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HOCHSCHILD COHOMOLOGY FOR ALGEBRAS OF SEMIDIHEDRAL TYPE. X. COHOMOLOGY ALGEBRA FOR THE EXCEPTIONAL LOCAL ALGEBRAS

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Hochschild cohomology algebra is described in term of generators and relations for a family of local algebras of semidihedral type. This family appears in famous K. Erdmann's classification only if the characteristic of the base field is equal to 2. Bibliography: 13 titles.

Introduction

The present paper is devoted to calculation of the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ for the so called "exceptional" family of local algebras of semidihedral type. Recall that the algebras of dihedral, semidihedral, and quaternion type appear in K. Erdmann's papers on classification of group blocks having a tame representation type (see [1]). The Hochschild cohomology groups for this "exceptional" family were earlier calculated in [2]; this family appears in the case when the base field has characteristic different from 2.

For another family of local algebras of semidihedral type, the Hochschild cohomology algebra $\mathrm{HH}^*(R)$ was calculated in [3, 4]. Moreover, the Hochschild cohomology was investigated for several families of algebras of semidihedral type with 2 or 3 simple modules in [5–11].

In order to calculate multiplication in $\mathrm{HH}^*(R)$, we use the minimal projective (= free) resolution for algebras under consideration, which was constructed in [2].

1. Formulation of the main result

Let K be an algebraically closed field of arbitrary characteristic $p := \operatorname{char} K$. For $k \in \mathbb{N}\setminus\{1\}$ and $c, d \in K$, we define a K-algebra $R_{k,c,d} = K\langle X, Y\rangle/I$, where I is an ideal of the free algebra $K\langle X, Y\rangle$, generated by the elements

$$X^2 - Y(XY)^{k-1} - c(XY)^k,\ Y^2 - d(XY)^k,\ (XY)^k - (YX)^k,\ X(YX)^k.$$

The images of X and Y under the canonical homomorphism from $K\langle X, Y\rangle$ to $R_{k,c,d}$ are denoted by x and y, respectively. The algebra $R_{k,c,d}$ is a symmetric local algebra of tame representation type [1, III.1.2]; moreover, $R_{k,c,d}$ is an algebra of semidihedral type, in the terminology of [1, Chap.VIII].

The algebra $R_{k,c,d}$ admits as a K-basis the set

$$\begin{aligned}\mathcal{B}_R = \{(xy)^i \mid 0 \le i \le k\} \cup \{(yx)^i \mid 1 \le i \le k-1\}\\ \cup \{x(yx)^i \mid 0 \le i \le k-1\} \cup \{y(xy)^i \mid 0 \le i \le k-1\}\end{aligned} \tag{1.1}$$

consisting of all nonzero paths of the quiver of R (it consists of a single vertex and two loops x and y).

The algebras $S_k := R_{k,0,0}$ (the parameters c, d are zero) form a family of local algebras that is included in the classification in [1]. If $p \neq 2$, then this family contains all local algebras of semidihedral type. The Hochschild cohomology of the algebras S_k (in any characteristic p) was studied in [3, 4].

But if $(c, d) \neq (0, 0)$ and $p = 2$, then the algebras $R_{k,c,d}$ form a one more family of local algebras in K. Erdmann's classification [1]. Note that if $c \neq 0$, then we may assume that $c = 1$.

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In the sequel, we assume that the base field K has characteristic 2.

In this section, we state the main result of the paper, namely, we describe the multiplicative structure of the Hochschild cohomology algebra for algebras under consideration. Since this description depends on the values of parameters that are included in the defining relations of the studied algebras, we first construct several auxiliary graded K-algebras.

Let

$$\mathcal{X}_1 = \{p_1, p_2, p_3, p_4, u_1, u_2, u_3, v_1, v_2, v_3, v_4, \widetilde{w}, z\}. \tag{1.2}$$

We introduce a grading on the algebra $K[\mathcal{X}_1]$, such that

$$\begin{gathered}\deg p_1 = \deg p_2 = \deg p_3 = \deg p_4 = 0,\\ \deg u_1 = \deg u_2 = \deg u_3 = 1,\\ \deg v_1 = \deg v_2 = \deg v_3 = \deg v_4 = 2,\\ \deg \widetilde{w} = 3,\ \deg z = 4.\end{gathered}$$

Then we define a graded K-algebra $\mathcal{A}_1 = \mathcal{A}_1(k, c, d) = K[\mathcal{X}_1]/I_1$, where the ideal I_1 is generated by the elements

$$\left.\begin{array}{r} p_1^k, p_2^2, p_3^2, p_4^2, \\ p_ip_j \ \text{ for } \ 1 \le i < j \le 4; \end{array}\right\} \tag{1.3}$$

$$cp_1u_1 + d(p_2u_1 + p_1^{k-2}u_2), p_1^{k-1}u_2, \tag{1.4}$$

$$p_2u_2, p_3u_2, p_4u_2; \tag{1.5}$$

$$p_ju_3 \ \text{ for } \ 1 \le j \le 4, \tag{1.6}$$

$$u_1u_2, u_2^2, p_1^{k-1}v_4,\ u_2u_3, \tag{1.7}$$

$$u_1u_3 + cp_4v_1,\ u_3^2;, \tag{1.8}$$

$$cp_1v_2 + dp_4v_1,\ p_4v_1 + p_2v_2, \tag{1.9}$$

$$p_4u_1^2 + p_1v_2,\ p_4u_1^2 + p_3v_3,\ p_4u_1^2 + p_2v_4, \tag{1.10}$$

$$p_1u_1^2,\ p_3u_1^2, \tag{1.11}$$

$$p_3v_2,\ p_4v_2,\ p_3v_4,\ p_4v_4, \tag{1.12}$$

$$p_1v_1,\ p_2v_1,\ p_3v_1,\ p_1v_3,\ p_2v_3,\ p_4v_3, \tag{1.13}$$

$$cu_1^3 + du_1v_1,\ p_1^2\widetilde{w} + u_2v_4, dp_2\widetilde{w} + u_1v_4, \tag{1.14}$$

$$u_1v_2 + p_1^{k-1}\widetilde{w} + cp_2\widetilde{w},\ p_4u_1^3, \tag{1.15}$$

$$p_3\widetilde{w},\ p_4\widetilde{w}, \tag{1.16}$$

$$u_2v_1,\ u_2v_2,\ u_2v_3, \tag{1.17}$$

$$u_3v_2,\ u_3v_3,\ u_3v_4, \tag{1.18}$$

$$v_2^2,\ v_3^2;\ v_iv_j \ \text{ if } i < j; \tag{1.19}$$

$$u_1^4, \tag{1.20}$$

$$u_1\widetilde{w} + (p_3 + cp_4)z, \tag{1.21}$$

$$v_4^2 + p_1^2z, \tag{1.22}$$

$$u_1^2v_j \text{ for } 1 \le j \le 4, \tag{1.23}$$

$$\text{for } \ c = 0 \ \ u_1v_1; \tag{1.24}$$

$$u_2\widetilde{w}, \tag{1.25}$$

$$v_3\widetilde{w} + p_4u_1z,\ v_4\widetilde{w} + u_2z,\ v_2\widetilde{w} \tag{1.26}$$

$$(\widetilde{w})^2 + cp_4u_1^2z. \tag{1.27}$$

Furthermore, we introduce a grading on the algebra $\mathcal{A}_1$, induced by the grading of the algebra $K[\mathcal{X}_1]$.

