($D = 0$), but for the plate ($D \neq 0$) they should be considered only if $m \neq 0$.

5. Some mechanical effects

For the plate, the left-hand side of equation (42) is a monotonically increasing function of the parameter ω^2 within the intervals $\left(\xi_{nm}\tilde{d}_{nm}/a\tilde{a}_{nm},\, \xi_{n+1,m}\,\tilde{d}_{n+1,m}/a\tilde{a}_{n+1,m}\right)$ $(n = 1, 2, ...)$ and varies there from $-\infty$ to ∞. Therefore, only one root of this equation lies between two consecutive values of $\xi_{nm}\tilde{d}_{nm}/a\tilde{a}_{nm}$. This fact predetermines the intervals where the eigenfrequencies of plate vibrations are located. For high frequencies ($n \gg 1$), the following approximate formula can be obtained:

$$\omega_n^2 \approx \xi_{nm}\tilde{d}_{nm}/a\tilde{a}_{nm}, \tag{43}$$

where $\tilde{d}_{nm} = \left(Dk_{nm}^2 + T\right)k_{nm}^2 + g\Delta\rho$ and $\tilde{a}_{nm} = a_{nm} + m_0 k_{nm}$. Formula (43) brings about the following approximate condition for the stability of common vibrations of the plate and the liquid:

$$\left(Dk_{nm}^2 + T\right)k_{nm}^2 + g\Delta\rho > 0. \tag{44}$$

This inequality is always obeyed if $T \geq 0$ and $\rho_2 \geq \rho_1$.

For the membrane ($D = 0$), only one root of equation (42) lies between two successive values of $\xi_{nm}\tilde{d}_{nm}/a\tilde{a}_{nm}$. For high frequencies ($n \gg 1$), we have the approximate formula

$$\omega_n^2 \approx \xi_{nm}\tilde{d}_{nm}/a\tilde{a}_{nm}, \tag{45}$$

where $\tilde{d}_{nm} = Tk_{nm}^2 + g\Delta\rho$. In turn, formula (45) brings about the following approximate condition for the stability of common vibrations of the membrane and the liquid:

$$Tk_{nm}^2 + g\Delta\rho > 0. \tag{46}$$

It should be noted that the obtained approximate stability conditions (44) and (46) coincide with the sufficient stability condition obtained in work [4] for a single plate or membrane.

In the absence of a plate or a membrane ($D = 0$, $T = 0$, $m_0 = 0$) from the equations (10)-(11) we have the frequency equation of natural vibrations of a two-layer ideal liquid in a cylindrical tank of arbitrary cross-section

$$(\rho_1 + \rho_2 \coth\kappa_{1n}\coth\kappa_{2n})\,x^2 - \rho_2\,(\coth\kappa_{1n} + \coth\kappa_{2n})\,x + \Delta\rho = 0. \tag{47}$$

Here $x = \omega_n^2/gk_n$.

For an infinitely deep bottom liquid ($h_2 = \infty$), equation (47) has the roots $x_1 = \Delta\rho/\left(\rho_1 + \rho_2 \coth \kappa_{1n}\right)$ and $x_2 = 1$, where the quantity $gk_n \Delta\rho/\left(\rho_1 + \rho_2 \coth \kappa_{1n}\right)$ is the eigenfrequency of vibrations of the internal liquid surface in the presence of a "lid" on the free surface, and the product gk_n is the eigenfrequency of vibrations of an infinitely deep ideal liquid with the free surface.

From the frequency equation (35) and the results of works [2, 3], it follows that the frequency spectrum of this equation consists of two frequency sets corresponding to the vibrations of the plate and the free surface of the liquid. Of high interest is the mutual influence of those two sets of vibrations. It follows from equations (35) and (36) that if $h_1/b \to \infty$, where b is a characteristic cavity size, then $a_n^* \to a_n$ ($\tilde{b}_n \to 0$). As the ratio h_1/b increases, the coefficient $\tilde{b}_n$ vanishes as $e^{-2k_1 h_1/b}$ because the eigenvalues k_n form an infinitely increasing numerical sequence. Therefore, if $h_1/b > 1$, the influence of the free surface on the frequency spectrum of the plate and liquid vibrations can be neglected.

Now let us obtain approximate stability conditions for the free vibrations of the plate and the liquid in a straight circular cylinder. For convenience of notation, we omit the subscript m in equation (41) and rewrite this equation as follows:

$$\sum_{n=1}^{\infty} \frac{\alpha_n k_n}{\omega^2 a_n^* - k_n d_n} = 0. \tag{48}$$

