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ON DOUBLY ALTERNATIVE ZERO DIVISORS IN CAYLEY–DICKSON ALGEBRAS

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Zero divisors of Cayley–Dickson algebras over an arbitrary field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$, are studied. It is shown that the zero divisors whose components alternate strongly pairwise and have nonzero norm form hexagonal structures in the zero-divisor graph of a Cayley–Dickson algebra. Properties of the doubly alternative zero divisors at least one of whose components has nonzero norm are established, and explicit forms of their annihilators, orthogonalizers, and centralizers are obtained. Properties of the zero divisors in Cayley–Dickson algebras with anisotropic norm are described, and it is shown that in this case, directed hexagons in the zero-divisor graph can be extended to undirected double hexagons in the orthogonality graph. A criterion of C-equivalence for elements of Cayley–Dickson algebras with anisotropic norm is obtained. Possible values of dimension for the annihilators of elements in Cayley–Dickson algebras are considered. Bibliography: 23 titles.

1. Introduction

A convenient method for visualizing a binary algebraic relation R is to define the corresponding graph. Its vertices represent the elements or their equivalence classes in an algebraic structure under consideration, and there is an edge from x to y if and only if xRy. The most popular relation graphs of various algebras are the commutativity, orthogonality, and zero-divisor graphs.

Studying relation graphs is a rapidly expanding branch of modern mathematics. Among the fields where relation graphs find particularly important applications, one should mention the problem of classifying relation preserving mappings, see [5], and the isomorphism problem, that is, exploring the interrelation between an isomorphism of algebraic structures and an isomorphism of the corresponding relation graphs, see [6, 12].

This work aims at studying the commutativity and orthogonality relations, the relation of forming a pair of zero divisors, and also the graphs induced by them for a particular class of nonassociative algebras, namely, the Cayley–Dickson algebras. The study of Cayley–Dickson algebras started in the theory of composition algebras, i.e., those algebras that possess a strictly nondegenerate quadratic form $n(\cdot)$ satisfying the identity $n(ab) = n(a)n(b)$ for all elements of the algebra.

In 1898, Hurwitz proved that the unital composition division algebras over $\mathbb{R}$ are exhausted by the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$. Later the Hurwitz theorem was extended by Jacobson to arbitrary unital composition algebras over an arbitrary field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$. He demonstrated that any such algebra $\mathcal{A}$ is isomorphic to a Cayley–Dickson algebra $\mathcal{A}_n$ of dimension 2^n, where $0 \leq n \leq 3$, see [11, p. 61, Theorem 1]. This result was generalized to a field $\mathbb{F}$ of arbitrary characteristic by Zhevlakov et al., see [20, p. 46, Theorem 1].

In general, the Cayley–Dickson algebras over a field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$, form a family of 2^n-dimensional algebras $\mathcal{A}_n$, $n \in \mathbb{N}_0$, that are defined inductively: $\mathcal{A}_0 = \mathbb{F}$, and at every step the algebra $\mathcal{A}_{n+1}$ is obtained from $\mathcal{A}_n$ by applying the Cayley–Dickson process with a certain parameter $\gamma_n \in \mathbb{F} \setminus \{0\}$. The elements of $\mathcal{A}_{n+1}$ are the ordered pairs of elements from $\mathcal{A}_n$, that is, they are of the form $(a, b) \in \mathcal{A}_n \times \mathcal{A}_n$. For $n \geq 4$, the algebras $\mathcal{A}_n$ are not alternative, whence

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they are not composition algebras. Consequently, even in the case where the norm on $\mathcal{A}_n$ is anisotropic, $\mathcal{A}_n$ contains zero divisors. The problems of classifying them and describing their annihilators are rather difficult, except for some particular cases.

At present, most authors restrict their attention to the real algebras of the main sequence, which we denote by $\mathcal{M}_n$. In this case, $\mathbb{F} = \mathbb{R}$, and all the Cayley–Dickson parameters are equal to -1. The most successful studies of zero divisors in these algebras are due to Moreno [14–16] and Biss, Dugger, and Isaksen [3, 4]. Particularly, in [3, 4] the dimensions of the annihilators of their elements were completely described, and the zero divisors with annihilators of the largest possible dimension were classified. Then Pixton [18] obtained a similar result for the dimensions of the alternators in these algebras.

It should be mentioned that Moreno was the first who studied doubly alternative elements in the real algebras of the main sequence, that is, the elements both of whose components are alternative in the preceding algebra of the sequence. He established several important properties of the doubly alternative zero divisors, see [14, pp. 25–27]. As it was shown in [15, p. 15], a reason why the doubly alternative elements can be successfully studied is that though the composition identity $n(ab) = n(ba) = n(a)n(b)$ does not hold in the entire algebra $\mathcal{M}_n$ for $n \geq 4$, it is valid if $a, b \in \mathcal{M}_n$ alternate with each other.

Among the recent works on the relation graphs of real Cayley–Dickson algebras one should mention the papers [9, 10, 21], where the relation graphs of the low-dimensional real Cayley–Dickson algebras, namely, of the split-complex numbers, split-quaternions, split-octonions, split-sedenions, and sedenions, have been described. In the author's papers [22, 23], the zero divisors in the real Cayley–Dickson algebras whose components satisfy additional conditions on their norm and alternativity were studied, and the isomorphism problem for the orthogonality graphs on pairs of basis elements of real Cayley–Dickson algebras was solved.

In the present paper, we generalize the results obtained in [22] for the real Cayley–Dickson algebras to the case of arbitrary Cayley–Dickson algebras over a field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$, and the results obtained in [3,14,22] for the real algebras of the main sequence are extended to the case of arbitrary Cayley–Dickson algebras with anisotropic norm. Corollary 3.3 and Lemma 5.6 correct some inaccuracies occurring in the proofs of Lemma 4.6 and Corollary 5.9 in [22]. Also we study the possible dimensions of the annihilators of elements in arbitrary Cayley–Dickson algebras.

The paper is organized as follows: In Sec. 2, we introduce the main definitions and notation, which are used throughout the paper. In particular, we give a detailed description of the Cayley–Dickson process in Sec. 2.2 and recall some properties of the Cayley–Dickson algebras in Sec. 2.3.

Section 3 extends some well-known results on subalgebras in the real algebras of the main sequence to the case of arbitrary Cayley–Dickson algebras. Namely, in Lemma 3.4, Corollary 3.8, and Theorems 3.7 and 3.9, we establish a sufficient condition for two or three elements to generate an associative or an alternative subalgebra and present the multiplication table for the elements of this subalgebra. The proofs of these results are based on constructing a homomorphism from $\mathcal{A}_2$ or $\mathcal{A}_3$ to the subalgebra in question.

In Sec. 4, we consider pairs of the zero divisors in arbitrary Cayley–Dickson algebras whose components have nonzero norm and alternate with each other. In Sec. 4.1, it is shown that they form hexagonal patterns in the zero divisor graph. Lemma 4.1 plays the key part in studying such elements because it allows one to construct a new pair of zero divisors given a pair of them. The main result of Sec. 4.1 is Theorem 4.13. In Sec. 4.2, we describe properties of the doubly alternative zero divisors at least one of whose components has nonzero norm. Lemma 4.19 and Theorem 4.22 establish an explicit form of their annihilators and orthogonalizers and describe the relationship between their centralizers and orthogonalizers.

In Sec. 5, we consider zero divisors in the Cayley–Dickson algebras with anisotropic norm. Lemmas 5.1 and 5.6 generalize the results on properties of zero divisors in the real algebras of the main sequence obtained in [14]. Corollary 5.4 shows that two noncentral elements of a Cayley–Dickson algebra with anisotropic norm are C-equivalent, i.e., their centralizers coincide if and only if their imaginary parts are proportional to each other. Theorem 5.11 states that in the case of Cayley–Dickson algebras with anisotropic norm, the directed hexagons in the zero divisor graph from Theorem 4.13 can be extended to undirected double hexagons in the orthogonality graph.

Section 6 is devoted to studying possible values of dimensions for the annihilators of elements in Cayley–Dickson algebras. Examples 6.3, 6.4, and 6.5 demonstrate that, in general, the dimension of an annihilator can be either divisible by four, or even but not divisible by four, or odd. However, in accordance with Theorem 6.11, in the case of Cayley–Dickson algebras with anisotropic norm, the dimension of an annihilator is always divisible by four. This result generalizes Theorem 9.8 in [3] on the dimensions of annihilators in real algebras of the main sequence.

2. Main definitions and notation

2.1. Algebraic relations and their graphs. Let $\mathbb{F}$ be an arbitrary field and let $(\mathcal{A}, +, \cdot)$ be an algebra over a field $\mathbb{F}$, possibly noncommutative and nonassociative. We say that $a, b \in \mathcal{A}$ *anticommute* if $ab + ba = 0$, and $a, b \in \mathcal{A}$ are *orthogonal* if $ab = ba = 0$. We denote the set of zero divisors (left, right, or two-sided) in $\mathcal{A}$ by $Z(\mathcal{A})$, the set of two-sided zero divisors in $\mathcal{A}$ by $Z_{LR}(\mathcal{A})$, and the (commutative) center of $\mathcal{A}$ by $C_{\mathcal{A}}$.

Definition 2.1. Let a be an arbitrary element of an algebra $\mathcal{A}$.

- *The centralizer* of a is the set $C_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = ba\}$ of all elements in $\mathcal{A}$ that commute with a.
- *The anticentralizer* of a is the set $\mathrm{Anc}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab + ba = 0\}$ of all elements in $\mathcal{A}$ that anticommute with a.
- *The orthogonalizer* of a is the set $O_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = ba = 0\}$ of all elements in $\mathcal{A}$ that are orthogonal to a.
- *The left annihilator* of a is the set $l.\mathrm{Ann}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ba = 0\}$.
- Similarly, *the right annihilator* of a is the set $r.\mathrm{Ann}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = 0\}$.

It is clear that $C_{\mathcal{A}}(a)$, $\mathrm{Anc}_{\mathcal{A}}(a)$, $O_{\mathcal{A}}(a)$, $l.\mathrm{Ann}_{\mathcal{A}}(a)$, and $r.\mathrm{Ann}_{\mathcal{A}}(a)$ are linear spaces over $\mathbb{F}$.

Definition 2.2. Elements $a, b \in \mathcal{A}$ are said to be *C-equivalent* if $C_{\mathcal{A}}(a) = C_{\mathcal{A}}(b)$ and *O-equivalent* if $O_{\mathcal{A}}(a) = O_{\mathcal{A}}(b)$.

Notation 2.3. Given a subset X of a linear space W over $\mathbb{F}$, we denote the set of lines passing through elements of X by

$$\mathbb{P}(X) = \{[x] = \mathbb{F}x \mid x \in X \setminus \{0\}\}.$$

Introduce some relation graphs, which will be studied in this paper.

Definition 2.4. Let $\mathcal{A}$ be an arbitrary algebra. Define the following relation graphs of $\mathcal{A}$:

- *The commutativity graph* $\Gamma_C(\mathcal{A})$: its vertices are the elements of
$$\mathbb{P}(\mathcal{A}/C_{\mathcal{A}}) = \{[a + C_{\mathcal{A}}] = \mathbb{F}a + C_{\mathcal{A}} \mid a \in \mathcal{A} \setminus C_{\mathcal{A}}\},$$
and distinct vertices $[a + C_{\mathcal{A}}]$ and $[b + C_{\mathcal{A}}]$ are adjacent if and only if $ab = ba$.
- *The orthogonality graph* $\Gamma_O(\mathcal{A})$: its vertices are the elements of $\mathbb{P}(Z_{LR}(\mathcal{A}))$, and distinct vertices $[a]$ and $[b]$ are adjacent if and only if $ab = ba = 0$.

- *The directed zero divisor graph* $\Gamma_Z(\mathcal{A})$: its vertices are the elements of $\mathbb{P}(Z(\mathcal{A}))$, and distinct vertices $[a]$ and $[b]$ are connected by a directed edge $([a],[b])$ if and only if $ab = 0$.

Note that the edges of $\Gamma_C(\mathcal{A})$, $\Gamma_O(\mathcal{A})$, and $\Gamma_Z(\mathcal{A})$ are well defined. When speaking of the vertices of these graphs, we will not distinguish between a nonzero element a and the line $[a] = \mathbb{F}a$ passing through it. Also we denote $\operatorname{span}(a_1,\ldots,a_k) = \mathbb{F}a_1 + \cdots + \mathbb{F}a_k$.

2.2. Constructing Cayley–Dickson algebras. We refer the reader to [13,19] for auxiliary definitions and general properties of the Cayley–Dickson algebras.

Definition 2.5. Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ with an involution $a \mapsto \bar{a}$. The algebra $\mathcal{A}\{\gamma\}$ produced by the Cayley–Dickson process applied to $\mathcal{A}$ with a parameter $\gamma \in \mathbb{F}$, $\gamma \neq 0$, is defined as the set of ordered pairs of elements of $\mathcal{A}$ with the operations

$$\begin{aligned} \alpha(a,b) &= (\alpha a, \alpha b);\\ (a,b)+(c,d) &= (a+c, b+d);\\ (a,b)(c,d) &= (ac+\gamma\bar{d}b, da+b\bar{c}) \end{aligned}$$

and the involution

$$\overline{(a,b)} = (\bar{a}, -b),$$

where $a,b,c,d \in \mathcal{A}$, $\alpha \in \mathbb{F}$. If the involution on $\mathcal{A}$ is regular, that is, $a+\bar{a} \in \mathbb{F}1_{\mathcal{A}}$ and $a\bar{a} = \bar{a}a \in \mathbb{F}1_{\mathcal{A}}$ for all $a \in \mathcal{A}$, then the involution on $\mathcal{A}\{\gamma\}$ also is regular, see [19, p. 435].

Proposition 2.6 ([13, p. 161, Exercise 2.5.1])**.** *Let $\gamma' = \alpha^2\gamma$ for an $\alpha \neq 0$. Then the algebras $\mathcal{A}\{\gamma\}$ and $\mathcal{A}\{\gamma'\}$ are isomorphic.*

In the sequel, we assume that $\operatorname{char}\mathbb{F} \neq 2$. Now we define an arbitrary Cayley–Dickson algebra, which is determined by a set of its parameters.

Definition 2.7. Given an integer $n \geq 0$ and nonzero numbers $\gamma_0,\ldots,\gamma_{n-1} \in \mathbb{F}$, we define the Cayley–Dickson algebra $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0,\ldots,\gamma_{n-1}\}$ by induction as follows:

(1) $\mathcal{A}_0 = \mathbb{F}$, and $e_0^{(0)} = 1$ is its single basis element.
(2) If $\mathcal{A}_n\{\gamma_0,\ldots,\gamma_{n-1}\}$ is constructed, then $\mathcal{A}_{n+1}\{\gamma_0,\ldots,\gamma_n\} = (\mathcal{A}_n\{\gamma_0,\ldots,\gamma_{n-1}\})\{\gamma_n\}$. Its basis elements are $e_0^{(n+1)},\ldots,e_{2^{n+1}-1}^{(n+1)}$, where

$$e_m^{(n+1)} = \begin{cases} (e_m^{(n)}, 0), & 0 \leq m \leq 2^n - 1,\\ (0, e_{m-2^n}^{(n)}), & 2^n \leq m \leq 2^{n+1}-1. \end{cases}$$

For every integer $n \geq 0$, the structure $\mathcal{A}_n$ in Definition 2.7 is a 2^n-dimensional algebra over $\mathbb{F}$ with the unit element $e_0^{(n)}$ and a regular involution. We denote $1 = e_0 = e_0^{(n)}$ and $k = ke_0^{(n)}$ for $k \in \mathbb{F}$.

