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A METHOD FOR SOLVING THE FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

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The paper considers a numerical method for solving the Fredholm integral equation of the first kind. The essence of the method is to replace the original equation with the corresponding regularized equation of the second kind, which is then solved by the modified spline collocation method. The solution in this case is represented as a linear combination of minimal splines. The coefficients at the splines are computed using local approximation (in some cases, quasi-interpolation) methods. Results of numerical experiments are presented and show that on model problems, the method proposed yields sufficiently accurate approximations, and the approximation accuracy can be improved by using minimal nonpolynomial splines and related functionals. Bibliography: 24 *titles.*

1. Introduction

In the last few decades, local approximation methods have actively been studied. Their main feature is that the coefficients at the basis functions are determined as values of approximation functionals, which are, for example, linear combinations of values of the function itself and its derivatives at certain points (see [1–6] for more detail). Local methods allowing one to obtain the highest order of accuracy are called *quasi-interpolation methods*, and the functionals used in constructing them are called *quasi-interpolation functionals*, or *quasi-interpolants*.

Methods based on quasi-interpolation have repeatedly been used in constructing algorithms for solving some integral equations. In particular, in [7, 8] it is shown that replacement of the solution of the Fredholm integral equation of the second kind with a linear combination of B-splines allows one to obtain accurate approximations, provided that the coefficients at the spline functions are constructed as quasi-interpolants. This approach is used in a number of solution methods (the Galerkin and Kantorovich methods, Sloan iterations [9], Kulkarni method [10], wavelet-Galerkin method [11], etc.). Construction of approximation functionals for minimal splines [12, 13] (spline functions with minimal support obtained from approximation relations with a complete chain of vectors and a generating vector function) makes it possible to construct the modified spline-collocation method for solving the Fredholm integral equation of the second kind [14].

The Fredholm integral equation of the first kind is an example of an ill-posed problem, and its solution cannot be obtained by applying an approach similar to the one used for equations of the second kind. The standard solution method is to reduce the original problem to a system of linear algebraic equations and to apply Tikhonov's regularization [15] or its more accurate version (see, e.g., [16, 17] and the references therein). An alternative approach is to regularize the original equation of the first kind, i.e., to replace it with a specially constructed equation of the second kind. This approach was suggested and tested on several model problems in [18], where it was recommended to solve the resulting equation of the second kind by an arbitrary known numerical method. This approach was mentioned as a perspective one in the survey [19].

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In this paper, we consider a method for solving the Fredholm integral equation of the first kind that combines the above-mentioned regularization approach and the modified spline-collocation method, suggested by the authors earlier. The results of numerical experiments presented demonstrate that the combined approach is quite accurate on several model problems; moreover, using the minimal splines instead of the polynomial B-splines allows one to improve the accuracy of the method.

2. The space of quadratic minimal splines

Let $\mathbb{Z}$ be the set of integers, $\mathbb{R}^3$ be the linear space of three-dimensional column vectors with components from the set of reals $\mathbb{R}^1$. The vector components are denoted by square brackets with integer subscripts, e.g., $\mathbf{a} = ([\mathbf{a}]_0, [\mathbf{a}]_1, [\mathbf{a}]_2)^T$, where $\mathbf{a} \in \mathbb{R}^3$ and T means transposition. On an interval $[a, b] \subset \mathbb{R}^1$, consider a grid X,

$$a = x_{-2} = x_{-1} = x_0 < x_1 < \ldots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = b. \tag{1}$$

Introduce the following notation:

$$J_{i,k} := \{i, i+1, \ldots, k\}, \quad i, k \in \mathbb{Z}, \quad i < k; \quad M := \cup_{j \in J_{0,n-1}} (x_j, x_{j+1});$$
$$S_j := [x_j, x_{j+3}], \quad j \in J_{-2,n-1}.$$

