DOI 10.1007/s10958-023-06487-x
Journal of Mathematical Sciences, Vol. 273, No. 1, June, 2023

NONCLASSICAL LINEAR THEORIES OF CONTINUUM MECHANICS

O. R. Hrytsyna

UDC 539.3

We present a brief survey of nonclassical linear theories of continuum mechanics. Thus, we give a concise characterization of the Eringen–Edelen nonlocal theory of elasticity, theories of polar and micropolar media, Toupin couple-stress theory of elasticity, Eringen–Suhubi–Mindlin micromorphic theory, Mindlin gradient theory of elasticity, and also the local gradient theory of deformation of elastic media that takes into account local mass displacements.

Keywords: linear elasticity, nonclassical models, nonlocal elasticity, gradient-type theories, theories with nonclassical kinematics.

Introduction

For more than one hundred years, attention of numerous researchers is directed toward the development of generalized (nonclassical) theories of mechanics that take into account the influence of the local structure of materials on its macroproperties and describe the observed effects and phenomena that cannot be substantiated within the framework of the classical theory. This corresponds, in particular, to the size effects of the mechanical characteristics of materials [28, 77, 78], the subsurface inhomogeneity of physicomechanical fields [41, 69, 111], the high-frequency dispersion of elastic waves [13, 36], etc. The influence of the microstructure is significant in the case of propagation of ultrasonic waves (elastic vibrations characterized by high frequencies and small wavelengths). The analysis of this influence reveals waves of new type that cannot be described within the framework of the classical theory. The development of nonclassical mathematical models was stimulated by the introduction of new composite and porous materials in the contemporary engineering, the necessity of controlling the structure of materials, miniaturization of the engineering devices, development of nanotechnologies, etc. The construction of new theories of mechanics is explained by the necessity of avoiding the singularities of solutions in problems with cracks, notches, dislocations, etc.

The aim of the present paper is to give a concise survey of the existing nonclassical (nonlocal and gradient) theories of continuum mechanics.

1. Brief Survey of the State-of-the-Art of the Problem

The nonclassical theory of elasticity was originated in the works by Piola [35, 42]. In 1909, E. Cosserat and F. Cosserat proposed the theory of polar continuum [40]. Twenty years later, Jaramillo formulated relations of the gradient theory of elastic media [68]. The next step in the development of nonclassical theories of elasticity was made in the 1960s, when the foundations of the Aéro–Kuvshinskii polar theory [2], couple stress theory [90,

Institute of Construction and Architecture, Slovak Academy of Sciences, Bratislava; Center of Mathematical Modeling, Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv; e-mail: gryt045@gmail.com.

Translated from Matematychni Metody ta Fizyko-Mekhanichni Polya, Vol. 63, No. 3, pp. 85–106, July–September, 2020. Original article submitted October 22, 2019.

114], and the theory of elastic media with microstructure [46, 52, 86, 109] (known as the theory of micromorphic materials) were laid. The micropolar theory was developed in [52, 109]. In 1964, Green and Rivlin proposed a mathematical model of multipolar continuum [61]. In 1987, Burak [4] developed the continuum thermodynamic approach to the construction of gradient-type mathematical models of mechanics. The theory constructed on the basis of this approach was called the theory of local gradient elasticity [6, 11, 20].

The development of generalized theories of the media with microstructure was, to a large extent, realized in the works by Eringen and his coauthors who developed the linear [45] and nonlinear [71, 72] theories of micropolar elasticity, micromorphic theory [46, 52, 109], and the theory of microstretched elastic bodies (microstretch continuum) [49].

Surveys of different directions in the development of nonclassical theories of elasticity were published almost simultaneously with the construction of new nonclassical mathematical models of elastic continua. Chapters in the monographs and survey papers on this subject were published as early as in 1960–70s [23, 25, 47]. In [18, 84, 101], the trends of development and the prospects of application of nonclassical theories were clarified. In [47, 49], Eringen formulated the mathematical foundations and traced the development of the theories of micropolar elastic and thermoelastic continua, microstretch media. In cooperation with Kafadar [51], he presented a general survey of microcontinuum theories. Nowacki [95] gave a series of solutions of the problems of micropolar elasticity. Chen, etc. [38] analyzed the relations of the nonlocal, micromorphic, micropolar, polar, and couple stress theories from the viewpoint of the theory of crystal lattice and molecular dynamics. Sarkisyan [26, 27] and J. Altenbach with coauthors [33] presented surveys of the existing models of micropolar elastic thin shells, plates, beams, and rods.

Aifantis [29–31] presented a brief survey of the nonlocal and gradient models of elasticity, diffusion, and plasticity developed by him together with his coauthors. He treated his generalized theories of elastic continua as a specific compromise between the classical theory of elasticity, which does not give correct description of the mechanical response of nanostructures to the external action and the molecular dynamics whose application requires significant computational resources [30].

Forest [55] established the relationships between several well-known gradient-type models of thermoelastic and viscoplastic bodies and analyzed, in cooperation with Papenfuss [97], the existing higher-order theories with internal variables and additional degrees of freedom. Jirásek [70] focused his attention on the application of strongly and weakly nonlocal theories of mechanics and models of media with internal degrees of freedom to the description of size effects and dispersion of short elastic waves in heterogeneous media. Belov and Lurie [3, 19] presented a comparative analysis of the Cosserat [40], Jaramillo [68], Aéro-Kuvshinskii [2], Mindlin [86], Tupin [114], and Belov–Lurie [3] theories. Erofeev [13] used the theory of media with microstructure (Leroux continua, Mindlin–Eringen micromorphic media, Cosserat continuum, and Cosserat pseudocontinuum) to study the regularities of propagation of elastic waves. In [101], Polizzotto focused on the investigation of relations of the gradient models of mechanics containing, in addition, two, three, or four constants of higher order and also presented a graphical scheme of relationships between the analyzed models for isotropic materials. The works [42, 112] give a retrospective view of the development of nonclassical models of mechanics that take into account the specific features of complex mechanical behaviors of materials with microstructure.

Numerous authors [28, 56, 59, 100] carried out the comparative analysis of various versions of the theories of media with microstructure and revealed their efficiency in the investigation of the mechanical behaviors of micro- and nanoobjects. The applicability of the methods of continuum mechanics to the analysis of the mechanical response of nanostructures was investigated in [18]. Moreover, in [91], it was indicated that the application of the determining relations of nonlocal elasticity to the description of the mechanical behavior of the analyzed objects should be performed with certain precautions.

The intense development of microelectronics stimulated the creation of integral- and gradient-type mathematical models of elastic micro- and nanosized tubes, shells, plates, beams, and rods. The results obtained in this direction were analyzed in [34, 43, 113, 116, 118, 119].

As a separate extremely important direction of investigations, we can mention the evaluation of higher-order material constants required to take into account nonlocal effects. In [75, 78, 85, 957, 111, etc.], the available experimental data were used for the determination of the mechanical characteristics of media (including the characteristic distance of the material, i.e., the scale parameter of material length). In these investigations, the researchers used the so-called method of size effects. In this method, for the evaluation of higher-order material constants, it is customary to use analytic solutions of test boundary-value problems of nonclassical elasticity that describe size effects in small-sized structures and the corresponding experimental results accumulated for these structures. Numerous works deal with the atomistic approach to the evaluation of the strain-gradient elasticity material constants [83, 95] and also the kernels of nonlocal continuum elasticity [110]. In [88, 89], Mindlin applied the lattice theory to find the additional moduli of elasticity connected with the microstructure of the material. Surveys of works dealing with the determination of higher-order constants in the nonclassical theories of elasticity with strain gradients can be found in [35, 76, 95].

In what follows, we briefly characterize the nonclassical theories of continuum mechanics with emphasis made on the conceptual works devoted to the foundations of mathematical description of these theories. We analyze two directions in the development of nonclassical mathematical models of the theory of elasticity, namely, gradient- and integral-type models.

2. Nonlocal Theories of Media with Space-Type Functional Constitutive Equations

According to the nonlocal theory, the stresses acting at a fixed point of the body depend not only on the level of strains at this point but also on the levels of strains at the other points. Within the framework of this theory, for the description of the long-range effects, it is customary to use the following representation for the density of strain energy:

$$W = \frac{1}{2} C_{ijk\ell} e_{ij} \int\limits_{(V)} \mathcal{K}\left(|\mathbf{r}-\mathbf{r}'|\right) e_{k\ell}(\mathbf{r}')\, dV(\mathbf{r}') \tag{1}$$

and the functional relationship between the conjugated parameters of state [99]:

$$\sigma_{ij}(\mathbf{r}) = C_{ijk\ell} \int\limits_{(V)} \mathcal{K}\left(|\mathbf{r}-\mathbf{r}'|\right) e_{kl}(\mathbf{r}')\, dV(\mathbf{r}'). \tag{2}$$

Here, $\mathbf{r}$ is the radius vector, $\hat{\boldsymbol{e}} = \{e_{ij}\}$ and $\hat{\boldsymbol{\sigma}} = \{\sigma_{ij}\}$ are the strain and stress tensors, respectively, $\hat{\mathbf{C}}^{(4)} = \{C_{ijk\ell}\}$ is a tensor of rank four whose components are material characteristics (elasticity moduli), and $\mathcal{K}\left(|\mathbf{r}-\mathbf{r}'|\right)$ is the relaxation kernel (damping function of influence). Here and in what follows, bold symbols denote the vector quantities, whereas bold symbols with hat correspond to tensors of the second and higher ranks. The superscript in parentheses located to the right of the symbol marks the rank of the tensor quantity (we use these superscript to denote tensors of the third and higher ranks).

For a certain class of relaxation kernels, the determining integral relations (2) can be represented in the differential form as follows [48]:

$$\left(1 - c\frac{\partial^2}{\partial x_m^2}\right)\sigma_{ij} = C_{ijk\ell}e_{k\ell}. \tag{3}$$

Here, $c = (e_0 a)^2$ is a nonlocal parameter, e_0 is a material constant (determined either by the laboratory methods or with the use of approaches of molecular dynamics), a is the internal characteristic length (namely, the lattice parameter, the grain size, or the molecule size). These mathematical models are sometimes called generalized *models of media with gradients (Laplacian) of the stress tensor* [35].

Askes and Aifantis [35] studied a more general relationship between the stress and strain tensors

$$\left(1 - c_1\frac{\partial^2}{\partial x_m^2}\right)\sigma_{ij} = C_{ijk\ell}\left(1 - c_2\frac{\partial^2}{\partial x_m^2}\right)e_{k\ell}, \tag{4}$$

where c_1 and c_2 are gradient coefficients. The determining relation (4) makes it possible to avoid singularities on lines of defects and at the crack tips.