Definition 2.8.

- Let $a \in \mathcal{A}_n$. Its *trace* is $t(a) = a + \bar{a}$, its *imaginary part* is $\mathfrak{Im}(a) = \frac{a-\bar{a}}{2}$, and its *norm* is $n(a) = a\bar{a} = \bar{a}a$. Since the involution on $\mathcal{A}_n$ is regular, we have $t(a), n(a) \in \mathbb{F}$.
- An element $a \in \mathcal{A}_n$ is said to be *pure* if $t(a) = 0$.
- An element $(a,b) \in \mathcal{A}_{n+1}$ is said to be *doubly pure* if $t(a) = t(b) = 0$.

Proposition 2.9 ([19, p. 435])**.** *The trace and norm of an element $(a,b) \in \mathcal{A}_{n+1}$ can be determined inductively by using the following relations:*

$$\begin{aligned} t((a,b)) &= t(a),\\ n((a,b)) &= n(a) - \gamma_n n(b). \end{aligned}$$

From Proposition 2.9 it follows that the norm $n(\cdot)$ is a nondegenerate quadratic form on $\mathcal{A}_n$.

2.3. Properties of the Cayley–Dickson algebras. In what follows, we assume that $\mathcal{A}$ is an arbitrary algebra over a field $\mathbb{F}$, and $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is an arbitrary Cayley–Dickson algebra over a field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$.

Proposition 2.10 ([19, p. 440]). *Let $\langle a, b\rangle$ denote the $\mathbb{F}$-valued symmetric bilinear form associated with the quadratic form $n(a)$. Then, for all $a, b \in \mathcal{A}_n$, $\langle a, a\rangle = n(a)$ and $2\langle a, b\rangle = a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a = t(a\bar{b})$. In addition, for arbitrary $a, b \in \mathcal{A}_n$ it holds that $\langle a, b\rangle = \langle \bar{a}, \bar{b}\rangle$ and $t(a) = 2\langle a, e_0\rangle$.*

Notation 2.11. We write $a \perp b$ if a and b are orthogonal with respect to $\langle\cdot,\cdot\rangle$, that is, $\langle a, b\rangle = 0$.

Lemma 2.12 ([19, Lemmas 2 and 6]). *For all $x, y, z \in \mathcal{A}_n$ we have*
(1) $t([x, y, z]) = 0$;
(2) $\langle x, yz\rangle = \langle x\bar{z}, y\rangle = \langle \bar{y}x, z\rangle$.

Definition 2.13. Let $\mathbb{F} = \mathbb{R}$.

- The algebra $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is called *a real Cayley–Dickson algebra of the main sequence* if $\gamma_k = -1$ for all $k = 0, \dots, n-1$. We denote this algebra by $\mathcal{M}_n$.
- The algebra $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is called *a real Cayley–Dickson split-algebra* if $\gamma_k = -1$ for all $k = 0, \dots, n-2$ and $\gamma_{n-1} = 1$. We denote it by $\mathcal{H}_n$ because the norm on $\mathcal{H}_n$ appears to be hyperbolic.

Proposition 2.14 ([10, Proposition 3.31]).

- *Let $a = \sum\limits_{m=0}^{2^n-1} a_m e_m^{(n)}$, $b = \sum\limits_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{M}_n$. Then $\langle a, b\rangle = \sum\limits_{m=0}^{2^n-1} a_m b_m$ is a Euclidean inner product. Particularly, $n(a) = \sum\limits_{m=0}^{2^n-1} a_m^2$, whence $n(a) = 0$ if and only if $a = 0$.*
- *Let $a = \sum\limits_{m=0}^{2^n-1} a_m e_m^{(n)}, b = \sum\limits_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{H}_n$. Then $\langle a, b\rangle = \sum\limits_{m=0}^{2^{n-1}-1} a_m b_m - \sum\limits_{m=2^{n-1}}^{2^n-1} a_m b_m$.*

Remark 2.15. In the case of real algebras of the main sequence, the norm of a is frequently defined as $\sqrt{a\bar{a}}$, in contrast with the definition $n(a) = a\bar{a}$ used in this paper. However, most results can readily be extended to the norm modified in this way.

Example 2.16.

- The complex numbers ($\mathbb{C}$), quaternions ($\mathbb{H}$), octonions ($\mathbb{O}$), and sedenions ($\mathbb{S}$) are the real algebras of the main sequence for $n = 1$, 2, 3, and 4, respectively, see [1].
- The split-complex numbers ($\hat{\mathbb{C}}$), split-quaternions ($\hat{\mathbb{H}}$), split-octonions ($\hat{\mathbb{O}}$), and split-sedenions ($\hat{\mathbb{S}}$) are the real split-algebras for $n = 1$, 2, 3, and 4, respectively, see [2, 21].

Now we proceed to some concepts related to associativity. Given $a, b, c \in \mathcal{A}$, we denote their associator by $[a, b, c] = (ab)c - a(bc)$ and their antiassociator by $\{a, b, c\} = (ab)c + a(bc)$. An algebra $\mathcal{A}$ is said to be *flexible* if $[a, b, a] = 0$ for all $a, b \in \mathcal{A}$. Clearly, in a flexible algebra $\mathcal{A}$ we have $[a, b, c] = -[c, b, a]$ for all $a, b, c \in \mathcal{A}$. An algebra $\mathcal{A}$ is said to be *alternative* if $[a, a, b] = [b, a, a] = 0$ for all $a, b \in \mathcal{A}$.

It is well known that the algebra $\mathcal{A}_n$ is alternative if and only if $n \leq 3$; however, $\mathcal{A}_n$ always is flexible, see, e.g., [19, p. 436, Theorem 1].

Definition 2.17 ([15, p. 12, p. 15]). Let $a, b \in \mathcal{A}_n$.

- We say that a *alternates* with b if $[a, a, b] = 0$.
- If a alternates with every $b \in \mathcal{A}_n$, then a is said to be *alternative*.

- We say that a *alternates strongly* with b if $[a,a,b]=0$ and $[b,b,a]=0$.
- If a alternates strongly with every $b\in\mathcal{A}_n$, then a is said to be *strongly alternative*.

The following three lemmas describe the anticentralizer of an arbitrary nonzero element of $\mathcal{A}_n$ and the relationship between the centralizer and orthogonalizer of an arbitrary pure element. In [8], they are stated for the real Cayley–Dickson algebras only. However, their proofs are valid verbatim in the case of an arbitrary field. Nevertheless, we provide the proofs of Lemmas 2.19 and 2.20 for completeness. In the statement of Lemma 2.20, the direct sum also implies that the direct summands are orthogonal to each other with respect to the symmetric bilinear form $\langle\cdot,\cdot\rangle$. By [8, Proposition 8.19], the condition $n\le 3$ is essential in Lemma 2.20(1).

Lemma 2.18 ([8, Lemma 5.8])**.** *Let $a\in\mathcal{A}_n$, $a\neq 0$.*

(1) *If $t(a)\neq 0$, $n(a)\neq 0$, then* $\operatorname{Anc}_{\mathcal{A}_n}(a)=\{0\}$.
(2) *If $t(a)\neq 0$, $n(a)=0$, then* $\operatorname{Anc}_{\mathcal{A}_n}(a)=\mathbb{F}\bar{a}$.
(3) *If $t(a)=0$, then* $\operatorname{Anc}_{\mathcal{A}_n}(a)=\{b\in\mathcal{A}_n\mid t(b)=\langle a,b\rangle=0\}=\operatorname{span}(e_0,a)^{\perp}$.

Lemma 2.19 ([8, Lemma 8.10])**.** *Let $x\in\mathcal{A}_n\setminus\{0\}$, $t(x)=0$. Then $C_{\mathcal{A}_n}(x)=\mathbb{F}\oplus O_{\mathcal{A}_n}(x)\oplus V$, where* $\dim(V)\le 1$.

Proof. Obviously, $\mathbb{F}\subseteq C_{\mathcal{A}_n}(x)$, and it is sufficient to show that $\mathfrak{Im}(C_{\mathcal{A}_n}(x))=O_{\mathcal{A}_n}(x)\oplus V$, where $\dim(V)\le 1$. By Lemma 2.18, $\operatorname{Anc}_{\mathcal{A}_n}(x)\subset\mathfrak{Im}(\mathcal{A}_n)$, whence

$$O_{\mathcal{A}_n}(x)=C_{\mathcal{A}_n}(x)\cap\operatorname{Anc}_{\mathcal{A}_n}(x)=\mathfrak{Im}(C_{\mathcal{A}_n}(x))\cap\operatorname{Anc}_{\mathcal{A}_n}(x).$$

Since for any $y\in\mathfrak{Im}(C_{\mathcal{A}_n}(x))$ (implying that $t(y)=0$) the condition $y\in\operatorname{Anc}_{\mathcal{A}_n}(x)$ is described by a single linear equation, we have $\dim(\mathfrak{Im}(C_{\mathcal{A}_n}(x)))-\dim(O_{\mathcal{A}_n}(x))\le 1$. □

Lemma 2.20 ([8, Lemma 8.11])**.** *Let $x\in\mathcal{A}_n\setminus\{0\}$, $t(x)=0$.*

(1) *If $n(x)=0$ and $n\le 3$, then $C_{\mathcal{A}_n}(x)=\mathbb{F}\oplus O_{\mathcal{A}_n}(x)$.*
(2) *If $n(x)\neq 0$, then $C_{\mathcal{A}_n}(x)=\mathbb{F}\oplus\mathbb{F}x\oplus O_{\mathcal{A}_n}(x)$.*

Proof. It is clear that $C_{\mathcal{A}_n}(x)\supseteq\mathbb{F}+\mathbb{F}x+O_{\mathcal{A}_n}(x)$. Observe that if $y\in O_{\mathcal{A}_n}(x)$, then $t(y)=0$, whence, by Proposition 2.10, $\langle x,y\rangle=\frac{1}{2}t(x\bar{y})=-\frac{1}{2}t(xy)=0$. Since $n(x)=x\bar{x}=-x^2$, the conditions $n(x)=0$ and $x\in O_{\mathcal{A}_n}(x)$ are equivalent. Consider the following two cases:

(1) If $n(x)=0$, then this inclusion takes the form $C_{\mathcal{A}_n}(x)\supseteq\mathbb{F}\oplus O_{\mathcal{A}_n}(x)$. Show that for $n\le 3$ the reverse inclusion relation also holds. Let $y\in C_{\mathcal{A}_n}(x)$ and let $t(y)=0$. Since $n\le 3$, we can use the alternativity of $\mathcal{A}_n$. Note that $\overline{xy}=\bar{y}\bar{x}=yx=xy$, i.e., $xy=k\in\mathbb{F}$. Then $0=x^2y=x(xy)=kx$, whence $k=0$, that is, $y\in O_{\mathcal{A}_n}(x)$.
(2) If $n(x)\neq 0$, then this inclusion takes the form $C_{\mathcal{A}_n}(x)\supseteq\mathbb{F}\oplus\mathbb{F}x\oplus O_{\mathcal{A}_n}(x)$. The reverse inclusion relation follows from Lemma 2.19 and dimension considerations. □

Example 2.21. If $\mathcal{A}_n=\mathcal{M}_n$ is a real algebra of the main sequence, then any element $x\in\mathcal{M}_n\setminus\{0\}$, $t(x)=0$, satisfies the assumption of Lemma 2.20(2).

3. Alternative subalgebras

In this section, we establish a sufficient condition for two or three elements to generate an associative or an alternative subalgebra in an arbitrary Cayley–Dickson algebra. Also we present the multiplication table for the elements of this subalgebra. It should be noted that Assertions 3.2, 3.4–3.7, and 3.9 were already partially proved in the author's paper [22] for the real Cayley–Dickson algebras. Corollary 3.3 was also stated in that paper (see [22, Corollary 5.9]), but there the elements x and y were assumed to alternate strongly, and the proof contained an inaccuracy, namely, orthogonal projections with respect to subspaces with possibly degenerate norms were considered.

For $n \geq 4$, the algebra $\mathcal{A}_n$ is not alternative, whence it is not a composition algebra. However, as the following lemma shows, the composition identity still holds for the elements that alternate with each other. In [10, 15], it is stated for the real Cayley–Dickson algebras only; however, its proof remains valid for an arbitrary field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$.

Lemma 3.1 ([15, p. 15], [10, Lemma 4.8])**.** *Let $a, b \in \mathcal{A}_n$, $[a, a, b] = 0$. Then $n(ab) = n(ba) = n(a)n(b)$.*

Following [14–16], we denote $\widetilde{e}_0 = (0, e_0) \in \mathcal{A}_n$ and $\widetilde{a} = a\widetilde{e}_0$ for all $a \in \mathcal{A}_n$.

Lemma 3.2. *Let $a, b \in \mathcal{A}_n$ and let b be doubly pure. Then*
(1) $\widetilde{\widetilde{a}} = \gamma_{n-1}a$;
(2) $\widetilde{a}b = -\widetilde{ab}$;
(3) $\widetilde{a} \perp a$.
If a also is doubly pure, then
(4) $\widetilde{a}b + \widetilde{b}a = 0$ *if and only if* $a \perp b$;
(5) $\gamma_{n-1}ab + \widetilde{b}\widetilde{a} = 0$ *if and only if* $\widetilde{a} \perp b$.

Proof. Let $a = (a_1, a_2)$, $b = (b_1, b_2)$. By definition, $\widetilde{a} = (a_1, a_2)(0, e_0) = (\gamma_{n-1}a_2, a_1)$.
(1) It holds that $\widetilde{\widetilde{a}} = \widetilde{(\gamma_{n-1}a_2, a_1)} = (\gamma_{n-1}a_1, \gamma_{n-1}a_2) = \gamma_{n-1}a$.
(2) Since b is doubly pure, we have

$$\begin{aligned}\widetilde{a}b = (\gamma_{n-1}a_2, a_1)(b_1, b_2) &= (\gamma_{n-1}a_2b_1 + \gamma_{n-1}\bar{b}_2a_1, \gamma_{n-1}b_2a_2 + a_1\bar{b}_1)\\ &= -(\gamma_{n-1}(b_2a_1 + a_2\bar{b}_1), a_1b_1 + \gamma_{n-1}\bar{b}_2a_2) = -\widetilde{ab}.\end{aligned}$$

(3) By Lemma 2.12(2), $\langle a, \widetilde{a}\rangle = \langle a, a\widetilde{e}_0\rangle = \langle \bar{a}a, \widetilde{e}_0\rangle = \langle n(a)e_0, \widetilde{e}_0\rangle = 0$.
(4) By Lemma 2.18, $a \perp b$ if and only if $ab = -ba$, which amounts to $-\widetilde{a}b = \widetilde{ab} = -\widetilde{ba} = \widetilde{b}a$.
(5) By Lemma 2.18, $\widetilde{a} \perp b$ if and only if $\widetilde{a}b = -b\widetilde{a}$, or equivalently $-\gamma_{n-1}ab = -\widetilde{\widetilde{ab}} = \widetilde{\widetilde{a}b} = -\widetilde{b\widetilde{a}} = \widetilde{b}\widetilde{a}$. □

Corollary 3.3. *Let $x, y \in \mathcal{A}_{n-1}$. Then in $\mathcal{A}_n$ the following relations hold:*

$$x\widetilde{y} = \widetilde{yx}, \quad \widetilde{x}y = \widetilde{x\bar{y}}, \quad \widetilde{x}\widetilde{y} = \gamma_{n-1}\bar{y}x.$$

Proof. For any $z \in \mathcal{A}_{n-1}$, we have $z = (z, 0)$ in $\mathcal{A}_n$. Therefore, by Lemma 3.2, $\widetilde{z} = (0, z)$. Then

$$\begin{aligned}x\widetilde{y} &= (x, 0)(0, y) = (0, yx) = \widetilde{yx},\\ \widetilde{x}y &= (0, x)(y, 0) = (0, x\bar{y}) = \widetilde{x\bar{y}},\\ \widetilde{x}\widetilde{y} &= (0, x)(0, y) = (\gamma_{n-1}\bar{y}x, 0) = \gamma_{n-1}\bar{y}x.\end{aligned}$$ □

Note that in Lemma 3.4 and Theorems 3.7 and 3.9, $n(a)$ and $n(b)$ are allowed to be zero, which is in contrast with the usual definition of Cayley–Dickson algebras.