Next, let

$$\mathcal{X}_2 = \{p_1, p_2, p_3, p_4, u_1', u_2', u_3, v_1, v_2, v_3, v_4, w, z\}. \tag{1.28}$$

We introduce a grading on the algebra $K[\mathcal{X}_2]$, such that

$$\left.\begin{array}{c} \deg p_1 = \deg p_2 = \deg p_3 = \deg p_4 = 0, \\ \deg u_1' = \deg u_2' = \deg u_3 = 1, \\ \deg v_1 = \deg v_2 = \deg v_3 = \deg v_4 = 2, \\ \deg w = 3,\ \deg z = 4. \end{array}\right\} \tag{1.29}$$

Then we define a graded K-algebra $\mathcal{A}_2 = K[\mathcal{X}_2]/I_2$, where the ideal I_2 is generated by the elements (1.3), (1.6), (1.8), (1.12), (1.13), (1.18), (1.19), (1.22), and by the following elements (here we use a constant θ_n of the field K $(n \in \mathbb{N})$, defined by the formula

$$\theta_n := \sum_{i=1}^{n} i\ ): \tag{1.30}$$

$$\begin{gathered} p_1u_1' + (dp_2 + cp_1)u_2',\ p_2u_1' + p_1^{k-1}u_2', \\ p_1^{k-1}v_4 + dp_4v_1 + cp_3v_3,\ p_4(u_1')^2 + p_1v_2,\ p_4(u_1')^2 + p_3v_3,\ p_4(u_1')^2 + p_2v_4, \\ p_1(u_1')^2,\ p_3(u_1')^2, \\ u_1'u_2' + \theta_{k-1}(cdp_4v_1 + c^2p_3v_3), \\ (u_1')^3 + du_2'v_1,\ u_1'v_2 + p_1^{k-1}w,\ u_2'v_2 + p_2w, \\ u_1'v_4 + u_2'(dv_2 + cv_4),\ u_2'v_4 + p_1w, \\ u_1'v_1,\ u_2'v_3, \\ p_3w,\ p_4w, \\ (u_1')^2v_j \text{ for } 2 \le j \le 4, \\ u_1'w,\ u_2'w, \\ v_2w + p_2u_2'z,\ v_4w + p_1u_2'z,\ v_3w, \\ w^2 + (d\theta_{k+1}p_4v_1 + c\theta_{k-1}p_3v_3)z. \end{gathered}$$

Furthermore, we introduce a grading on the algebra $\mathcal{A}_2$, induced by the grading of the algebra $K[\mathcal{X}_2]$.

Let

$$\mathcal{X}_3 = \{p_1, p_2, p_3, p_4, u_0, u_1, u_2, v_2, v_3, v_4, w_0, w_1, z\}. \tag{1.31}$$

We introduce a grading on the algebra $K[\mathcal{X}_3]$, such that

$$\begin{gathered} \deg p_1 = \deg p_2 = \deg p_3 = \deg p_4 = 0, \\ \deg u_0 = \deg u_1 = \deg u_2 = 1, \\ \deg v_2 = \deg v_3 = \deg v_4 = 2, \\ \deg w_0 = \deg w_1 = 3,\ \deg z = 4. \end{gathered}$$

Then we define a graded K-algebra $\mathcal{A}_3 = K[\mathcal{X}_3]/I_3$, where the ideal I_3 is generated by the elements (1.3), (1.5), (1.7), (1.12), (1.19), (1.22), and by the elements

$$\begin{gathered} p_1u_0 + p_3u_1,\ p_3u_0 + p_1^{k-2}u_2, \\ p_2u_1 + (p_3 + cp_4)u_0,\ p_1u_1; \\ u_0u_2,\ p_4u_1^2,\ p_4u_0u_1, \end{gathered}$$

$$p_1v_2,\ p_2v_4,\ p_jv_3 \ \text{ for } \ 1\le j\le 4;$$
$$u_0v_3+p_3w_0,\ u_0v_3+p_1^{k-1}w_1,\ p_2w_1+p_4w_0,$$
$$p_3w_1,\ p_4w_1,$$
$$u_1u_0^2+cu_0v_2+cp_2w_0,\ u_0u_1^2,\ u_1^3,$$
$$u_0v_4+u_1v_3,\ u_0v_4+p_1w_0,\ u_2v_4+p_1^2w_1,$$
$$u_1v_2+(p_1^{k-1}+cp_2)w_1,$$
$$u_1v_4,\ u_2v_2,\ u_2v_3,$$
$$u_1w_1+(p_3+cp_4)z,\ u_1w_0+cu_0w_1+cp_2z+p_1^{k-1}z;$$
$$v_2w_0+u_0^2w_1,\ v_3w_0+p_1^{k-2}u_2z,$$
$$v_4w_0+p_3u_1z,\ v_3w_1+p_4u_1z,\ v_4w_1+u_2z,\ v_2w_1;$$
$$w_0^2+(1+c^3p_4)u_0^2z,$$
$$w_0w_1+v_2z,\ w_1^2.$$

We introduce a grading on the algebra $\mathcal{A}_3$, induced by the grading of the algebra $K[\mathcal{X}_3]$.

Let

$$\mathcal{X}_4=\{p_1,p_2,p_3,p_4,u_0,u_1',u_2',v_2,v_3,v_4,w_0,w_1,z\}. \tag{1.32}$$

We introduce a grading on the algebra $K[\mathcal{X}_4]$ such that

$$\left.\begin{array}{c}\deg p_1=\deg p_2=\deg p_3=\deg p_4=0,\\ \deg u_0=\deg u_1'=\deg u_2'=1,\\ \deg v_2=\deg v_3=\deg v_4=2,\\ \deg w_0=\deg w_1=3,\ \deg z=4.\end{array}\right\} \tag{1.33}$$

We define a graded K-algebra $\mathcal{A}_4=K[\mathcal{X}_4]/I_4$, where the ideal I_4 is generated by the elements (1.3), (1.12), (1.19), (1.22), and by the following elements (again we use the constant θ_n, see (1.30)):

$$p_3u_0+p_2u_1',\ p_3u_0+p_1^{k-1}u_2',\ p_1u_1'+cp_1u_2',$$
$$p_3u_1'+p_1u_0,\ p_2u_2'+p_4u_0,\ p_3u_2',\ p_4u_2';$$
$$u_1'u_2'+c^2\theta_{k-1}p_3v_3,$$
$$p_1v_2+p_3v_3,\ p_1v_2+p_2v_4,\ p_1v_2+p_4(u_1')^2,$$
$$u_0u_1',\ p_2v_2+p_4u_0^2,\ (u_2')^2+c\theta_{k-1}p_3v_3,$$
$$p_1^{k-1}v_4+cp_3v_3,$$
$$p_1v_3,\ p_2v_3,\ p_4v_3,$$
$$u_1'v_4+u_2'v_3,\ u_1'v_4+p_1w_1,\ u_0v_4+p_1w_0,$$
$$u_0v_3+p_1^{k-2}u_2'v_3,\ u_0v_3+p_3w_0,\ u_0v_3+u_1'v_2,\ (u_1')^3,$$
$$u_0v_2+u_0^2u_2'+p_2w_0,\ p_2w_1+p_4w_0,\ p_2w_1+u_2'v_2,$$
$$u_2'v_4,\ u_1'v_3,\ p_3w_1,\ p_4w_1,$$
$$u_2'w_0+u_0w_1,\ u_1'w_0,\ u_1'w_1,\ u_2'w_1,$$
$$v_2w_0+u_0^2w_1+p_2u_0z,\ v_3w_0+p_1^{k-1}u_2'z,$$
$$v_2w_1+p_4u_0z,\ v_4w_0+(cp_1^{k-1}u_2'+p_1u_0)z,$$
$$v_4w_1+p_1u_2'z,\ v_3w_1;$$
$$w_0^2+(1+c^3p_4)u_0^2z,\ w_0w_1+u_0u_2'z,\ w_1^2+c\theta_{k-1}p_3v_3z.$$

Furthermore, we introduce a grading on the algebra $\mathcal{A}_4$, induced by the grading of the algebra $K[\mathcal{X}_4]$.

Next, we consider a set $\mathcal{X}_5$ that coincides with $\mathcal{X}_2$ (see (1.28)) and introduce the same grading on the algebra $K[\mathcal{X}_5]$ as in (1.29). Then we define a graded K-algebra $\mathcal{A}_5 = K[\mathcal{X}_5]/I_5$, where the ideal I_5 is generated by the elements

$$\begin{gathered}
p_ip_j \ \text{ for all } i,j \in \{1,2,3,4\};\\
p_1u_1' + (dp_2 + cp_1)u_2',\ p_2u_1' + p_1u_2',\\
p_ju_3 \ \text{ for } \ 1 \le j \le 4,\ p_3u_2',\ p_4u_2',\\
p_1v_4 + dp_4v_1 + cp_3v_3,\ u_3^2,\\
p_3v_1 + p_1v_2,\ p_3v_1 + p_3v_3,\ p_3v_1 + p_2v_4,\ p_3v_1 + u_1'u_3,\ p_3v_1 + p_4(u_1')^2,\\
p_2v_2 + p_4v_1,\ p_2v_2 + u_2'u_3,\ p_3(u_1')^2,\\
u_1'u_2' + cdp_4v_1 + c^2p_3v_1,\ (u_2')^2 + cp_3v_1,\\
u_1'v_4 + u_2'(dv_2 + cv_4),\ u_2'v_4 + u_1'v_2, u_2'v_4 + p_1w,\\
u_1'v_1 + u_1'v_3,\ p_2w + u_2'v_2,\\
(u_1')^3 + du_2'v_1,\ p_3w,\ p_4w,\\
v_2^2,\ v_3^2,\ v_4^2;\ v_iv_j \ \text{ if } i<j;\\
u_1'w,\ u_2'w,\\
v_2w + p_2u_2'z,\ v_4w + p_1u_2'z,\ v_3w,\\
w^2 + cp_3v_3z.
\end{gathered}$$

Furthermore, we introduce a grading on the algebra $\mathcal{A}_5$, induced by the grading of the algebra $K[\mathcal{X}_5]$.