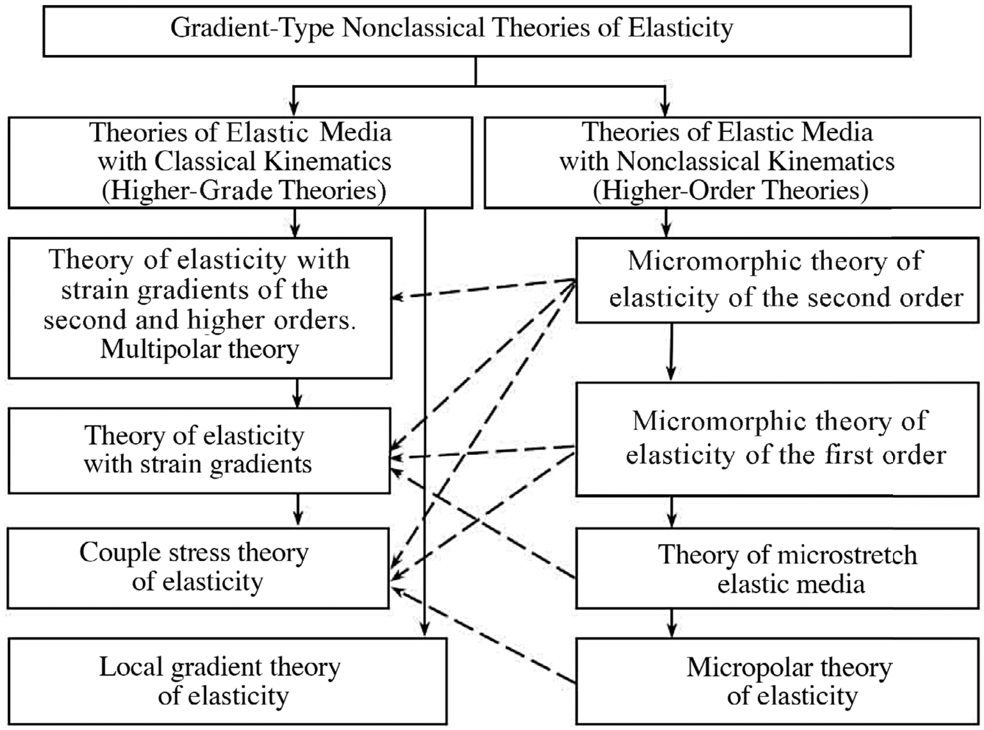
The differential form (3) of the equations of state of nonlocal elasticity was obtained for an infinite medium, which imposes certain restrictions on the field of their applicability. In particular, these equations of state cannot correctly describe nonlocal effects at all points of the nanosized body. In more detail, this was described in [53, 91]. Aifantis [29, 30, 35] repeatedly considered the problem of equivalence of the integral form of determining relations of the Eringen nonlocal theory and the differential equations (3), (4). The model of elastic medium based on the equation of state (3) and its relationship with the Eringen nonlocal theory was also analyzed by Polizzotto [100], Romano and Barretta [105], and other researchers.

The theories characterized by the functional relationships between the stress and strain tensors were called "*nonlocal theories of the integral type,*" "*nonlocal theories,*" or "*strongly nonlocal theories*." Since these mathematical models describe the space dispersion of short waves, they are sometimes called "*nonlocal models of media with space dispersion*."

The differential form of determining equations is much more convenient for solving the problems of mathematical physics. Therefore, the differential equations (3) and (4) are widely used for the investigation of the mechanical response of elastic bodies to static and dynamic loading. Nonlocal equations of state were used for the investigation of the mechanical behavior of beams [43, 103, 105, 116], wires [73], nanoplates [81, 113], nanotubes [117], spherical shells [115], and many other objects. The surveys of investigations of the static and dynamic behaviors of nanobeams carried out within the framework of the Euler–Bernoulli, Timoshenko, Reddy, Levinson, and other nonlocal theories of beams were presented in [103, 119]. The papers [34, 37, 79] contain surveys of the results of modern investigations performed from the viewpoint of application of nonlocal elasticity to the analysis of graphene sheets and carbon nanotubes. These structures are extensively used in practice as elements of transistors, sensors, gauges, nanocapacitors, and other devices, as well as for the design of new superstrong composite nanomaterials. Engelbrecht and Braun [44] and Eltaher with colleagues [43] presented surveys of the nonlocal theories of mechanics from the viewpoint of their application to the analysis of wave motion.

Lim, et al. [80] proposed a modified nonlocal theory of beams based on the following representation of the strain energy more general than (1):

$$W = \frac{1}{2}C_{ijk\ell}e_{ij}\int_{(V)}\mathcal{K}_0(|\mathbf{r}-\mathbf{r}'|)\,e_{k\ell}(\mathbf{r}')\,dV(\mathbf{r}')$$



Scheme 1. Relationships between the gradient-type generalized theories.

$$+\frac{\ell^2}{2} C_{ijk\ell} e_{ij,m} \int\limits_{(V)} \mathcal{K}_1(|\mathbf{r}-\mathbf{r}'|)\, e_{k\ell,m}(\mathbf{r}')\, dV(\mathbf{r}'),$$

where ℓ and $\mathcal{K}_1(|\mathbf{r}-\mathbf{r}'|)$ are the characteristic distance and additional kernel introduced to take into account the influence of the gradient of strain tensor. Here and in what follows, the subscript after comma stands for the operation of differentiation with respect to the corresponding space coordinate. The determining equations of this model and the kinematic hypotheses of the Euler–Bernoulli and Timoshenko beam theories were applied to the investigation of the dynamic behavior of elastic nanobeams.

In Ukraine, the nonlocal theory of elastic continuum was developed by Pidstryhach [22], Povstenko [102], and other researchers.

3. Gradient-Type Theories

3.1. General Characteristic. Theories in which the phase space of the parameters of state is extended by the gradients of some physical quantities are called either gradient-type theories or weakly nonlocal theories. Gradient theories can be conventionally split into two subgroups (see Scheme 1). One of these subgroups is characterized by the use of classical kinematic characteristics. In these theories, as in the classical elasticity, the vector of displacements $\mathbf{u}(\mathbf{x},t)$ plays the role of a single kinematic characteristic. Sometimes, these theories are called either higher-grade continuum theories of elasticity [112], or gradient theories of materials with simple structure, or the Leroux media [13, 17], or are classified as theories with long-range effects [38]. The family of gradient theories with classical kinematics includes the Toupin couple stress theory [114], the Mindlin theory of elasticity with strain gradients [87], the Jaramillo [68] and Aéro–Kuvshinskii [2] theories, etc. The Mindlin–Green–Rivlin [87, 60] theory with strain gradients of higher orders proves to be the most general among the mentioned theories. As limit cases, it contains the Toupin, Aéro–Kuvshinskii, and Jaramillo mathematical models of the media.

The local gradient theory of elasticity also belongs to the group of gradient-type theories [4, 6, 11, 20]. This theory is based on the model description that takes into account the relationship between the processes of deformation and local mass displacements. In this case, local mass displacements are associated with the mass flow $\mathbf{J}_{ms}$ of nondiffusion and nonconvective nature caused by the changes in the structure of the material of a physically small element of the body. The polarization current is an analog of a flow of this kind in electroelasticity.

Mathematical models of media with additional degrees of freedom form another group of theories. Sometimes they are called higher-order continuum theories of elasticity [112], gradient theories of materials with complex structure [17], media with microstructure [86], or media with additional degrees of freedom [17]. In these theories, the medium is modeled by the collection of a large number of representative elements (macroelements) each of which is characterized by a finite size, certain structure, and orientation. Macroelements are formed by systems of interacting particles (microelements), which, in turn, may deform, rotate about the center of mass of the macroelement and, as a final result, affect the macroscopic behavior of the body. Thus, in the micromorphic theories, parallel with the convective (translational) motion of a microelement, we also consider the rotational motion and deformation of its microparticles. Depending on the forms of motion of microelements that are taken into account (translational or rotational), it is necessary to construct different versions of the theories of media with microstructure. At the same time, a macroelement is regarded either as a rigid oriented body (in the polar and micropolar theories) or as a solid deformable body (in the micromorphic theory and theory of elastic media with dislocations). This group of theories is formed by the polar and micropolar theories [49], micromorphic theory [52, 86, 109], theories of microstretch media [49], elastic media with constant numbers of dislocations [3], and various modifications of the indicated theories. Among the mentioned theories, the micromorphic theory proves to be most general. As special cases, it contains polar and micropolar theories, mathematical models of microstretch media, etc.

The development of the static theories of gradient elasticity was stimulated by the necessity of investigation of size and subsurface effects and the requirement of prevention of singularities of the solutions in problems with cracks, dislocations, etc. The description of the dynamic behaviors of elastic continua (in particular, their dispersive properties) under the action of rapidly varying loads requires the construction of nonclassical theories of continuum mechanics that take into account either the inertia of strain gradients [86] (within the framework of the gradient theory of elasticity), or the kinetic energy of local displacements of masses, or the irreversible component of the gradient of modified chemical potential [11, 14] (within the framework of local gradient elasticity). The resolving systems of equations of these models contain the terms proportional to higher-order mixed (space-time) derivatives of the key functions [11, 14, 35, 120]. The indicated mathematical models make it possible to describe the propagation of new types of elastic waves in solid bodies. A survey of these mathematical models can be found in [35]. In what follows, we briefly characterize the gradient models of continuum mechanics.

3.2. Cosserat Media. Micropolar Theory. The Cosserat continuum theory (1909) was the first step in the development of the nonclassical theory of elasticity [40]. In this theory, a representative element is regarded as an oriented solid body. This is why it is often called the polar theory. In [17], Kunin classified polar and micropolar continua as models of continuum media with weak nonlocality of elastic properties.

The kinematic properties of the Cosserat media are described by two independent vector quantities: the vector of displacements $\mathbf{u}(\mathbf{x},t)$ and the vector of rotations $\boldsymbol{\omega}(\mathbf{x},t)$. These vectors specify the nonsymmetric strain tensor $\hat{\boldsymbol{\gamma}} = \{\gamma_{ij}\}$ and the bending-torsion tensor $\hat{\boldsymbol{\kappa}} = \{\kappa_{ij}\}$ [21]:

$$\gamma_{ji} = u_{i,j} + \mathrm{E}_{ijk}\omega_k, \quad \kappa_{ji} = \omega_{i,j}.$$

Here, E_{ijk} are the components of the Levi-Civita pseudotensor and $\hat{\mathbf{E}}^{(3)} = \{\mathrm{E}_{ijk}\}$.

In polar media, we observe the formation not only of the force stresses $\hat{\boldsymbol{\sigma}} = \{\sigma_{ij}\}$ but also of the couple stresses $\hat{\mathbf{m}} = \{m_{ij}\}$ given by the following equations of state:

$$\sigma_{ij} = \frac{\partial W}{\partial \gamma_{ij}}, \quad m_{ij} = \frac{\partial W}{\partial \kappa_{ij}}.$$

Unlike the classical (symmetric) theory of elasticity, the stressed state of the Cosserat medium is described by the nonsymmetric stress tensor $\hat{\boldsymbol{\sigma}}$. Thus, the Cosserat theory is also often called the nonsymmetric elasticity theory [21]. The dynamic behavior of the Cosserat elastic medium is characterized by the following equations:

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i, \tag{5}$$

$$m_{ji,j} + \mathrm{E}_{ijk}\sigma_{jk} + M_i = J\ddot{\omega}_i, \tag{6}$$

where $\mathbf{F} = \{F_i\}$ and $\mathbf{M} = \{M_i\}$ are the vectors of volume forces and moments and J is the measure of inertia in the course of rotation (dynamic characteristic of the medium).

