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ASYMPTOTIC BEHAVIOR OF SOME TYPES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DIFFERENT TYPES OF NONLINEARITIES

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We consider a differential equation of the n th order containing a sum of terms with regularly and rapidly varying nonlinearities on the right-hand side and determine the asymptotic behavior of some types of solutions to this equation.

Keywords: nonlinearities of different types, n th order differential equations, regularly varying nonlinearities, asymptotic behavior.

Introduction

We consider a differential equation

$$y^{(n)} = \sum_{i=1}^{m} \alpha_i p_i(t)\varphi_i(y), \tag{1}$$

where $\alpha_i \in \{-1,1\}$, $i = 1,\ldots,m$; $p_i : [a,\omega[\to]0,+\infty[$, $i = 1,\ldots,m$, are continuous functions, $-\infty < a < \omega \le +\infty$; $\varphi_i : \Delta_{Y_0} \to]0,+\infty[$, where Δ_{Y_0} is a unilateral neighborhood of Y_0 (Y_0 is either equal to zero or to $\pm\infty$), are continuous functions for $i = 1,\ldots,\ell$, twice continuously differentiable for $i = \ell+1,\ldots,m$, and such that

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i}, \quad i = 1,\ldots,\ell, \quad \text{for any} \quad \lambda > 0, \tag{2}$$

$$\varphi_i'(y) \ne 0, \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \varphi_i(y) \in \{0,+\infty\}, \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)} = 1, \quad i = \ell+1,\ldots,m. \tag{3}$$

It follows from conditions (2) that every function φ_i, $i = 1,\ldots,\ell$, is a function of order σ_i regularly varying as $y \to Y_0$ (see the monograph by Seneta [15, Chap. 1, Sec. 1, p. 9]). Moreover, it follows from conditions (3) that, in particular, the functions φ_i, $i = \ell+1,\ldots,m$, are rapidly varying as $y \to Y_0$ (see the monograph by Marić [21, Chap. 3, Sec. 3.4, Lemmas 3.2 and 3.3, pp. 91–92]). Thus, the right-hand side of the differential equation (1) contains both terms with regularly varying nonlinearities and terms with rapidly varying nonlinearities.

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Definition 1. *The solution* y *of Eq. (1) is called a* $P_\omega(Y_0,\lambda_0)$*-solution, where* $-\infty \le \lambda_0 \le +\infty$ *if it is defined in the interval* $[t_0,\omega[\subset [a,\omega[$ *and satisfies the following conditions*:

$$\lim_{t\uparrow\omega} y(t) = Y_0, \qquad \lim_{t\uparrow\omega} y^{(k)}(t) = 0 \quad or \quad \lim_{t\uparrow\omega} y^{(k)}(t) = \pm\infty, \qquad k = 1,\ldots,n-1, \tag{4}$$

$$\lim_{t\uparrow\omega} \frac{\left[y^{(n-1)}(t)\right]^2}{y^{(n)}(t)y^{(n-2)}(t)} = \lambda_0. \tag{5}$$

In the case where $n = 2$, the asymptotic properties of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1) were studied in [5–7, 12–14, 18] under the condition that, for each solution of this kind, the right-hand side of Eq. (1) is equivalent, for $t\uparrow\omega$, to one sth term, where $s \in \{1,\ldots,m\}$, i.e., in the case where

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = 0 \qquad \text{for all} \qquad i \in \{1,\ldots,m\}\backslash\{s\}. \tag{6}$$

In [8], the authors proposed an approach that made it possible to extend the results known for $n = 2$ in the case of two-term differential equations with regularly varying nonlinearity on the right-hand side (see [2, 16, 19–25]) to the case where $n \ge 2$. The works [3, 4, 10, 11, 17] were devoted to the investigation of a differential equation of the nth order containing the sum of terms with regularly varying nonlinearities on the right-hand side. The problem of asymptotic behavior of the solutions of differential equations of the nth order containing the sum of terms with nonlinearities of different types on the right-hand side is now quite urgent. The present work is devoted to the investigation of this type of equations.

The aim of the present work is to extend the results obtained in [6] to the case $n \ge 2$; i.e., to establish the conditions of existence of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1) for $n \ge 2$ and to construct asymptotic representations of these solutions as $t\uparrow\omega$ and their derivatives of the $(n-1)$th order, inclusively, for

$$\lambda_0 \in \mathbb{R}\backslash\left\{0,\frac{1}{2},\frac{2}{3},\ldots,\frac{n-2}{n-1},1\right\}$$

in the case where conditions (6) are satisfied and $s \in \{1,\ldots,m\}$, i.e., the sth term contains a regularly varying nonlinearity. In the case where conditions (6) are satisfied, we say that the sth term is the *principal* term on the right-hand side of the differential equation (1).