Lemma 3.4. *Let $a \in \mathcal{A}_n$ be doubly pure. Consider $\mathbb{H}_a = \operatorname{span}(e_0, a, \widetilde{e}_0, \widetilde{a})$. Then there exists a surjective homomorphism $\varphi_a : \mathcal{A}_2\{-n(a), \gamma_{n-1}\} \to \mathbb{H}_a$, whence $\mathbb{H}_a$ is an associative subalgebra in $\mathcal{A}_n$. Moreover, if $n(a) \neq 0$, then φ_a is an isomorphism.*

Proof. Denote $\mu_1 = n(a)$ and $\mu_2 = n(\widetilde{e}_0) = -\gamma_{n-1}$. Since a and $\widetilde{e}_0$ are pure, we have $a^2 = -n(a) = -\mu_1$ and $(\widetilde{e}_0)^2 = -n(\widetilde{e}_0) = -\mu_2$. The condition $a \in \operatorname{span}(e_0, \widetilde{e}_0)^\perp$ implies that $\widetilde{a} \in \operatorname{span}(e_0, \widetilde{e}_0)^\perp$. By Lemma 3.2(3), $\widetilde{a} \perp a$, whence $a, \widetilde{e}_0, \widetilde{a}$ anticommute pairwise. It remains to observe that $\widetilde{a}a = -\widetilde{aa} = \widetilde{\mu_1 e_0} = \mu_1\widetilde{e}_0$ by Lemma 3.2(2), $\widetilde{a}\widetilde{e}_0 = \widetilde{\widetilde{a}} = \gamma_{n-1}a = -\mu_2 a$ by Lemma 3.2(1), and $(\widetilde{a})^2 = -n(\widetilde{a}) = -n(a\widetilde{e}_0) = -n(a)n(\widetilde{e}_0) = -\mu_1\mu_2$ by Lemma 3.1. Thus, we have the following multiplication table in $\mathbb{H}_a$:

$\times$	e_0	a	$\widetilde{e}_0$	$\widetilde{a}$
e_0	e_0	a	$\widetilde{e}_0$	$\widetilde{a}$
a	a	$-\mu_1$	$\widetilde{a}$	$-\mu_1\widetilde{e}_0$
$\widetilde{e}_0$	$\widetilde{e}_0$	$-\widetilde{a}$	$-\mu_2$	$\mu_2 a$
$\widetilde{a}$	$\widetilde{a}$	$\mu_1\widetilde{e}_0$	$-\mu_2 a$	$-\mu_1\mu_2$

Table 1. Multiplication table in $\mathbb{H}_a$.

Now we may define $\varphi_a : \mathcal{A}_2\{-\mu_1,-\mu_2\}\to\mathbb{H}_a$ by setting $\varphi_a(e_0)=e_0$, $\varphi_a(e_1)=a$, $\varphi_a(e_2)=\widetilde{e}_0$, and $\varphi_a(e_3) = \widetilde{a}$. Table 1 coincides with the multiplication table of $\mathcal{A}_2\{-\mu_1,-\mu_2\}$, and e_0, e_1, e_2, e_3 form a basis in $\mathcal{A}_2\{-\mu_1,-\mu_2\}$. Thus, every nontrivial relation in $\mathcal{A}_2\{-\mu_1,-\mu_2\}$ is preserved under φ_a, and φ_a actually is a homomorphism. Clearly, φ_a is surjective because $\mathbb{H}_a = \operatorname{span}(e_0, a, \widetilde{e}_0, \widetilde{a})$.

In order to prove the last assertion of the lemma, we use the fact that $e_0, a, \widetilde{e}_0, \widetilde{a}$ form an orthogonal system with respect to the symmetric bilinear form $\langle\cdot,\cdot\rangle$. If $n(a) \neq 0$, then $n(\widetilde{a}) = n(a)n(\widetilde{e}_0) \neq 0$, whence $e_0, a, \widetilde{e}_0, \widetilde{a}$ are linearly independent. Thus, φ_a is an isomorphism. □

Remark 3.5. Note that if $n(a) = 0$, then φ_a in Lemma 3.4 can have a nontrivial kernel even for $a \neq 0$ because it is possible that $a = \widetilde{a}$.

Lemma 3.4 immediately implies the well-known assertion on the strong alternativity of the element $\widetilde{e}_0$, see [7, Lemma 1.2].

Corollary 3.6. *The element $\widetilde{e}_0$ is strongly alternative in $\mathcal{A}_n$.*

Proof. Let $a \in \mathcal{A}_n$ and let a' be the orthogonal projection of a onto $\operatorname{span}(e_0, \widetilde{e}_0)^\perp$. By Lemma 3.4, a' and $\widetilde{e}_0$ generate an associative subalgebra $\mathbb{H}_{a'} \subset \mathcal{A}_n$. Clearly, $a \in \mathbb{H}_{a'}$, whence $[a, a, \widetilde{e}_0] = [\widetilde{e}_0, \widetilde{e}_0, a] = 0$. □

Theorem 3.7. *Let $a, b \in \mathcal{A}_n$ alternate strongly, $t(a) = t(b) = 0$. Then $\mathbb{H}_{a,b} = \operatorname{span}(e_0, a, b, ab)$ is an associative subalgebra in $\mathcal{A}_n$ closed with respect to involution, and multiplication of its elements is described by Table* **??**, *where $\mu_1 = n(a)$, $\mu_2 = n(b)$, and $k = -2\langle a, b\rangle$. In the case where $k = 0$, there exists a surjective homomorphism $\psi_{a,b} : \mathcal{A}_2\{-n(a), -n(b)\} \to \mathbb{H}_{a,b}$. Moreover, if $n(a) \neq 0$ and $n(b) \neq 0$, then $\psi_{a,b}$ is an isomorphism.*

Proof. Since a and b are pure elements, we have $a^2 = -n(a) = -\mu_1$ and $b^2 = -n(b) = -\mu_2$. From $k = -2\langle a, b\rangle = -t(a\bar{b}) = t(ab)$ it follows that $ba = \bar{b}\bar{a} = \overline{ab} = k - ab$.

Since the elements a and b alternate strongly, we have $a(ab) = a^2b = -\mu_1 b$, $b(ab) = b(k-ba) = kb-b^2a = kb+\mu_2 a$, $(ab)a = (k-ba)a = ka-ba^2 = ka+\mu_1 b$, and $(ab)b = ab^2 = -\mu_2 a$. Finally, Lemma 3.1 implies that $n(ab) = n(a)n(b) = \mu_1\mu_2$, whence $(ab)^2 = (ab)(k - \overline{ab}) = kab - n(ab) = kab - \mu_1\mu_2$. Thus, we have the following multiplication table in $\mathbb{H}_{a,b}$:

Table 2. Multiplication table in $\mathbb{H}_{a,b}$.

$\times$	e_0	a	b	ab
e_0	e_0	a	b	ab
a	a	$-\mu_1$	ab	$-\mu_1 b$
b	b	$k - ab$	$-\mu_2$	$kb + \mu_2 a$
ab	ab	$ka + \mu_1 b$	$-\mu_2 a$	$kab - \mu_1\mu_2$

If $k = 0$, that is, $a \perp b$, then we may define $\psi_{a,b} : \mathcal{A}_2\{-\mu_1, -\mu_2\} \to \mathbb{H}_{a,b}$ by the relations $\psi_{a,b}(e_0) = e_0$, $\psi_{a,b}(e_1) = a$, $\psi_{a,b}(e_2) = b$, and $\psi_{a,b}(e_3) = ab$. Then the associativity of $\mathbb{H}_{a,b}$ follows from the associativity of $\mathcal{A}_2\{-\mu_1, -\mu_2\}$, and the rest of the proof is similar to that of Lemma 3.4.

Now assume that $k \neq 0$. First consider the case where $n(a) \neq 0$ or $n(b) \neq 0$. Without loss of generality, we may assume that $n(a) \neq 0$. Let $b' = b - qa$, where $q = \frac{\langle a,b\rangle}{n(a)}$. Then $a \perp b'$ and $\mathbb{H}_{a,b} = \operatorname{span}(e_0, a, b, ab) = \operatorname{span}(e_0, a, b', ab') = \mathbb{H}_{a,b'}$. Moreover, $[a, a, b'] = [a, a, b] - q[a, a, a] = 0$ and $[b', b', a] = [b, b, a] - q[a, b, a] - q[b, a, a] + q^2[a, a, a] = 0$, that is, a and b' alternate strongly. Consequently, $\mathbb{H}_{a,b'}$ is an associative subalgebra in $\mathcal{A}_n$ closed with respect to involution, as desired.

Now let $n(a) = n(b) = 0$. Consider the elements $x = e_1 + e_2$, $y = -\frac{k}{2}(e_1 + e_3)$, and $xy = \frac{k}{2}(e_0 + e_1 + e_2 + e_3)$ in $\mathcal{A}_2\{-1, 1\}$. Then e_0, x, y, xy are linearly independent in $\mathcal{A}_2\{-1, 1\}$, and their products satisfy the same relations as the products of e_0, a, b, ab because $\mathcal{A}_2\{-1, 1\}$ is associative, $n(x) = n(y) = 0$, and $t(xy) = k$. Thus, we may define the homomorphism $\theta_{a,b} : \mathcal{A}_2\{-1, 1\} \to \mathbb{H}_{a,b}$ by the relations $\theta_{a,b}(e_0) = e_0$, $\theta_{a,b}(x) = a$, $\theta_{a,b}(y) = b$, and $\theta_{a,b}(xy) = ab$, and the associativity of $\mathbb{H}_{a,b}$ follows from that of $\mathcal{A}_2\{-1, 1\}$. □

Corollary 3.8. *Let $a, b \in \mathcal{A}_n$ alternate strongly. Then the set* $\operatorname{span}(e_0, a, b, ab)$ *is an associative subalgebra in $\mathcal{A}_n$ closed with respect to involution.*

Proof. Let $a' = \mathfrak{Im}(a)$ and let $b' = \mathfrak{Im}(b)$. It is clear that a' and b' alternate strongly and $\operatorname{span}(e_0, a, b, ab) = \operatorname{span}(e_0, a', b', a'b')$. Then the desired assertion immediately follows from Theorem 3.7 applied to the elements a' and b'. □

The theorem below is a generalization of Theorem 5.1 in [15].

Theorem 3.9. *Let elements $a, b \in \mathcal{A}_n$ be doubly pure, $b \perp \operatorname{span}(a, \widetilde{a})$. Also let a alternate strongly with b. Denote $\mathbb{O}_{a,b} = \operatorname{span}(e_0, a, b, ab, \widetilde{e}_0, \widetilde{a}, \widetilde{b}, \widetilde{ab})$. Then there exists a surjective homomorphism $\varphi_{a,b} : \mathcal{A}_3\{-n(a), -n(b), \gamma_{n-1}\} \to \mathbb{O}_{a,b}$, whence $\mathbb{O}_{a,b}$ is an alternative subalgebra in $\mathcal{A}_n$. Moreover, if $n(a) \neq 0$ or $n(b) \neq 0$, then $\varphi_{a,b}$ is an isomorphism.*

Proof. Denote $\mu_1 = n(a)$, $\mu_2 = n(b)$, and $\mu_3 = n(\widetilde{e}_0) = -\gamma_{n-1}$. Since a, b, and $\widetilde{e}_0$ are pure, we have $a^2 = -n(a) = -\mu_1$, $b^2 = -n(b) = -\mu_2$, and $(\widetilde{e}_0)^2 = -n(\widetilde{e}_0) = -\mu_3$. Using Lemma 2.12(2), we can show that from $a \perp \operatorname{span}(e_0, \widetilde{e}_0)$ and $b \perp \operatorname{span}(e_0, a, \widetilde{e}_0, \widetilde{a})$ it follows that $\{e_0, a, b, ab, \widetilde{e}_0, \widetilde{a}, \widetilde{b}, \widetilde{ab}\}$ is an orthogonal system with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$. Then, by Lemma 2.18, $a, b, ab, \widetilde{e}_0, \widetilde{a}, \widetilde{b}, \widetilde{ab}$ anticommute pairwise. Note that the element ab also is doubly pure.

By Theorem 3.7, there exists a surjective homomorphism $\psi_{a,b} : \mathcal{A}_2\{-\mu_1, -\mu_2\} \to \mathbb{H}_{a,b}$. Now we extend it to $\varphi_{a,b} : \mathcal{A}_3\{-\mu_1, -\mu_2, -\mu_3\} \to \mathbb{O}_{a,b}$. We may apply Lemma 3.4 to a, b, and ab separately. Then, by using Lemma 3.2(2), we obtain that $\widetilde{a}b = -\widetilde{ab}$, $\widetilde{a}(ab) = -\widetilde{a(ab)} = \mu_1\widetilde{b}$, $\widetilde{b}a = -\widetilde{ba} = \widetilde{ab}$, $\widetilde{b}(ab) = -\widetilde{b(ab)} = -\mu_2\widetilde{a}$, $\widetilde{ab} \cdot a = -\widetilde{(ab)a} = -\mu_1\widetilde{b}$, and $\widetilde{ab} \cdot b = -\widetilde{(ab)b} = \mu_2\widetilde{a}$. By Lemma 3.2(5), we obtain that $\widetilde{b}\widetilde{a} = -\gamma_{n-1}ab = \mu_3 ab$, $\widetilde{ab} \cdot \widetilde{a} = -\gamma_{n-1}a(ab) = -\mu_1\mu_3 b$, and $\widetilde{ab} \cdot \widetilde{b} = -\gamma_{n-1}b(ab) = \mu_2\mu_3 a$. Thus, the multiplication table in $\mathbb{O}_{a,b}$ is provided by Table 3.

Now we may define $\varphi_{a,b}$ by setting $\varphi_{a,b}((e_j, 0)) = \psi_{a,b}(e_j)$ and $\varphi_{a,b}((0, e_j)) = \widetilde{\psi_{a,b}(e_j)}$ for all $0 \leq j \leq 3$. The rest of the proof is similar to that of Lemma 3.4. □

4. Zero divisors with alternativity conditions on their components

This section is devoted to studying the zero divisors in arbitrary Cayley–Dickson algebras whose components satisfy some additional conditions on the norm and alternativity. We generalize and strengthen the results obtained in Sec. 3 of the author's paper [22] for the real Cayley–Dickson algebras.