If, for the density of strain energy W of the isotropic continuum, we use the bilinear expansion

$$W(\gamma_{ij}, \kappa_{ij}) = \frac{1}{2}\lambda\gamma_{ii}\gamma_{jj} + \frac{1}{2}(\mu+\alpha)\gamma_{ij}\gamma_{ij} + \frac{1}{2}(\mu-\alpha)\gamma_{ij}\gamma_{ji}$$

$$+ \frac{1}{2}\beta\kappa_{ii}\kappa_{jj} + \frac{1}{2}(\gamma+\varepsilon)\kappa_{ij}\kappa_{ij} + \frac{1}{2}(\gamma-\varepsilon)\kappa_{ij}\kappa_{ji},$$

then we get the following explicit form of the equation of state [21]:

$$\sigma_{ij} = 2\mu\gamma_{ij}^S + 2\alpha\gamma_{ij}^A + \lambda\gamma_{kk}\delta_{ij}, \quad m_{ij} = 2\gamma\kappa_{ij}^S + 2\varepsilon\kappa_{ij}^A + \beta\kappa_{kk}\delta_{ij}. \tag{7}$$

Here, the superscripts "S" and "A" denote the symmetric and skew-symmetric parts of the strain tensor $\hat{\boldsymbol{\gamma}}$ and the bending-torsion tensor $\hat{\boldsymbol{\kappa}}$:

$$\gamma_{ij}^S = \frac{1}{2}(u_{i,j} + u_{j,i}) = e_{ij}, \quad \kappa_{ij}^S = \frac{1}{2}(\omega_{i,j} + \omega_{j,i}),$$

$$\gamma_{ij}^A = \frac{1}{2}(u_{i,j} - u_{j,i}) - \mathrm{E}_{ijk}\omega_k, \quad \kappa_{ij}^A = \frac{1}{2}(\omega_{i,j} - \omega_{j,i}).$$

The mechanical behavior of the isotropic elastic material within the framework of the polar theory is described by six moduli of elasticity, namely, by two Lamé constants λ and μ and by four higher-order material constants α, β, γ, and ε.

Substituting the equation of state (7) in the balance equations (5) and (6), for the evaluation of displacements and rotations, we get the following two vector (six scalar) key differential equations of the second order:

$$(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - (\mu+\alpha)\nabla\times(\nabla\times\mathbf{u}) + 2\alpha\nabla\times\boldsymbol{\omega}+\mathbf{F} = \rho\ddot{\mathbf{u}},$$

$$(\beta+2\gamma)\nabla(\nabla\cdot\boldsymbol{\omega}) - (\gamma+\varepsilon)\nabla\times(\nabla\times\boldsymbol{\omega}) + 2\alpha\nabla\times\mathbf{u} - 4\alpha\boldsymbol{\omega} + \mathbf{M} = J\ddot{\boldsymbol{\omega}}.$$

Here, ∇ is the nabla operator, "$\times$" and "$\cdot$" denote, respectively, the operations of vector and scalar product, and overdots stand for the time derivatives.

The Cosserat continuum was used as the base for the mathematical description of micropolar media in which, parallel with translational motion, a particle has additional degrees of freedom, namely, the possibility of rigid rotations [45]. If the microinertia of rotations is regarded as a constant quantity, then the micropolar theory is reduced to the theory of Cosserat media [38].

The elastic characteristics of polar and micropolar continua were determined by Lakes [75, 76], Hassanpour and Heppler [65], McFarland and Colton [85], and other researchers.

The relationships of the Cosserat continuum and micropolar elasticity were extended to the theory of elastic shells, plates, and rods [1, 106], beams [85], and other small-sized structures.

Y. Chen, et al. [38] demonstrated, from the viewpoint of molecular dynamics and lattice theory, that it is reasonable to apply the polar theory to the investigation of the mechanical behaviors of materials admitting significant changes in the orientation of microstructures (liquid crystals and ferroelectrics) and the micropolar theory to the investigation of molecular crystals, granulated materials, and some types of composites.

3.3. Cosserat Pseudocontinuum (1960). The theory of Cosserat pseudocontinuum is a simplified version of the theory of Cosserat continuum. In this theory, the vectors of rotations and displacements are connected by the formula $\boldsymbol{\omega} = \operatorname{rot}\mathbf{u}/2$. This theory is sometimes called the Cosserat theory with "restrained rotations" or the couple stress theory. The analyzed version of the couple stress theory of elasticity contains only one independent kinematic characteristic, namely, the vector of displacements $\mathbf{u}$, which determines the following strain measures:

$$e_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i}), \qquad \kappa_{ij} = \frac{1}{2}\mathrm{E}_{k\ell j}u_{\ell,ki}.$$

The number of constants obtained for the isotropic elastic body within the framework of this theory decreases from six to four. In this case, the vector of displacements satisfies a fourth-order vector differential equation. In the absence of mass forces, this equation takes the following form [21]:

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \frac{1}{4}(\gamma+\varepsilon)\nabla\times(\nabla\times\Delta\mathbf{u}) - \frac{I}{4}\nabla\times(\nabla\times\ddot{\mathbf{u}}) = \rho\ddot{\mathbf{u}}.$$

Here, λ, μ, γ, and ε are material constants characterizing the elastic properties of the Cosserat pseudocontinuum, I is a material constant that characterizes the inertial properties of the macrovolume, and Δ is the

Laplace operator.

The Cosserat theory with restrained rotations belongs to the group of higher-rank theories of elasticity. As compared with the classical theory of elasticity, it additionally takes into account coupled stresses and operates with the asymmetric stress tensor. In view of its relative simplicity, this theory is well developed and frequently used in applied investigations.

Thorough analyses of the systems of equations of the Cosserat continuum and Cosserat pseudocontinuum can be found in the monographs by Nowacki [21] and Savin [23] and in the works by Aéro and Kuvshinskii [2, 16], Mindlin and Tiersten [90], Toupin [114], Savin, et al. [24], Hadjesfandiari and Dargush [63], etc.

3.4. Mindlin–Eringen–Suhubi Micromorphic Theory (1964). The micropolar theory treats the structural elements of a macrovolume as solid bodies and takes into account only rigid rotations of these elements about the center of mass of the macrovolume. The micromorphic theory whose kinematics is described not only by a vector quantity but also by tensor quantities proves to be more general. This theory takes into account all components of the tensor of free distortion but not only its skew-symmetric part (free rotations), as in the Cosserat theory.

The foundations of the micromorphic theory were proposed by Eringen, Suhubi, and Mindlin in [52, 86, 109]. The micromorphic theory treats the body as a continuous set of a large number of deformable particles each of which is characterized by finite sizes and internal structure. Deformable particles (macroelements) consist of structural elements (microelements). The motion of macroelement can be described as a result of its convective transfer, rotation about the center of mass, and deformation. As a result of deformation of micromorphic continuum, we observe the formation of *macrostrains* and *microscopic internal strains*.

In what follows, we present the principal relations of the first-order micromorphic theory. The kinematic behaviors of points of this medium are described by the vector field of displacements $\mathbf{u}$ and the second-rank tensor $\hat{\boldsymbol{\chi}} = \{\chi_{ij}\}$ whose components characterize microstrains. Within the framework of this theory, every point of the three-dimensional medium has twelve degrees of freedom.

The first-order micromorphic medium is characterized by the following force factors, including the "classical" stress tensor $\hat{\boldsymbol{\sigma}} = \{\sigma_{ij}\}$, the tensor of *double stresses* $\hat{\boldsymbol{\tau}}^{(3)} = \{\tau_{ijk}\}$, and the *tensor of relative stress* $\hat{\mathbf{s}} = \{s_{ij}\}$. In the stationary case, in the absence of mass forces, these stresses satisfy the following balance relations [39]:

$$\sigma_{ij,j} + s_{ij,j} = 0, \qquad s_{ij} + \tau_{ijk,k} = 0. \tag{8}$$

The density of strain energy is described by the function $W = W(e_{ij}, \gamma_{ij}, \kappa_{ijk})$, where the tensors $\hat{\boldsymbol{e}} = \{e_{ij}\}$, $\hat{\boldsymbol{\gamma}} = \{\gamma_{ij}\}$, and $\hat{\boldsymbol{\kappa}}^{(3)} = \{\kappa_{ijk}\}$ are given by the formulas

$$e_{ij} = \frac{1}{2}(\chi_{ij} + \chi_{ji}), \qquad \gamma_{ij} = u_{j,i} - \chi_{ji}, \qquad \kappa_{ijk} = \chi_{ij,k},$$

$\hat{\boldsymbol{e}}$ is the tensor of macrostrains, which characterizes relative displacements of the centers of masses of macrovolumes (it coincides with the Green strain tensor in the classical theory of elasticity), the tensor $\hat{\boldsymbol{\kappa}}^{(3)} = \nabla \otimes \hat{\boldsymbol{\chi}}$ was introduced to describe the gradients of microstrains, "$\otimes$" denotes the dyadic product, $\hat{\boldsymbol{\gamma}}$ is the relative strain (measure of difference between the macrostrain $\hat{\boldsymbol{e}}$ and the microstrain $\hat{\boldsymbol{\chi}}$). The tensors $\hat{\boldsymbol{\kappa}}^{(3)}$ and $\hat{\boldsymbol{\gamma}}$

are new strain measures absent in the classical theory.

The corresponding equations of state take the form

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}}, \qquad s_{ij} = \frac{\partial W}{\partial \gamma_{ij}}, \qquad \tau_{ijk} = \frac{\partial W}{\partial æ_{ijk}}. \tag{9}$$

The tensors of ordinary $\hat{\boldsymbol{\sigma}}$ [N/m^2], relative $\hat{\mathbf{s}}$ [N/m^2], and double $\hat{\boldsymbol{\tau}}^{(3)}$ [N/m] stresses have the same properties of symmetry as the corresponding strain tensors $\hat{\boldsymbol{e}}$, $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\kappa}}^{(3)}$.

Within the framework of the linear theory, the density of strain energy is described by the following bilinear form:

$$W(e_{ij},\gamma_{ij},\kappa_{ijk}) = \frac{1}{2} C_{ijk\ell} e_{ij} e_{k\ell} + \frac{1}{2} B_{ijk\ell} \gamma_{ij} \gamma_{k\ell} + \frac{1}{2} A_{ijk\ell mn} \kappa_{ijk} \kappa_{\ell mn}$$

$$+ G_{ijk\ell} e_{ij} \gamma_{k\ell} + F_{ijk\ell m} e_{ij} \kappa_{k\ell m} + D_{ijk\ell m} \gamma_{ij} \kappa_{k\ell m}, \tag{10}$$

where $C_{ijk\ell}$, $B_{ijk\ell}$, $A_{ijk\ell mn}$, $G_{ijk\ell}$, $F_{ijk\ell m}$, and $D_{ijk\ell m}$ are material characteristics. The mechanical behavior of an anisotropic medium is characterized by 903 moduli. For isotropic materials, the linear theory has 18 materials constants. For this continuum, the bilinear form (10) can be represented as follows:

$$W = \frac{1}{2}\lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij} + \frac{1}{2} b_1 \gamma_{ii} \gamma_{jj} + \frac{1}{2} b_2 \gamma_{ij} \gamma_{ij} + g_1 \gamma_{ii} e_{jj}$$

$$+ g_2 (\gamma_{ij} + \gamma_{ji}) e_{ij} + a_1 \kappa_{iik} \kappa_{kjj} + a_2 \kappa_{iik} \kappa_{jkj}$$

$$+ \frac{1}{2} a_3 \kappa_{iik} \kappa_{jjk} + \frac{1}{2} a_4 \kappa_{ijj} \kappa_{ikk} + a_5 \kappa_{ijj} \kappa_{kik}$$

$$+ \frac{1}{2} a_8 \kappa_{iji} \kappa_{kjk} + \frac{1}{2} a_{10} \kappa_{ijk} \kappa_{ijk} + a_{11} \kappa_{ijk} \kappa_{jki}$$

$$+ \frac{1}{2} a_{13} \kappa_{ijk} \kappa_{ikj} + \frac{1}{2} a_{14} \kappa_{ijk} \kappa_{jik} + \frac{1}{2} a_{15} \kappa_{ijk} \kappa_{kji}, \tag{11}$$

where λ and μ are Lamé constants and b_i, a_i, and g_i are 16 higher-order material constants.