Next, we consider a set $\mathcal{X}_6$ that coincides with $\mathcal{X}_4$ (see (1.32)) and introduce the same grading on the algebra $K[\mathcal{X}_6]$ as in (1.33). We define a graded K-algebra $\mathcal{A}_6 = K[\mathcal{X}_6]/I_6$, where the ideal I_6 is generated by the elements

$$\begin{gathered}
p_ip_j \ \text{ for all } i,j \in \{1,2,3,4\};\\
p_1u_1' + cp_3u_0,\ p_2u_1' + p_3u_0,\ p_2u_1' + p_1u_2',\\
p_3u_1' + p_1u_0,\ p_2u_2' + p_4u_0,\ p_3u_2',\ p_4u_2';\\
p_1v_2 + p_3v_3,\ p_1v_2 + p_2v_4,\ p_1v_2 + c^{-1}p_1v_4,\ p_1v_2 + c^{-1}(u_2')^2,\\
p_1v_2 + c^{-1}u_1'u_2',\ p_1v_2 + p_4(u_1')^2,\\
p_2v_2 + p_4u_0^2,\ p_3v_2,\ p_4v_2,\\
p_1v_3,\ p_2v_3,\ p_4v_3,\ p_3v_4,\ p_4v_4,\ u_0u_1';\\
u_0v_3 + u_1'v_2,\ u_0v_3 + c^{-1}u_1'v_4,\ u_0v_3 + u_2'v_4,\ u_0v_3 + p_3w_0,\ u_0v_3 + p_1w_1,\\
u_0v_4 + u_1'v_3,\ u_0v_4 + p_1w_0,\ (u_1')^3,\ u_2'v_3,\\
p_2w_0 + u_0v_2 + u_0^2u_2',\\
p_2w_1 + p_4w_0,\ p_2w_1 + u_2'v_2,\ p_3w_1,\ p_4w_1;\\
v_iv_j \ \text{ for all } i,j \in \{2,3,4\},\\
u_2'w_0 + u_0w_1,\ u_1'w_0,\ u_1'w_1,\ u_2'w_1,\\
v_2w_0 + u_0^2w_1 + p_2u_0z,\ v_3w_0 + v_4w_1,\ v_3w_0 + p_1u_2'z,\\
v_2w_1 + p_4u_0z,\ v_4w_0 + p_1u_0z,\ v_3w_1,\\
w_0^2 + (1 + c^3p_4)u_0^2z,\ w_0w_1 + u_0u_2'z,\ w_1^2 + cp_3v_3z.
\end{gathered}$$

Furthermore, we introduce a grading on the algebra $\mathcal{A}_6$, induced by the grading of the algebra $K[\mathcal{X}_6]$.

The main result of the paper is the following theorem.

Theorem 1.1. *Assume that* $\operatorname{char} K = 2$, *and let* $R = R_{k,c,d}$ *with* $k \geq 2$, $c, d \in K$, $(c,d) \neq (0,0)$.

(1) *If k is odd and $d \neq 0$, then the Hochschild cohomology algebra* $\mathrm{HH}^*(R)$ *is isomorphic, as graded K-algebra, to the algebra* $\mathcal{A}_1$.

(2) *If k is even, $k > 2$, and $d \neq 0$, then, as graded K-algebra,* $\mathrm{HH}^*(R) \simeq \mathcal{A}_2$.

(3) *If k is odd and $d = 0$, then, as graded K-algebra,* $\mathrm{HH}^*(R) \simeq \mathcal{A}_3$.

(4) *If k is even, $k > 2$, and $d = 0$, then, as graded K-algebra,* $\mathrm{HH}^*(R) \simeq \mathcal{A}_4$.

(5) *If $k = 2$ and $d \neq 0$, then, as graded K-algebra,* $\mathrm{HH}^*(R) \simeq \mathcal{A}_5$.

(6) *If $k = 2$ and $d = 0$, then, as graded K-algebra,* $\mathrm{HH}^*(R) \simeq \mathcal{A}_6$.

2. Auxiliary results

Let $R = R_{k,c,d}$ with $k \in \mathbb{N} \setminus \{1\}, (c,d) \neq (0,0)$, and let $\Lambda := R \otimes_K R^{\mathrm{op}}$ be the enveloping algebra of the algebra R. Let $\mu\colon Q_\bullet \to R$ be the minimal Λ-projective resolution of the bimodule R, that was constructed in [2]. Recall that in the complex $Q_\bullet$, we have

$$\begin{aligned} &Q_0 = \Lambda, \quad Q_1 = Q_2 = Q_3 = \Lambda^2, \\ &Q_n = \Lambda^2 \oplus Q_{n-4} \ \text{for} \quad n \geq 4; \end{aligned} \tag{2.1}$$

the description of the differentials d_n^Q in $Q_\bullet$ is more complicated, see [2]. Moreover, in the rest of the paper, we fix the decompositions of the modules $Q_n = \Lambda^{t_n}$ with respect to which the matrices of the differentials are described in [2]; these decompositions are called *standard.* We use also another notation in [2], in particular, we put

$$\widetilde{y} := y + dx(yx)^{k-1},$$

and

$$\delta^n := \mathrm{Hom}_\Lambda(d_n^Q, R).$$

Let $X_\bullet$ be a subcomplex of the complex $Q_\bullet$, such that

$$\left.\begin{array}{l} \text{for} \quad n \geq 4, \qquad X_n = \Lambda^2 \\ \text{the sum of the first two direct summands in decomposition of } Q_n \text{ in (2.1),} \\ \text{for } \ 0 \leq n \leq 3, \qquad X_n = Q_n. \end{array}\right\} \tag{2.2}$$

The following statement was proved in [2, Proposition 3.2].

Proposition 2.1. *There is a short exact sequence of complexes*

$$0 \to X_\bullet \xrightarrow{i} Q_\bullet \xrightarrow{\pi} Q_\bullet[-4] \to 0, \tag{2.3}$$

which splits in each degree.

The following statement is derived from [1, III.14].

Proposition 2.2. $\mathrm{HH}^0(R)$ *admits as a K-basis the set*

$$\{1, xy + yx, (xy)^2 + (yx)^2, \ldots, (xy)^{k-1} + (yx)^{k-1}, x(yx)^{k-1}, y(xy)^{k-1}, (xy)^k\}.$$

When studying the Hochschild cohomology groups of higher degrees, we impose additional conditions on the parameters k, c, d, namely, we distinguish the cases of even and odd k, and also of $d \neq 0$ and $d = 0$.

Although the dimensions of the groups $\mathrm{HH}^i(R)$, $i > 0$, were calculated in [2], we need an explicit description of bases (over K) of these groups. In order to obtain it, we first present bases of the vector spaces $\operatorname{Ker}\delta^i$, $i \in \{1, 2, 3\}$. We obtain these bases as in [13, Proposition 4.4], namely, we represent elements in R as linear combinations of basis elements from (1.1) and obtain systems of linear equations on coefficients of those linear combinations substituting them in the formulas for δ^i. We leave the details of such computation to the reader.