An ordered set $\mathbf{A} := \{\mathbf{a}_j\}_{j \in J_{-2,n-1}}$ of vectors $\mathbf{a}_j \in \mathbb{R}^3$ is called *a vector chain*. Assume that the square matrices $(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j)$ formed of the vectors $\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j \in \mathbb{R}^3$ are invertible, i.e.,

$$\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j) \neq 0, \quad j \in J_{0,n-1}. \tag{2}$$

Consider a three-component (column) vector function $\boldsymbol{\varphi} : [a, b] \to \mathbb{R}^3$ with components in the space $C^2[a, b]$ and nonzero Wronskian determinant

$$\det(\boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t), \boldsymbol{\varphi}''(t)) \neq 0, \quad t \in [a, b]. \tag{3}$$

The linear space of real-valued functions defined on M is denoted by $\mathbb{X}(M)$.

Assume that functions $\omega_j \in \mathbb{X}(M)$, $j \in J_{-2,n-1}$, satisfy the identities

$$\sum_{j'=k-2}^{k} \mathbf{a}_{j'}\, \omega_{j'}(t) \equiv \boldsymbol{\varphi}(t), \quad t \in (x_k, x_{k+1}),\ k \in J_{0,n-1};$$
$$\omega_j(t) \equiv 0, \qquad t \in M \backslash S_j,\ j \in J_{-2,n-1}. \tag{4}$$

For any fixed $t \in (x_k, x_{k+1})$, where $k \in J_{0,n-1}$, relations (4) can be regarded as a system of linear algebraic equations with respect to the unknowns $\omega_j(t)$. By assumption (2), system (4) has a unique solution, and $\operatorname{supp} \omega_j \subset S_j$.

The linear span of the functions $\omega_j(t)$ is called *the space of quadratic minimal coordinate* $(\mathbf{A}, \boldsymbol{\varphi})$*-splines*. Identities (4) are called *the approximation relations*. The vector function $\boldsymbol{\varphi}$ is said to be *generating*.

Let $\boldsymbol{\varphi}_j := \boldsymbol{\varphi}(x_j)$, $\boldsymbol{\varphi}'_j := \boldsymbol{\varphi}'(x_j)$, $\boldsymbol{\varphi}''_j := \boldsymbol{\varphi}''(x_j)$, $j \in J_{-2,n+2}$, and consider the chain of vectors $\{\mathbf{a}^N_j\}_{j \in J_{-2,n-1}}$ defined by the formula

$$\mathbf{a}^N_j := \boldsymbol{\varphi}_{j+1} - \frac{\mathbf{d}^T_{j+2}\, \boldsymbol{\varphi}_{j+1}}{\mathbf{d}^T_{j+2}\, \boldsymbol{\varphi}'_{j+1}}\, \boldsymbol{\varphi}'_{j+1}, \tag{5}$$

where the vectors $\mathbf{d}_j \in \mathbb{R}^3$ are determined by the identity

$$\mathbf{d}^T_j \mathbf{x} \equiv \det(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}'_j, \boldsymbol{\varphi}''_j, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

As is known [20], if condition (3) is fulfilled for a chain of vectors $\mathbf{A}^N := \{\mathbf{a}_j^N\}$, $j \in J_{0,n-1}$, then inequality (2) holds, and the functions satisfy the condition $\omega_j \in C^1[a,b]$ for all $j \in J_{-2,n-1}$. The possibility of using multiple nodes in the grid (1) was studied in [21].