If representations (10) [or (11) in the case of an isotropic medium] are substituted in the determining relations (9), then we get the explicit form of the equations of state. To guarantee the uniqueness of solutions of the corresponding boundary-value problems of mathematical physics, the balance equations (8) and the determining relations (9) are complemented by boundary conditions specified on the surface of the body either by the vector $\boldsymbol{\tau}^a = \{\tau_i^a\}$ or by the second-rank tensor $\hat{\boldsymbol{\tau}}^a = \{\tau_{ij}^a\}$ [39]:

$$(\sigma_{ij} + s_{ij}) n_j = \tau_i^a, \qquad \tau_{ijk} n_k = \tau_{ij}^a.$$

The basic concepts of the micromorphic and micropolar theories and the theory of microstretch media were thoroughly described in the monograph [50] in which the relationships between these theories were also established.

In analyzing some types of internal motion, we can get different versions of gradient-type nonclassical models of mechanics from the micromorphic theory [50, 58]. If we do not take into account the changes in the microstructure of physically small elements of the body, then the micropolar theory is obtained from the micromorphic theory. The micropolar continuum takes into account solely the rigid rotations of structural elements. If we describe the motion of representative elements by microrotations and microstretching (i.e., take into account the action of compressive and tensile strains), then we get the theory of microstretch media (microstretch continuum). This theory considers the following strain measures:

$$e_{ij} = u_{j,i} + \mathrm{E}_{jik}\omega_k , \quad \kappa_{ij} = \omega_{i,j} , \quad \gamma_k = \varphi_{,k} , \quad e = \varphi ,$$

where φ is microstretching and ω_i are microrotations. If we assume that particles deform in the same way as the entire continuum, then the micromorphic theory turns into the gradient Mindlin theory of elasticity. If particles participate in the rotational motion that coincides with the rotational motion of the entire continuum, then we get the couple-stress theory of elasticity. If particles are reduced to points equipped with masses, then all theories of media with microstructures are reduced to the classical theory of elasticity.

It is reasonable to use the micromorphic theory for the investigation of covalent, molecular, and ionic crystals [38]. As a disadvantage of this theory, we can mention a large number of elastic constants and the absence of an experimental program required for their evaluation. This clearly restricts the practical applicability of the analyzed theory.

3.5. Stress-Gradient Theories of Elasticity. In the stress-gradient theories of elasticity, the variations of microstructures of the materials are associated with strain gradients.

The presence of a large number of versions of the gradient-type theories of elasticity is explained by the principal possibility of introducing three types of components in the representation for the density of strain energy. These are terms proportional to the following factors:

(1) second-order gradients of the vector of displacements, $\hat{\eta}^{(3)} = \{\eta_{ijk}\}$ (microdistortion gradient), and $\eta_{ijk} = u_{k,ij}$;

(2) first-order gradients of the macrostrain tensor,

$$\hat{e} = \{e_{ij}\} , \quad \gamma_{ijk} = e_{kj,i} = (u_{k,ji} + u_{j,ki})/2 = (\eta_{ijk} + \eta_{kij})/2 , \quad e_{ij} = (u_{i,j} + u_{j,i})/2 ;$$

(3) gradients of rotations $\kappa_{ij} = \mathrm{E}_{i\ell k} u_{k,\ell j}/2$ and the symmetric tensor of the second gradient of the vector of displacements $\eta^S_{ijk} = (u_{k,ij} + u_{i,jk} + u_{j,ki})/3$.

Note that the strain tensor $\hat{e} = \{e_{ij}\}$ has six independent components, whereas the tensor $\hat{\eta}^{(3)} = \{\eta_{ijk}\}$ is symmetric with respect to the first two indices and has 18 independent components. The presence of bilinear combinations of these components in the expansion for the energy density makes it possible to construct various types of the theories of elastic media with microstructures.

3.5.1. Theory of Elastic Continuum with the First Strain Gradient. In 1962, Toupin [114] and Mindlin and Tiersten [90] formulated a system of equations of gradient elasticity that takes into account the dependence of the density of strain energy on the strain gradient. This theory is known as the couple stress theory of elasticity. In 1964, Mindlin [86] proposed a mathematically more complicated linear theory of elastic continuum with microstructure. Unlike the couple stress theory, the mathematical model of a medium with microstructure includes the symmetric part of the strain gradient. Therefore, this theory contains more material constants than the couple stress theory. In particular, the elastic behaviors of centrally symmetric and isotropic materials are characterized by two Lamé constants and five additional materials constants (seven constants in total) [86]. Note that, for the density of strain energy, Mindlin used the following representation:

$$W(e_{ij}, \gamma_{ijk}) = \frac{1}{2}\lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij} + g_1 \gamma_{iik} \gamma_{kjj} + g_2 \gamma_{ijj} \gamma_{ikk}$$

$$+ g_3 \gamma_{iik} \gamma_{jjk} + g_4 \gamma_{ijk} \gamma_{ijk} + g_5 \gamma_{ijk} \gamma_{kji}, \qquad (12)$$

where g_i, $i = 1, 2, \ldots, 5$, are additional material constants connected with taking into account the strain gradients. Within the framework of this theory, we get the following equations of state:

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}}, \qquad \tau_{ijk} = \tau_{jik} = \frac{\partial W}{\partial \gamma_{ijk}}.$$

Here, $\hat{\boldsymbol{\tau}}^{(3)} = \{\tau_{ijk}\}$ is the tensor of double stresses. Parallel with relation (12), Mindlin considered another form of representation of the strain energy, namely,

$$W(e_{ij}, \eta_{ijk}) = \frac{1}{2}\lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij} + a_1 \eta_{iik} \eta_{kjj} + a_2 \eta_{ijj} \eta_{ikk}$$

$$+ a_3 \eta_{iik} \eta_{jjk} + a_4 \eta_{ijk} \eta_{ijk} + a_5 \eta_{ijk} \eta_{kji}.$$

In this mathematical model, the “classical” and double stresses are given by the formulas

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}}, \qquad \tau_{ijk} = \tau_{jik} = \frac{\partial W}{\partial \eta_{ijk}}$$

and satisfy the balance equation

$$\sigma_{ji,j} - \tau_{kji,kj} + F_i = 0.$$

The gradient theory of elasticity was used as basic by numerous researchers who proposed to use either simplified or, on the contrary, generalized versions of this theory. In what follows, we discuss some of these versions.

Lam, et al. [77] reduced the number of additional elastic constants to three. They considered the following expansion for the potential W:

$$W\left(e_{ij}, e_{mm,i}, \eta_{ijk}^{SD}, \kappa_{ij}^{S}\right) = \frac{1}{2}\lambda e_{ii}e_{jj} + \mu e_{ij}e_{ij}$$

$$+ \mu\ell_0^2 e_{mm,i}e_{nn,i} + \mu\ell_1^2\eta_{ijk}^{SD}\eta_{ijk}^{SD} + \mu\ell_2^2\kappa_{ij}^{S}\kappa_{ij}^{S}, \tag{13}$$

where ℓ_0, ℓ_1, and ℓ_2 are the characteristic distances connected with the dilatation gradient, deviatoric component of the gradient of tensile strains, and symmetric component of the gradient of rotations, respectively:

$$\eta_{ijk}^{SD} = \eta_{ijk}^{S} - \left(\delta_{ij}\eta_{\ell\ell k}^{S} + \delta_{jk}\eta_{\ell\ell i}^{S} + \delta_{ik}\eta_{\ell\ell j}^{S}\right),$$

$$\kappa_{ij}^{S} = (\kappa_{ij} + \kappa_{ij})/2 = (\mathrm{E}_{ip\ell}e_{\ell j,p} + \mathrm{E}_{jp\ell}e_{\ell i,\ell})/2.$$

This theory is often called the *modified theory of gradient elasticity*. It considers the following force factors:

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}} = 2\mu e_{ij} + \lambda e_{\ell\ell}\delta_{ij}, \qquad \gamma_i = \frac{\partial W}{\partial e_{nn,i}} = 2\mu\ell_0^2 e_{nn,i},$$

$$\tau_{ijk}^{SD} = \frac{\partial W}{\partial \eta_{ijk}^{SD}} = 2\mu\ell_1^2\eta_{ijk}^{SD}, \qquad m_{ij}^{S} = \frac{\partial W}{\partial \chi_{ij}^{S}} = 2\mu\ell_2^2\chi_{ij}^{S}.$$

Here, σ_{ij} are the components of the tensor of Cauchy stresses, γ_i is the pressure gradient, τ_{ijk}^{SD} are the components of the tensor of double stresses, and m_{ij}^{S} is the symmetric component of couple stresses.

Gusev and Lurie [62] constructed a simpler version of the gradient theory of elasticity containing, for isotropic materials, two classical moduli of elasticity and two higher-order material constants (four constants in total). In the Gusev–Lurie theory, the components of the tensor of double stresses are given by the formula

$$\tau_{ijk} = a_1\gamma_{ijk}^{SH} + a_2\gamma_{ijk}^{S},$$

where γ_{ijk}^{S} and γ_{ijk}^{SH} are the symmetric part and the spherical component of the tensor $\hat{\gamma}^{(3)}$:

$$\gamma_{ijk}^{S} = (\gamma_{ijk} + \gamma_{jki} + \gamma_{kij})/3 = \eta_{ijk}^{S} \quad \text{and} \quad \gamma_{ijk}^{SH} = \left(\gamma_{i\ell\ell}^{S}\delta_{jk} + \gamma_{j\ell\ell}^{S}\delta_{ki} + \gamma_{k\ell\ell}^{S}\delta_{ij}\right)/5.$$

In the expansion for the density of strain energy, Tekoğlu and Onck [112] took into account the terms proportional to the divergence of the strain gradient

$$\tilde{\eta}_i = \frac{1}{2}(u_{j,ij} + u_{i,jj}) = e_{ij,j}.$$

For anisotropic materials, they used the bilinear form

$$W(e_{ij}, \tilde{\eta}_i) = \frac{1}{2} C_{ijk\ell} e_{ij} e_{k\ell} + B_{ijk} e_{ij} \tilde{\eta}_k + D_{ij} \tilde{\eta}_i \tilde{\eta}_j .$$

For centrally symmetric materials, the components of the pseudotensor B_{ijk} are equal to zero and the equations of state take the following form:

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}} = C_{ijk\ell} e_{k\ell}, \qquad \tau_i = \frac{\partial W}{\partial \tilde{\eta}_i} = D_{ij} \tilde{\eta}_j,$$

where τ_i is the stress vector, which is a parameter conjugate to the divergence of the strain gradient. For isotropic materials, $C_{ijk\ell} = \lambda \delta_{ij} \delta_{k\ell} + 2\mu \delta_{ik} \delta_{j\ell}$ and $D_{ij} = a\delta_{ij}$, where a is an additional material constant connected with the divergence of strain gradient. The authors called this theory the generalized *strain divergence theory* of elasticity. Within the framework of this theory, the tensor and vector of stresses introduced above satisfy the following balance equation:

$$\sigma_{ij,i} - \frac{1}{2}(\tau_{i,ji} + \tau_{j,ii}) + F_i = 0.$$

Altan and Aifantis [32] offered another representation for the strain energy

$$W(e_{ij}, e_{ij,k}) = \frac{1}{2} \lambda e_{ii} e_{jj} + \mu e_{ij} e_{ij} + \ell^2 \left(\frac{1}{2} \lambda e_{ii,k} e_{jj,k} + \mu e_{ij,k} e_{ij,k} \right),$$

where ℓ is the characteristic distance in the microstructure of the material. Then the components of the tensor of double stresses $\hat{\tau}^{(3)}$ are given by the formulas:

$$\tau_{ijk} = \tau_{jik} = \frac{\partial W}{\partial e_{ij,k}} = \ell^2 (2\mu e_{ij,k} + \lambda e_{\ell\ell,k} \delta_{ij}) = \ell^2 \sigma_{ij,k},$$

and the balance equation for the isotropic and anisotropic materials takes the form

$$\lambda u_{j,ij} + 2\mu u_{i,jj} - \ell^2 (\lambda u_{j,ijmm} + 2\mu u_{i,jjmm}) + F_i = 0,$$

$$C_{ijk\ell} \left(u_{k,j\ell} - \ell^2 u_{k,j\ell mm} \right) + F_i = 0.$$

The variational formulation of this simplified version of gradient elasticity (with three material constants two of which are the Lamé constants and the remaining constant is the characteristic length of the material) was proposed by Gao and Park [57].