Note that, in our investigations, we use the fact [which follows from conditions (2)] that the functions φ_i admit the following representation:

$$\varphi_i(y) = |y|^{\sigma_i} L_i(y), \qquad i = 1,\ldots,\ell, \tag{7}$$

where L_i, $i = 1,\ldots,\ell$, are slowly varying functions, i.e., regularly varying functions of order zero.

In the study of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1), we use some *a priori* asymptotic properties. We introduce a function $\pi_\omega : [a,\omega[\to \mathbb{R}$ as follows:

$$\pi_\omega(t) = \begin{cases} t, & \omega = +\infty, \\ t-\omega, & \omega < +\infty. \end{cases}$$

The next assertion follows directly from [1, Corollary 10.1].

Lemma 1. *If* $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}, 1\right\}$, *then the following asymptotic relations are true for any* $P_\omega(Y_0, \lambda_0)$*-solution of Eq. (1):*

$$y^{(k-1)}(t) \sim \left[(\lambda_0 - 1)\pi_\omega(t)\right]^{n-k} \prod_{i=k}^{n-1} a_{0i}^{-1} y^{(n-1)}(t), \qquad k = 1, \ldots, n-1, \qquad t \uparrow \omega, \tag{8}$$

where $a_{0i} = (n-i)\lambda_0 - (n-i-1)$, $i = 1, \ldots, n-1$.

1. Main Results

Assume that

$$\Delta_{Y_0} = \Delta_{Y_0}(b),$$

where $\Delta_{Y_0}(b) = [b, Y_0[$ if Δ_{Y_0} is a left neighborhood of Y_0 and $\Delta_{Y_0}(b) =]Y_0, b]$ if Δ_{Y_0} is a right neighborhood of Y_0, and the number b satisfies the inequalities

$$|b| < 1 \quad \text{for} \quad Y_0 = 0,$$

$$b > 1 \quad \text{for} \quad Y_0 = +\infty,$$

$$b < -1 \quad \text{for} \quad Y_0 = -\infty.$$

It follows from the definition of $P_\omega(Y_0, \lambda_0)$-solutions of Eq. (1) that each solution of this kind and its derivatives up to the nth order, inclusively, are nonzero in a certain interval $[t_1, \omega[\subset [t_0, \omega[$. Moreover, in this interval, the first derivative of the solution is positive if Δ_{Y_0} is a left neighborhood of Y_0 and negative if Δ_{Y_0} is a right neighborhood of Y_0. We introduce two numbers ν_0 and ν_1 specifying, respectively, the signs of the $P_\omega(Y_0, \lambda_0)$-solution and its first derivative in the interval $[t_1, \omega[$:

$$\nu_0 = \operatorname{sgn} b, \qquad \nu_1 = \pm 1.$$

Furthermore, $\nu_1 = 1$ if Δ_{Y_0} is a left neighborhood of Y_0, and $\nu_1 = -1$ if Δ_{Y_0} is a right neighborhood of Y_0.

To formulate the accumulated results, we introduce the following auxiliary notation:

$$q(t) = \alpha_s(\lambda_0 - 1)^{n-1}\prod_{i=2}^{n-1} a_{0i}^{-1} J_s(t),$$

$$J_s(t) = \int_{A_s}^{t} [\pi_\omega(\tau)]^{n-1} p_s(\tau)\,d\tau, \qquad H_s(y) = \int_{B_s}^{y} \frac{dx}{\varphi_s(x)}, \qquad s \in \{1,\ldots,m\},$$

$$A_s = \begin{cases} a, & \int_a^\omega [\pi_\omega(\tau)]^{n-1} p_s(\tau)\,d\tau = \pm\infty, \\ \omega, & \int_a^\omega [\pi_\omega(\tau)]^{n-1} p_s(\tau)\,d\tau = \text{const}, \end{cases} \qquad B_s = \begin{cases} b, & \int_b^{Y_0} \frac{dy}{\varphi_s(y)} = \pm\infty, \\ Y_0, & \int_b^{Y_0} \frac{dy}{\varphi_s(y)} = \text{const}. \end{cases}$$

We now mention some properties of the function H_s.