Proposition 2.3. *Assume that k is odd and $d \neq 0$. Then*

(a) $\operatorname{Ker}\delta^1$ *admits a K-basis formed by the elements*

$$\big((xy)^i + (yx)^i, 0\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.4}$$
$$\big(y(xy)^i, 0\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.5}$$
$$\big(0, (xy)^i + (yx)^i\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.6}$$
$$\big(0, x(yx)^i\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.7}$$
$$\big(d(xy)^{k-1} + cy, 1 + cx\big),\ \big(x(yx)^{k-1}, 0\big),\ \big((xy)^k, 0\big), \tag{2.8}$$
$$\big(0, y(xy)^{k-1}\big),\ \big(0, (xy)^k\big); \tag{2.9}$$

(b) $\operatorname{Ker}\delta^2$ *admits a K-basis formed by the elements*

$$\big((xy)^i + (yx)^i, 0\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.10}$$
$$\big(d(xy)^{i+1}, y(xy)^i + c(xy)^{i+1}\big) \ \text{for}\ 0 \le i \le k-2, \tag{2.11}$$
$$\big(y(xy)^i, 0\big) \ \text{for}\ 0 \le i \le k-1, \tag{2.12}$$
$$\big(0, (xy)^i + (yx)^i\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.13}$$
$$\big(0, x(yx)^i\big) \ \text{for}\ 0 \le i \le k-1, \tag{2.14}$$
$$\big(1, 0\big),\ \big(x(yx)^{k-1}, 0\big),\ \big((xy)^k, 0\big), \tag{2.15}$$
$$\big(0, 1\big),\ \big(0, y(xy)^{k-1}\big),\ \big(0, (xy)^k\big); \tag{2.16}$$

(c) $\operatorname{Ker}\delta^3$ *admits a K-basis formed by the elements*

$$\big((xy)^i + (yx)^i, 0\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.17}$$
$$\big(d(yx)^i, (xy)^i + (yx)^i\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.18}$$
$$\big(y(xy)^i, 0\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.19}$$
$$\big(0, y(xy)^i\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.20}$$
$$\big(y, x(yx)^{k-1}\big),\ \big(dy, y\big),\ \big(x(yx)^{k-1}, 0\big), \tag{2.21}$$
$$\big((xy)^k, 0\big),\ \big(0, (xy)^k\big). \tag{2.22}$$

Corollary 2.4. *Assume that k is odd and $d \neq 0$. Then*

(a) $\mathrm{HH}^1(R)$ *admits a K-basis formed by the elements*

$$\big(y(xy)^i, 0\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.23}$$
$$\big(d(xy)^{k-1} + cy, 1 + cx\big),\ \big(x(yx)^{k-1}, 0\big),\ \big((xy)^k, 0\big), \tag{2.24}$$
$$\big(0, y(xy)^{k-1}\big),\ \big(0, (xy)^k\big); \tag{2.25}$$

(b) $\mathrm{HH}^2(R)$ *admits a* K*-basis formed by the elements*

$$\big(d(xy)^{i+1}, y(xy)^i + c(xy)^{i+1}\big) \ \text{for}\ 0 \le i \le k-2, \tag{2.26}$$

$$\big(1,0\big),\ \big(y,0\big), \tag{2.27}$$

$$\big(0,1\big),\ \big(0,x\big),\ \big(0,(xy)^k\big); \tag{2.28}$$

(c) $\mathrm{HH}^3(R)$ *admits a* K*-basis formed by the elements*

$$\big(0, y(xy)^i\big) \ \text{for}\ 1 \le i \le k-1, \tag{2.29}$$

$$\big(dyx, xy + yx\big),\ \big(y, x(yx)^{k-1}\big),\ \big(0,\widetilde{y}\big), \tag{2.30}$$

$$\big(x(yx)^{k-1}, 0\big),\ \big(0,(xy)^k\big). \tag{2.31}$$

Proof. The statements follow immediately from Proposition 2.3 and the description of bases for $\operatorname{Im}\delta^i$, $0 \le i \le 2$, in [2]. □

Proposition 2.5. *Assume that* k *is even and* $d \neq 0$. *Then*

(a) *to get a basis of* $\operatorname{Ker}\delta^1$, *we need to replace in the set in Proposition* 2.3 (a), *the element* $\big(d(xy)^{k-1}+cy, 1+cx\big)$ *in* (2.8) *by the element* $\big(d(xy)^{k-1}, 1+cx\big)$ *and to add the element* $\big(y,0\big)$ *to this set;*

(b) $\operatorname{Ker}\delta^2$ *admits as a* K*-basis the set in* 2.3 (b)*;*

(c) *to get a basis of* $\operatorname{Ker}\delta^3$, *we need to replace in the set in Proposition* 2.3 (c), *the element* $\big(dy, y\big)$ *in* (2.21) *by the element* $\big(0,y\big)$.

Corollary 2.6. *Assume that* k *is even and* $d \neq 0$. *Then*

(a) *to get a basis of* $\mathrm{HH}^1(R)$, *we need to add the element*$\big(y,0\big)$ *to the set in Corollary* 2.4 (a)*;*

(b) *to get a basis of* $\mathrm{HH}^2(R)$, *we need to add the element* $\big((xy)^k,0\big)$ *to the set in Corollary* 2.4 (b)*;*

(c) *to get a basis of* $\mathrm{HH}^3(R)$, *we need to replace in the set in Corollary* 2.4 (c), *the element* $\big(0,\widetilde{y}\big)$ *in* (2.30) *by the element* $\big(0,y\big)$.

Proof. Again the statement follows from the description of bases for $\operatorname{Im}\delta^i$, $0 \le i \le 2$, in [2], and Proposition 2.5. □

Proposition 2.7. *Assume that* k *is odd and* $d = 0$. *Then*

(a) *to get a basis of the vector space* $\operatorname{Ker}\delta^1$, *we need to add the element* $\big(1, y(xy)^{k-2} + c(xy)^{k-1}\big)$ *to the set in Proposition* 2.3, (a)*;*

(b) *to get a basis of the vector space* $\operatorname{Ker}\delta^2$, *we need to replace in the set in Proposition* 2.3, (b), *the element* $\big(d(yx)^{k-1}, c(xy)^{k-1} + y(xy)^{k-2}\big)$ *in* (2.11) *by* $\big(0, y(xy)^{k-2}\big)$, *and the element* $\big(0,(xy)^{k-1} + (yx)^{k-1}\big)$ *in* (2.13) *by a pair of the elements* $\big(0,(xy)^{k-1}\big)$, $\big(0,(yx)^{k-1}\big)$*;*

(c) *to get a basis of the vector space* $\operatorname{Ker}\delta^3$, *we need to replace in the set in Proposition* 2.3, (c), *the element* $\big(y, x(yx)^{k-1}\big)$ *in* (2.21) *by the element* $\big(0, x(yx)^{k-1}\big)$, *and to add the elements*

$$(1,0),\ (y,0),\ (0,1)$$

to this set.

Corollary 2.8. *Assume that* k *is odd and* $d = 0$. *Then*

(a) *to get a basis of* $\mathrm{HH}^1(R)$, *we need to add the element* $\big(1, y(xy)^{k-2} + c(xy)^{k-1}\big)$ *to the set in Corollary* 2.4 (a),

(b) *to get a basis of* $\mathrm{HH}^2(R)$, *we need to replace in the set in Corollary* 2.4 (b), *the element* $\big(d(yx)^{k-1}, c(xy)^{k-1} + y(xy)^{k-2}\big)$ *in* (2.26) *by a pair of the elements* $\big(0, y(xy)^{k-2}\big), \big(0,(xy)^{k-1}\big)$, *and the element* $\big(0,(xy)^k\big)$ *in* (2.28) *by a pair of the elements* $\big(x(yx)^{k-1}, 0\big), \big((xy)^k, 0\big)$*;*

(c) *to get a basis of* $\mathrm{HH}^3(R)$, *we need to replace in the set in Corollary* 2.4 (c), *the element* $\big(y, x(yx)^{k-1}\big)$ *in* (2.30) *by a pair of the elements* $(y,0)$, $\big(0, x(yx)^{k-1}\big)$, *and to add the elements*

$$(1,0),\ (x(yx)^{k-1}, 0),\ (0,1)$$

to this set.

Proof. Again the statement follows from Proposition 2.7 (and from the description of bases for $\operatorname{Im}\delta^i$, $0 \le i \le 2$, in [2]). □

Proposition 2.9. *Assume that k is even and $d = 0$. Then*

(a) *to get a basis of* $\operatorname{Ker}\delta^1$, *we need to add the element* $\big(1, y(xy)^{k-2} + c(xy)^{k-1}\big)$ *to the set in Proposition* 2.5 (a)*;*

(b) $\operatorname{Ker}\delta^2$ *admits as a K-basis the set in Proposition* 2.7 (b)*;*

(c) $\operatorname{Ker}\delta^3$ *admits as a K-basis the set in Proposition* 2.7 (c).

Corollary 2.10. *Assume that k is even and $d = 0$. Then*

(a) *to get a basis* $\mathrm{HH}^1(R)$, *we need to replace in the set in Corollary* 2.8 (a), *the element* $\big(cy, 1 + cx\big)$ *by the pair*

$$\big(y, 0\big),\ \big(0, 1 + cx\big);$$

(b) *to get a basis of* $\mathrm{HH}^2(R)$, *we need to add the element* $\big(0, (xy)^k\big)$ *to the set in Corollary* 2.8 (a)*;*

(c) $\mathrm{HH}^3(R)$ *admits as a K-basis the set in Corollary* 2.8 (c).