The skew-symmetric part of the tensor of double stresses is the tensor of couple stresses. Within the framework of the couple stress theory of elasticity formulated in 1962 [90, 114], the determining relations for isotropic materials contain two additional constants characterizing the microstructure of the material (together with the Lamé constants, the total number of constants is equal to four). Somewhat later, the researchers proposed new modified versions of the couple stress theory of elasticity containing larger or smaller numbers of material constants. Thus, in particular, in 1993, Fleck and Hutchinson [54] formulated the relations of a modified couple stress theory characterizing an isotropic continuum by three more additional material constants connected with taking into account the gradients of rotation and tension (five constants in total).

In 2002, Yang, et al. [122] proposed a simplified version of the couple stress elasticity, which contained only three material constants in total (two elastic moduli and one characteristic distance). In this theory, parallel with the equations of balance of the mechanical momentum and angular momentum, the researchers used an additional balance equation for the higher-order stresses. The equations of this model can be obtained if, in relation (13), we take $\ell_0 = 0$ and $\ell_1 = 0$. In view of a quite small number of material constants, this theory was fairly extensively used for practical purposes [82, 98].

Abazari, et al. [28] proposed a generalized couple stress theory based on the postulate of variation of the mechanical properties of the material in the near-surface layers of the body. The authors applied this theory to substantiate the size effect of Young's modulus.

The relationship between the theories based on different representations of the density of strain energy was analyzed in [35, 59, 93, 101, 112], etc. The couple stress theories were thoroughly investigated in [63, 64].

The couple stress theory and the theory of gradient elasticity and their modified versions are used to investigate the stress-strain state around cracks [108], cavities [112], etc. They were extended to the mathematical models of elastic plates [96, 104], beams [77, 93], and other small-sized structures. Thorough analyses of the gradient models of beams and plates can be found in [35, 113].

By analyzing the process of bending of nanobeams, Shokrieh and Zibaei [107] indicated the advantages of gradient elasticity over the theory of beams with functional determining relations.

Within the framework of the gradient theories, the deflections of elastic beams and plates are described by a six-order equation. At the same time, the classical theories are based on a fourth-order equation. For micro- and nanobeams whose thickness is comparable with the characteristic length of the material, the gradient models state that the frequency of natural vibrations increases and the deflection of the beam decreases (as compared with the classical theory) under the action of concentrated forces or distributed loads [74]. The indicated strengthening of the material is in good agreement with the results of experimental measurements. At the same time, it is not confirmed by the classical theory.

The theory of elasticity with the first strain gradient describes the experimentally observed high-frequency dispersion of elastic mechanical longitudinal waves but gives the incorrect sign of curvature of the dispersion curve [87, 89]. To remove this disadvantage, it was proposed to use mathematical models of elastic media taking into account the dependence of the internal energy on the strain gradients of the second and higher orders.

3.5.2. Theory of the Green–Rivlin Multipolar Continua (1964). Mindlin Gradient Theory of Elasticity (1965). In 1964, Green and Rivlin [60, 61] developed the foundations of a very general mathematical model of elastic continuum that includes strain gradients of any order. The number of physical constants in this model is determined by the order of the theory. The indicated theory was called the theory of multipolar continua. It is based on the use of the following strain measures:

$$e_{ij} = (u_{i,j} + u_{j,i})/2, \quad \gamma_{ijk} = e_{kj,i}, \quad \gamma_{ijk\ell} = e_{kj,i\ell}, \quad \gamma_{ijk\ell m} = e_{kj,i\ell m}.$$

A year later, Mindlin [87] proposed a simpler linear dependence taking into account the dependence of the energy density on the strain tensor and also on the second and third gradients of the vector of displacements: $W = W(e_{ij}, \eta_{ijk}, \eta_{ijk\ell})$. Here, $\eta_{ijk\ell} = u_{\ell,ijk}$ and $\hat{\eta}^{(4)} = \{\eta_{ijk\ell}\}$. In order to get linear determining relations, Mindlin represented the strain energy in the form of a polynomial of the second order with all possible combinations of quadratic terms from the components of the strain tensor and the second and third gradients of the vector of displacements. For this description, he obtained the following equation of state:

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}}, \qquad \tau_{ijk} = \frac{\partial W}{\partial \eta_{ijk}}, \qquad \tau_{ijk\ell} = \frac{\partial W}{\partial \eta_{ijk\ell}},$$

where $\hat{\boldsymbol{\sigma}} = \{\sigma_{ij}\}$ is the classical tensor of macrostresses with dimension N/m^2; $\hat{\boldsymbol{\tau}}^{(3)} = \{\tau_{pqr}\}$ is the tensor of double stresses with dimension N/m, and $\hat{\boldsymbol{\tau}}^{(4)} = \{\tau_{pqrs}\}$ is the fourth-rank tensor with dimension N, which was called the tensor of ternary stresses.

The tensors of classical, double, and ternary stresses satisfy the following single higher-order differential equilibrium equation:

$$\sigma_{ij,j} - \tau_{ijk,jk} + \tau_{ijk\ell,jk\ell} + F_i = 0.$$

This equation should be supplemented with the boundary conditions specifying the surface values of projections of the vector of displacements and its first and second derivatives onto the normal to the surface of the body [in this case, vector quantities are specified on the surface unlike the micromorphic theory in which, in addition to the vector of displacements, tensor quantities (additional degrees of freedom) are given on the surface] [39].

Mindlin analyzed the relationship between the system of equations of gradient elasticity and the results of the lattice theory [87, 89] and, on the basis of linear relations, deduced the formulas for the surface tension in liquids, as well as the formulas for the surface strain energy required to decompose an elastic solid body into two parts along a certain surface [87]. He showed that the theory with the second strain gradient correctly describes the subsurface inhomogeneity of mechanical fields and the dispersion of elastic longitudinal waves in the high-frequency range.

The efficiency of application of the theories taking into account the gradients of the strain tensor of the second, fourth, sixth, and eighth orders was analyzed in [121] by analyzing an example of elastic nanorod.

3.6. ***Gradient Theory of Elastic Media With Regard for Local Mass Displacements.*** The investigations in this direction were originated by Burak [4] and developed in [5–7, 14, 20, 92, etc.]. The mathematical model of thermoelastic continuum proposed in 1987 was based on taking into account the mass flow $\mathbf{J}_{ms} = -\partial \boldsymbol{\Pi}_m / \partial t$ of nondiffusion and nonconvective nature in the mass balance equation. Burak associated this mass flow with changes in the structure of the material of a physically small element of the body [4]. The vector $\boldsymbol{\Pi}_m = \{\Pi_i^m\}$ (with dimension of the density of mass dipole moment, kg·m/m^3) was called the vector of local mass displacements. The analysis of the influence of mass flow $\mathbf{J}_{ms}$ performed under the assumption that it is responsible for the energy flow $\mu \partial \boldsymbol{\Pi}_m / \partial t$, where μ is the chemical potential of the material, made it possible to construct a gradient-type theory of physicomechanical processes in solid elastic and thermoelastic bodies, which was called the local gradient theory [4, 10]. In this theory, the phase space of the parameters of state was additionally extended by a pair of conjugate parameters: the gradient of chemical potential $\nabla\mu$ and the vector of local mass displacements [10]:

$$\Pi_i^m = \frac{\partial W}{\partial \mu_{,i}}.$$

The list of works carried out within the framework of this approach was presented in [10].

The continuum-thermodynamic approach to the construction of local gradient models of the continuum mechanics was additionally developed in [6, 7, 14]. In these works, for the description of the process of local mass displacements, the authors introduced two more objective physical quantities. These are the density of induced mass $\rho_{m\pi} = -\nabla \cdot \mathbf{\Pi}_m$ and the potential μ_π introduced as a measure of influence of local mass displacements on the internal energy of the system. In electroelasticity, the density of induced electric charge serves as an analog of density of the induced mass. For these physical quantities, we get the following balance equation [11]:

$$\frac{\partial \rho_{m\pi}}{\partial t} + \nabla \cdot \mathbf{J}_{ms} = 0. \tag{14}$$

In this model description, the stressed state of the body is determined by the modified stress tensor $\hat{\boldsymbol{\sigma}}_* = \{\sigma_{ij}^*\}$ expressed via the Cauchy stress tensor $\hat{\boldsymbol{\sigma}}$ by the formula

$$\hat{\boldsymbol{\sigma}}_* = \hat{\boldsymbol{\sigma}} - (\rho_{m\pi}\mu'_\pi - \mathbf{\Pi}_m \cdot \nabla\mu'_\pi)\,\hat{\mathbf{I}}.$$

Here, $\mu'_\pi = \mu_\pi - \mu$ is a modified chemical potential. In this mathematical model, parallel with the stress tensor $\hat{\boldsymbol{\sigma}}_*$ and strain tensor $\hat{\boldsymbol{e}}$, two additional pairs of conjugate parameters connected with the influence of local mass displacements are introduced in the space of parameters of state. The modified chemical potential μ'_π and the specific density of induced mass ρ_m form one pair of parameters, whereas the second pair is formed by the specific vector of local mass displacements $\boldsymbol{\pi}_m = \{\pi_{mi}\}$ and the gradient of modified chemical potential $\nabla\mu'_\pi = \{\mu'_{\pi,i}\}$. The equations of state of this model have the form [11]

$$\sigma_{ij}^* = \rho\frac{\partial W}{\partial e_{ij}}, \qquad \mu'_\pi = \frac{\partial W}{\partial \rho_m}, \qquad \pi_i^m = \frac{\partial W}{\partial \mu'_{\pi,i}},$$

where ρ is the mass density, $\boldsymbol{\pi}_m = \mathbf{\Pi}_m/\rho$, $\rho_m = \rho_{m\pi}/\rho$, $\boldsymbol{\pi}_m = \{\pi_i^m\}$, and $\hat{\boldsymbol{e}}$ is the Green strain tensor. In the linear approximation, the function W is represented by the following bilinear form:

$$W(e_{ij}, \rho_m, \mu'_{\pi,i}) = \mu'_{\pi 0}\rho_m + \frac{\underline{\lambda}}{2\rho_0} e_{ii}e_{jj} + \frac{\underline{\mu}}{\rho_0} I_2 + \frac{d_\rho}{2}\rho_m^2$$

$$- \frac{\alpha_\rho}{\rho_0}\left(\underline{\lambda} + \frac{2}{3}\underline{\mu}\right) e_{ii}\rho_m - \frac{\chi_m}{2}(\mu'_{\pi,i})^2,$$

where $\underline{\lambda}$ and $\underline{\mu}$ are the Lamé constants, d_ρ, α_ρ, and χ_m are higher-order elastic constants, and $\mu'_{\pi 0}$ are the values of the modified chemical potential μ'_π in the infinite medium.