Since

$$H_s'(y) = \frac{1}{\varphi_s(y)} > 0$$

for $y \in \Delta_{Y_0}(b)$, the function H_s increases on $\Delta_{Y_0}(b)$, and there exists an increasing inverse function

$$H_s^{-1} : \Delta_{Z_s}(c_s) \to \Delta_{Y_0}(b)$$

such that

$$\lim_{\substack{z \to Z_s \\ z \in \Delta_{Z_s}(c_s)}} H_s^{-1}(z) = Y_0, \tag{9}$$

where

$$c_s = \int_{B_s}^{b} \frac{dx}{\varphi_s(x)}, \qquad Z_s = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}(b)}} H_s(y) = \begin{cases} 0, & B_s = Y_0, \\ +\infty, & B_s = b < Y_0, \\ -\infty, & B_s = b > Y_0, \end{cases}$$

$$\Delta_{Z_s}(c_s) = \begin{cases} [c_s, Z_s[, & \Delta_{Y_0}(b) = [b, Y_0[, \\ ]Z_s, c_s], & \Delta_{Y_0}(b) =]Y_0, b]. \end{cases}$$

In view of representation (7) and Propositions 1 and 2 in [21] (see Appendix on p. 115)], we get

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}(b)}} \frac{y}{H_s(y)\varphi_s(y)} = 1 - \sigma_s. \tag{10}$$

Theorem 1. *Let* $\lambda_0 \in \mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},\ldots,\frac{n-2}{n-1},1\right\}$ *and let* $\sigma_s \neq 1$ *for some* $s \in \{1,\ldots,\ell\}$. *Then, for the existence of* $P_\omega(Y_0,\lambda_0)$*-solutions of the differential equation (1) satisfying conditions (6), it is necessary that the following inequalities be true:*

$$\alpha_s \nu_0 [(\lambda_0 - 1)\pi_\omega(t)]^n \prod_{i=1}^{n-1} a_{0i} > 0, \qquad \nu_0 \nu_1 a_{01}(\lambda_0 - 1)\pi_\omega(t) > 0, \qquad t \in]a,\omega[, \tag{11}$$

and the following conditions be satisfied:

$$\alpha_s(\lambda_0 - 1)^{n-1}\prod_{i=2}^{n-1} a_{0i}^{-1} \lim_{t\uparrow\omega} J_s(t) = \mathrm{Z}_s, \qquad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_s'(t)}{J_s(t)} = \frac{(1-\sigma_s)a_{01}}{\lambda_0 - 1}, \tag{12}$$

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(H_s^{-1}(q(t)))}{p_s(t)\varphi_s(H_s^{-1}(q(t)))} = 0 \qquad \text{for all} \qquad i \in \{1,\ldots,\ell\}\setminus\{s\}, \tag{13}$$

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i\left(H_s^{-1}\left(q(t)(1+\delta_i)\right)\right)}{p_s(t)\varphi_s\left(H_s^{-1}(q(t))\right)} = 0 \qquad \text{for all} \qquad i \in \{\ell+1,\ldots,m\}, \tag{14}$$

where δ_i *are any numbers from some unilateral neighborhood of zero. Moreover, for each* $P_\omega(Y_0,\lambda_0)$*-solution of this kind, the following asymptotic representation is true:*

$$y(t) = H_s^{-1}(q(t))[1+o(1)], \qquad t\uparrow\omega, \tag{15}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0 - 1)\pi_\omega(t)}[1+o(1)], \qquad k = 1,\ldots,n-1, \qquad t\uparrow\omega. \tag{16}$$

Proof. Let $y:[t_0,\omega[\,\to\mathbb{R}$ be an arbitrary $P_\omega(Y_0,\lambda_0)$-solution of the differential equation (1) satisfying conditions (6). Then it follows from (1) and (6) that

$$y^{(n)}(t) = \alpha_s p_s(t)\varphi_s(y(t))[1+o(1)], \qquad t\uparrow\omega, \tag{17}$$

and there exists $t_1 \in [t_0,\omega[$ such that

$$\operatorname{sgn} y(t) = \nu_0, \qquad \operatorname{sgn} y'(t) = \nu_1, \qquad \operatorname{sgn} y^{(n)}(t) = \alpha_s, \qquad t \in [t_1,\omega[.$$

Hence, in view of conditions (8) in Lemma 1, we get the second inequality in (11). In addition, according to Lemma 1, the asymptotic relations (8) are true for $a_{0i} \neq 0$, $i = 1,\ldots,n-1$. These formulas yield the following asymptotic relations:

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0 - 1)\pi_\omega(t)}[1+o(1)], \qquad k = 1,\ldots,n-1, \qquad t \uparrow \omega. \tag{18}$$

In view of (5), we can write

$$y^{(n)}(t) \sim \frac{1}{\lambda_0}\left[\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\right]^2 y^{(n-2)}(t)$$

$$= \frac{1}{\lambda_0}\left[\frac{y^{(n-1)}(t)}{y^{(n-2)}(t)}\right]^2 \frac{y^{(n-2)}(t)}{y^{(n-3)}(t)} \cdots \frac{y''(t)}{y'(t)} y'(t)$$

$$\sim \frac{\lambda_0}{(\lambda_0-1)^2\pi_\omega^2(t)} \frac{a_{0n-2}}{(\lambda_0-1)\pi_\omega(t)} \cdots \frac{a_{02}}{(\lambda_0-1)\pi_\omega(t)} y'(t)$$

$$= \frac{\prod_{i=2}^{n-1} a_{0i}}{[(\lambda_0-1)\pi_\omega(t)]^{n-1}} y'(t), \qquad t \uparrow \omega.$$