If a vector function $\boldsymbol{\varphi}^N$ is such that $\boldsymbol{\varphi}^N = \boldsymbol{\varphi}$, where $[\boldsymbol{\varphi}(t)]_0 \equiv 1$, then the *partition of unity* property is valid, i.e.,

$$\sum_{j=-2}^{n-1} \omega_j(t) = 1, \quad t \in [a,b].$$

In this case, the functions $\omega_j(t)$ are called the *normalized quadratic minimal coordinate* $B_{\boldsymbol{\varphi}}$*-splines*, and they can be represented as follows:

$$\omega_j(t) = \begin{cases} \dfrac{\mathbf{d}_j^T \boldsymbol{\varphi}^N(t)}{\mathbf{d}_j^T \mathbf{a}_j^N}, & t \in [x_j, x_{j+1}), \\ \dfrac{\mathbf{d}_j^T \boldsymbol{\varphi}^N(t)}{\mathbf{d}_j^T \mathbf{a}_j^N} - \dfrac{\mathbf{d}_j^T \mathbf{a}_{j+1}^N}{\mathbf{d}_j^T \mathbf{a}_j^N} \dfrac{\mathbf{d}_{j+1}^T \boldsymbol{\varphi}^N(t)}{\mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^N}, & t \in [x_{j+1}, x_{j+2}), \\ \dfrac{\mathbf{d}_{j+3}^T \boldsymbol{\varphi}^N(t)}{\mathbf{d}_{j+3}^T \mathbf{a}_j^N}, & t \in [x_{j+2}, x_{j+3}). \end{cases} \tag{6}$$

Their linear span is denoted by

$$\mathbb{S}(X, \mathbf{A}^N, \boldsymbol{\varphi}^N) := \left\{ u(t) = \sum_{j=-2}^{n-1} c_j \, \omega_j(t), \quad c_j \in \mathbb{R}^1, t \in [a,b] \right\}.$$

3. Averaging approximation functionals

In this section, we consider different grids. For convenience, all the necessary objects defined on a grid X are supplied, if necessary, with the superscript X, e.g., we write ω_j^X.

Let $\boldsymbol{\varphi}(t) := (1, \rho(t), \sigma(t))^T$, where $\rho, \sigma \in C^2[a,b]$. Introduce the notation (see [22] for more detail)

$$\Delta_j(\rho, \sigma) := \begin{vmatrix} \rho_j & \rho'_j \\ \sigma_j & \sigma'_j \end{vmatrix}, \quad S_j^X(\rho, \sigma, \tau) := -\frac{\begin{vmatrix} \Delta_j(\rho, \sigma) & \Delta_{j+1}(\rho, \sigma) \\ \tau_j & \tau_{j+1} \end{vmatrix}}{\begin{vmatrix} \rho_j & \rho_{j+1} \\ \sigma_j & \sigma_{j+1} \end{vmatrix}},$$

where $\rho_j := \rho(x_j)$, $\sigma_j := \sigma(x_j)$, $\tau_j := \tau(x_j)$.

Consider another grid Y with the nodes

$$y_j := \begin{cases} x_0, & j = -2; \\ x_{j+1} + \theta(x_{j+2} - x_{j+1}), & \theta \in [0,1],\ j = -1, \ldots, n-2; \\ x_n, & j = n-1. \end{cases} \tag{7}$$

We construct an approximation $\mathfrak{Q} f$ of a given function f in the form

$$\mathfrak{Q} f = \sum_{j=-2}^{n-1} \mu_j^Y(f) \, \omega_j^X, \tag{8}$$

where the approximation functionals $\mu_j^Y(f)$ are defined as follows:

$$\mu_j^Y(f) := \begin{cases} f(y_{-2}), & j = -2; \\ a_j f(y_{j-1}) + b_j f(y_j) + c_j f(y_{j+1}), & j = -1, \ldots, n-2; \\ f(y_{n-1}), & j = n-1. \end{cases} \tag{9}$$