The balance equations of the model include the balance equation for the induced mass (11) and the balance equation for the mechanical momentum

$$\nabla \cdot \hat{\boldsymbol{\sigma}}_* + \mathbf{F} + \rho \mathbf{F}'_* = \rho \ddot{\mathbf{u}} . \tag{15}$$

Parallel with the mass force $\mathbf{F}$, Eq. (15) contains an additional nonlinear mass force

$$\mathbf{F}'_* = \rho_m \nabla \mu'_\pi - \boldsymbol{\pi}_m \cdot \nabla \otimes \nabla \mu'_\pi$$

caused by the changes in the microstructure of the material (local mass displacements).

The key system of equations of local gradient elasticity includes a second-order vector differential equation for the vector of displacements (equation of motion) and a second-order scalar differential equation for the modified chemical potential [11]. If the mathematical model of this medium does not take into account the irreversibility and inertia of local mass displacements, then we get a dynamically uncoupled key system of equations of local gradient elasticity. This theory made it possible to explain numerous phenomena that were not covered by the classical theory of elasticity. In particular, the indicated theory describes the near-surface inhomogeneities of the stress-strain state of solid bodies [12], the dispersion of short mechanical waves [12], the size effect of the moduli of elasticity [66, 92], the process of propagation of SH waves in a homogeneous isotropic half space [9], the appearance of wedging pressure in thin solid films [12], the effect of hardening of elastic nanobeams [67], and other effects.

The relations of the local gradient theory were used to determine the levels of surface tension and surface strain energy in solid elastic bodies [8, 12]. The analysis of effect of irreversibility of local mass displacements made it possible to describe transient processes of formation of the near-surface inhomogeneity of fields in solids with plane boundaries [15], which was not done within the framework of the other theories.

The relationship between the local gradient elasticity and other nonclassical theories of elastic continua was studied in a series of works. Note that it is possible to exclude the parameters connected with local mass displacements from the equations of state for the stress tensor $\hat{\boldsymbol{\sigma}}_*$ [11]. As a final result, we obtain a more general relationship between the tensors $\hat{\boldsymbol{\sigma}}_*$ and $\hat{\boldsymbol{e}}$:

$$(1 - \ell_*^2 \Delta)\hat{\boldsymbol{\sigma}} = 2\underline{\mu}(1 - \ell_*^2 \Delta)\,\hat{\boldsymbol{e}} + \underline{\lambda}(1 - \ell_{1*}^2 \Delta)\, e \hat{\mathbf{I}} ,$$

than the formula predicted by the Aifantis theory [see the equation of state (4)]. Here, ℓ_* is the characteristic length of the material, $\ell_{1*}^2 = \ell_*^2 \mathfrak{M}/(1+\mathfrak{M})$, where $\mathfrak{M}$ is the parameter of correlation between the processes of deformation and local mass displacements [11].

In [11], by analyzing an example of infinite medium, it was shown that the effect exerted by the local mass displacements in the model description is, in a certain sense, equivalent to the use of space-type integral determining relations with exponential relaxation kernels.

The present work was financially supported by the Slovak Science and Technology Assistance Agency (Project No. SK-CN-RD-18–0005) and by the National Academy of Sciences of Ukraine (Project No. 0117 U004156).

REFERENCES

1. S. A. Ambartsumyan, *Micropolar Theory of Shells and Plates* [in Russian], National Academy of Sciences of Armenia, Yerevan (1999).
2. É. L. Aéro and E. V. Kuvshinskii, “Main equations of the theory of elasticity of media with rotation interaction of particles,” *Fiz. Tverd. Tela,* **2**, No. 7, 1399–1409 (1960).
3. P. A. Belov and S. A. Lurie, “A continuum model of microheterogeneous media,” *Prikl. Mat. Mekh.*, **73**, Issue 5, 833–848 (2009); ***English translation:*** *J. Appl. Math. Mech.*, **73**, No. 5, 599–608 (2009); https://doi.org/10.1016/j.jappmathmech.2009.11.013.
4. Ya. I. Burak, Ye. Ya. Chaplya, V. F. Kondrat, and O. R. Hrytsyna, “Mathematical modeling of thermomechanical processes in elastic bodies with regard for local mass displacements,” *Dop. Nats. Akad. Nauk Ukr.*, No. 6, 45–49 (2007).
5. Ya. I. Burak and T. S. Nagirnii, “Theoretical principles for computing local-gradient thermomechanical systems with allowance for subsurface phenomena,” *Fiz.-Khim. Mekh. Mater.*, **29**, No. 4, 24–30 (1993); ***English translation:*** *Mater. Sci.*, **29**, No. 4, 349–354 (1994); https://doi.org/10.1007/BF00566442.
6. Ya. I. Burak, “Determining relations of local-gradient thermomechanics,” *Dop. Akad. Nauk Ukr. RSR. Ser. A*, No. 12, 19–23 (1987).
7. O. Hrytsyna, “On the description of the influence of local mass displacements on shear stresses,” *Fiz.-Mat. Model. Inf. Tekhnol.*, Issue 16, 61–75 (2012).
8. O. Hrytsyna, “Determination of the surface energy of solid bodies,” *Fiz.-Mat. Model. Inf. Tekhnol.*, Issue 17, 43–54 (2013).
9. O. R. Hrytsyna, “Influence of subsurface inhomogeneity on the propagation of SH waves in isotropic materials,” *Fiz.-Khim. Mekh. Mater.*, **53**, No. 2, 128–134 (2017); ***English translation:*** *Mater. Sci.*, **53**, No. 2, 273–281 (2017); https://doi.org/10.1007/s11003-017-0072-0.
10. O. Hrytsyna, T. Nahirnyi, and K. Chervinka, “Local-gradient approach in thermomechanics,” *Fiz.-Mat. Model. Inf. Tekhnol.*, Issue 3, 72–83 (2006).
11. O. Hrytsyna and V. Kondrat, *Thermomechanics of Condensed Systems with Regard for Local Mass Displacements: I. Fundamentals of the Theory* [in Ukrainian], Rastr-7, Lviv (2017).
12. O. Hrytsyna and V. Kondrat, *Thermomechanics of Condensed Systems with Regard for Local Mass Displacements: II. Applied Investigations* [in Ukrainian], Rastr-7, Lviv (2019).
13. V. I. Erofeev, *Wave Processes in Solid Bodies with Microstructures* [in Russian], Moscow University, Moscow (1999).
14. V. F. Kondrat and O. R. Hrytsyna, “Relations of gradient thermomechanics taking into account the irreversibility and inertia of local mass displacement,” *Mat. Met. Fiz.-Mekh. Polya*, **54**, No. 1, 91–100 (2011); ***English translation:*** *J. Math. Sci.*, **183**, No. 1, 100–111 (2012); https://doi.org/10.1007/s10958-012-0800-9.
15. V. Kondrat, T. Nahirnyi, and O. Hrytsyna, “Formation and mutual influence of near-surface inhomogeneities in the elastic layer with regard for the irreversibility of local mass displacements,” *Mashynoznavstvo*, No. 3 (129), 31–36 (2008).
16. E. V. Kuvshinskii and É. L. Aéro, “Continuum theory of asymmetric elasticity. Influence of 'internal rotation',” *Fiz. Tverd. Tela.,* **5**, No. 9, 2591–2598 (1963).
17. I. A. Kunin, *Theory of Elastic Bodies with Microstructure. Nonlocal Theory of Elasticity* [in Russian], Nauka, Moscow (1975).
18. S. A. Lisina and A. I. Potapov, “Generalized continuum models in nanomechanics,” *Dokl. Ros. Akad. Nauk*, **420**, No. 3, 328–330 (2008).
19. S. A. Lurie and P. A. Belov, “Theory of media with preserved dislocations: Special cases: Cosserat and Aéro–Kuvshinskii media, porous media, and media with ‘twinning',” in: *Proc. of the Conf. “Modern Problems of the Mechanics of Heterogeneous Media,”* Institute of Applied Mechanics, Russian Academy of Sciences (2005), pp. 235–267.
20. T. Nahirnyi and K. Chervinka, Thermodynamic *Models and Methods of Thermomechanics with Regard for the Near-Surface and Structural Inhomogeneities. Fundamentals of Nanomechanics. I* [in Ukrainian], Spolom, Lviv (2012).
21. W. Nowacki, *Teoria Sprężystości,* PWN, Warszawa (1970).
22. Ya. S. Podstrigach, “On a nonlocal theory of solid body deformation,” *Prikl. Mekh.*, **3**, No. 2, 71–76 (1967); ***English translation:*** *Sov. Appl. Mech.*, **3**, No. 2, 44–46 (1967); https://doi.org/10.1007/BF00885584.
23. G. N. Savin, *Fundamentals of the Plane Problem of the Couple Stress Theory of Elasticity* [in Russian], Kyiv State University, Kyiv (1965).
24. G. N. Savin, A. A. Lukashev, E. M. Lysko, S. V. Veremeenko, and G. G. Agas’ev, “Elastic wave propagation in a Cosserat continuum with constrained particle rotation,” *Prikl. Mekh.*, **6**, No. 6, 27–41 (1970); ***English translation:*** *Sov. Appl. Mech.*, **6**, No. 6, 599–602 (1970), https://doi.org/10.1007/BF00888458.
25. G. N. Savin and Y. N. Nemish, “Investigations into stress concentration in the moment theory of elasticity (a survey),” *Prikl. Mekh.,* **4**, No. 12, 1–17 (1968); ***English translation:*** *Sov. Appl. Mech.*, **4**, No. 12, 1–15 (1968); https://doi.org/10.1007/BF00886725.
26. S. O. Sarkisyan, “Micropolar theory of thin rods, plates, and shells,” *Izv. Nats. Akad. Nauk Armen., Mekh.*, **58**, No. 2, 84–95 (2005).
27. S. O. Sarkisyan, “General mathematical models of micropolar elastic thin plates,” *Izv. Nats. Akad. Nauk Armen. Mekh.*, **64**, No. 1, 58–67 (2011).
28. A. M. Abazari, S. M. Safavi, G. Rezazadeh, and L. G. Villanueva, “Modelling the size effects on the mechanical properties of micro/nano structures,” *Sensors*, **15**, No. 11, 28543–28562 (2015); https://doi.org/10.3390/s151128543.