Table 3. Multiplication table in $\mathbb{O}_{a,b}$.

$\times$	e_0	a	b	ab	$\widetilde{e}_0$	$\widetilde{a}$	$\widetilde{b}$	$\widetilde{ab}$
e_0	e_0	a	b	ab	$\widetilde{e}_0$	$\widetilde{a}$	$\widetilde{b}$	$\widetilde{ab}$
a	a	$-\mu_1$	ab	$-\mu_1 b$	$\widetilde{a}$	$-\mu_1\widetilde{e}_0$	$-\widetilde{ab}$	$\mu_1\widetilde{b}$
b	b	$-ab$	$-\mu_2$	$\mu_2 a$	$\widetilde{b}$	$\widetilde{ab}$	$-\mu_2\widetilde{e}_0$	$-\mu_2\widetilde{a}$
ab	ab	$\mu_1 b$	$-\mu_2 a$	$-\mu_1\mu_2$	$\widetilde{ab}$	$-\mu_1\widetilde{b}$	$\mu_2\widetilde{a}$	$-\mu_1\mu_2\widetilde{e}_0$
$\widetilde{e}_0$	$\widetilde{e}_0$	$-\widetilde{a}$	$-\widetilde{b}$	$-\widetilde{ab}$	$-\mu_3$	$\mu_3 a$	$\mu_3 b$	$\mu_3 ab$
$\widetilde{a}$	$\widetilde{a}$	$\mu_1\widetilde{e}_0$	$-\widetilde{ab}$	$\mu_1\widetilde{b}$	$-\mu_3 a$	$-\mu_1\mu_3$	$-\mu_3 ab$	$\mu_1\mu_3 b$
$\widetilde{b}$	$\widetilde{b}$	$\widetilde{ab}$	$\mu_2\widetilde{e}_0$	$-\mu_2\widetilde{a}$	$-\mu_3 b$	$\mu_3 ab$	$-\mu_2\mu_3$	$-\mu_2\mu_3 a$
$\widetilde{ab}$	$\widetilde{ab}$	$-\mu_1\widetilde{b}$	$\mu_2\widetilde{a}$	$\mu_1\mu_2\widetilde{e}_0$	$-\mu_3 ab$	$-\mu_1\mu_3 b$	$\mu_2\mu_3 a$	$-\mu_1\mu_2\mu_3$

4.1. Hexagons in zero-divisor graphs

Lemma 4.1. *Let $(a,b),(c,d)\in\mathcal{A}_{n+1}$ and let the elements $c,d\in\mathcal{A}_n$ alternate (not strongly) with $a,b\in\mathcal{A}_n$. Also assume that $n(c)-\chi\gamma_n n(d)=\chi n(c)-\gamma_n n(d)=0$ for a certain $\chi\in\mathbb{F}$. Then*

(1) *$(a,b)(c,d)=0$ implies $(c,d)(\overline{ac},-\chi da)=0$;*
(2) *$(c,d)(a,b)=0$ implies $(\overline{ca},-\chi d\bar{a})(c,d)=0$.*

Proof. (1) We have the following string of equalities:

$$\begin{aligned}(c,d)(\overline{ac},-\chi da)&=\big(c(\overline{ac})+\gamma_n(\overline{-\chi da})d,(-\chi da)c+d(ac)\big)\\&=\big(c(\bar{c}\bar{a})-\chi\gamma_n(\bar{a}\bar{d})d,\chi(b\bar{c})c-\gamma_n d(\bar{d}b)\big)\\&=\big((c\bar{c})\bar{a}-\chi\gamma_n\bar{a}(\bar{d}d),\chi b(\bar{c}c)-\gamma_n(d\bar{d})b\big)\\&=((n(c)-\chi\gamma_n n(d))\bar{a},(\chi n(c)-\gamma_n n(d))b)=0.\end{aligned}$$

(2) Similarly,

$$\begin{aligned}(\overline{ca},-\chi d\bar{a})(c,d)&=\big((\overline{ca})c+\gamma_n\bar{d}(-\chi d\bar{a}),d(\overline{ca})+(-\chi d\bar{a})\bar{c}\big)\\&=\Big((\bar{a}\bar{c})c-\chi\gamma_n\bar{d}(d\bar{a}),d(\overline{-\gamma_n\bar{b}d})+\chi(bc)\bar{c}\Big)\\&=\big(\bar{a}(\bar{c}c)-\chi\gamma_n(\bar{d}d)\bar{a},-\gamma_n(d\bar{d})b+\chi b(c\bar{c})\big)\\&=((n(c)-\chi\gamma_n n(d))\bar{a},(\chi n(c)-\gamma_n n(d))b)=0.\end{aligned}$$ ☐

Remark 4.2. If, in Lemma 4.1, $n(c)=n(d)=0$, then one can take an arbitrary $\chi\in\mathbb{F}$. Otherwise we immediately obtain

$$\begin{cases}n(c)=\pm\gamma_n n(d)\neq 0;\\ \chi=\dfrac{n(c)}{\gamma_n n(d)}=\dfrac{\gamma_n n(d)}{n(c)}=\pm 1.\end{cases}\tag{$*$}$$

Condition $(*)$ is fulfilled automatically whenever $\mathcal{A}_{n+1}$ is a real algebra of the main sequence, see [14, pp. 25–27], or $\mathcal{A}_{n+1}$ is a real Cayley–Dickson split-algebra, see [10, Lemma 4.1]. One can verify that the proofs in those papers only require that $c,d\in\mathcal{A}_n$ alternate (not strongly) with $a,b\in\mathcal{A}_n$, and then $n(c)=n(d)$. Thus, the values of χ are equal to -1 and 1, respectively. From Lemma 4.6 it follows that condition $(*)$ also is satisfied if $\mathcal{A}_n$ is a Cayley–Dickson algebra with anisotropic norm over an arbitrary field $\mathbb{F}$, $\operatorname{char}\mathbb{F}\neq 2$. However, in the general case, condition $(*)$ is not necessarily fulfilled, see [10, Example 4.17] and Example 6.3 below.

Notation 4.3. Let $(a,b) \in \mathcal{A}_{n+1}$ and let $n(a) \neq 0$. Then the value $\chi((a,b)) = \dfrac{\gamma_n n(b)}{n(a)}$ is called the *characteristic* of (a,b).

Remark 4.4. Note that we could define the characteristic as the reciprocal of $\chi((a,b))$, i.e., $\frac{n(a)}{\gamma_n n(b)}$. Then the condition that $n(a) \neq 0$ should be replaced by $n(b) \neq 0$. Most of the results of this section can readily be transferred to the case of this modified definition. In particular, in Lemma 4.19, the element c should be expressed in terms of d, not vice versa.

Proposition 4.5. *If $(x,y) \in \mathcal{A}_{n+1}$ is pure and $\chi((x,y)) = 1$, then (x,y) is (strongly) orthogonal to itself.*

Proof. By definition, $n((x,y)) = n(x) - \gamma_n n(y) = \gamma_n n(y) - \gamma_n n(y) = 0$, whence $(x,y)(x,y) = -(x,y)\overline{(x,y)} = -n((x,y)) = 0$. □

Lemma 4.6. *Let elements $c, d \in \mathcal{A}_n$ alternate with $a, b \in \mathcal{A}_n$ and let $(a,b)(c,d) = 0$ or $(c,d)(a,b) = 0$ in $\mathcal{A}_{n+1}$. Assume that $n(a) \neq 0$ or $n(b) \neq 0$ and that $n(c) \neq 0$ or $n(d) \neq 0$. Then $\chi = \chi((a,b)) = \chi((c,d)) = \pm 1$ and, moreover, $\chi((\overline{ac}, -\chi da)) = \chi((\overline{ca}, -\chi d\bar{a})) = \chi$. In other terms, the elements (a,b), (c,d), $(\overline{ac}, -\chi da)$, and $(\overline{ca}, -\chi d\bar{a})$ satisfy condition $(*)$ with the same value of χ.*

Proof. Without loss of generality, we may assume that $(a,b)(c,d) = (ac + \gamma_n \bar{d}b, da + b\bar{c}) = 0$; the case where $(c,d)(a,b) = 0$ is considered similarly. By Lemma 3.1,

$$\begin{aligned}
n(a)n(c) &= n(ac) = n(-\gamma_n \bar{d}b) = \gamma_n^2 n(\bar{d}b) = \gamma_n^2 n(b)n(\bar{d}) = \gamma_n^2 n(b)n(d),\\
n(a)n(d) &= n(da) = n(-b\bar{c}) = n(b\bar{c}) = n(b)n(\bar{c}) = n(b)n(c),\\
(n(c))^2 n(a) &= n(c)(n(a)n(c)) = \gamma_n^2 n(c)(n(b)n(d)) = \gamma_n^2 n(d)(n(b)n(c))\\
&= \gamma_n^2 n(d)(n(a)n(d)) = (\gamma_n n(d))^2 n(a),\\
(n(c))^2 n(b) &= n(c)(n(b)n(c)) = n(c)(n(a)n(d)) = n(d)(n(a)n(c))\\
&= \gamma_n^2 n(d)(n(b)n(d)) = (\gamma_n n(d))^2 n(b).
\end{aligned}$$

From $n(a) \neq 0$ or $n(b) \neq 0$ it follows that $(n(c))^2 = (\gamma_n n(d))^2$, whence $n(c) = \pm\gamma_n n(d) \neq 0$. Similarly, from $n(c) \neq 0$ or $n(d) \neq 0$ it follows that $n(a) = \pm\gamma_n n(b) \neq 0$. Thus, $\chi = \chi((a,b)) = \gamma_n \dfrac{n(b)}{n(a)} = \gamma_n \dfrac{n(d)}{n(c)} = \chi((c,d)) = \pm 1$. Moreover, Lemma 3.1 implies that

$$\chi((\overline{ac}, -\chi da)) = \gamma_n \frac{n(-\chi da)}{n(\overline{ac})} = \gamma_n \frac{n(da)}{n(ac)} = \gamma_n \frac{n(a)n(d)}{n(a)n(c)} = \gamma_n \frac{n(d)}{n(c)} = \chi.$$

The relation $\chi((\overline{ca}, -\chi d\bar{a})) = \chi$ is established similarly. □

Below, in Assertions 4.7–4.12 and in Fig. 1, we assume that $(a,b)(c,d) = 0$ in $\mathcal{A}_{n+1}$ and that the elements $a, b \in \mathcal{A}_n$ alternate strongly with $c, d \in \mathcal{A}_n$, that is, $[x,x,y] = [y,y,x] = 0$ for $x \in \{a,b\}$ and $y \in \{c,d\}$. Everywhere, except for Lemma 4.7, we also assume that (a,b) and (c,d) satisfy condition $(*)$.

Lemma 4.7. *The elements ac and da alternate strongly with a, b, c, d.*

Proof. From $(a,b)(c,d) = (ac + \gamma_n \bar{d}b, da + b\bar{c}) = 0$ it follows that $ac = -\gamma_n \bar{d}b$ and $da = -b\bar{c}$. It remains to apply Corollary 3.8 to a and c, to b and c, to a and d, and to b and d.

Note that this assertion can readily be proved directly:

$$\begin{aligned}
[a,a,ac] &= -[a,\bar{a},ac] = -(a\bar{a})(ac) + a(\bar{a}(ac))\\
&= -(a\bar{a})(ac) + a((\bar{a}a)c) = -n(a)ac + n(a)ac = 0,
\end{aligned}$$

$$[b,b,ac]=[b,b,-\gamma_n\bar{d}b]=\gamma_n[\bar{d}b,b,b]=-\gamma_n[\bar{d}b,\bar{b},b]=0.$$

Similarly, one can show that the elements a,b,c,d alternate with ac and ad. Conversely, we have

$$\begin{aligned}[ac,ac,a]&=-[ac,\overline{ac},a]=-((ac)(\overline{ac}))a+(ac)((\bar{c}\bar{a})a)\\&=-n(ac)a+(ac)(\bar{c}(\bar{a}a))=-n(ac)a+n(a)(ac)\bar{c}\\&=-n(ac)a+n(a)a(c\bar{c})=-n(ac)a+n(a)n(c)a=0\end{aligned}$$

because from Lemma 3.1 it follows that $n(ac)=n(a)n(c)$. Thus, the elements ac and ad alternate with a,b,c, and d. □

Corollary 4.8. *In* $\Gamma_Z(\mathcal{A}_{n+1})$*, there is the following* 6*-cycle:*

$$(a,b)\to(c,d)\to(\overline{ac},-\chi da)\to(a,-b)\to(c,-d)\to(\overline{ac},\chi da)\to(a,b).$$

Proof. By Lemma 4.6, $\chi=\chi((a,b))=\chi((c,d))=\chi((\overline{ac},-\chi da))=\pm1$. On the other hand, by Lemma 4.7, the elements ac,da alternate strongly with a,b,c,d. We obtain the desired cycle by successively applying Lemma 4.1:

- $(a,b)(c,d)=0$ implies $(c,d)(\overline{ac},-\chi da)=0$;
- since $c(\overline{ac})=c(\overline{\bar{c}\bar{a}})=\overline{(c\bar{c})\bar{a}}=n(c)a$ and $-\chi(-\chi da)c=(da)c=(-b\bar{c})c=-b(\bar{c}c)=-n(c)b$, we conclude that the relations $(c,d)(\overline{ac},-\chi da)=0$ and $n(c)\neq0$ imply that $(\overline{ac},-\chi da)(a,-b)=0$;
- since $(\overline{ac})a=\overline{(\bar{c}\bar{a})a}=\overline{\bar{c}(\bar{a}a)}=n(a)c$ and $-\chi(-b)(\overline{ac})=\chi b(\overline{-\gamma_n\bar{d}b})=-\chi\gamma_n b(\bar{b}d)=-\chi\gamma_n(b\bar{b})d=-\chi\gamma_n n(b)d=-n(a)d$, we conclude that $(\overline{ac},-\chi da)(a,-b)=0$ and $n(a)\neq0$ imply $(a,-b)(c,-d)=0$;
- from $(a,-b)(c,-d)=0$ it follows that $(c,-d)(\overline{ac},\chi da)=0$;
- $(c,-d)(\overline{ac},\chi da)=0$ implies $(\overline{ac},\chi da)(a,b)=0$. □

Proposition 4.9. *Let* $(x,y)(z,w)=0$ *in* $\mathcal{A}_{n+1}$*. Then*

$$(\bar{x},\bar{y})(\gamma_n\bar{w},\bar{z})=(\gamma_n\bar{y},\bar{x})(\gamma_n w,z)=(\gamma_n y,x)(\bar{z},\bar{w})=0.$$

Proof. By assumption, $(x,y)(z,w)=(xz+\gamma_n\bar{w}y,wx+y\bar{z})=0$. Therefore,

$$\begin{aligned}(\bar{x},\bar{y})(\gamma_n\bar{w},\bar{z})&=(\gamma_n\bar{x}\bar{w}+\gamma_n z\bar{y},\bar{z}\bar{x}+\gamma_n\bar{y}w)=(\gamma_n(\overline{wx+y\bar{z}}),\overline{xz+\gamma_n\bar{w}y})=0,\\(\gamma_n\bar{y},\bar{x})(\gamma_n w,z)&=(\gamma_n^2\bar{y}w+\gamma_n\bar{z}\bar{x},\gamma_n z\bar{y}+\gamma_n\bar{x}\bar{w})=\gamma_n(\overline{xz+\gamma_n\bar{w}y},\overline{wx+y\bar{z}})=0,\\(\gamma_n y,x)(\bar{z},\bar{w})&=(\gamma_n y\bar{z}+\gamma_n wx,\gamma_n\bar{w}y+xz)=0.\end{aligned}$$

□

Corollary 4.10. *In* $\Gamma_Z(\mathcal{A}_{n+1})$ *there are the following* 6*-cycles:*

$$(\bar{a},\bar{b})\to(\gamma_n\bar{d},\bar{c})\to(-\chi\gamma_n da,\overline{ac})\to(\bar{a},-\bar{b})\to(\gamma_n\bar{d},-\bar{c})\to(\chi\gamma_n da,\overline{ac})\to(\bar{a},\bar{b}),$$

$$(\gamma_n b,a)\to(\bar{c},\bar{d})\to(-\chi\gamma_n\overline{da},ac)\to(\gamma_n b,-a)\to(\bar{c},-\bar{d})\to(\chi\gamma_n\overline{da},ac)\to(\gamma_n b,a),$$

$$(\gamma_n\bar{b},\bar{a})\to(\gamma_n d,c)\to(ac,-\chi\overline{da})\to(\gamma_n\bar{b},-\bar{a})\to(\gamma_n d,-c)\to(ac,\chi\overline{da})\to(\gamma_n\bar{b},\bar{a}).$$

Proof. The assertions immediately follow from Corollary 4.8 and Proposition 4.9. □

Remark 4.11. The cycles in Corollary 4.10 can also be obtained by using Corollary 4.8 if the pairs $(\bar{a},\bar{b})$ and $(\gamma_n\bar{d},\bar{c})$, $(\gamma_n b,a)$ and $(\bar{c},\bar{d})$, $(\gamma_n\bar{b},\bar{a})$ and $(\gamma_n d,c)$ are chosen as the initial pairs of zero divisors.