Proof. The statement follows from Proposition 2.9 (and from the description of bases for $\operatorname{Im}\delta^i$, $0 \le i \le 2$, in [2]). □

The above results imply the following statement obtained in [2], and we state it for convenience of the reader.

Corollary 2.11. (I) *Assume that $d \neq 0$. Then*

$$\text{(Ia)}\qquad \dim_K \mathrm{HH}^1(R) = \dim_K \mathrm{HH}^2(R) = \begin{cases} k+5 & \text{if } k \text{ is even,}\\ k+4 & \text{if } k \text{ is odd;}\end{cases}$$

$$\text{(Ib)}\qquad \dim_K \mathrm{HH}^3(R) = k + 4.$$

(II) *Assume that $d = 0$. Then*

$$\text{(IIa)}\qquad \dim_K \mathrm{HH}^1(R) = \begin{cases} k+6 & \text{if } k \text{ is even,}\\ k+5 & \text{if } k \text{ is odd;}\end{cases}$$

$$\text{(IIb)}\qquad \dim_K \mathrm{HH}^2(R) = \begin{cases} k+7 & \text{if } k \text{ is even,}\\ k+6 & \text{if } k \text{ is odd;}\end{cases}$$

$$\text{(IIc)}\qquad \dim_K \mathrm{HH}^3(R) = k + 8.$$

Remark 2.12. The short exact sequence (2.3) induces a long cohomology sequence in which the connecting homomorphisms are zero starting at some place (see the proof of Proposition 4.10 in [2]), and hence, for any $n \ge 4$,

$$\mathrm{HH}^n(R) \simeq \mathrm{HH}^{n-4}(R) \oplus \mathrm{H}^n(\mathcal{X}^\bullet),$$

where $\mathcal{X}^\bullet = \operatorname{Hom}_\Lambda(X_\bullet, R)$ (with $X_\bullet$ as in (2.2)).

Remark 2.13. From the form of the differentials d_n^Q (see [2]), one can easily see that if $n \geq 4$ and

$$f = (r_1, \dots, r_{t_n}) \in \mathrm{Hom}_\Lambda(Q_n, R) (\simeq R^{t_n})$$

is a cocycle (i.e., $\delta^n(f) = 0$), then its "pieces"

$$(r_1, r_2) \in \mathcal{X}^n \simeq R^2 \quad \text{and} \quad (r_3, \dots, r_{t_n}) \in \mathrm{Hom}_\Lambda(Q_{n-4}, R)$$

are also cocycles in the corresponding complexes. With the help of Corollaries 2.4–2.10 (and Proposition 2.2), this observation allows us to write down representatives of the cohomology classes that form bases in the groups $\mathrm{HH}^n(R)$, $n \geq 4$.

Remark 2.14. In the sequel, if $x \in \operatorname{Ker} \delta^n$ is a n-cocycle, we keep notation x for its cohomology class $\operatorname{cl} x \in \mathrm{HH}^n(R)$.

3. Generators and relations

We present briefly an interpretation of the Yoneda product in the algebra $\mathrm{HH}^*(R) = \bigoplus_{m \geq 0} \mathrm{Ext}_\Lambda^m(R, R)$ used earlier in [12]. Let $\mu\colon Q_\bullet \to R$ be the minimal Λ-projective resolution (see Sec. 2). Consider the complex

$$\mathrm{Hom}_\Lambda(Q_\bullet, R) = \big(\mathrm{Hom}_\Lambda(Q_n, R), \delta^n\big);$$

here, as above, the δ^n are the differentials induced by the differentials of the resolution $Q_\bullet$. If $f \in \operatorname{Ker} \delta^n$, $g \in \operatorname{Ker} \delta^t$ are cocycles, then $\operatorname{cl} g \cdot \operatorname{cl} f = \operatorname{cl}(\mu \mathrm{T}^0(g) \mathrm{T}^t(f))$, where $\mathrm{T}^i(h)$ is the ith translate of the cocycle h. In the sequel, we define the translates $\mathrm{T}^i(h)$ $(i \geq 0)$ with the help of matrices that correspond to the standard decompositions of the modules Q_n.

Now, we begin calculation of the multiplicative structure of the algebra $\mathrm{HH}^*(R)$ for algebras from the family under consideration. This structure depends essentially on the fact that k is even or odd. Moreover, the case where $k = 2$ must be studied separately.

Case 1. First, assume that k is odd and $d \neq 0$.
We consider the following homogeneous elements of $\mathrm{HH}^*(R)$:

— of degree 0 : $$\begin{cases} p_1 := xy + yx,\ p_2 := x(yx)^{k-1}, \\ p_3 := y(xy)^{k-1},\ p_4 := (xy)^k; \end{cases} \tag{3.1}$$

— of degree 1 : $$\begin{cases} u_1 := (cy + d(xy)^{k-1}, 1 + cx), \\ u_2 := (yxy, 0),\ u_3 := (x(yx)^{k-1}, 0); \end{cases} \tag{3.2}$$

— of degree 2 : $$\begin{cases} v_1 := (1, 0),\ v_2 := (y, 0),\ v_3 := (0, x), \\ v_4 := (dyx, y + cxy); \end{cases} \tag{3.3}$$

— of degree 3 : $$\widetilde{w} := (0, \widetilde{y}); \tag{3.4}$$

— of degree 4 : $$z := (0, 0, 1). \tag{3.5}$$

Proposition 3.1. *Assume that k is odd and $d \neq 0$. In the algebra $\mathrm{HH}^*(R)$, the elements of the set*

$$\mathcal{Y}_1 = \{p_1, p_2, p_3, p_4, u_1, u_2, u_3, v_1, v_2, v_3, v_4, \widetilde{w}, z\} \tag{3.6}$$

satisfy the following relations:

$$\left.\begin{array}{c} p_1^k = p_2^2 = p_3^2 = p_4^2 = 0, \\ p_i p_j = 0 \ \ \textit{for} \ \ 1 \le i < j \le 4; \end{array}\right\} \tag{3.7}$$

$$cp_1u_1 = d(p_2u_1 + p_1^{k-2}u_2),\ p_1^{k-1}u_2 = 0, \tag{3.8}$$

$$p_2u_2 = p_3u_2 = p_4u_2 = 0; \tag{3.9}$$

$$p_ju_3 = 0 \ \ \textit{for} \ \ 1 \le j \le 4;, \tag{3.10}$$

$$u_1u_2 = u_2^2 = p_1^{k-1}v_4 = u_2u_3 = 0, \tag{3.11}$$

$$u_1u_3 = cp_4v_1,\ u_3^2 = 0;, \tag{3.12}$$

$$cp_1v_2 = dp_4v_1,\ p_4v_1 = p_2v_2, \tag{3.13}$$

$$p_4u_1^2 = p_1v_2 = p_3v_3 = p_2v_4, \tag{3.14}$$

$$p_1u_1^2 = p_3u_1^2 = 0, \tag{3.15}$$

$$p_3v_2 = p_4v_2 = p_3v_4 = p_4v_4 = 0, \tag{3.16}$$

$$p_1v_1 = p_2v_1 = p_3v_1 = p_1v_3 = p_2v_3 = p_4v_3 = 0, \tag{3.17}$$

$$cu_1^3 = du_1v_1,\ p_1^2\widetilde{w} = u_2v_4, dp_2\widetilde{w} = u_1v_4, \tag{3.18}$$

$$u_1v_2 = p_1^{k-1}\widetilde{w} + cp_2\widetilde{w},\ p_4u_1^3 = 0, \tag{3.19}$$

$$p_3\widetilde{w} = p_4\widetilde{w} = 0, \tag{3.20}$$

$$u_2v_1 = u_2v_2 = u_2v_3 = 0, \tag{3.21}$$

$$u_3v_2 = u_3v_3 = u_3v_4 = 0, \tag{3.22}$$

$$v_2^2 = v_3^2 = 0;\ v_iv_j = 0 \ \ \textit{for} \ \ i < j; \tag{3.23}$$

$$u_1^4 = 0, \tag{3.24}$$

$$u_1\widetilde{w} = (p_3 + cp_4)z, \tag{3.25}$$

$$v_4^2 = p_1^2z, \tag{3.26}$$

$$u_1^2v_j = 0\, \textit{for}\, 1 \le j \le 4, \tag{3.27}$$

$$\textit{for} \ \ c = 0 \ \ u_1v_1 = 0; \tag{3.28}$$

$$u_2\widetilde{w} = 0, \tag{3.29}$$

$$v_3\widetilde{w} = p_4u_1z,\ v_4\widetilde{w} = u_2z,\ v_2\widetilde{w} = 0; \tag{3.30}$$

$$(\widetilde{w})^2 = cp_4u_1^2z. \tag{3.31}$$

Proof. Relations (3.7), (3.8), (3.9), (3.10), (3.14), (3.16), (3.17), (3.20) are verified directly. In proving of the remaining relations we have to compute the translates of suitable orders for elements in $\mathcal{Y}_1$, which have positive degree.