29. E. C. Aifantis, "Update on a class of gradient theories," *Mech. Mater.*, **35**, No. 3-6, 259–280 (2003); https://doi.org/10.1016/S0167-6636(02)00278-8.
30. E. C. Aifantis, "Exploring the applicability of gradient elasticity to certain micro/nano reliability problems," *Microsyst. Technol.*, **15**, No. 1, 109–115 (2009); https://doi.org/10.1007/s00542-008-0699-8.
31. E. C. Aifantis, "A concise review of gradient models in mechanics and physics," *Front. Phys.*, **7**, 239 (2020); https://doi.org/10.3389/fphy.2019.00239.
32. B. S. Altan and E. C. Aifantis, "On some aspects in the special theory of gradient elasticity," *J. Mech. Behav. Mater.*, **8**, No. 3, 231–282 (1997); https://doi.org/10.1515/JMBM.1997.8.3.231.
33. J. Altenbach, H. Altenbach, and V. Eremeyev, "On generalized Cosserat-type theories of plates and shells: a short review and bibliography," *Arch. Appl. Mech.*, **80**, No. 1, 73–92 (2010); https://doi.org/10.1007/s00419-009-0365-3.
34. B. Arash and Q. Wang, "A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes," *Comput. Mater. Sci.*, **51**, No. 1, 303–313 (2012); https://doi.org/10.1016/j.commatsci.2011.07.040.
35. H. Askes and E. C. Aifantis, "Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results," *Int. J. Solids Struct.*, **48**, No. 13, 1962–1990 (2011); https://doi.org/10.1016/j.ijsolstr.2011.03.006.
36. J. D. Axe, J. Harada, and G. Shirane, "Anomalous acoustic dispersion in centrosymmetric crystals with soft optic phonons," *Phys. Rev. B*, **1**, No. 3, 1227–1234 (1970); https://doi.org/10.1103/PhysRevB.1.1227.
37. L. Behera and S. Chakraverty, "Recent researches on nonlocal elasticity theory in the vibration of carbon nanotubes using beam models: A review," *Arch. Comput. Meth. Eng.*, **24**, No. 3, 481–494 (2017); https://doi.org/10.1007/s11831-016-9179-y.
38. Y. Chen, J. D. Lee, and A. Eskandarian, "Atomistic viewpoint of the applicability of microcontinuum theories," *Int. J. Solids Struct.*, **41**, No. 8, 2085–2097 (2004); https://doi.org/10.1016/j.ijsolstr.2003.11.030.
39. N. Cordero, *A Strain Gradient Approach to the Mechanics of Micro and Nanocrystals*, Materials. Thèse de Doctorat, *École Nationale Supérieure des Mines de Paris*, Paris (2011).
40. E. Cosserat and F. Cosserat, *Théorie des Corps Déformable*, Hermann et Fils, Paris (1909).
41. S. Cuenot, C. Frétigny, S. Demoustier-Champagne, and B. Nysten, "Surface tension effect on the mechanical properties of nanomaterials measured by atomic force microscopy," *Phys. Rev. B*, **69**, No. 16, 165410 (2004); https://doi.org/10.1103/PhysRevB.69.165410.
42. F. Dell'Isola, A. D. Corte, and I. Giorgio, "Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives," *Math. Mech. Solids*, **22**, No. 4, 852–872 (2017); https://doi.org/10.1177/1081286515616034.
43. M. A. Eltaher, M. E. Khater, and S. A. Emam, "A review on nonlocal elastic models for bending, buckling, vibrations, and wave propagation of nanoscale beams," *Appl. Math. Model.*, **40**, No. 5-6, 4109–4128 (2016); https://doi.org/10.1016/j.apm.2015.11.026.
44. J. Engelbrecht and M. Braun, "Nonlinear waves in nonlocal media," *Appl. Mech. Rev.*, **51**, No. 8, 475–488 (1998); https://doi.org/10.1115/1.3099016.
45. A. C. Eringen, "Linear theory of micropolar elasticity," *Indiana J. Math. Mech.*, **15**, No. 6, 909–923 (1966); https://doi.org/10.1512/iumj.1966.15.15060.
46. A. C. Eringen, "Mechanics of micromorphic continua," in: E. Kröner (editor) *Mechanics of Generalized Continua*, Springer, Berlin (1968), pp. 18–35; https://doi.org/10.1007/978-3-662-30257-6_2.
47. A. C. Eringen, *Theory of Micropolar Elasticity*, in: H. Liebowitz (editor), *Fracture*, *An Advanced Treatise, Vol. II: Mathematical Fundamentals*, Academic Press, New York (1968), pp. 621–729.
48. A. C. Eringen, "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves," *J. Appl. Phys.*, **54**, No. 9, 4703–4710 (1983); https://doi.org/10.1063/1.332803.
49. A. C. Eringen, *Microcontinuum Field Theories. 1. Foundation and Solids*, Springer, New York (1999).
50. A. C. Eringen, *Nonlocal Continuum Field Theories*, Springer, New York (2002).
51. A. C. Eringen and C. B. Kafadar, "Polar field theories," in: *Continuum Physics*, Vol. IV: A. C. Eringen (editor), *Polar and Nonlocal Field Theories*, Academic Press, New York (1976), pp. 1–73; https://doi.org/10.1016/B978-0-12-240804-5.50007-5.
52. A. C. Eringen and E. S. Suhubi, "Nonlinear theory of simple microelastic solids — I," *Int. J. Eng. Sci.*, **2**, No. 2, 189–203 (1964); https://doi.org/10.1016/0020-7225(64)90004-7.
53. J. Fernández-Sáez, R. Zaera, J. A. Loya, and J. N. Reddy, "Bending of Euler–Bernoulli beams using Eringen's integral formulation: A paradox resolved," *Int. J. Eng. Sci.*, **99**, 107–116 (2016); https://doi.org/10.1016/j.ijengsci.2015.10.013.
54. N. A. Fleck and J. W. Hutchinson, "A phenomenological theory for strain gradient effects in plasticity," *J. Mech. Phys. Solids*, **41**, No. 12, 1825–1857 (1993); https://doi.org/10.1016/0022-5096(93)90072-N.
55. S. Forest, "Micromorphic approach for gradient elasticity, viscoplasticity, and damage," *J. Eng. Mech.*, **135**, No. 3, 117–131 (2009); https://doi.org/10.1061/(ASCE)0733-9399(2009)135:3(117).
56. S. Forest, N. M. Cordero, and E. P. Busso, "First vs. second gradient of strain theory for capillarity effects in an elastic fluid at small length scales," *Comput. Mater. Sci.*, **50**, No. 4, 1299–1304 (2011); https://doi.org/10.1016/j.commatsci.2010.03.048.
57. X.-L. Gao and S. K. Park, "Variational formulation of a simplified strain gradient elasticity theory and its application to a pressurized thick-walled cylinder problem," *Int. J. Solids Struct.*, **44**, No. 22–23, 7486–7499 (2007); https://doi.org/10.1016/j.ijsolstr.2007.04.022.

58. P. Germain, “The method of virtual power in continuum mechanics. Part 2. Microstructure,” *SIAM J. Appl. Math.*, **25**, No. 3, 556–575 (1973); https://doi.org/10.1137/0125053.
59. I.-D. Ghiba, P. Neff, A. Madeo, and I. Münch, “A variant of the linear isotropic indeterminate couple-stress model with symmetric local force-stress, symmetric nonlocal force-stress, symmetric couple-stresses and orthogonal boundary conditions,” *Math. Mech. Solids*, **22**, No. 6, 1221–1266 (2017).
60. A. E. Green and R. S. Rivlin, “Simple force and stress multipoles,” *Arch. Ration. Mech. Anal.*, **16**, No. 5, 325–353 (1964); https://doi.org/ 10.1007/BF00281725.
61. A. E. Green and R. S. Rivlin, “Multipolar continuum mechanics,” *Arch. Ration. Mech. Anal.*, **17**, No. 2, 113–147 (1964); https://doi.org/10.1007/BF00253051.
62. A. A. Gusev and S. A. Lurie, “Symmetry conditions in strain gradient elasticity,” *Math. Mech. Solids*, **22**, No. 4, 683–691 (2017); https://doi.org/10.1177/1081286515606960.
63. A. R. Hadjesfandiari and G. F. Dargush, “Couple stress theory for solids,” *Int. J. Solids Struct.*, **48**, No. 18, 2496–2510 (2011); https://doi.org/10.1016/j.ijsolstr.2011.05.002.
64. A. R. Hadjesfandiari and G. F. Dargush, *Evolution of Generalized Couple-Stress Continuum Theories: A Critical Analysis*. Preprint arXiv: 1501.03112 (2015).
65. S. Hassanpour and G. R. Heppler, “Micropolar elasticity theory: A survey of linear isotropic equations, representative notations, and experimental investigations,” *Math. Mech. Solids*, **22**, No. 2, 224–242 (2017); https://doi.org/10.1177/1081286515581183.
66. O. R. Hrytsyna, “Applications of the local gradient elasticity to the description of the size effect of shear modulus,” *SN Appl. Sci.*, **2**, No. 8, 1453 (2020); https://doi.org/10.1007/s42452-020-03217-9.
67. O. Hrytsyna, “A Bernoulli–Euler beam model based on the local gradient theory of elasticity,” *J. Mech. Mater. Struct.*, **15**, No. 4, 471–487 (2020); https://doi.org/10.2140/jomms.2020.15.471.
68. T. J. Jaramillo, *A Generalization of the Energy Function of Elasticity Theory*, Dissertation, University of Chicago (1929).
69. G. Y. Jing, H. L. Duan, X. M. Sun, Z. S. Zhang, J. Xu, Y. D. Li, J. X. Wang, and D. P. Yu, “Surface effects on elastic properties of silver nanowires: Contact atomic-force microscopy,” *Phys. Rev.* ***B***, **73**, No. 23, 235409 (2006); https://doi.org/10.1103/PhysRevB.73.235409.
70. M. Jirásek, “Nonlocal theories in continuum mechanics,” *Acta Polytech.*, **44**, Nos. 5-6, 16–34 (2004); https://doi.org/10.14311/610.
71. C. B. Kafadar and A. C. Eringen, “Micropolar media — I: The classical theory,” *Int. J. Eng. Sci.*, **9**, No. 3, 271–305 (1971); https://doi.org/10.1016/0020-7225(71)90040-1.
72. C. B. Kafadar and A. C. Eringen, “Micropolar media — II: The relativistic theory,” *Int. J. Eng. Sci.*, **9**, No. 3, 307–329 (1971); https://doi.org/10.1016/0020-7225(71)90041-3.
73. K. Kiani, “Free longitudinal vibration of tapered nanowires in the context of nonlocal continuum theory via a perturbation technique,” *Phys. E: Low Dimens. Syst. Nanostructures*, **43**, No. 1, 387–397 (2010), https://doi.org/10.1016/j.physe.2010.08.022.
74. S. Kong, S. Zhou, Z. Nie, and K. Wang, “Static and dynamic analysis of micro beams based on strain gradient elasticity theory,” *Int. J. Eng. Sci.*, **47**, No. 4, 487–498 (2009); https://doi.org/10.1016/j.ijengsci.2008.08.008.
75. R. Lakes, “Cosserat micromechanics of structured media experimental methods,” in: *Proc. of the American Society for Composites. The 3rd Technical Conf. “Integrated Composites Technology” (September 25–29, 1988)*, Seattle, Washington (1988), pp. 505–516.
76. R. Lakes, “Experimental methods for study of Cosserat elastic solids and other generalized elastic continua,” in: H. Mühlaus (editor), *Continuum Models for Materials with Microstructure*, Wiley, New York (1995), pp. 1–22.
77. D. C. C. Lam, F. Yang, A. C. M. Chong, J. Wang, and P. Tong, “Experiments and theory in strain gradient elasticity,” *J. Mech. Phys. Solids*, **51**, No. 8, 1477–1508 (2003); https://doi.org/10.1016/S0022-5096(03)00053-X.
78. C. Liebold and W. H. Müller, “Applications of strain gradient theories to the size effect in submicro-structures incl. experimental analysis of elastic material parameters,” *Bull. TICMI*, **19**, No. 1, 45–55 (2015).
79. K. M. Liew, Y. Zhang, and L. W. Zhang, “Nonlocal elasticity theory for graphene modeling and simulation: prospects and challenges,” *J. Model. Mech. Mater.*, **1**, No. 1, 20160159 (2017); https://doi.org/10.1515/jmmm-2016-0159.
80. C. W. Lim, G. Zhang, and J. N. Reddy, “A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation,” *J. Mech. Phys. Solids*, **78**, 298–313 (2015); https://doi.org/10.1016/j.jmps.2015.02.001.
81. P. Lu, P. Q. Zhang, H. P. Lee, C. M. Wang, and J. N. Reddy, “Non-local elastic plate theories,” *Proc. R. Soc. A*, **463**, No. 2088, 3225–3240 (2007); https://doi.org/10.1098/rspa.2007.1903.
82. H. M. Ma, X.-L. Gao, and J. N. Reddy, “A microstructure-dependent Timoshenko beam model based on a modified couple stress theory,” *J. Mech. Phys. Solids*, **56**, No. 12, 3379–3391 (2008); https://doi.org/10.1016/j.jmps.2008.09.007.
83. R. Maranganti and P. Sharma, “A novel atomistic approach to determine strain gradient elasticity constants: tabulation and comparison for various metals, semiconductors, silica, polymers and the (ir)relevance for nanotechnologies,” *J. Mech. Phys. Solids*, **55**, No. 9, 1823–1852 (2007); https://doi.org/10.1016/j.jmps.2007.02.011.
84. G. A. Maugin, “Generalized continuum mechanics: What do we mean by that?” in: G. Maugin and A. Metrikine (editors), *Mechanics of Generalized Continua. Advances in Mechanics and Mathematics*, Vol. 21, Springer (2010), pp. 3–13; https://doi.org/10.1007/978-1-4419-5695-8_1.
85. A. W. McFarland and J. S. Colton, “Role of material microstructure in plate stiffness with relevance to microcantilever sensors,” *J. Micromech. Microeng.*, **15**, No. 5, 1060–1067 (2005); https://doi.org/10.1088/0960-1317/15/5/024.