Description 4.12. By using Corollaries 4.8 and 4.10, we obtain subgraphs of $\Gamma_Z(\mathcal{A}_{n+1})$, which will be called *hexagons*. They are shown in Fig. 1 below.

By combining the results of Lemmas 4.6 and 4.7 and Corollaries 4.8 and 4.10, we obtain the following theorem.

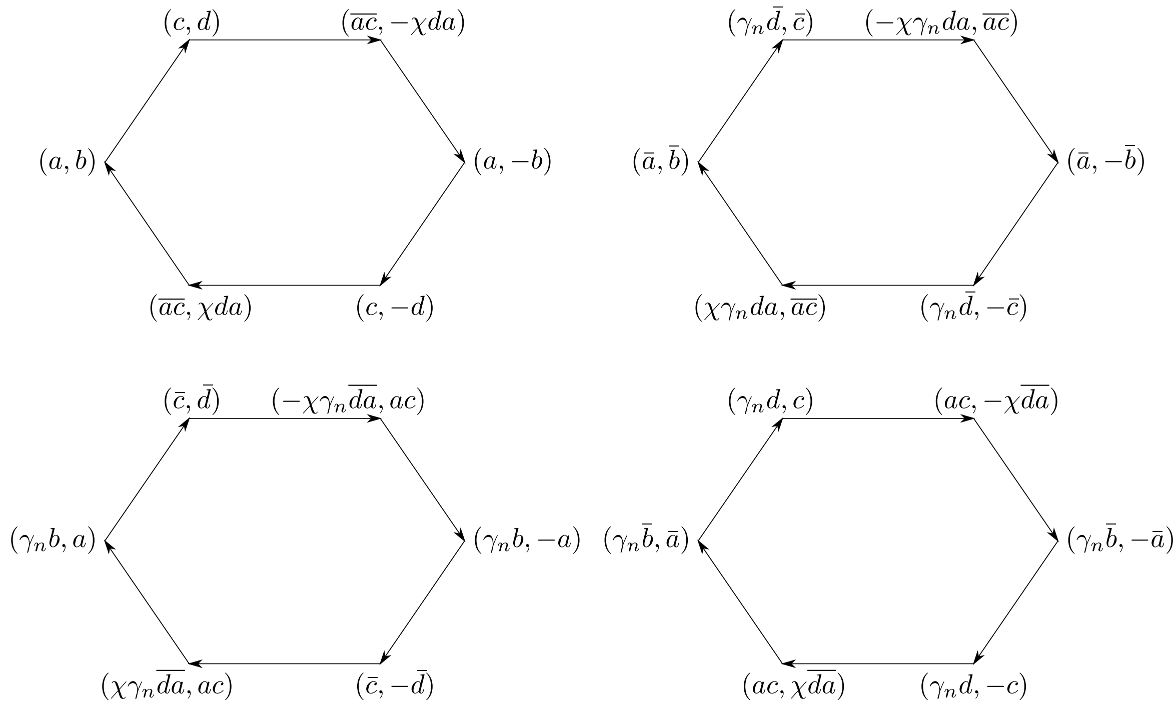

Fig. 1. Hexagons.

Theorem 4.13. *Let elements $a, b \in \mathcal{A}_n$ alternate strongly with $c, d \in \mathcal{A}_n$ and let $(a, b)(c, d) = 0$ in $\mathcal{A}_{n+1}$. Then the following assertions are valid:*

(1) *The elements ac, da alternate strongly with each of the elements a, b, c, d.*
(2) *Let $n(a) \neq 0$ or $n(b) \neq 0$, and also let $n(c) \neq 0$ or $n(d) \neq 0$. Then (a, b), (c, d), and $(\overline{ac}, -\chi da)$ satisfy condition $(*)$ with the same value of χ.*
(3) *In this case, in $\Gamma_Z(\mathcal{A}_{n+1})$ there are the following 6-cycles:*

$$(a, b) \to (c, d) \to (\overline{ac}, -\chi da) \to (a, -b) \to (c, -d) \to (\overline{ac}, \chi da) \to (a, b),$$
$$(\bar{a}, \bar{b}) \to (\gamma_n \bar{d}, \bar{c}) \to (-\chi\gamma_n da, \overline{ac}) \to (\bar{a}, -\bar{b}) \to (\gamma_n \bar{d}, -\bar{c}) \to (\chi\gamma_n da, \overline{ac}) \to (\bar{a}, \bar{b}),$$
$$(\gamma_n b, a) \to (\bar{c}, \bar{d}) \to (-\chi\gamma_n \overline{da}, ac) \to (\gamma_n b, -a) \to (\bar{c}, -\bar{d}) \to (\chi\gamma_n \overline{da}, ac) \to (\gamma_n b, a),$$
$$(\gamma_n \bar{b}, \bar{a}) \to (\gamma_n d, c) \to (ac, -\chi\overline{da}) \to (\gamma_n \bar{b}, -\bar{a}) \to (\gamma_n d, -c) \to (ac, \chi\overline{da}) \to (\gamma_n \bar{b}, \bar{a}).$$

4.2. Doubly alternative zero divisors

Notation 4.14. Let $a \in \mathcal{A}_n$. The mappings $L_a, R_a : \mathcal{A}_n \to \mathcal{A}_n$ are defined as follows: for all $x \in \mathcal{A}_n$, we set

$$L_a(x) = ax, \quad R_a(x) = xa. \tag{4.1}$$

Obviously, L_a and R_a are linear operators on the 2^n-dimensional linear space $\mathcal{A}_n$.

Lemma 4.15. *Let $a \in \mathcal{A}_n$. Then* $\dim(\operatorname{Ker} L_a) = \dim(\operatorname{Ker} R_a)$*, that is,* $\dim(l.\operatorname{Ann}_{\mathcal{A}}(a)) = \dim(r.\operatorname{Ann}_{\mathcal{A}}(a))$.

Proof. By Lemma 2.12(2), the linear operators L_a and $L_{\bar{a}}$ are conjugate with respect to the symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ in the sense that for all $x, y \in \mathcal{A}_n$ we have $\langle L_a(x), y \rangle = \langle x, L_{\bar{a}}(y) \rangle$. Thus, $\dim(\operatorname{Ker} L_a) = \dim(\operatorname{Ker} L_{\bar{a}})$. On the other hand, $L_{\bar{a}}(\bar{x}) = \bar{a}\bar{x} = \overline{xa} = \overline{R_a(x)}$ for all $x \in \mathcal{A}_n$, whence $\dim(\operatorname{Ker} L_{\bar{a}}) = \dim(\operatorname{Ker} R_a)$. This completes the proof. $\square$

Corollary 4.16. $Z(\mathcal{A}_n) = Z_{LR}(\mathcal{A}_n)$.

Proof. Let $a \in \mathcal{A}_n$, $a \neq 0$. Then, by Lemma 4.15, $\operatorname{Ker} L_a \neq \{0\}$ if and only if $\operatorname{Ker} R_a \neq \{0\}$. In other terms, a is a right zero divisor if and only if a is a left zero divisor. Thus, the sets of left and right zero divisors in $\mathcal{A}_n$ coincide, that is, $Z(\mathcal{A}_n) = Z_{LR}(\mathcal{A}_n)$. □

Thus, in the case of Cayley–Dickson algebras, all zero divisors are two-sided zero divisors. The following proposition describes the interrelation between the orthogonality and zero-divisor graphs of these algebras. Note that in [22] this assertion is stated for the real Cayley–Dickson algebras only; however, its proof remains valid in the case of an arbitrary field as well.

Proposition 4.17 ([22, Proposition 3.10])**.** *An edge $([a],[b])$ in $\Gamma_Z(\mathcal{A}_n)$ is an edge in $\Gamma_O(\mathcal{A}_n)$ if and only if one of the following conditions is fulfilled:*
(1) $[b] = [\bar{a}]$ *and* $n(a) = 0$;
(2) $t(a) = t(b) = 0$.

From Proposition 4.17 it follows that any zero divisor $a \in \mathcal{A}_n$ with a nontrivial orthogonalizer either is pure or has zero norm. If a is not pure, then its connected component in $\Gamma_O(\mathcal{A}_n)$ consists of two vertices $[a]$ and $[\bar{a}]$. Hence, in the context of orthogonality graphs, we are only interested in pure zero divisors.

Now consider the zero divisors $(a,b) \in \mathcal{A}_{n+1}$ such that both a and b are alternative elements in $\mathcal{A}_n$.

Definition 4.18. The set of *doubly alternative elements* of $\mathcal{A}_{n+1}$ is the set

$$DA(\mathcal{A}_{n+1}) = \{(a,b) \in \mathcal{A}_{n+1} \mid \text{ both } a \text{ and } b \text{ are alternative in } \mathcal{A}_n\}.$$

An algebra $\mathcal{A}_n$ is alternative only for $n \leq 3$. Therefore, all elements of $\mathcal{A}_{n+1}$ are doubly alternative if and only if $n \leq 3$. Note that doubly alternative elements need not be alternative, see [15, Theorem 3.3] and [10, Lemma 4.16].

We also note that, in accordance with [10, Example 4.17], doubly alternative elements need not satisfy condition $(*)$ even in the case where both of their components have nonzero norms. In other terms, their characteristic χ can be well defined but not equal to 0 or ± 1. However, by Lemma 4.6, if the left or right annihilator of a doubly alternative element (a,b) contains an element (c,d) and if $n(a) \neq 0$ or $n(b) \neq 0$ and also $n(c) \neq 0$ or $n(d) \neq 0$, then (a,b) satisfies condition $(*)$.

Lemma 4.19. *Let $(a,b) \in DA(\mathcal{A}_{n+1})$ and let $n(a) \neq 0$. Denote $\chi = \chi((a,b))$. Then*

$$l.\operatorname{Ann}_{\mathcal{A}_{n+1}}((a,b)) = \left\{ \left(c, -\frac{(bc)a}{n(a)} \right) \;\middle|\; b(ca) = \chi(bc)a \right\},$$

$$r.\operatorname{Ann}_{\mathcal{A}_{n+1}}((a,b)) = \left\{ \left(c, -\frac{(b\bar{c})\bar{a}}{n(a)} \right) \;\middle|\; b(\bar{c}\bar{a}) = \chi(b\bar{c})\bar{a} \right\}.$$

Moreover, if $t((a,b)) = 0$, then

$$O_{\mathcal{A}_{n+1}}((a,b)) = \left\{ \left(c, -\frac{(bc)a}{n(a)} \right) \;\middle|\; t(c) = 0,\ b(ca) = \chi(bc)a \right\}.$$

Proof. First consider $l.\operatorname{Ann}_{\mathcal{A}_{n+1}}((a,b))$. Let $(c,d) \in \mathcal{A}_{n+1}$ be such that $(c,d)(a,b) = (ca + \gamma_n \bar{b}d, bc + d\bar{a}) = 0$. Then $bc + d\bar{a} = 0$, whence $n(a)d = d(\bar{a}a) = (d\bar{a})a = -(bc)a$. Moreover, $ca + \gamma_n \bar{b}d = 0$ and $\chi n(a) = \gamma_n n(b)$, whence $b(ca) = -\gamma_n b(\bar{b}d) = -\gamma_n (b\bar{b})d = -\gamma_n n(b)d = -\chi n(a)d = \chi(bc)a$. Arguing in the reverse direction, we conclude that for any $c \in \mathcal{A}_n$ such that $b(ca) = \chi(bc)a$ we have $\left(c, -\frac{(bc)a}{n(a)}\right) \in l.\operatorname{Ann}_{\mathcal{A}_{n+1}}((a,b))$. Thus, the converse also is true.

Proceed to $r.\operatorname{Ann}_{\mathcal{A}_{n+1}}((a,b))$. Let $(c,d) \in \mathcal{A}_{n+1}$ be such that $(a,b)(c,d) = (ac + \gamma_n \bar{d}b, da + b\bar{c}) = 0$. Then $da + b\bar{c} = 0$, whence $n(a)d = d(a\bar{a}) = (da)\bar{a} = -(b\bar{c})\bar{a}$. Since $ac + \gamma_n \bar{d}b = 0$ and $\chi n(a) = \gamma_n n(b)$, we have $b(\bar{c}\bar{a}) = b(\overline{ac}) = b(\overline{-\gamma_n \bar{d}b}) = -\gamma_n b(\bar{b}d) = -\gamma_n(b\bar{b})d = -\gamma_n n(b)d = -\chi n(a)d = \chi(b\bar{c})\bar{a}$. Clearly, in this case, the converse also is true, that is, for any $c \in \mathcal{A}_n$ such that $b(\bar{c}\bar{a}) = \chi(b\bar{c})\bar{a}$ we have $\left(c, -\frac{(b\bar{c})\bar{a}}{n(a)}\right) \in r.\operatorname{Ann}_{\mathcal{A}_{n+1}}((a,b))$.

Finally, the formula for $O_{\mathcal{A}_{n+1}}((a,b))$ follows from Proposition 4.17. □

Corollary 4.20. *Let $a, b, c, d, ac, ad \in \mathcal{A}_n$ alternate strongly pairwise, $(a,b)(c,d) = 0$ in $\mathcal{A}_{n+1}$, (a,b) and (c,d) satisfy condition $(*)$. Also let $t(a) = t(c) = t(ac) = 0$. Then the left upper hexagon in Fig. 1 is an undirected hexagon in $\Gamma_O(\mathcal{A}_{n+1})$, and in $\Gamma_O(\mathcal{A}_{n+1})$ there are no other edges connecting its vertices, that is, there are no chords in it.*

Proof. Since $t(a) = t(c) = t(ac) = 0$, from Proposition 4.17 it follows that this hexagon is a hexagon not only in $\Gamma_Z(\mathcal{A}_{n+1})$ but also in $\Gamma_O(\mathcal{A}_{n+1})$.