By using (17), we get

$$\frac{y'(t)}{\varphi_s(y(t))} = \alpha_s[(\lambda_0-1)\pi_\omega(t)]^{n-1}\prod_{i=2}^{n-1} a_{0i}^{-1} p_s(t)[1+o(1)], \qquad t \uparrow \omega, \tag{19}$$

whence, in particular, we arrive at the inequality

$$\alpha_s \nu_1 [(\lambda_0-1)\pi_\omega(t)]^{n-1}\prod_{i=2}^{n-1} a_{0i} > 0, \qquad t \in\,]a,\omega[\,.$$

Thus, in view of the second sign condition in (11), we obtain the first inequality in (11). Integrating relation (19) over the interval from t_1 to t, where $t \in\,]t_1,\omega[$, we get

$$\int_{y(t_1)}^{y(t)} \frac{ds}{\varphi_s(s)} = \alpha_s(\lambda_0-1)^{n-1}\prod_{i=2}^{n-1}\frac{1}{a_{0i}}\int_{t_1}^{t}[\pi_\omega(\tau)]^{n-1} p_s(\tau)[1+o(1)]\,d\tau, \qquad t \uparrow \omega. \tag{20}$$

Since, according to the first condition in (4), $\lim_{t\uparrow\omega} y(t) = Y_0$, it follows from (20) that the improper integrals

$$\int_{y(t_1)}^{Y_0} \frac{ds}{\varphi_s(s)} \quad \text{and} \quad \int_{t_1}^{t}[\pi_\omega(\tau)]^{n-1} p_s(\tau)[1+o(1)]\,d\tau$$

either simultaneously converge or simultaneously diverge. Therefore, in view of the choice of the limits of integration A_s and B_s in the functions J_s and H_s, relation (20) can be represented in the form

$$H_s(y(t)) = q(t)[1+o(1)], \qquad t \uparrow \omega. \tag{21}$$

This yields the validity of the first condition in (12). Moreover, in view of (10), it follows from (21) that

$$\frac{y(t)}{\varphi_s(y(t))} = \alpha_s(1-\sigma_s)(\lambda_0-1)^{n-1}\prod_{i=2}^{n-1} a_{0i}^{-1} J_s(t)[1+o(1)], \qquad t \uparrow \omega. \tag{22}$$

By virtue of (19), (22), and condition (8) in Lemma 1, we get the second limit relation in (12).
Further, it follows from (21) that

$$y(t) = H_s^{-1}(q(t)[1+o(1)]), \qquad t \uparrow \omega. \tag{23}$$

The function H_s^{-1} is regularly varying function of order $\dfrac{1}{1-\sigma_s}$ for $z \to Z_s$ as the inverse function of a regularly varying (as $y \to Y_0$) function H_s of order $1-\sigma_s \neq 0$. Moreover, according to conditions (12), there exists $t_2 \in [t_1, \omega[$ for which the function

$$z(t) = q(t)[1+o(1)]$$

is such that $\lim\limits_{t\uparrow\omega} z(t) = \mathrm{Z}_s$ and $z(t) \in \Delta_{\mathrm{Z}_s}(c_s)$ for $t \in [t_2, \omega[$. Therefore, in view of the properties of regularly varying functions, we can rewrite relation (23) in the form (15). Moreover, it follows from conditions (8) of Lemma 1 and asymptotics (15) that the asymptotic relations (16) are true.

Since $s \in \{1,\ldots,\ell\}$, the functions φ_i, $i = 1,\ldots,\ell$, are regularly varying as $y \to Y_0$ and the function $z(t)$ satisfies the conditions presented above, we can write

$$\varphi_i(H_s^{-1}(q(t)[1+o(1)])) = \varphi_i(H_s^{-1}(q(t)))[1+o(1)], \qquad t \uparrow \omega.$$

Then it follows from the asymptotic representations (15) that

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(q(t)))[1+o(1)]}{p_s(t)\varphi_s(H_s^{-1}(q(t)))[1+o(1)]}$$

$$= \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(q(t)))}{p_s(t)\varphi_s(H_s^{-1}(q(t)))}, \qquad i = 1,\ldots,\ell,$$

whence, in view of (6), we arrive at conditions (14).