86. R. D. Mindlin, "Micro-structure in linear elasticity," *Arch. Ration. Mech. Anal.*, **16**, No. 1, 51–78 (1964); https://doi.org/10.1007/BF00248490.
87. R. D. Mindlin, "Second gradient of strain and surface-tension in linear elasticity," *Int. J. Solids Struct.*, **1**, No. 4, 417–438 (1965); https://doi.org/10.1016/0020-7683(65)90006-5.
88. R. D. Mindlin, "Theories of elastic continua and crystal lattice theories," in: E. Kröner (editor), *Mechanics of Generalized Continua. Proc of IUTAM Symposia (Stuttgart, 1967)*, Springer, Berlin (1968), pp. 312–320; https://doi.org/10.1007/978-3-662-30257-6_38.
89. R. D. Mindlin, "Elasticity, piezoelectricity and crystal lattice dynamics," *J. Elast.*, **2**, No. 4, 217–282 (1972); https://doi.org/10.1007/BF00045712.
90. R. D. Mindlin and H. F. Tiersten, "Effects of couple-stresses in linear elasticity," *Arch. Ration. Mech. Anal.*, **11**, No. 1, 415–448 (1962); https://doi.org/10.1007/BF00253946.
91. A. Naderi and A. R. Saidi, "Common nonlocal elastic constitutive relation and material-behavior modeling of nanostructures," *Proc. Inst. Mech. Eng. N: J. Nanomater. Nanoeng. Nanosyst.*, **23**1, No. 2, 83–87 (2017); https://doi.org/10.1177/2397791417712870.
92. T. Nahirnyj and K. Tchervinka, "Mathematical modeling of structural and near-surface non-homogeneities in thermoelastic thin films," *Int. J. Eng. Sci.*, **91**, 49–62 (2015); https://doi.org/10.1016/j.ijengsci.2015.02.001.
93. J. Niiranen, V. Balobanov, J. Kiendl, and S. B. Hosseini, "Variational formulations, model comparisons and numerical methods for Euler–Bernoulli micro- and nano-beam models," *Math. Mech. Solids*. **24**, No. 1, 312–335 (2019); https://doi.org/10.1177/1081286517739669.
94. W. Nowacki, *Theory of Micropolar Elasticity*, Springer, Wien (1970).
95. F. Ojaghnezhad and H. M. Shodja, "A combined first principles and analytical determination of the modulus of cohesion, surface energy, and the additional constants in the second strain gradient elasticity," *Int. J. Solids Struct.*, **50**, No. 24, 3967–3974 (2013); https://doi.org/10.1016/j.ijsolstr.2013.08.004.
96. S. Papargyri-Beskou and D. E. Beskos, "Static, stability and dynamic analysis of gradient elastic flexural Kirchhoff plates," *Arch. Appl. Mech.*, **78**, No. 8, 625–635 (2008); https://doi.org/10.1007/s00419-007-0166-5.
97. C. Papenfuss and S. Forest, "Thermodynamical frameworks for higher grade material theories with internal variables or additional degrees of freedom," *J. Non-Equilib. Thermodyn.*, **31**, No. 4, 319–353 (2006); https://doi.org/10.1515/jnetdy.2006.014.
98. S. K. Park and X.-L. Gao, "Bernoulli–Euler beam model based on a modified couple stress theory," *J. Micromech. Microeng.*, **16**, No. 11, 2355–2359 (2006); https://doi.org/10.1088/0960-1317/16/11/015.
99. C. Polizzotto, "Nonlocal elasticity and related variational principles," *Int. J. Solids Struct.*, **38**, No. 42–43, 7359–7380 (2001); https://doi.org/10.1016/S0020-7683(01)00039-7.
100. C. Polizzotto, "A unifying variational framework for stress gradient and strain gradient elasticity theories," *Eur. J. Mech. A/Solids*, **49**, 430–440 (2015); https://doi.org/10.1016/j.euromechsol.2014.08.013.
101. C. Polizzotto, "A hierarchy of simplified constitutive models within isotropic strain gradient elasticity," *Eur. J. Mech. A/Solids*, **61**, 92–109 (2017); https://doi.org/10.1016/j.euromechsol.2016.09.006.
102. Yu. Z. Povstenko, "Straight disclinations in nonlocal elasticity," *Int. J. Eng. Sci.*, **33**, No. 4, 575–582 (1995); https://doi.org/10.1016/0020-7225(94)00070-0.
103. J. N. Reddy, "Nonlocal theories for bending, buckling and vibration of beams," *Int. J. Eng. Sci.*, **45**, Nos. 2-8, 288–307 (2007); https://doi.org/10.1016/j.ijengsci.2007.04.004.
104. M. Repka, V. Sladek, and J. Sladek, "Gradient elasticity theory enrichment of plate bending theories," *Compos. Struct.*, **202**, 447–457 (2018); https://doi.org/10.1016/j.compstruct.2018.02.065.
105. G. Romano and R. Barretta, "Nonlocal elasticity in nanobeams: the stress-driven integral model," *Int. J. Eng. Sci.*, **115**, 14–27 (2017); https://doi.org/10.1016/j.ijengsci.2017.03.002.
106. M. B. Rubin, *Cosserat Theories: Shells, Rods and Points*, in: G. M. L. Gladwell (editor), Ser. *Solid Mechanics and Its Applications*, Vol. 79, Kluwer, Dordrecht (2000); https://doi.org/10.1007/978-94-015-9379-3.
107. M. M. Shokrieh and I. Zibaei, "Determination of the appropriate gradient elasticity theory for bending analysis of nano-beams by considering boundary conditions effect," *Lat. Am. J. Solids Struct.*, **12**, No. 12, 2208–2230 (2015); https://doi.org/10.1590/1679-78251589.
108. J. Sladek, V. Sladek, M. Repka, and S. Schmauder, "Gradient theory for crack problems in quasicrystals," *Eur. J. Mech. A/Solids*, **77**, 103813 (2019); https://doi.org/10.1016/j.euromechsol.2019.103813.
109. E. S. Suhubl and A. C. Eringen, "Nonlinear theory of simple microelastic solids — II," *Int. J. Eng. Sci.*, **2**, No. 4, 389–404 (1964); https://doi.org/10.1016/0020-7225(64)90017-5.
110. V. Sundararaghavan and A. Waas, "Non-local continuum modeling of carbon nanotubes: Physical interpretation of non-local kernels using atomistic simulations," *J. Mech. Phys. Solids*, **59**, No. 6, 1191–1203 (2011); https://doi.org/10.1016/j.jmps.2011.03.009.
111. C. Tang and G. Alici, "Evaluation of length-scale effects for mechanical behaviour of micro- and nanocantilevers: I. Experimental determination of length-scale factors," *J. Phys. D: Appl. Phys.*, **44**, No. 33, 335501 (2011); https://doi.org/10.1088/0022-3727/44/33/335501.
112. C. Tekoğlu and P. R. Onck, "Size effects in two-dimensional Voronoi foams: A comparison between generalized continua and discrete models," *J. Mech. Phys. Solids*, **56**, No. 12, 3541–3564 (2008); https://doi.org/10.1016/j.jmps.2008.06.007.

113. H.-T. Thai, T. P. Vo, T.-K. Nguyen, and S.-E. Kim, "A review of continuum mechanics models for size-dependent analysis of beams and plates," *Compos. Struct.*, **177**, 196–219 (2017); https://doi.org/10.1016/j.compstruct.2017.06.040.
114. R. A. Toupin, "Elastic materials with couple-stresses," *Arch. Rat. Mech. Anal.*, **11**, No. 1, 385–414 (1962); https://doi.org/10.1007/BF00253945.
115. J. Vila, R. Zaera, and J. Fernández-Sáez, "Axisymmetric free vibration of closed thin spherical nanoshells with bending effects," *J. Vibr. Control.*, **22**, No. 18, 3789–3806 (2016); https://doi.org/10.1177/1077546314565808.
116. Q. Wang and K. M. Liew, "Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures," *Phys. Lett. A*, **363**, No. 3, 236–242 (2007); https://doi.org/10.1016/j.physleta.2006.10.093.
117. Q. Wang and C. M. Wang, "The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes," *Nanotechnology*, **18**, No. 7, 075702 (2007); https://doi.org/10.1088/0957-4484/18/7/075702.
118. K. F. Wang, B. L. Wang, and T. Kitamura, "A review on the application of modified continuum models in modeling and simulation of nanostructures," *Acta Mech. Sin.*, **32**, No. 1, 83–100 (2016); https://doi.org/10.1007/s10409-015-0508-4.
119. C.-P. Wu and J.-J. Yu, "A review of mechanical analyses of rectangular nanobeams and single-, double-, and multi-walled carbon nanotubes using Eringen's nonlocal elasticity theory," *Arch. Appl. Mech.*, **89**, No. 9, 1761–1792 (2019); https://doi.org/10.1007/s00419-019-01542-z.
120. S. T. Yaghoubi, V. Balobanov, S. M. Mousavi, and J. Niiranen, "Variational formulations and isogeometric analysis for the dynamics of anisotropic gradient-elastic Euler–Bernoulli and shear-deformable beams," *Eur. J. Mech. A/Solids*, **69**, 113–123 (2018); https://doi.org/10.1016/j.euromechsol.2017.11.012.
121. B. Yakaiah and A. Srihari Rao, "Higher order nonlocal strain gradient approach for wave characteristics of carbon nanorod," *Nonlinear Anal.: Model. Control*, **19**, No. 4, 660–668 (2014). https://doi.org/10.15388/NA.2014.4.10.
122. F. Yang, A. C. M. Chong, D. C. C. Lam, and P. Tong, "Couple stress based strain gradient theory for elasticity," *Int. J. Solids Struct.*, **39**, No. 10, 2731–2743 (2002); https://doi.org/10.1016/S0020–7683(02)00152-X.