Proposition 2.1 implies immediately the following statement.

Lemma 3.2. *For any $i \ge 0$, the projection onto the direct summand $\pi_{i+4}\colon Q_{i+4} = X_{i+4} \oplus Q_i \to Q_i$ is the ith translate $\mathrm{T}^i(z)$ of the cocycle z.*

Suitable translates of the other elements in $\mathcal{Y}_1$ with positive degree are presented in the following lemma.

Lemma 3.3. *As translates of the elements in $\mathcal{Y}_1 \setminus \{z\}$ of positive degree, one can take the homomorphisms determined by the matrices*

$$\mathrm{T}^0(u_1) = \big(cy \otimes 1 + d(xy)^{k-1} \otimes 1, 1 \otimes 1 + cx \otimes 1\big),$$

$$\mathrm{T}^1(u_1) = \begin{pmatrix} * & * \\ d\sum\limits_{i=0}^{k-2} y(xy)^i \otimes y(xy)^{k-2-i} & * \end{pmatrix}$$

with

$$\begin{aligned}
(\mathrm{T}^1(u_1))_{11} &= cd\sum_{i=0}^{k-1} x(yx)^i \otimes (xy)^{k-1-i} + d\sum_{i=0}^{k-1}(yx)^i \otimes (xy)^{k-1-i} \\
&\quad + d\sum_{i=0}^{k-3} x(yx)^i \otimes y(xy)^{k-2-i} + cy \otimes 1, \\
(\mathrm{T}^1(u_1))_{12} &= c\sum_{i=0}^{k-1}(yx)^i \otimes (xy)^{k-1-i} + c\sum_{i=1}^{k-1}(xy)^i \otimes (xy)^{k-1-i} \\
&\quad + c^2\sum_{i=0}^{k-1} x(yx)^i \otimes (xy)^{k-1-i} + \sum_{i=0}^{k-2}(yx)^i \otimes y(xy)^{k-2-i} \\
&\quad + dx(yx)^{k-1} \otimes (xy)^{k-2}, \\
(\mathrm{T}^1(u_1))_{22} &= 1 \otimes 1 + cx \otimes 1 + d(yx)^{k-2} \otimes y(xy)^{k-1} \\
&\quad + d(xy)^{k-1} \otimes y(xy)^{k-2} + dy(xy)^{k-1} \otimes (xy)^{k-2}; \\
\mathrm{T}^2(u_1) &= \begin{pmatrix} 1 \otimes 1 & (yx)^{3k-5} \otimes y \\ 0 & x \otimes 1 + 1 \otimes x + y \otimes y(xy)^{k-2} \end{pmatrix}, \\
\mathrm{T}^3(u_1) &= \begin{pmatrix} 1 \otimes 1 & 0 & 0 \\ 0 & 1 \otimes 1 & * \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
(\mathrm{T}^3(u_1))_{23} &= \sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} \\
&\quad + (yx)^{k-1} \otimes y(xy)^{k-2} + y(xy)^{k-2} \otimes (xy)^{k-1}; \\
\mathrm{T}^0(u_2) &= \big(yxy \otimes 1, 0\big), \\
\mathrm{T}^1(u_2) &= \begin{pmatrix} yxy \otimes 1 + d\sum\limits_{i=2}^{k-1}(i-1)x(yx)^i \otimes (xy)^{k-i} & * \\ d\sum\limits_{i=2}^{k-1}(i-1)(xy)^i \otimes y(xy)^{k-i} & * \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
(\mathrm{T}^1(u_2))_{12} &= \sum_{i=1}^{k-2} i(yx)^{i+1} \otimes (xy)^{k-1-i} + c\sum_{i=1}^{k-2} ix(yx)^{i+1} \otimes (xy)^{k-1-i}, \\
(\mathrm{T}^1(u_2))_{22} &= \sum_{i=1}^{k-2} iy(xy)^i \otimes y(xy)^{k-1-i} + c\sum_{i=1}^{k-2} i(xy)^{i+1} \otimes y(xy)^{k-1-i},
\end{aligned}$$

$$\mathrm{T}^2(u_2) = \begin{pmatrix} 0 & 1\otimes(xy)^{k-1} + 1\otimes(yx)^{k-1} \\ 0 & y\otimes 1 + 1\otimes y \end{pmatrix};$$

$$\mathrm{T}^0(u_3) = \big(x(yx)^{k-1}\otimes 1, 0\big),$$

$$\mathrm{T}^1(u_3) = \begin{pmatrix} x(yx)^{k-1}\otimes 1 & 0 \\ 0 & x(yx)^{k-1}\otimes y(xy)^{k-2} \end{pmatrix};$$

$$\mathrm{T}^0(v_1) = \big(1\otimes 1, 0\big), \quad \mathrm{T}^1(v_1) = \begin{pmatrix} 1\otimes 1 & \sum_{i=0}^{k-2} x(yx)^i\otimes x(yx)^{k-2-i} \\ 0 & * \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^1(v_1))_{22} &= \sum_{i=0}^{k-1}(xy)^i\otimes(yx)^{k-1-i} \\ &\quad + c^2 y(xy)^{k-2}\otimes(xy)^k + c^3(xy)^{k-1}\otimes(xy)^k;\end{aligned}$$

$$\mathrm{T}^2(v_1) = \begin{pmatrix} 1\otimes 1 & * & 0 \\ 0 & * & (xy)^{k-1}\otimes(yx)^{k-1} \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^2(v_1))_{12} &= c^2(yx)^{k-1}\otimes x(yx)^{k-1} + c^2 x(yx)^{k-1}\otimes(xy)^{k-1}, \\ (\mathrm{T}^2(v_1))_{22} &= x\otimes 1 + 1\otimes x + cx\otimes x + cx^2\otimes 1 \\ &\quad + c^2\cdot 1\otimes(xy)^k + c^2 x\otimes y(xy)^{k-1} + c^3 yx\otimes x(yx)^{k-1};\end{aligned}$$

$$\mathrm{T}^0(v_2) = \big(y\otimes 1, 0\big), \quad \mathrm{T}^1(v_2) = \begin{pmatrix} y\otimes 1 & * \\ 0 & \sum_{i=0}^{k-1} y(xy)^i\otimes(yx)^{k-1-i} \end{pmatrix}$$

with

$$(\mathrm{T}^1(v_2))_{12} = \sum_{i=1}^{k-1}(yx)^i\otimes x(yx)^{k-1-i} + c^2(yx)^{k-1}\otimes(xy)^k + c^3 x(yx)^{k-1}\otimes(xy)^k;$$

$$\mathrm{T}^2(v_2) = \begin{pmatrix} y\otimes 1 & \rho + c^2x^2\otimes x(yx)^{k-1} & 0 \\ d^2(xy)^k\otimes y(xy)^{k-2} & * & * \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^2(v_2))_{22} &= 1\otimes xy + x\otimes y + cx\otimes xy + cx^2\otimes y \\ &\quad + d(xy)^k\otimes y(xy)^{k-2} + dx(yx)^{k-1}\otimes x, \\ (\mathrm{T}^2(v_2))_{23} &= y(xy)^{k-1}\otimes(xy)^{k-1} + y(xy)^{k-1}\otimes(yx)^{k-1};\end{aligned}$$

$$\mathrm{T}^0(v_3) = \big(0, x\otimes 1\big), \quad \mathrm{T}^1(v_3) = \begin{pmatrix} 0 & * \\ dx\otimes y + cdy(xy)^{k-1}\otimes y & * \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^1(v_3))_{12} &= dx(yx)^{k-1}\otimes(xy)^{k-1} + d(xy)^k\otimes x(yx)^{k-2}, \\ (\mathrm{T}^1(v_3))_{22} &= yx\otimes 1 + x\otimes y + cx^2\otimes y + dx\otimes x(yx)^{k-1} \\ &\quad + d(xy)^k\otimes y(xy)^{k-2} + cd(xy)^k\otimes(yx)^{k-1};\end{aligned}$$