For $i \in \{\ell+1,\ldots,m\}$, in view of (23) and the properties of the function $z(t)$, we get

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(H_s^{-1}(q(t)[1+o(1)]))}{p_s(t)\varphi_s(H_s^{-1}(q(t)))}. \tag{24}$$

Since the functions $\varphi_i(H_s^{-1}(z))$, $i=\ell+1,\ldots,m$, are monotone on the interval $\Delta_{Z_s}(c_s)$ for any δ_i from a certain unilateral neighborhood of zero, there exists $t_3\in[t_2,\omega[$ such that, for $t\in[t_3,\omega[$, the inequalities

$$\frac{p_i(t)\varphi_i(H_s^{-1}(q(t)[1+o(1)]))}{p_s(t)\varphi_s(H_s^{-1}(q(t)))}\ge\frac{p_i(t)\varphi_i(H_s^{-1}(q(t)[1+\delta_i]))}{p_s(t)\varphi_s(H_s^{-1}(q(t)))}\ge 0 \tag{25}$$

are true. Then conditions (14) follow from relations (6), (24), and (25).

The theorem is proved.

Theorem 2. *Let* $\lambda_0\in\mathbb{R}\setminus\left\{0,\frac{1}{2},\frac{2}{3},\ldots,\frac{n-2}{n-1},1\right\}$ *and let* $\sigma_s\ne 1$ *for some* $s\in\{1,\ldots,\ell\}$. *Suppose that conditions (11)–(13) are satisfied and that, for any* $i\in\{\ell+1,\ldots,m\}$,

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(H_s^{-1}(q(t)(1+u)))}{p_s(t)\varphi_s(H_s^{-1}(q(t)))}=0 \tag{26}$$

uniformly in $u\in[-\delta,\delta]$ *for some* $0<\delta<1$. *Moreover, assume that the following algebraic equation for* ρ:

$$(1+\rho)\prod_{i=1}^{n-1}(a_{0i}+\rho)=\sigma_s\prod_{i=1}^{n-1}a_{0i}, \tag{27}$$

where $a_{0i}=(n-i)\lambda_0-(n-i-1)$, $i=1,\ldots,n-1$, *does not have roots whose real part is equal to zero.*

Then there exist $P_\omega(Y_0,\lambda_0)$*-solutions of the differential equation (1) for which the asymptotic representations (15) and (16) are true and there exists an* m*-parameter family of solutions of this kind if the collection of roots of equation (27) contains* m *roots (with regard for multiplicities) for which the sign of the real part coincides with the sign of the function* $(1-\lambda_0)\pi_\omega(t)$.

Proof. By applying the replacement

$$\begin{gathered}H_s(y(t))=q(t)[1+u_1(t)],\\ \frac{y^{(k)}(t)}{y^{(k-1)}(t)}=\frac{a_{0k}}{(\lambda_0-1)\pi_\omega(t)}[1+u_{k+1}(t)],\qquad k=1,\ldots,n-1,\end{gathered} \tag{28}$$

to Eq. (1), we get the following system of differential equations:

$$u_1'=\frac{1}{\pi_\omega(t)}\left[-\frac{\pi_\omega(t)J'_s(t)}{J_s(t)}(1+u_1)+\frac{a_{01}}{\lambda_0-1}\frac{G(t,u_1)}{q(t)}(1+u_2)\right],$$

$$u_k'=\frac{1}{\pi_\omega(t)}\left[1+u_k+\frac{a_{0k}}{\lambda_0-1}(1+u_k)(1+u_{k+1})-\frac{a_{0k-1}}{\lambda_0-1}(1+u_k)^2\right],\qquad k=2,\ldots,n-1, \tag{29}$$

$$u_n' = \frac{1}{\pi_\omega(t)}\left[1+u_n - \frac{a_{0n-1}}{\lambda_0-1}(1+u_n)^2 + \frac{p_s(t)\pi_\omega^2(t)q(t)}{a_{01}J_s(t)G(Y(t,u_1))}\right.$$

$$\left.\times\left(\prod_{k=2}^{n-1}(1+u_k)\right)^{-1}(1+R(t,u_1))\right],$$

in which

$$G(t,u_1) = \frac{Y(t,u_1)}{\varphi_s(Y(t,u_1))}, \qquad Y(t,u_1) = H_s^{-1}(q(t)(1+u_1)), \tag{30}$$

$$R(t,u_1) = \sum_{\substack{i=1\\ i\neq s}}^{m} \frac{\alpha_i p_i(t)\varphi_i(Y(t,u_1))}{\alpha_s p_s(t)\varphi_s(Y(t,u_1))}. \tag{31}$$

In view of conditions (12), we select the number $t_0 \in [a,\omega[$ such that, for $|u_1| \le \delta$,

$$q(t)(1+u_1) \in \Delta_{Z_s}(c_s), \qquad Y(t,u_1) \in \Delta_{Y_0}(b).$$

We now consider system (29) on the set

$$\Omega = [t_0,\omega[\times \mathbb{R}^n_\delta, \qquad \mathbb{R}^n_\delta = \{(u_1,\ldots,u_n)\in\mathbb{R}^n : |u_k| \le \delta,\ k=1,\ldots,n\}.$$