Now let (x,y) and (z,w) be two of its vertices, possibly coinciding. First show that (x,y) cannot be orthogonal to both (z,w) and $(z,-w)$. Suppose the contrary. Then, as in the proof of Lemma 4.19, $(x,y)(z,w) = 0$ implies $w = -\frac{(yz)x}{n(x)}$, and $(x,y)(z,-w) = 0$ implies $-w = -\frac{(yz)x}{n(x)}$, whence $w = -w$. But $w \neq 0$, a contradiction.

Let $t(x) = 0$. In general, it is possible that $(x,y)(x,-y) = (-n(x) - \gamma_n n(y), -2yx) = 0$, i.e., (x,y) and $(x,-y)$ are orthogonal. However, if x alternates strongly with y and (x,y) satisfies condition $(*)$, then (x,y) cannot be orthogonal to $(x,-y)$. Indeed, by Lemma 3.1, $n(yx) = n(y)n(x) \neq 0$. Thus, $yx \neq 0$, implying that $(x,y)(x,-y) \neq 0$. □

The proof of the following lemma uses arguments from [14, p. 21].

Lemma 4.21. *Let elements $a, b, c \in \mathcal{A}_n$ satisfy the relations $t(a) = t(b) = 0$, $[a,b,b] = 0$, and $b = [a,c,b]$. Then $n(b) = 0$.*

Proof. Consider the mapping $S : \mathcal{A}_n \to \mathcal{A}_n$ defined by $S(x) = [a,x,b]$ for all $x \in \mathcal{A}_n$. Then $S = R_b L_a - L_a R_b$, where L_a and R_b are defined in (4.1). The linear operators L_a and R_b are skew-symmetric with respect to the symmetric bilinear form $\langle \cdot\,, \cdot \rangle$ because, by Lemma 2.12(2), $\langle L_a(x), y \rangle = \langle ax, y \rangle = \langle x, \bar{a}y \rangle = \langle x, -ay \rangle = -\langle x, L_a y \rangle$. Therefore, S also is skew-symmetric, and $S(S(x)) = 0$ implies that $0 = \langle x, -S(S(x)) \rangle = \langle S(x), S(x) \rangle = n(S(x))$. Since $b = S(c)$ and $0 = [a,b,b] = S(b) = S(S(c))$, we conclude that $n(b) = n(S(c)) = 0$. □

The next theorem is a generalization of Lemma 2.20(1) to the case of doubly alternative zero divisors whose characteristic χ equals 1. If the characteristic is well defined but not equal to 1, then the norm of such an element is nonzero, and Lemma 2.20(2) can be applied. Thus, we know the centralizer of an arbitrary pure doubly alternative zero divisor whose first component has nonzero norm.

Theorem 4.22. *Let $(a,b) \in DA(\mathcal{A}_{n+1})$ be pure and let $\chi((a,b)) = 1$. Then $C_{\mathcal{A}_{n+1}}((a,b)) = \mathbb{F} \oplus O_{\mathcal{A}_{n+1}}((a,b))$.*

Proof. Since $\chi((a,b)) = 1$, we have $n((a,b)) = n(a) - \gamma_n n(b) = 0$. Suppose there exists $(c,d) \in C_{\mathcal{A}_{n+1}}((a,b)) \setminus (\mathbb{F} \oplus O_{\mathcal{A}_{n+1}}((a,b)))$. Without loss of generality, we may assume that $t(c) = 0$. Then

$$\overline{(a,b)(c,d)} = \overline{(c,d)} \cdot \overline{(a,b)} = (c,d)(a,b) = (a,b)(c,d),$$

that is, $(a,b)(c,d) = k \in \mathbb{F}$. Since $(c,d) \notin O_{\mathcal{A}_{n+1}}((a,b))$, we have $k \neq 0$. Assume, without loss of generality, that $k = 1$. Then

$$(1,0) = (a,b)(c,d) = (ac + \gamma_n \bar{d}b, da + b\bar{c}) = (ac + \gamma_n \bar{d}b, da - bc).$$

The condition $da = bc$ implies that $n(a)d = d(a\bar{a}) = (da)\bar{a} = (bc)\bar{a}$, whence $n(a)\bar{d} = a(\bar{c}\bar{b}) = -a(c\bar{b})$. By multiplying the equality $1 = ac + \gamma_n \bar{d}b$ by $\bar{b}$ on the right and substituting the expression for $n(a)\bar{d}$, we obtain

$$\begin{aligned}\bar{b} &= (ac)\bar{b} + \gamma_n(\bar{d}b)\bar{b} = (ac)\bar{b} + \gamma_n \bar{d}(b\bar{b}) = (ac)\bar{b} + \gamma_n n(b)\bar{d} \\ &= (ac)\bar{b} + n(a)\bar{d} = (ac)\bar{b} - a(c\bar{b}) = [a, c, \bar{b}].\end{aligned}$$

From Lemma 2.12(1) it follows that $t(\bar{b}) = t([a, c, \bar{b}]) = 0$, whence $\bar{b} = -b$ and $b = [a, c, b]$. By Lemma 4.21, we obtain that $n(b) = 0$, which contradicts the assumption that $\chi((a, b)) = 1$. □

5. Zero divisors in algebras with anisotropic norm

In Lemma 5.1, we extend the Moreno's results obtained in Sec. 1 of [14] for the real algebras of the main sequence to the case of Cayley–Dickson algebras with anisotropic norm over an arbitrary field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$. Recall that we denote $\widetilde{e}_0 = (0, e_0) \in \mathcal{A}_n$ and $\widetilde{a} = a\widetilde{e}_0$ for all $a \in \mathcal{A}_n$.

Corollary 4.16 claims that all zero divisors in the Cayley–Dickson algebras prove to be two-sided zero divisors, that is, $Z(\mathcal{A}_n) = Z_{LR}(\mathcal{A}_n)$. In the case of a Cayley–Dickson algebra with anisotropic norm, the following stronger result holds.

Lemma 5.1 ([14, Corollaries 1.5, 1.6, 1.9, and 1.12])**.** *Let $\mathcal{A}_n$ be a Cayley–Dickson algebra with anisotropic norm and let $a, b \in \mathcal{A}_n$. Then*
(1) $n(ab) = n(\bar{a}b) = n(a\bar{b}) = n(ba)$*;*
(2) *the elements ab, ba, $\bar{a}b$, $a\bar{b}$ are zero or nonzero simultaneously;*
(3) *if $a \in Z(\mathcal{A}_n)$, then $t(a) = 0$;*
(4) *if $a \in Z(\mathcal{A}_n)$, then a is doubly pure;*
(5) *$ab = 0$ if and only if $a\widetilde{b} = 0$.*

Proof. (1) By Lemma 2.12(2), $n(ab) = \langle ab, ab\rangle = \langle \bar{a}(ab), b\rangle = \langle \bar{b}(\bar{a}(ab)), e_0\rangle = \frac{1}{2}t(\bar{b}(\bar{a}(ab)))$. Note that $\bar{a}(ab) = (t(a) - a)(ab) = a((t(a) - a)b) = a(\bar{a}b)$, hence $n(ab) = \frac{1}{2}t(\bar{b}(\bar{a}(ab))) = \frac{1}{2}t(\bar{b}(a(\bar{a}b))) = n(\bar{a}b)$. Moreover, $n(\bar{a}b) = n(\overline{\bar{a}b}) = n(\bar{b}a) = n(ba) = n(\overline{ba}) = n(\bar{a}\bar{b}) = n(a\bar{b})$.

(2) This assertion immediately follows from item (1) and the fact that the norm on $\mathcal{A}_n$ is anisotropic.

(3) Let $a \in Z(\mathcal{A}_n)$, that is, $ab = 0$ for a certain $b \in \mathcal{A}_n$, $b \neq 0$. By item (2), we have $ab = \bar{a}b = 0$, whence $t(a)b = (a + \bar{a})b = 0$. Since $t(a) \in \mathbb{F}$ and $b \neq 0$, we conclude that $t(a) = 0$.

(4) From Proposition 4.9 it follows that if $a = (a_1, a_2) \in Z(\mathcal{A}_n)$, then $\widetilde{a} = (\gamma_{n-1}a_2, a_1) \in Z(\mathcal{A}_n)$. Therefore, by item (3), we have $t(a) = t(\widetilde{a}) = 0$, that is, $t(a_1) = t(a_2) = 0$.

(5) If $a = 0$ or $b = 0$, then $ab = a\widetilde{b} = 0$. Otherwise, by item (4), a and b are doubly pure, and the desired assertion follows from the first equality in Proposition 4.9. □

Corollary 5.2. *Let $\mathcal{A}_n$ be a Cayley–Dickson algebra with anisotropic norm. Then the graph $\Gamma_Z(\mathcal{A}_n)$ can be obtained from the graph $\Gamma_O(\mathcal{A}_n)$ by replacing every undirected edge with a pair of directed edges.*

Proof. This assertion immediately follows from Lemma 5.1(2). □

Theorem 5.3. *Let $\mathcal{A}_n$ be a Cayley–Dickson algebra with anisotropic norm and let $a, b \in \mathcal{A}_n \setminus \{0\}$, $t(a) = t(b) = 0$. If a and b are C-equivalent, i.e., $C_{\mathcal{A}_n}(a) = C_{\mathcal{A}_n}(b)$, then $[a] = [b]$.*

Proof. By Lemma 2.20(2), $C_{\mathcal{A}_n}(a) = \mathbb{F} \oplus \mathbb{F}a \oplus O_{\mathcal{A}_n}(a)$. Moreover, by Proposition 4.17, for all $x \in O_{\mathcal{A}_n}(a)$ it holds that $t(x) = 0$. Since $b \in C_{\mathcal{A}_n}(b) = C_{\mathcal{A}_n}(a)$ and $t(b) = 0$, we have $b = ka + x$ for some $k \in \mathbb{F}$ and $x \in O_{\mathcal{A}_n}(a)$. Therefore, $C_{\mathcal{A}_n}(a) \subseteq C_{\mathcal{A}_n}(x)$.

Suppose $[a] \neq [b]$, that is, $x \neq 0$. Since the norm on $\mathcal{A}_n$ is anisotropic, this means that $n(x) \neq 0$. By Lemma 5.1(5), $ax = 0$ implies $a\widetilde{x} = 0$, i.e., $\widetilde{x} \in O_{\mathcal{A}_n}(a)$. Hence $\widetilde{x} \in C_{\mathcal{A}_n}(a) \subseteq$

$C_{\mathcal{A}_n}(x)$. Moreover, by Lemma 5.1(4), the element x is doubly pure. From Lemma 3.4 it follows that $\widetilde{x}x = -x\widetilde{x} = n(x)\widetilde{e}_0 \neq 0$, which contradicts the equality $\widetilde{x}x = x\widetilde{x}$. $\square$

Corollary 5.4. *Let $\mathcal{A}_n$ be a Cayley–Dickson algebra with anisotropic norm and let $a, b \in \mathcal{A}_n$, $\mathfrak{Im}(a) \neq 0$, $\mathfrak{Im}(b) \neq 0$. If a and b are C-equivalent, then $[\mathfrak{Im}(a)] = [\mathfrak{Im}(b)]$.*

Proof. This assertion immediately follows from Theorem 5.3 because $C_{\mathcal{A}_n}(a) = C_{\mathcal{A}_n}(\mathfrak{Im}(a))$ and $C_{\mathcal{A}_n}(b) = C_{\mathcal{A}_n}(\mathfrak{Im}(b))$. $\square$

Remark 5.5. Let $\mathcal{A}_n$ be a Cayley–Dickson algebra with anisotropic norm and let $a \in Z(\mathcal{A}_n)$. Then, by Lemmas 3.2(3) and 5.1(4)–5.1(5), the elements a and $\widetilde{a}$ are doubly pure and linearly independent but O-equivalent. On the other hand, if $b \in Z(\mathcal{A}_{n-1})$, then $O_{\mathcal{A}_n}((b, 0)) = O_{\mathcal{A}_n}((0, b)) = \{(c, d) \mid c, d \in O_{\mathcal{A}_{n-1}}(b)\}$, whence $(b, 0)$ and $(0, b)$ also are doubly pure and linearly independent but O-equivalent.

Items (1)–(5) of Lemma 5.6 were proved by Moreno in [14, pp. 25–27] for the real algebras of the main sequence. In his proof, Moreno assumed that c and d were alternative elements in $\mathcal{M}_n$. However, his proof is valid verbatim in the case where the elements $c, d \in \mathcal{M}_n$ alternate with $a, b \in \mathcal{M}_n$ only. Also Moreno proved Lemma 5.6(6), but his proof required that the elements c and d alternate strongly.

Recall that the antiassociator of elements a, b, c in $\mathcal{A}$ is $\{a, b, c\} = (ab)c + a(bc)$.

Lemma 5.6. *Let $\mathcal{A}_{n+1}$ be a Cayley–Dickson algebra with anisotropic norm and let elements $c, d \in \mathcal{A}_n$ alternate with elements $a, b \in \mathcal{A}_n$, $(a, b), (c, d) \in Z(\mathcal{A}_{n+1})$, $(a, b)(c, d) = 0$. Then*

(1) $t(a) = t(b) = t(c) = t(d) = 0$;
(2) $n(a) = -\gamma_n n(b)$ and $n(c) = -\gamma_n n(d)$, i.e., $\chi((a, b)) = \chi((c, d)) = -1$;
(3) $[c, a, d] = 2n(c)b$, $[c, b, d] = -2n(d)a$;
(4) $\{c, a, d\} = \{c, b, d\} = 0$;
(5) $a \perp b$;
(6) $a, b \in \operatorname{span}(e_0, c, d, cd)^{\perp}$;
(7) $(c, d)(ac, ad) = 0$.

Proof. Item (1) immediately follows from Lemma 5.1(4).

(2) Since $(a, b) \neq 0$, $(c, d) \neq 0$, and the norm on $\mathcal{A}_n$ is anisotropic, we have $n(a) \neq 0$ or $n(b) \neq 0$, and also $n(c) \neq 0$ or $n(d) \neq 0$. By Lemma 4.6, $\chi = \chi((a, b)) = \chi((c, d)) = \pm 1$. However, if $\chi = 1$, then $n((a, b)) = n((c, d)) = 0$, which contradicts the fact that the norm on $\mathcal{A}_{n+1}$ is anisotropic. Therefore, $\chi = -1$.

(3) Since $(a, b)(c, d) = (ac + \gamma_n \bar{d}b, da + b\bar{c}) = (ac - \gamma_n db, da - bc) = 0$, we have $ac = \gamma_n db$ and $da = bc$. Thus,

$$
\begin{aligned}
n(d)a &= \bar{d}(da) = -d(bc),\\
n(d)a &= -\tfrac{1}{\gamma_n} n(c)a = -\tfrac{1}{\gamma_n}(ac)\bar{c} = \tfrac{1}{\gamma_n}(\gamma_n db)c = (db)c,\\
n(c)b &= (bc)\bar{c} = -(da)c,\\
n(c)b &= -\gamma_n n(d)b = -\bar{d}(\gamma_n db) = d(ac).
\end{aligned}
$$

Applying the involution to both sides of every equality, we obtain that $n(d)a = -(cb)d = c(bd)$ and $n(c)b = -c(ad) = (ca)d$. It follows that $[c, a, d] = 2n(c)b$, $[c, b, d] = -2n(d)a$, and $\{c, a, d\} = \{c, b, d\} = 0$, which also proves item (4).