As is shown in [13], the approximation (8) reproduces all the functions $f \in \{[\boldsymbol{\varphi}]_i \,|\, i = 0, 1, 2\}$, provided that the values of the coefficients a_j, b_j, c_j in (9) are determined by the formulas

$$a_j = 1 - b_j - c_j,$$

$$b_j = \frac{N_j(y_{j+1})}{D_j},$$

$$c_j = -\frac{N_j(y_j)}{D_j},$$

where

$$N_j(y) := \begin{vmatrix} \rho(y) - \rho(y_{j-1}) & S^X_{j+1}(\rho,\, \sigma,\, \rho') - \rho(y_{j-1}) \\ \sigma(y) - \sigma(y_{j-1}) & S^X_{j+1}(\rho,\, \sigma,\, \sigma') - \sigma(y_{j-1}) \end{vmatrix},$$

$$D_j := \begin{vmatrix} \rho(y_{j+1}) - \rho(y_{j-1}) & \rho(y_j) - \rho(y_{j-1}) \\ \sigma(y_{j+1}) - \sigma(y_{j-1}) & \sigma(y_j) - \sigma(y_{j-1}) \end{vmatrix}.$$

If $\boldsymbol{\varphi}(t) = (1, t, t^2)^T$ and $\theta = \frac{1}{2}$, then, on a uniform grid, the functional (9) takes the form

$$\mu^Y_j(f) = -\frac{1}{8}\left(f(y_{j-1}) - 10f(y_j) + f(y_{j+1})\right) \tag{10}$$

and coincides with the well-known functional for the quadratic B-splines (see [23]).

4. Regularization and solution of integral equations

Consider the Fredholm integral equation of the first kind

$$\int_a^b K(t, x)\, u(x)\, dx = f(t). \tag{11}$$

In order to solve it, we are going to use regularization, which replaces solution of the original equation (11) by solving the following auxiliary Fredholm integral equation of the second kind:

$$\alpha u_\alpha(t) + \int_a^b K(t, x)\, u_\alpha(x)\, dx = f(t). \tag{12}$$

Equation (12) is parametrized by a real small parameter $\alpha > 0$. As is known [15, 18], the solution $u_\alpha(t)$ of Eq. (12) tends to the solution $u(t)$ of the original equation (11) as $\alpha \to 0$.

We solve Eq. (12) using the modified spline-collocation approach, suggested by the authors in [14], assuming that the Fredholm kernel $K(t, x)$ and the function $f(t)$ satisfy the corresponding requirements. Note that the Fredholm equation of the first kind is an ill-posed problem, whence it can happen that it has no solutions or a few of them.

Let $t \in [a, b]$. We write the Fredholm integral equation of the second kind as

$$u(t) - \mathcal{K}u(t) = f(t). \tag{13}$$

Assume that $f \in C[a, b]$ and the operator $\mathcal{I} - \mathcal{K}$ is invertible; here, $\mathcal{I}$ is the identity operator. The Fredholm integral equation of the second kind is a well-posed problem, whence, for any function $f \in C[a, b]$, Eq. (13) has a unique solution $u \in C[a, b]$.

If $K \in C([a, b] \times [a, b])$, then we define the linear compact operator $\mathcal{K}$ as follows:

$$\mathcal{K}u(t) := \int_a^b K(t, x)\, u(x)\, dx, \quad t \in [a, b].$$

An approximate solution of Eq. (13) on the grid (1) is constructed in the form

$$u^h(t) = \sum_{j=-2}^{n-1} c_j\,\omega_j(t), \tag{14}$$

where $\mathcal{K}$ and f are replaced with their approximations $\mathfrak{Q}\mathcal{K}$ and $\mathfrak{Q}f$, respectively, which are constructed by approximating them in accordance with formulas (8) and (9). Thus, the original equation (13) is written as

$$u^h = \mathfrak{Q}f + \mathfrak{Q}\mathcal{K}u^h, \tag{15}$$

or, in the notation $\widetilde{\omega}_j := \mathcal{K}\,\omega_j$, as

$$\sum_{j=-2}^{n-1} c_j\,\omega_j(t) = \sum_{j=-2}^{n-1} \mu_j^Y(f)\,\omega_j(t) + \mathfrak{Q}\sum_{i=-2}^{n-1} c_i\,\widetilde{\omega}_i(t),$$

which implies that

$$c_j = \mu_j^Y(f) + \sum_{i=-2}^{n-1} c_i\,\mu_j^Y(\widetilde{\omega}_i), \quad j \in J_{-2,n-1}. \tag{16}$$