Thus, it follows from (9) and (12) that

$$\lim_{t\uparrow\omega} Y(t,u_1) = Y_0 \qquad \text{uniformly in} \qquad u_1 \in [-\delta,\delta]. \tag{32}$$

Hence, by using (11) and the form of the function G, we obtain

$$\lim_{t\uparrow\omega}\frac{G(t,u_1)}{H_s(Y(t,u_1))} = 1-\sigma_s \qquad \text{uniformly in} \qquad u_1\in[-\delta,\delta],$$

i.e.,

$$G(t,u_1) = [1-\sigma_s + R_1(t,u_1)]H_s(Y(t,u_1))$$

and

$$\frac{1}{G(t,u_1)} = \frac{1/(1-\sigma_s)+R_2(t,u_1)}{H_s(Y(t,u_1))},$$

where the functions R_i, $i=1,2$, satisfy the conditions

$$\lim_{t\uparrow\omega} R_i(t,u_1) = 0 \quad \text{uniformly in} \quad u_1 \in [-\delta,\delta]. \tag{33}$$

Thus, with regard for the form of the function $Y(t,u_1)$, we get the representations

$$G(t,u_1) = \left[1-\sigma_s + R_1(t,u_1)\right] q(t)(1+u_1), \tag{34}$$

$$\frac{1}{G(t,u_1)} = \frac{1/(1-\sigma_s) + R_2(t,u_1)}{q(t)(1+u_1)}. \tag{35}$$

Furthermore, we can show that

$$\lim_{t\uparrow\omega} R(t,u_1) = 0 \quad \text{uniformly in} \quad u_1 \in [-\delta,\delta]. \tag{36}$$

Since the functions φ_i, $i = 1,\ldots,\ell$, are regularly varying functions of orders σ_i as $y \to Y_0$, $y \in \Delta_{Y_0}(b)$, in view of representations (7) and the property of slowly varying functions [15], we have

$$\begin{aligned}
\varphi_i(Y(t,u_1)) &= \varphi_i(H_s^{-1}(q(t)(1+u_1))) \\
&= \left| H_s^{-1}(q(t))(1+u_1) \right|^{\sigma_i} L_i(H_s^{-1}(q(t)(1+u_1))) \\
&= \left| H_s^{-1}(q(t))(1+u_1) \right|^{\sigma_i} L_i(H_s^{-1}(q(t)))(1+r_i(t,u_1)) \\
&= \varphi_i(H_s^{-1}(q(t)))(1+u_1)^{\sigma_i}(1+r_i(t,u_1)), \qquad i = 1,\ldots,\ell,
\end{aligned}$$

where the functions $r_i(t,u_1)$ are such that

$$\lim_{t\uparrow\omega} r_i(t,u_1) = 0 \quad \text{uniformly in} \quad u_1 \in [-\delta,\delta].$$

Therefore,

$$\lim_{t\uparrow\omega} \sum_{\substack{i=1 \\ i\neq s}}^{\ell} \frac{\alpha_i p_i(t)\varphi_i(Y(t,u_1))}{\alpha_s p_s(t)\varphi_s(Y(t,u_1))} = 0 \quad \text{uniformly in} \quad u_1 \in [-\delta,\delta] \tag{37}$$

because it follows from conditions (14) that

$$\lim_{t\uparrow\omega} \sum_{\substack{i=1 \\ i\neq s}}^{\ell} \frac{\alpha_i p_i(t)\varphi_i(Y(t,u_1))}{\alpha_s p_s(t)\varphi_s(Y(t,u_1))}$$

$$= \lim_{t\uparrow\omega} \sum_{\substack{i=1\\ i\neq s}}^{\ell} \frac{\alpha_i p_i(t)\varphi_i(H_s^{-1}(q(t)))(1+u_1)^{\sigma_i}[1+r_i(t,u_1)]}{\alpha_s p_s(t)\varphi_s(H_s^{-1}(q(t)))(1+u_1)^{\sigma_s}[1+r_s(t,u_1)]}$$

$$= \lim_{t\uparrow\omega} \sum_{\substack{i=1\\ i\neq s}}^{\ell} \frac{\alpha_i p_i(t)\varphi_i(H_s^{-1}(q(t)))}{\alpha_s p_s(t)\varphi_s(H_s^{-1}(q(t)))} = 0 \quad \text{uniformly in} \quad u_1 \in [-\delta,\delta].$$