(5) As in the proof of Lemma 4.21, consider the skew-symmetric linear operator $S : \mathcal{A}_n \to \mathcal{A}_n$ defined by the formula $S(x) = [c, x, d]$ for all $x \in \mathcal{A}_n$. Then $a \perp S(a) = [c, a, d] = 2n(c)b$, implying that $a \perp b$.

(6) By item (1), $e_0 \in \operatorname{span}(a, b, c, d)^{\perp}$. We apply Lemma 4.1(1) to the elements (a, b) and (c, d) with $\chi = \chi((c, d)) = -1$ and use the relations $\overline{ac} = \bar{c}\bar{a} = ca$. Then $(c, d)(ca, da) = 0$ and,

by Lemma 5.1(4), $t(ca) = t(da) = 0$. From Proposition 2.10 it follows that $\langle a, c\rangle = -\langle a, \bar{c}\rangle = -\frac{1}{2}t(ca) = 0$ and, similarly, $\langle a, d\rangle = 0$, that is, $a \perp c$ and $a \perp d$. By using the equalities $ca = \gamma_n bd$ and $da = bc$, we similarly obtain that $b \perp c$ and $b \perp d$.

From Lemma 2.18 it follows that the elements a, b anticommute with c, d. Then $ac = -ca$ and $ad = -da$, which immediately proves item (7). It remains to observe that, by Lemma 2.12(2),

$$\langle a, cd\rangle = \langle a\bar{d}, c\rangle = \langle da, c\rangle = \langle bc, c\rangle = \langle b, c\bar{c}\rangle = n(c)\langle b, e_0\rangle = 0,$$

$$\langle b, cd\rangle = \langle \bar{c}b, d\rangle = \langle bc, d\rangle = \langle da, d\rangle = \langle a, \bar{d}d\rangle = n(d)\langle a, e_0\rangle = 0,$$

whence $a \perp cd$ and $b \perp cd$. □

Now we will generalize some results obtained in Sec. 4.2 of the author's paper [22] for the real algebras of the main sequence. Note that in [22] there is an inaccuracy, namely, the proof of the pairwise orthogonality of the elements a, b, c, d with respect to $\langle\cdot,\cdot\rangle$ uses the fact that the elements a, b, c, d alternate strongly pairwise. However, all other assertions are stated under the assumption that the elements a, b alternate strongly with c, d only (see [22, Corollary 4.4, Lemma 4.6]).

Below, in Assertions 5.7–5.10 and in Fig. 2, we assume that $\mathcal{A}_{n+1}$ is a Cayley–Dickson algebra with anisotropic norm, $(a, b), (c, d) \in Z(\mathcal{A}_{n+1})$, $(a, b)(c, d) = 0$, and the elements $a, b \in \mathcal{A}_n$ alternate strongly with $c, d \in \mathcal{A}_n$, i.e., $[x, x, y] = [y, y, x] = 0$ for $x \in \{a, b\}$ and $y \in \{c, d\}$.

Corollary 5.7. *In $\Gamma_O(\mathcal{A}_{n+1})$ there is the following 6-cycle:*

$$(a, b) \leftrightarrow (c, d) \leftrightarrow (ac, ad) \leftrightarrow (a, -b) \leftrightarrow (c, -d) \leftrightarrow (ac, -ad) \leftrightarrow (a, b).$$

Proof. Apply Corollary 4.8 with $\chi = \chi((c, d)) = -1$. By Lemma 5.6(6), we have $\overline{ac} = -ac$ and $da = -ad$. Finally, Corollary 5.2 implies that the directed edges of the hexagon in $\Gamma_Z(\mathcal{A}_{n+1})$ correspond to the undirected ones in $\Gamma_O(\mathcal{A}_{n+1})$. □

Description 5.8. By using Lemma 5.1(5) and Corollary 5.7, we obtain a subgraph of $\Gamma_O(\mathcal{A}_{n+1})$, which we call a *double hexagon*. It is shown in Fig. 2. The double hexagon consists of 6 bipartite graphs $K_{2,2}$ glued together. Note that it contains all hexagons from Fig. 1.

Lemma 5.9.

(1) *The elements e_0, a, b, c, d are orthogonal with respect to $\langle\cdot,\cdot\rangle$.*

(2) *The elements e_0, a, b, c, d, ac, ad are orthogonal with respect to $\langle\cdot,\cdot\rangle$.*

Proof. (1) This assertion immediately follows from items (5)–(6) of Lemma 5.6 applied to the pairs of elements $(a, b)(c, d) = 0$ and $(c, d)(a, b) = 0$.

(2) By Lemma 4.7, the elements ac, ad alternate strongly with a, b, c, d. By Corollary 5.7, $(a, b)(c, d) = (c, d)(ac, ad) = (ac, ad)(a, -b) = 0$. It remains to use item (1) thrice. □

Corollary 5.10. *All elements at the vertices of the double hexagon in Fig. 2 are linearly independent.*

Proof. This assertion immediately follows from Lemma 5.9(2) because, by Lemma 3.1, $n(ac) = n(a)n(c) \neq 0$ and $n(ad) = n(a)n(d) \neq 0$. □

By combining the results of Lemmas 4.7 and 5.9(2) and of Corollaries 5.7 and 5.10 with Description 5.8, we obtain the following theorem.

Theorem 5.11. *Let $\mathcal{A}_{n+1}$ be a Cayley–Dickson algebra with anisotropic norm and let elements $a, b \in \mathcal{A}_n$ alternate strongly with $c, d \in \mathcal{A}_n$; $(a, b), (c, d) \in Z(\mathcal{A}_{n+1})$; $(a, b)(c, d) = 0$. Then*

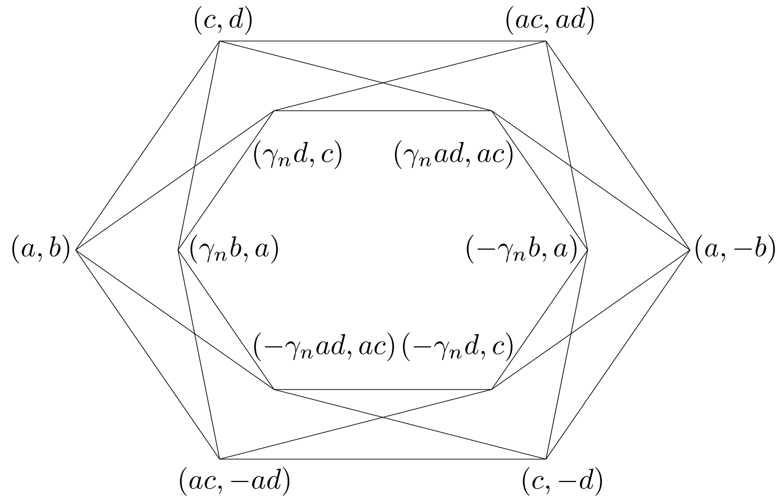

Fig. 2. The double hexagon.

(1) *the elements* ac, ad *alternate strongly with* a, b, c, d;
(2) *the elements* e_0, a, b, c, d, ac, ad *are orthogonal with respect to* $\langle \cdot, \cdot \rangle$;
(3) *there exists a subgraph of* $\Gamma_O(\mathcal{A}_{n+1})$ *that is shown in Fig. 2 and called the double hexagon*;
(4) *all elements at the vertices of the double hexagon are linearly independent.*

In [22, Theorem 4.11], the multiplication table for the vertices of the double hexagon is obtained in the case where $\mathcal{A}_{n+1}$ is a real algebra of the main sequence $\mathcal{M}_{n+1}$. This result can be generalized to an arbitrary Cayley–Dickson algebra with anisotropic norm, but the new multiplication table will depend on the parameters $n(a)$, $n(c)$, and γ_n. If $\mathcal{A}_{n+1} = \mathcal{M}_{n+1}$, then $\gamma_n = -1$, and we may assume, without loss of generality, that $n(a) = n(c) = 1$. Therefore, all coefficients in the multiplication table of the vertices of the double hexagon [22, p. 677, Table 1] are constant.

6. Dimensions of annihilators

In this section, we extend the proof of Theorem 9.8 from [3], which claims that the dimension of the annihilator of an arbitrary element in a real algebra of the main sequence is divisible by four, to the case of an arbitrary Cayley–Dickson algebra $\mathcal{A}_n$ with anisotropic norm over a field $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$. First we show that this assertion can fail for Cayley–Dickson algebras with isotropic norm. Recall that by Lemma 4.15, for any $a \in \mathcal{A}_n$ we have $\dim(l.\operatorname{Ann}_{\mathcal{A}}(a)) = \dim(r.\operatorname{Ann}_{\mathcal{A}}(a))$.

Lemma 6.1. *Let* $n \geq 1$, $a \in \mathcal{A}_n$, $t(a) = 0$. *Then* $\dim(r.\operatorname{Ann}_{\mathcal{A}_n}(a))$ *is even.*

Proof. Since $t(a) = 0$, by Lemma 2.12(2) we have $\langle L_a(b), c\rangle = \langle ab, c\rangle = \langle b, \bar{a}c\rangle = -\langle b, L_a(c)\rangle$ for all $b, c \in \mathcal{A}_n$, that is, L_a is a skew-symmetric linear operator with respect to the nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Since $\operatorname{char} \mathbb{F} \neq 2$, it follows that the rank of L_a is even. But $\dim \mathcal{A}_n = 2^n$ is even, whence $\dim(r.\operatorname{Ann}_{\mathcal{A}_n}(a)) = \dim(\operatorname{Ker} L_a)$ also is even. □

Proposition 6.2 ([10, Lemma 4.18], [21, Corollary 4.5]). *Let* $\mathcal{A}_n$ *be a real low-dimensional Cayley–Dickson split-algebra, i.e.,* $\mathbb{F} = \mathbb{R}$, $\mathcal{A}_n = \mathcal{H}_n$, *where* $1 \leq n \leq 4$, *and let* $a \in \mathcal{A}_n$.

(1) *If* $1 \leq n \leq 3$, *then* $\dim(r.\operatorname{Ann}_{\mathcal{A}_n}(a)) \in \{0, 2^{n-1}, 2^n\}$.
(2) *If* $n = 4$, *then* $\dim(r.\operatorname{Ann}_{\mathcal{A}_n}(a)) \in \{0, 4, 8, 16\}$.

Thus, for $n \in \{3, 4\}$, $\dim(r.\operatorname{Ann}_{\mathcal{A}_n}(a))$ *is divisible by four.*

The two examples below show that for $n \geq 4$ the algebra $\mathcal{A}_n$ can contain pure doubly alternative elements such that the dimensions of their annihilators are even but not divisible by four.

Example 6.3 ([10, Example 4.17])**.** Let $n \geq 4$, $\mathbb{F} = \mathbb{R}$, $\mathcal{A}_n = \mathcal{H}_{n-1}\{1\}$. Consider $a = e_1^{(n-2)} \in \mathcal{M}_{n-2}$. Then, by [10, Lemma 4.16], the element $A = (2a + \widetilde{a}, a) \in \mathcal{A}_n$ is pure and doubly alternative. However, $n(a) = 1$ and $n(2a + \widetilde{a}) = 3$, whence $\chi(A) = \gamma_{n-1}\frac{n(a)}{n(2a+\widetilde{a})} = \frac{1}{3} \neq \pm 1$. Thus, A does not satisfy condition $(*)$. Moreover, from Theorem 3.9 and Lemma 4.19 one can obtain that

$$r.\operatorname{Ann}_{\mathcal{A}_n}(A) = \left\{(c, -c) \;\middle|\; c = (x, -x),\ x \in \operatorname{span}(e_0, a)^{\perp} \subseteq \mathcal{M}_{n-2}\right\}.$$

Therefore, $\dim(r.\operatorname{Ann}_{\mathcal{A}_n}(A)) = \dim(\operatorname{span}(e_0, a)^{\perp}) = 2^{n-2} - 2$ is even but not divisible by four.

Example 6.4. Let $\mathbb{F} = \mathbb{R}$, $\mathcal{A}_4 = \mathcal{H}_3\{1\}$. Consider $a = e_2 + e_5 + e_6$, $b = e_0 + e_1 + e_5 \in \mathcal{H}_3$. Since $\mathcal{H}_3$ is an alternative algebra, the element $(a, b) \in \mathcal{A}_4$ is pure and doubly alternative. Furthermore, $n(b) = -n(a) = 1$, whence $\chi((a, b)) = \gamma_{n-1}\frac{n(b)}{n(a)} = -1$, that is, (a, b) satisfies condition $(*)$. However,

$$\begin{aligned} r.\operatorname{Ann}_{\mathcal{A}_4}((a, b)) = \operatorname{span}\big(&(-e_2 + e_3 - e_6 - e_7, e_2 + e_3 + e_6 - e_7),\\ &(e_1 + e_2 + 2e_3 - 2e_4 + e_5 - e_7, e_1 - 3e_3 + 2e_4 + e_5 - e_6 + 2e_7)\big). \end{aligned}$$

Thus, $\dim(r.\operatorname{Ann}_{\mathcal{A}_4}((a, b))) = 2$ is even but not divisible by four.

The following example shows that if an element of $\mathcal{A}_n$ is not pure, then, for n sufficiently large, the dimension of its annihilator can be odd.

Example 6.5. Let $\mathbb{F} = \mathbb{R}$ and let $\mathcal{A}_5 = \mathcal{H}_5 = \mathcal{M}_4\{-1\}$. Consider the elements

$$\begin{aligned} a &= e_0 - e_1 + 2e_2 + 2e_5 - e_6 + 2e_7 + e_9 - e_{10} - e_{11} - 2e_{12} - 2e_{13} - 2e_{14} - 2e_{15},\\ b &= -e_0 - 2e_1 + e_2 - 2e_3 - 2e_4 - 2e_6 + 2e_7 + e_9 - 2e_{10} + 2e_{11} + e_{12} + e_{13} - e_{15} \in \mathcal{M}_4. \end{aligned}$$

One can verify that in this case, $\dim(r.\operatorname{Ann}_{\mathcal{H}_5}((a, b))) = 3$ is odd.

Now let $\mathcal{A}_n$ be a Cayley–Dickson algebra with anisotropic norm over a field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$. Recall that in this case, by Lemma 5.1(2), we have $l.\operatorname{Ann}_{\mathcal{A}_n}(a) = r.\operatorname{Ann}_{\mathcal{A}_n}(a) = O_{\mathcal{A}_n}(a)$ for all $a \in \mathcal{A}_n$.

Lemma 6.6. *Let $n \geq 1$. Denote $\mathbb{K} = \operatorname{span}(e_0, \widetilde{e}_0)$. Then*

(1) *$\mathbb{K}$ is a field;*

(2) *$\mathcal{A}_n$ is a left vector space over $\mathbb{K}$.*

Proof. (1) Since $(\widetilde{e}_0)^2 = \gamma_{n-1}$, the set $\mathbb{K}$ is closed with respect to addition and multiplication, that is, it is a subalgebra in $\mathcal{A}_n$. Moreover, multiplication on $\mathbb{K}$ is commutative and associative. Since the norm on $\mathcal{A}_n$ is anisotropic, its restriction to $\mathbb{K}$ also is anisotropic, whence $\mathbb{K}$ contains no zero divisors. Therefore, $\mathbb{K}$ is a field.