Compose the vector $\mathbf{c} := (c_{-2}, c_{-1}, \ldots, c_{n-1})^T$ of the coefficients c_j, the vector $\boldsymbol{\mu} := (\mu_{-2}^Y(f), \mu_{-1}^Y(f), \ldots, \mu_{n-1}^Y(f))^T$ of the functionals $\mu_j^Y(f)$, and the matrix $\boldsymbol{M} := (M_{j,i}) = \left(\mu_j^Y(\widetilde{\omega}_i)\right)$ of the functionals $\mu_j^Y(\widetilde{\omega}_i)$. where $j, i \in J_{-2,\ldots,n-1}$. Then we obtain the system of linear algebraic equations

$$(\boldsymbol{I} - \boldsymbol{M})\,\mathbf{c} = \boldsymbol{\mu},$$

where $\boldsymbol{I}$ is the identity matrix. By solving this system, we obtain the coefficients c_j in the approximation (14).

Note [9] that using representation (15), the solution (14) can be refined by performing the following iterations:

$$\widetilde{u}^h := \mathcal{K}u^h + f.$$

Thus, an approximate solution of the Fredholm integral equation of the second kind can ultimately be represented in the form

$$\widetilde{u}^h(t) = f(t) + \sum_{j=-2}^{n-1} c_j\,\widetilde{\omega}_j(t). \tag{17}$$

Note that

$$\widetilde{\omega}_j(t) = \mathcal{K}\,\omega_j(t) = \int_a^b K(t,x)\,\omega_j(x)\,dx,$$

and this integral can be computed by replacing the function $K(t,x)$ with its approximation, whereas the integral

$$W_{ij} := \int_a^b \omega_i(t)\,\omega_j(t)\,dt$$

is computed either exactly or using an arbitrary standard method of numerical integration.

5. Numerical experiments

Below, in all the numerical experiments, approximations are constructed on the interval $[a,b]=[0,1]$ and on a grid of the type (1). Auxiliary grids are considered on the same interval.

For $\boldsymbol{\varphi}^B(t):=(1,t,t^2)^T$, the functions (6) coincide with the well-known quadratic polynomial B-splines of the third order

$$\omega_j^B(t)=\begin{cases}\dfrac{(t-x_j)^2}{(x_{j+1}-x_j)(x_{j+2}-x_j)}, & t\in[x_j,x_{j+1}),\\ \dfrac{1}{x_{j+1}-x_j}\left(\dfrac{(t-x_j)^2}{x_{j+2}-x_j}-\dfrac{(t-x_{j+1})^2(x_{j+3}-x_j)}{(x_{j+2}-x_{j+1})(x_{j+3}-x_{j+1})}\right), & t\in[x_{j+1},x_{j+2}),\\ \dfrac{(t-x_{j+3})^2}{(x_{j+3}-x_{j+1})\,(x_{j+3}-x_{j+2})}, & t\in[x_{j+2},x_{j+3}),\end{cases}$$

and for $\boldsymbol{\varphi}^H(t):=(1,\sinh t,\cosh t)^T$ the functions (6) are of the following form (see [24] for detail):

$$\omega_j^H(t)=\begin{cases}\dfrac{\sinh^2\left(\frac{t-x_j}{2}\right)\cosh\frac{x_{j+2}-x_{j+1}}{2}}{\sinh\frac{x_{j+1}-x_j}{2}\sinh\frac{x_{j+2}-x_j}{2}}, & t\in[x_j,x_{j+1});\\ \dfrac{\cosh\frac{x_{j+2}-x_{j+1}}{2}}{\sinh\frac{x_{j+1}-x_j}{2}}\left(\dfrac{\sinh^2\left(\frac{t-x_j}{2}\right)}{\sinh\frac{x_{j+2}-x_j}{2}}-\dfrac{\sinh^2\left(\frac{t-x_{j+1}}{2}\right)\sinh\frac{x_{j+3}-x_j}{2}}{\sinh\frac{x_{j+3}-x_{j+1}}{2}\sinh\frac{x_{j+2}-x_{j+1}}{2}}\right), & t\in[x_{j+1},x_{j+2});\\ \dfrac{\sinh^2\left(\frac{x_{j+3}-t}{2}\right)\cosh\frac{x_{j+2}-x_{j+1}}{2}}{\sinh\frac{x_{j+3}-x_{j+1}}{2}\sinh\frac{x_{j+3}-x_{j+2}}{2}}, & t\in[x_{j+2},x_{j+3}).\end{cases}$$