In view of the form of the function $R(t,u_1)$, by virtue of (37) and (26), we obtain (36). By using relations (31), (34), and (35) and the functions

$$h(t) = \frac{1}{(\lambda_0 - 1)\pi_\omega(t)}, \qquad g(t) = \frac{(\lambda_0 - 1)\pi_\omega(t) J_s{}'(t)}{a_{01}(1-\sigma_s)J_s(t)},$$

we represent the system of differential equations (31) in the form

$$\begin{aligned} u_1' &= h(t)[f_1(t,u_1,u_2) + a_{01}(1-\sigma_s)u_2 + V_1(u_1,u_2)], \\ u_k' &= h(t)[-a_{0k-1}u_k + a_{0k}u_{k+1} + V_k(u_k,u_{k+1})], \qquad k = 2,\ldots,n-1, \\ u_n' &= h(t)\left[f_2(t,u_1,\ldots,u_{n-1}) - \sum_{i=1}^{n-1} u_i - (\lambda_0+1)u_n + V_n(t,u_1,\ldots,u_n)\right], \end{aligned} \tag{38}$$

where

$$f_1(t,u_1,u_2) = a_{01}\big[R_1(t,u_1)(1+u_2) + (1-\sigma_s) - (1-\sigma_s)g(t)\big](1+u_1),$$

$$V_1(u_1,u_2) = a_{01}(1-\sigma_s)u_1u_2,$$

$$V_k(u_k,u_{k+1}) = a_{0k}u_ku_{k+1} - a_{0k-1}u_k^2, \qquad k = 2,\ldots,n-1,$$

$$f_2(t,u_1,\ldots,u_{n-1}) = (1-\sigma_s)g(t)R_2(t,u_1)\prod_{k=1}^{n-1}\frac{1}{1+u_k}(1+R(t,u_1)) + \big(g(t)-1\big)\left(1-\sum_{k=1}^{n-1}u_k\right),$$

$$V_n(t,u_1,\ldots,u_n) = -a_{0n-1}u_n^2 + g(t)\left(\prod_{k=1}^{n-1}\frac{1}{1+u_k} - 1 + \sum_{k=1}^{n-1}u_k\right).$$

We consider the obtained system of differential equations on the set Ω introduced above. On this set, the right-hand sides of the system are continuous. Thus, according to the second condition in (12), in view of (33) and (36), we find

$$\lim_{t\uparrow\omega} f_1(t,u_1,u_2) = 0 \quad \text{uniformly in} \quad (u_1,u_2)\in\mathbb{R}^2_\delta,$$

$$\lim_{t\uparrow\omega} f_2(t,u_1,\ldots,u_{n-1}) = 0 \quad \text{uniformly in} \quad (u_1,\ldots,u_{n-1}) \in \mathbb{R}_\delta^{n-1},$$

$$\lim_{|u_k|+|u_{k+1}|\to 0} \frac{V_k(u_k,u_{k+1})}{|u_k|+|u_{k+1}|} = 0, \qquad k=1,\ldots,n-1,$$

$$\lim_{|u_1|+\ldots+|u_n|\to 0} \frac{V_n(t,u_1,\ldots,u_n)}{|u_1|+\ldots+|u_n|} = 0 \quad \text{uniformly in} \quad t\in[t_0,\omega[.$$

In addition,

$$\int_{t_0}^{\omega} h(\tau)d\tau = \frac{1}{\lambda_0-1}\int_{t_0}^{\omega}\frac{d\tau}{\pi_\omega(\tau)} = \frac{1}{\lambda_0-1}\ln|\pi_\omega(\tau)|\Big|_{t_0}^{\omega} = \pm\infty.$$

We now write the characteristic equation of the matrix formed by the coefficients of $u_1,\ldots,u_n$ in the square brackets in system (38), i.e., of the matrix

$$C = \begin{pmatrix} 0 & a_{01}(1-\sigma_s) & 0 & \ldots & 0 & 0 \\ 0 & -a_{01} & a_{02} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -a_{0n-2} & a_{0n-1} \\ -1 & -1 & -1 & \ldots & -1 & -(\lambda_0+1) \end{pmatrix}.$$

As a result, we get the algebraic equation (27). However, in view of the conditions of the theorem, this equation does not have roots with real part equal to zero.

Thus, for system (38), all conditions of Theorem 2.2 from [9] are satisfied. According to this theorem, system (38) possesses at least one solution $(u_k)_{k=1}^n : [t_1,\omega[\to \mathbb{R}^n$, $t_1 \ge t_0$, approaching zero as $t\uparrow\omega$. Moreover, there exists an m-parameter family of solutions of this kind if the family of roots of the algebraic equation (27) contains m roots whose signs coincide with the sign of $\pi_\omega(t)(1-\lambda_0)$ in the left neighborhood of ω. In view of relation (28) and conditions (10) and (12), each of these solutions of system (38) is associated with a solution of the differential equation (1) with asymptotic representations (15) and (16), which is a $P_\omega(Y_0,\lambda_0)$-solution of Eq. (1).