(2) It is sufficient to show that $[k_1, k_2, a] = 0$ for all $k_1, k_2 \in \mathbb{K}$ and $a \in \mathcal{A}_n$. This is equivalent to the equality $[\widetilde{e}_0, \widetilde{e}_0, a] = 0$ for all $a \in \mathcal{A}_n$, that is, to the alternativity of the element $\widetilde{e}_0$. But $\widetilde{e}_0 = (0, e_0)$ is one of the standard basis elements, whence it is alternative by [19, Lemma 4]. □

By Lemma 2.12(2), $\langle a, b\rangle = \langle a\bar{b}, e_0\rangle$, that is, $\langle a, b\rangle$ is an orthogonal projection of $a\bar{b}$ onto $\mathbb{F}$ with respect to $\langle\cdot,\cdot\rangle$. We can use this observation in defining a $\mathbb{K}$-valued Hermitian inner product on $\mathcal{A}_n$.

Notation 6.7. Given $a, b \in \mathcal{A}_n$, by $\langle a, b\rangle_{\mathbb{K}}$ we denote the orthogonal projection of $a\bar{b}$ onto $\mathbb{K}$ with respect to $\langle\cdot,\cdot\rangle$.

Lemma 6.8. (1) *For arbitrary* $a, b \in \mathcal{A}_n$ *we have* $\langle a, b\rangle_{\mathbb{K}} = \langle a, b\rangle + \frac{1}{\gamma_{n-1}}\langle \widetilde{e}_0 a, b\rangle \widetilde{e}_0$.

(2) $\langle a, b\rangle_{\mathbb{K}}$ *is a Hermitian inner product, that is, an anisotropic (in particular, nondegenerate) Hermitian sesquilinear form.*

Proof. (1) By Lemma 2.12(2), $\langle a\bar{b}, e_0\rangle = \langle a, b\rangle$ and $\langle a\bar{b}, \widetilde{e}_0\rangle = \langle a, \widetilde{e}_0 b\rangle = -\langle \widetilde{e}_0 a, b\rangle$. Since $e_0 \perp \widetilde{e}_0$, $n(e_0) = 1$, and $n(\widetilde{e}_0) = -\gamma_{n-1}$, we have

$$\langle a, b\rangle_{\mathbb{K}} = \frac{\langle a\bar{b}, e_0\rangle}{n(e_0)} e_0 + \frac{\langle a\bar{b}, \widetilde{e}_0\rangle}{n(\widetilde{e}_0)} \widetilde{e}_0 = \langle a, b\rangle + \frac{\langle \widetilde{e}_0 a, b\rangle}{\gamma_{n-1}} \widetilde{e}_0.$$

(2) Additivity in both arguments is obvious. Show that $\langle ka, b\rangle_{\mathbb{K}} = k\langle a, b\rangle_{\mathbb{K}}$ for all $k \in \mathbb{K}$ and $a, b \in \mathcal{A}_n$. By the $\mathbb{F}$-linearity, it is sufficient to prove this assertion for $k = e_0$ and $k = \widetilde{e}_0$ only. The first case is evident, whereas in the second case we derive

$$\begin{aligned}\langle \widetilde{e}_0 a, b\rangle_{\mathbb{K}} &= \langle \widetilde{e}_0 a, b\rangle + \frac{\langle \widetilde{e}_0(\widetilde{e}_0 a), b\rangle}{\gamma_{n-1}} \widetilde{e}_0 = \langle \widetilde{e}_0 a, b\rangle + \frac{\langle (\widetilde{e}_0)^2 a, b\rangle}{\gamma_{n-1}} \widetilde{e}_0 \\ &= \langle \widetilde{e}_0 a, b\rangle + \langle a, b\rangle \widetilde{e}_0 = \langle a, b\rangle \widetilde{e}_0 + \frac{\langle \widetilde{e}_0 a, b\rangle}{\gamma_{n-1}} (\widetilde{e}_0)^2 = \widetilde{e}_0 \langle a, b\rangle_{\mathbb{K}}.\end{aligned}$$

Moreover, from $a\bar{b} = \overline{b\bar{a}}$ it follows that $\langle a, b\rangle_{\mathbb{K}} = \overline{\langle b, a\rangle_{\mathbb{K}}}$. Since $a\bar{a} = n(a) \in \mathbb{F} \subseteq \mathbb{K}$, we have $\langle a, a\rangle_{\mathbb{K}} = n(a)$ for $a \in \mathcal{A}_n$, whence $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{K}}$ is anisotropic. ☐

Lemma 6.9. *Let* $a \in \mathcal{A}_n$ *be doubly pure. Then* L_a *is a conjugate-linear skew-Hermitian mapping in the sense that*

(1) $L_a(b + c) = L_a(b) + L_a(c)$,
(2) $L_a(kb) = \bar{k} L_a(b)$,
(3) $\langle L_a(b), c\rangle_{\mathbb{K}} = -\overline{\langle b, L_a(c)\rangle_{\mathbb{K}}}$ *for all* $k \in \mathbb{K}$, $b, c \in \mathcal{A}_n$.

Proof. (1) The additivity of L_a is obvious.

(2) By virtue of the $\mathbb{F}$-linearity, it is sufficient to consider the cases $k = e_0$ and $k = \widetilde{e}_0$. The first case is evident, whereas in the second case we have to show that $a(\widetilde{e}_0 b) = -\widetilde{e}_0(ab)$. Applying the involution to both sides of the desired equality and using the fact that a is pure, we conclude that it is equivalent to $(\bar{b}\widetilde{e}_0)a = -(\bar{b}a)\widetilde{e}_0$. Since a is doubly pure, from Lemma 3.2(2) it follows that $\widetilde{\bar{b}}a = -\widetilde{\bar{b}a}$, as desired.

(3) Using Lemma 2.12(2) and items (1) and (2) of Lemma 6.8, we derive

$$\begin{aligned}\langle L_a(b), c\rangle_{\mathbb{K}} &= \langle ab, c\rangle_{\mathbb{K}} = \langle ab, c\rangle + \frac{\langle \widetilde{e}_0(ab), c\rangle}{\gamma_{n-1}} \widetilde{e}_0 = \langle b, \bar{a}c\rangle + \frac{\langle b, \bar{a}(\overline{\widetilde{e}_0} c)\rangle}{\gamma_{n-1}} \widetilde{e}_0 \\ &= -\langle b, ac\rangle + \frac{\langle b, a(\widetilde{e}_0 c)\rangle}{\gamma_{n-1}} \widetilde{e}_0 = -\langle b, ac\rangle + \frac{\langle b, -\widetilde{e}_0(ac)\rangle}{\gamma_{n-1}} \widetilde{e}_0 \\ &= -\langle b, ac\rangle + \frac{\langle \widetilde{e}_0 b, (ac)\rangle}{\gamma_{n-1}} \widetilde{e}_0 = -\overline{\langle b, ac\rangle_{\mathbb{K}}} = -\overline{\langle b, L_a(c)\rangle_{\mathbb{K}}}.\end{aligned}$$

☐

In [3], the following lemma is stated for the field of complex numbers only; however, the proof also holds in the case of an arbitrary field $\mathbb{K}$, $\operatorname{char}\mathbb{K} \neq 2$, with an involution $a \mapsto \bar{a}$.

Lemma 6.10 ([3, Lemma 6.7]). *Let* V *be a finite-dimensional linear space over* $\mathbb{K}$ *with a nondegenerate Hermitian form and let* L *be a conjugate-linear skew-Hermitian endomorphism of* V. *Then the* $\mathbb{K}$*-codimension of* $\operatorname{Ker} L$ *in* V *is even.*

Proof. We identify V with $\mathbb{K}^m$, where $m = \dim V$. Since L is conjugate-linear, there exists a matrix $A \in M_m(\mathbb{K})$ such that $Lx = \overline{Ax}$ for all $x \in V$. Let $H \in M_m(\mathbb{K})$ be the matrix of the Hermitian form on V, that is, $\langle x, y\rangle_{\mathbb{K}} = x^t H \bar{y}$ for all $x, y \in V$. Then we have $\bar{H} = H^t$. The mapping L is skew-Hermitian, hence for all $x, y \in V$ we have

$$\begin{aligned} x^t H A y &= \langle x, \overline{Ay}\rangle_{\mathbb{K}} = \langle x, Ly\rangle_{\mathbb{K}} = -\overline{\langle Lx, y\rangle_{\mathbb{K}}} = -\overline{\langle \overline{Ax}, y\rangle_{\mathbb{K}}} \\ &= -\overline{(\overline{Ax})^t H \bar{y}} = -(Ax)^t \bar{H} y = -x^t A^t H^t y = -x^t (HA)^t y. \end{aligned}$$

Thus, $HA = -(HA)^t$. Since $\operatorname{char}\mathbb{K} \neq 2$, it follows that the rank of HA is even, whence the nondegeneracy of H implies that the rank of A is even. Therefore, the $\mathbb{K}$-codimension of $\operatorname{Ker} L = \operatorname{Ker} A$ in V is even. ☐

Theorem 6.11. *Let $n \geq 2$, $a \in \mathcal{A}_n$. Then $\dim_{\mathbb{F}}(r.\operatorname{Ann}_{\mathcal{A}_n}(a))$ is divisible by four.*

Proof. If $a = 0$, then $\dim_{\mathbb{F}}(r.\operatorname{Ann}_{\mathcal{A}_n}(a)) = \dim_{\mathbb{F}}(\mathcal{A}_n) = 2^n$ is divisible by four, and if $a \neq 0$ and $a \notin Z(\mathcal{A}_n)$, then $\dim_{\mathbb{F}}(r.\operatorname{Ann}_{\mathcal{A}_n}(a)) = 0$ is as well divisible by four. Now consider the case where $a \in Z(\mathcal{A}_n)$. By Lemma 5.1(4), the element a is doubly pure. Hence, by Lemma 6.9, L_a is a conjugate-linear skew-Hermitian mapping. From Lemma 6.10 it follows that $\operatorname{codim}_{\mathbb{K}} \operatorname{Ker} L_a$ is even. Since $\dim_{\mathbb{K}} \mathcal{A}_n = 2^{n-1}$ is even, this means that $\dim_{\mathbb{K}} \operatorname{Ker} L_a$ also is even. Hence $\dim_{\mathbb{F}}(r.\operatorname{Ann}_{\mathcal{A}_n}(a)) = \dim_{\mathbb{F}} \operatorname{Ker} L_a = 2 \dim_{\mathbb{K}} \operatorname{Ker} L_a$ is divisible by four. ☐

Some other results on the dimensions of annihilators obtained in [3, 4] for the real algebras of the main sequence can also be generalized to the case of an arbitrary Cayley–Dickson algebra with anisotropic norm. In particular, this concerns Lemma 8.4, Proposition 8.11, and Theorem 13.2 from [3]. Notice that in the case of an arbitrary Cayley–Dickson algebra with anisotropic norm, Definition 3.1 of the element $\{a, b\} \in \mathcal{A}_{n+1}$ from [4] should be modified as follows: the factor $\frac{1}{\sqrt{2}}$ should be deleted, and a new factor γ_n should appear in the first component of $\{a, b\}$.

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REFERENCES

1. J. C. Baez, “The octonions,” *Bull. Amer. Math. Soc.*, **39**, 145–205 (2001).
2. L. Bentz and D. Tray, “Subalgebras of the split octonions,” *Adv. Appl. Clifford Alg.*, **28**, No. 2, 40 (2018).
3. D. K. Biss, D. Dugger, and D. C. Isaksen, “Large annihilators in Cayley–Dickson algebras,” *Comm. Algebra*, **36**, No. 2, 632–664 (2008).
4. D. K. Biss, D. Dugger, and D. C. Isaksen, “Large annihilators in Cayley–Dickson algebras II,” *Bol. Soc. Mat. Mex.*, **13**, No. 2, 269–292 (2007).
5. G. Dolinar, B. Kuzma, and N. Stopar, “Characterization of orthomaps on the Cayley plane,” *Aeq. Math.*, **92**, No. 2, 243–265 (2018).
6. G. Dolinar, B. Kuzma, and N. Stopar, “The orthogonality relation classifies formally real simple Jordan algebras,” *Comm. Algebra*, **48**, No. 6, 2274–2292 (2020).
7. P. Eakin and A. Sathaye, “On automorphisms and derivations of Cayley–Dickson algebras,” *J. Algebra*, **129**, No. 2, 263–278 (1990).

8. A. E. Guterman and S. A. Zhilina, "Relationship graphs of real Cayley–Dickson algebras," *Zap. Nauchn. Semin. POMI*, **472**, 44–75 (2018); English transl., *J. Math. Sci.*, **240**, No. 6, 733–753 (2019).
9. A. E. Guterman and S. A. Zhilina, "Relation graphs of the sedenion algebra," *Zap. Nauchn. Semin. POMI*, **496**, 61–86 (2020); English transl., *J. Math. Sci.*, **255**, No. 3, 254–270 (2021).
10. A. E. Guterman and S. A. Zhilina, "Cayley–Dickson split-algebras: doubly alternative zero divisors and relation graphs," *Fundam. Prikl. Mat.*, **23**, No. 3, 95–129 (2020).
11. N. Jacobson, "Composition algebras and their automorphisms," *Rend. Circ. Mat. Palermo*, **7**, 55–80 (1958).
12. B. Kuzma, "Dimensions of complex Hilbert spaces are determined by the commutativity relation," *J. Operator Theory*, **79**, No. 1, 201–211 (2018).
13. K. McCrimmon, *A Taste of Jordan Algebras*, Springer-Verlag, New York (2004).
14. G. Moreno, "The zero divisors of the Cayley–Dickson algebras over the real numbers," *Bol. Soc. Mat. Mex. (tercera ser.)*, **4**, No. 1, 13–28 (1998).
15. G. Moreno, "Alternative elements in the Cayley–Dickson algebras," in: *Topics in Mathematical Physics, General Relativity and Cosmology in Honor of Jerzy Plebañski*, World Sci. Publ. (2006), pp. 333–346.
16. G. Moreno, "Constructing zero divisors in the higher dimensional Cayley–Dickson algebras," `arXiv:math/0512517` (2005).
17. R. Moufang, "Zur Structur von Alternativekörpern," *Math. Ann.*, **110**, 416–430 (1935).
18. A. Pixton, "Alternators in the Cayley–Dickson algebras," *Forum Math.*, **21**, No. 5, 853–869 (2009).
19. R. D. Schafer, "On the algebras formed by the Cayley–Dickson process," *Amer. J. Math.*, **76**, No. 2, 435–446 (1954).
20. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, *Rings That Are Nearly Associative* [in Russian], Nauka, Moscow (1978).
21. S. A. Zhilina, "Relation graphs of the split-sedenion algebra," *Zap. Nauchn. Semin. POMI*, **482**, 87–113 (2019); English transl., *J. Math. Sci.*, **249**, No. 2, 167–184 (2020).
22. S. Zhilina, "Orthogonality graphs of real Cayley–Dickson algebras. Part I: Doubly alternative zero divisors and their hexagons," *Int. J. Algebra Comput.*, **31**, No. 4, 663–689 (2021).
23. S. Zhilina, "Orthogonality graphs of real Cayley–Dickson algebras. Part II: The subgraph on pairs of basis elements," *Int. J. Algebra Comput.*, **31**, No. 4, 691–725 (2021).