In our numerical experiments, given an original function g, we construct an approximation $\widetilde{g}\in\mathbb{S}(X,\mathbf{A}^N,\boldsymbol{\varphi}^N)$ in accordance with (8), which is written in the form

$$\widetilde{g}=\sum_{j=-2}^{n-1}c_j\,\omega_j, \tag{18}$$

where the functions ω_j are constructed on the uniform grid (1), and the coefficients c_j are computed on the grid (7) and are determined by the choice of the generating vector function $\boldsymbol{\varphi}(t)$ and the approximation functional. The three variants considered are as follows:

(1) $\boldsymbol{\varphi}(t)=\boldsymbol{\varphi}^B(t)$, $\omega_j(t)=\omega_j^B(t)$, and the coefficients c_j are determined by the values of g at Greville's points;

(2) $\boldsymbol{\varphi}(t)=\boldsymbol{\varphi}^B(t)$, $\omega_j(t)=\omega_j^B(t)$, and c_j are determined by relation (10);

(3) $\boldsymbol{\varphi}(t)=\boldsymbol{\varphi}^H(t)$, $\omega_j(t)=\omega_j^H(t)$, and c_j are determined by relation (9).

The error E is estimated by the absolute value of the largest deviation of the approximation $\widetilde{u}^h$ constructed using (17) from the value of the function u at the nodes of the ten times finer auxiliary grid, i.e.,

$$E=\max_{t\in[a,b]}|\widetilde{u}^h(t)-u(t)|.$$

The following experiments show how the error depends on the choice of the generating vector function and approximation functional in (18).

Example 1. Consider the Fredholm integral equation of the first kind

$$\int_0^1 t\,x\,u(x)\,dx=2t. \tag{19}$$

The function $u(t) = 6t$ is known to be one of its solutions.

An approximate solution of Eq. (19) is constructed in the form (17). A preliminary regularization is performed with the parameter value $\alpha = 10^{-10}$. The results of computations are presented in Table 1.

Table 1. Approximation error for Eq. (19) as a function of the number n of grid points.

type	$n = 16$	$n = 32$	$n = 64$
1	0.0139	0.0043	0.0013
2	**0.0069**	**0.0024**	**0.0008**

We see that the method suggested for solving the Fredholm integral equation of the first kind allows one to construct a sufficiently accurate approximation of the solution function.

Now we are going to study how the approximation accuracy can be improved by using the same approximation approach but replacing the polynomial B-splines with the nonpolynomial minimal splines.

Example 2. Consider the Fredholm integral equation of the first kind

$$\int_0^1 \mathrm{e}^{3t-4x}\, u(x)\, dx = (\mathrm{e}-1)\mathrm{e}^{3t}. \tag{20}$$

One of its solutions is the function $u(t) = \mathrm{e}^{3t+1}$. We approximate Eq. (20) as in the previous example. The results of the numerical experiments are presented in Table 2.

Table 2. Approximation error for Eq. (20) as a function of the number n of grid points.

type	$n = 32$	$n = 64$
2	0.00514	0.00110
3	**0.00067**	**0.00036**

Note that the model problems (19) and (20) are borrowed from [18], where some iterative methods for solving the regularized equations are used but no numerical data characterizing the convergence to the solution function are provided.

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