If two terms ($n = 1, 2$) in the series are made allowance for, this equation takes the form

$$c_0 x^3 + c_1 x^2 + c_2 x + c_3 = 0, \quad x = \omega^2 > 0. \tag{49}$$

Here

$$\begin{gathered}
c_0 = \alpha_1 k_1 a_2^* + k_2 a_1^* > 0, \quad c_1 = -\left(\sigma_1^2 + \sigma_2^2\right) c_0 - k_1 k_2 \left(\alpha_2 \tilde{d}_1 + \alpha_1 \tilde{d}_2\right), \\
c_2 = \sigma_1^2 \sigma_2^2 \left[\alpha_1 k_1 a_2 + \alpha_2 k_2 a_1 + m_0 k_1 k_2 \left(\alpha_1 + \alpha_2\right)\right] + \\
+ \left(\sigma_1^2 + \sigma_2^2\right) \left(\alpha_2 \tilde{d}_1 + \alpha_1 \tilde{d}_2\right) k_1 k_2, \; c_3 = -\sigma_1^2 \sigma_2^2 \left(\alpha_2 \tilde{d}_1 + \alpha_1 \tilde{d}_2\right) k_1 k_2, \\
a_n = \rho_1 \coth \kappa_{1n} + \rho_2 \coth \kappa_{2n}, \; \tilde{a}_n = a_n + m_0 k_n, \\
a_n^* = \tilde{a}_n - \frac{2\rho_1}{\sinh 2\kappa_{1n}} = \rho_1 \tanh \kappa_{1n} + \rho_2 \coth \kappa_{2n} + m_0 k_n > 0, \\
\alpha_n = \xi_n^2/\left(\xi_n^2 - m^2\right) > 0 \\
\tilde{a}_n = a_n - \frac{2\rho_1}{\sinh 2\kappa_{1n}} + m_0 k_n = \rho_1 \tanh \kappa_{1n} + \rho_2 \coth \kappa_{2n} + m_0 k_n > 0, \\
\tilde{d}_n = \left(D k_n^2 + T\right) k_n^2 + g\Delta\rho, \sigma_n^2 = g k_n \tanh \kappa_{1n}.
\end{gathered} \tag{50}$$

According to Descartes' rule of signs about the number of positive roots of polynomials, all roots of equation (49) are positive if

$$c_1 < 0, \quad c_2 > 0, \;\; c_3 < 0. \tag{51}$$

Hence it follows that in order to satisfy inequalities (51), it is sufficient that the inequality

$$\alpha_2 \tilde{d}_1 + \alpha_1 \tilde{d}_2 > 0$$

or

$$D\left(\alpha_{1m} k_{2m}^4 + \alpha_{2m} k_{1m}^4\right) + T\left(\alpha_{1m} k_{2m}^2 + \alpha_{2m} k_{1m}^2\right) + g\Delta\rho\left(\alpha_{1m} + \alpha_{2m}\right) > 0 \tag{52}$$

holds. Inequality (52) does not depend on the liquid filling depth and the plate mass. It follows from this inequality that if the forces are compressive ($T < 0$) or the density of the upper liquid is higher than that of the lower one ($\Delta\rho < 0$), the common vibrations of the plate and the liquid may become unstable. It also follows from inequality (52) that by preliminary increasing the tension ($T > 0$), it is possible to stabilize unstable vibrations of the plate and the liquid.

The approximate stability condition for free vibrations of the membrane ($D = 0$) and the liquid follows from inequality (52) and takes the form

$$T\left(\alpha_{1m}k_{2m}^2 + \alpha_{2m}k_{1m}^2\right) + g\Delta\rho\left(\alpha_{1m} + \alpha_{2m}\right) > 0. \tag{53}$$

The stability conditions (52) and (53) supplement the previously obtained conditions (44) and (46).

Inequality (52) also follows from equation (42) with two series terms ($n = 1, 2$) and the condition $\omega^2 > 0$. Thus, the approximate stability conditions are identical irrespective of whether the free surface is present or absent.

In papers [5,7,8], it was pointed out that the approximate stability conditions obtained taking two terms of the series into account are sufficient for practical accuracy.

It should be noted that the case $m = 1$ is of the highest theoretical and practical interest because in this case the plate and liquid vibrations affect the motion of the rigid body and this action is mutual [1].

6. Conclusions

The hydroelastic problem of free vibrations of a thin plate that horizontally separates ideal incompressible liquids of different densities in a rigid cylindrical tank of arbitrary cross-section has been solved in the linear formulation. To solve the complicated inhomogeneous biharmonic equation, the fundamental system of solutions of biharmonic equation (FSS) and the eigenforms of vibrations of an ideal liquid in a cylindrical cavity were used. The frequency equation was obtained for arbitrary methods of the plate contour fixation.

On the example of a clamped plate, the frequency equation was simplified by decomposing the homogeneous biharmonic equation into two harmonic equations and using Green's formula for the Laplace operator. It was shown that in this case the frequency equation does not depend on the FSS, which considerably simplified this equation and made it possible to carry out analytical studies.

It was shown that if the filling depth of the upper liquid exceeds the characteristic size of the cavity, the influence of the free surface on the frequency spectrum of plate vibrations can be neglected. The obtained simplified frequency equation has a uniform form in the cases of a straight circular cylinder and a rectangular channel, and in some cases coincides with the previously obtained equations.

The frequencies of asymmetric vibrations of the plate and the membrane and axisymmetric vibrations of the membrane in a circular cylinder have been studied. An approximation formula valid at high frequencies and approximate conditions for the stability of the plate and membrane vibrations were obtained. The stability conditions do not depend on the liquid filling depth and the plate or membrane mass. It follows from these conditions that under compressive forces or when the density of the upper liquid is greater than the density of the lower liquid, the coupled oscillations of the plate and the liquid can become unstable. Furthermore, by preliminary increasing the tension, unstable vibrations of the plate and the liquid can be stabilized. It was shown that the approximate sufficient conditions for the stability of vibrations are identical whether the free surface is available or not.

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