$\mathrm{T}^2(v_3)$ *is a* 2×3*-matrix, in which*

$$(\mathrm{T}^2(v_3))_{11} = dx^2\otimes 1 + d(yx)^{k-1}\otimes y + cdx(yx)^{k-1}\otimes y$$

$$
\begin{aligned}
&\quad + d^2x(yx)^{k-2} \otimes (xy)^k,\\
(\mathrm{T}^2(v_3))_{12} &= dx(yx)^{k-2} \otimes (xy)^k + dx(yx)^{k-1} \otimes (xy)^{k-1},\\
(\mathrm{T}^2(v_3))_{13} &= x(yx)^{k-2} \otimes (xy)^k + x(yx)^{k-1} \otimes (xy)^{k-1}\\
&\quad + (yx)^{k-1} \otimes x(yx)^{k-1},\\
(\mathrm{T}^2(v_3))_{21} &= c^2d^2y(xy)^{k-1} \otimes (xy)^k + d^2\sum_{i=0}^{k-1}(xy)^i \otimes (xy)^{k-i},\\
(\mathrm{T}^2(v_3))_{22} &= d\sum_{i=1}^{k}(xy)^i \otimes (xy)^{k-i} + d\sum_{i=1}^{k-1} y(xy)^{i-1} \otimes x(yx)^{k-i}\\
&\quad + dx \otimes y(xy)^{k-1} + c^2d(xy)^k \otimes y(xy)^{k-1},\\
(\mathrm{T}^2(v_3))_{23} &= \sum_{i=1}^{k} y(xy)^{i-1} \otimes x(yx)^{k-i} + \sum_{i=1}^{k-1}(xy)^i \otimes (xy)^{k-i}\\
&\quad + x \otimes y(xy)^{k-1} + (yx)^{k-1} \otimes xy\\
&\quad + cyx \otimes x(yx)^{k-1} + c^2(xy)^k \otimes y(xy)^{k-1};
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}^0(v_4) &= \big(dyx \otimes 1, y \otimes 1 + cxy \otimes 1\big),\\
\mathrm{T}^1(v_4) &= \begin{pmatrix} dx \otimes y & * \\ dy \otimes y + d^2 \cdot 1 \otimes (xy)^k & * \end{pmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
(\mathrm{T}^1(v_4))_{12} &= yx \otimes 1 + xy \otimes 1 + dx \otimes x(yx)^{k-1} + cd(xy)^k \otimes (xy)^{k-1},\\
(\mathrm{T}^1(v_4))_{22} &= y \otimes y + d(xy)^k \otimes 1 + dy \otimes x(yx)^{k-1} + d \cdot 1 \otimes (xy)^k\\
&\quad + c^2d(xy)^k \otimes y(xy)^{k-1};
\end{aligned}
$$

$$
\mathrm{T}^2(v_4) = \begin{pmatrix} dxy \otimes 1 & * & 0 \\ d^3(xy)^{k-1} \otimes (xy)^k & * & * \end{pmatrix}
$$

with

$$
\begin{aligned}
(\mathrm{T}^2(v_4))_{12} &= yx \otimes 1 + xy \otimes 1 + cd(xy)^k \otimes (xy)^{k-1},\\
(\mathrm{T}^2(v_4))_{22} &= dyx \otimes x + d^2(xy)^{k-1} \otimes (xy)^k + d^2(xy)^k \otimes (yx)^{k-1},\\
(\mathrm{T}^2(v_4))_{23} &= xyx \otimes 1 + yx \otimes x + xy \otimes x + y \otimes y(xy)^{k-1}\\
&\quad + cxyx \otimes x + dx(yx)^{k-1} \otimes y(xy)^{k-1},
\end{aligned}
$$

$$
\mathrm{T}^0(\widetilde{w}) = \big(0, \widetilde{y} \otimes 1\big),
$$

$\mathrm{T}^1(\widetilde{w})$ *is a* 2×3*-matrix, in which*

$$
\begin{aligned}
(\mathrm{T}^1(\widetilde{w}))_{11} &= d\widetilde{y} \otimes 1 + d^2\sum_{i=1}^{k-2} ix(yx)^i \otimes (xy)^{k-1-i},\\
(\mathrm{T}^1(\widetilde{w}))_{12} &= \widetilde{y} \otimes 1 + d\sum_{i=1}^{k-2} ix(yx)^i \otimes (xy)^{k-1-i}
\end{aligned}
$$

$$+ d\sum_{i=1}^{k-2} i(yx)^{i+1}\otimes x(yx)^{k-2-i} + cdy(xy)^{k-1}\otimes (xy)^{k-1},$$

$$(\mathrm{T}^1(\widetilde{w})_{13} = \sum_{i=1}^{k-2} ix(yx)^i\otimes (xy)^{k-1-i} + \sum_{i=1}^{k-2} i(yx)^i\otimes x(yx)^{k-1-i}$$
$$+ cy(xy)^{k-1}\otimes (xy)^{k-1},$$

$$(\mathrm{T}^1(\widetilde{w})_{21} = d^2\sum_{i=1}^{k-2} i(xy)^i\otimes y(xy)^{k-1-i},$$

$$(\mathrm{T}^1(\widetilde{w})_{22} = d\sum_{i=1}^{k-2} i(xy)^i\otimes y(xy)^{k-1-i} + d\sum_{i=1}^{k-2} iy(xy)^i\otimes (yx)^{k-1-i},$$

$$(\mathrm{T}^1(\widetilde{w})_{23} = \sum_{i=1}^{k-2} i(xy)^i\otimes y(xy)^{k-1-i} + \sum_{i=0}^{k-1} (i+1)y(xy)^i\otimes (yx)^{k-1-i} +$$
$$+ dx(yx)^{k-1}\otimes (yx)^{k-1};$$

$$\mathrm{T}^2(\widetilde{w}) = \begin{pmatrix} 0 & * & 0 & 0\\ 0 & 0 & * & * \end{pmatrix}$$

with

$$(\mathrm{T}^2(\widetilde{w}))_{12} = 1\otimes\widetilde{y} + c^3dx(yx)^{k-1}\otimes (xy)^k,$$
$$(\mathrm{T}^2(\widetilde{w}))_{23} = yx\otimes 1 + y\otimes x + cyx\otimes x + dx\otimes x(yx)^{k-1}$$
$$+ dy(xy)^{k-1}\otimes (yx)^{k-1} + d(xy)^{k-1}\otimes y(xy)^{k-1}$$
$$+ cd(xy)^k\otimes (yx)^{k-1},$$
$$(\mathrm{T}^2(\widetilde{w})))_{24} = y(xy)^{k-1}\otimes 1 + c(xy)^k\otimes 1 + cy(xy)^{k-1}\otimes x$$
$$+ c^2(xy)^k\otimes x;$$

$$\mathrm{T}^3(\widetilde{w}) = \begin{pmatrix} 0 & * & 0 & 0\\ 0 & 0 & * & * \end{pmatrix}$$

with

$$(\mathrm{T}^3(\widetilde{w}))_{12} = 1\otimes\widetilde{y} + c^3dx(yx)^{k-1}\otimes (xy)^k,$$

$$(\mathrm{T}^3(\widetilde{w}))_{23} = 1\otimes\widetilde{y} + d\sum_{i=1}^{k-2} ix(yx)^i\otimes (xy)^{k-1-i} + c^2dx(yx)^{k-1}\otimes y(xy)^{k-1},$$

$$(\mathrm{T}^3(\widetilde{w}))_{24} = \sum_{i=1}^{k-2} i(yx)^i\otimes (xy)^{k-1-i} + c\sum_{i=1}^{k-2} ix(yx)^i\otimes (xy)^{k-1-i}.$$

The *proof* of the lemma is a direct verification of the relations $\mu\mathrm{T}^0(b) = b$, $d_{i-1}\mathrm{T}^i(b) = \mathrm{T}^{i-1}(b)d_{i+\deg b-1}$ $(i>0)$, where $b\in\mathcal{Y}_1\setminus\{z\}$ with $\deg b>0$.

Now the proof of Proposition 3.1 is completed with using direct calculations with matrices described in Lemma 3.3, and we leave it to the reader. □

Remark 3.4. It should be noted that the formulas for the translates of u_3, v_1, v_4 (see Lemma 3.3) remain valid for any even k (if $d \neq 0$), and also the translates of v_2 are valid for even $k > 2$ (if these elements are included in the set of generators of the algebra $\mathrm{HH}^*(R)$).