The theorem is proved.

2. Example

To illustrate the accumulated results, we consider a differential equation of the form

$$y^{(n)} = \alpha_1 p_1(t)|y|^\sigma + \alpha_2 p_2(t)e^{\mu y}, \tag{39}$$

where $\alpha_i \in \{-1,1\}$, $i=1,2$, and $p_i : [a,\omega[\to]0,+\infty[$, $i=1,2$, are continuous functions, $-\infty < a < \omega \le +\infty$,

$\sigma \neq 1$, and $\mu \neq 0$.

We now clarify the problem of existence of $P_\omega(Y_0, \lambda_0)$-solutions of Eq. (39) for

$$\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}, 1\right\}$$

such that the following conditions are satisfied:

$$\lim_{t \uparrow \omega} y(t) = \pm\infty, \qquad Y_0 = \pm\infty, \qquad \lim_{t \uparrow \omega} \frac{p_2(t) e^{\mu y(t)}}{p_1(t) |y(t)|^\sigma} = 0. \tag{40}$$

Theorems 1 and 2 imply the following corollary:

Corollary 1. *Let* $\lambda_0 \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}, 1\right\}$ *and let* $\sigma \neq 1$. *Then, for the existence of* $P_\omega(Y_0, \lambda_0)$-*solutions of the differential equation (39) satisfying conditions (40), it is necessary and if*

$$p_2(t) = o\left(\frac{p_1(t) \left| (1-\sigma)\left(\alpha_1 (\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{a_{0i}} J_1(t) + C \right) \right|^{\sigma/(1-\sigma)}}{\exp\left(\mu \nu_0 \left| (1-\sigma)\left(\alpha_1 (\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{a_{0i}} J_1(t)(1+u) + C \right) \right|^{1/(1-\sigma)} \right)} \right), \qquad t \uparrow \omega,$$

uniformly in $u \in [-\delta, \delta]$ *for some* $0 < \delta < 1$, *where*

$$C = \begin{cases} 0, & \sigma > 1, \\ \dfrac{\nu_0 |b|^{1-\sigma}}{\sigma - 1}, & \sigma < 1, \end{cases}$$

and the following algebraic equation for ρ:

$$(1+\rho) \prod_{i=1}^{n-1} (a_{0i} + \rho) = \sigma \prod_{i=1}^{n-1} a_{0i} \tag{41}$$

in which $a_{0i} = (n-i)\lambda_0 - (n-i-1)$, $i = 1, \ldots, n-1$, *does not have roots with zero real part, then it is also sufficient that the following conditions be satisfied:*

$$\alpha_1 \nu_0 [(\lambda_0 - 1)\pi_\omega(t)]^n \prod_{i=1}^{n-1} a_{0i} > 0, \qquad \nu_0 \nu_1 a_{01} (\lambda_0 - 1) \pi_\omega(t) > 0, \qquad t \in]a, \omega[,$$

$$\alpha_1 (\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{a_{0i}} \lim_{t \uparrow \omega} J_1(t) = \mathrm{Z}_1, \qquad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_1'(t)}{J_1(t)} = \frac{(1-\sigma_1) a_{01}}{\lambda_0 - 1}.$$

Moreover, any solution of this kind has the following the asymptotic representations:

$$y(t) = \left| (1-\sigma)\left(\alpha_1 (\lambda_0 - 1)^{n-1} \prod_{i=2}^{n-1} \frac{1}{a_{0i}} J_1(t) \right) \right|^{1/(1-\sigma)} [1 + o(1)], \qquad t \uparrow \omega,$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{a_{0k}}{(\lambda_0 - 1)\pi_\omega(t)} [1 + o(1)], \qquad k = 1, \ldots, n-1, \qquad t \uparrow \omega.$$

Furthermore, there exists an m*-parameter family of these solutions if the family of roots of Eq. (41) contains* m *roots (counting multiplicities) for which the sign of their real part coincides with the sign of the function* $(1-\lambda_0)\pi_\omega(t)$.

CONCLUSIONS

For Eq. (1) with $\lambda_0 \in \mathbb{R} \setminus \left\{ 0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1}, 1 \right\}$, we establish the conditions for the existence of $P_\omega(Y_0, \lambda_0)$-solutions in the case where the term with regularly varying nonlinearity is the principal term of equation (1). We also determine the asymptotic representations of these solutions and their derivatives up to the $(n-1)$th order, inclusively, as $t \uparrow \omega$, $\omega \le +\infty$, and analyze the problem of the number of solutions admitting the indicated representations.

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