Proposition 3.5. *For any $\ell \in \mathbb{N}$, we have*

$$v_1^\ell = \big(1, \mathrm{O}\,\big) \in \mathrm{HH}^{2\ell}(R), \quad v_1^\ell u_3 = \big(x(yx)^{k-1}, \mathrm{O}\,\big) \in \mathrm{HH}^{2\ell+1}(R), \tag{3.32}$$

$$v_1^{\ell-1} u_3 \widetilde{w} = \big(0, (xy)^k, \mathrm{O}\,\big) \in \mathrm{HH}^{2\ell+2}(R), \quad v_1^{\ell-1} \widetilde{w} = \big(0, \widetilde{y}, \mathrm{O}\,\big) \in \mathrm{HH}^{2\ell+1}(R). \tag{3.33}$$

Remark 3.6. Owing to the description of the bases for the groups $\mathrm{H}^{2\ell}(\mathcal{X}^\bullet)$ (with $\ell \geq 2$) given in [2, Lemma 4.11], the elements

$$\big(1, \mathrm{O}_{r+1}\,\big), \big(0, (xy)^k, \mathrm{O}_r\,\big) \tag{3.34}$$

(for suitable r) form a basis of the image of $\mathrm{H}^{2\ell}(\mathcal{X}^\bullet)$ in the group $\mathrm{HH}^{2\ell}(R)$ (see Remark 2.12). Similarly, the elements

$$\big(0, \widetilde{y}, \mathrm{O}_r\,\big), \big(x(yx)^{k-1}, \mathrm{O}_{r+1}\,\big) \tag{3.35}$$

form a basis of the image of $\mathrm{H}^{2\ell+1}(\mathcal{X}^\bullet)$ in the group $\mathrm{HH}^{2\ell+1}(R)$.

Proof of Proposition 3.5. 1) A simple verification (with using Lemma 3.3) shows that $v_1^2 = \big(1, \mathrm{O}_2\,\big)$. Next, assuming $v_1^\ell = \big(1, \mathrm{O}\,\big)$ ($\ell \geq 2$), we get the following translates of this element:

$$\mathrm{T}^0(v_1^\ell) = \big(1 \otimes 1, \mathrm{O}\,\big),$$

$$\mathrm{T}^1(v_1^\ell) = \begin{pmatrix} 1 \otimes 1 & \sum\limits_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} & \mathrm{O} \\ 0 & \sum\limits_{i=0}^{k-1} (xy)^i \otimes (yx)^{k-1-i} + c^3 (xy)^{k-1} \otimes (xy)^k & \mathrm{O} \end{pmatrix},$$

$$\mathrm{T}^2(v_1^\ell) = \begin{pmatrix} 1 \otimes 1 & c^3 x(yx)^{k-1} \otimes x(yx)^{k-1} & 0 & \mathrm{O} \\ 0 & * & (xy)^{k-1} \otimes (yx)^{k-1} & \mathrm{O} \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^2(v_1^\ell))_{22} &= x \otimes 1 + 1 \otimes x + cx \otimes x + cx^2 \otimes 1 \\ &\quad + c^2 x^2 \otimes x + c^2 x^3 \otimes 1 + c^3 y(xy)^{k-1} \otimes y(xy)^{k-1}.\end{aligned}$$

Hence, we have

$$v_1^{\ell+1} = \mu \mathrm{T}^0(v_1) \mathrm{T}^2(v_1^\ell) = \big(1, \mathrm{O}\,\big).$$

2) We have seen that $v_1 u_3 = \big(x(yx)^{k-1}, 0\big)$. Next, assuming that $v_1^\ell u_3 = \big(x(yx)^{k-1}, \mathrm{O}\big)$ ($\ell \geq 0$), we get the following translates:

$$\mathrm{T}^0(v_1^\ell u_3) = \big(x(yx)^{k-1} \otimes 1, \mathrm{O}\,\big),$$

$$\mathrm{T}^1(v_1^\ell u_3) = \begin{pmatrix} x(yx)^{k-1} \otimes 1 & 0 & \mathrm{O} \\ 0 & x(yx)^{k-1} \otimes (yx)^{k-1} & \mathrm{O} \end{pmatrix},$$

$$\mathrm{T}^2(v_1^\ell u_3) = \begin{pmatrix} x(yx)^{k-1} \otimes 1 & 0 & \mathrm{O} \\ 0 & x(yx)^{k-1} \otimes x & \mathrm{O} \end{pmatrix}.$$

Then

$$v_1^{\ell+1} u_3 = \mu \mathrm{T}^0(v_1) \mathrm{T}^2(v_1^\ell u_3) = \big(x(yx)^{k-1}, \mathrm{O}\,\big).$$

3) A direct calculation gives $u_3\widetilde{w} = \big(0, (xy)^k, \mathrm{O}\big)$. Next, assuming $v_1^{\ell-1}u_3\widetilde{w} = \big(0, (xy)^k, \mathrm{O}\big)$ (for $\ell \in \mathbb{N}$), we obtain

$$\mathrm{T}^0(v_1^{\ell-1}u_3\widetilde{w}) = \big(0, (xy)^k \otimes 1, \mathrm{O}\big),$$
$$\mathrm{T}^1(v_1^{\ell-1}u_3\widetilde{w}) = \begin{pmatrix} d(xy)^k \otimes 1 & (xy)^k \otimes 1 & 0 & \mathrm{O} \\ 0 & 0 & (xy)^k \otimes (yx)^{k-1} & \mathrm{O} \end{pmatrix},$$
$$\mathrm{T}^2(v_1^{\ell-1}u_3\widetilde{w}) = \begin{pmatrix} 0 & (xy)^k \otimes 1 & 0 & \mathrm{O} \\ 0 & d(xy)^k \otimes x & (xy)^k \otimes x & \mathrm{O} \end{pmatrix}.$$

Then

$$v_1^{\ell}u_3\widetilde{w} = \mu \mathrm{T}^0(v_1)\mathrm{T}^2(v_1^{\ell-1}u_3\widetilde{w}) = \big(0, (xy)^k, \mathrm{O}\big).$$

4) Assuming $v_1^{\ell-1}\widetilde{w} = \big(0, \widetilde{y}, \mathrm{O}\big)$ ($\ell \in \mathbb{N}$), we obtain the following translates:

$$\mathrm{T}^0(v_1^{\ell-1}\widetilde{w}) = \big(0, \widetilde{w} \otimes 1, \mathrm{O}\big),$$
$$\mathrm{T}^1(v_1^{\ell-1}\widetilde{w}) = \begin{pmatrix} d\widetilde{w} \otimes 1 & \widetilde{w} \otimes 1 & \sum\limits_{i=1}^{k-1}(yx)^i \otimes x(yx)^{k-1-i} & \mathrm{O} \\ 0 & 0 & * & \mathrm{O} \end{pmatrix}$$

with

$$(\mathrm{T}^1(v_1^{\ell-1}\widetilde{w}))_{23} = \sum_{i=0}^{k-1} y(xy)^i \otimes (yx)^{k-1-i} + dx(yx)^{k-1} \otimes (yx)^{k-1};$$
$$\mathrm{T}^2(v_1^{\ell-1}\widetilde{w}) = \begin{pmatrix} 0 & \widetilde{y} \otimes 1 + c^3 d(xy)^k \otimes x(yx)^{k-1} & 0 & 0 & \mathrm{O} \\ 0 & dyx \otimes 1 + d\widetilde{y} \otimes x + cdyx \otimes x & * & * & \mathrm{O} \end{pmatrix}$$

with

$$(\mathrm{T}^2(v_1^{\ell-1}\widetilde{w}))_{23} = yx \otimes 1 + \widetilde{y} \otimes x + cyx \otimes x,$$
$$(\mathrm{T}^2(v_1^{\ell-1}\widetilde{w}))_{24} = y(xy)^{k-1} \otimes (yx)^{k-1}.$$

Then

$$v_1^{\ell}\widetilde{w} = \mu \mathrm{T}^0(v_1)\mathrm{T}^2(v_1^{\ell-1}\widetilde{w}) = \big(0, \widetilde{y}, \mathrm{O}\big).$$

□

Proposition 3.7. *Assume that k is odd and $d \neq 0$. The set $\mathcal{Y}_1$ in (3.6) generates $\mathrm{HH}^*(R)$ as a K-algebra.*

Proof. Let $\mathcal{H}$ denote a K-subalgebra of $\mathrm{HH}^*(R)$, generated by the set $\mathcal{Y}_1 \cup \{1\}$ (here, 1 denotes the unity of the algebra $\mathrm{HH}^*(R)$). First, we prove that $\bigcup\limits_{i=0}^{3} \mathrm{HH}^i(R) \subset \mathcal{H}$, and then the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ follows by induction on n. Since $p_1^i = (xy)^i + (yx)^i$, $1 \le i \le k-1$, we obtain the inclusion $\mathrm{HH}^0(R) \subset \mathcal{H}$.

The basis elements of $\mathrm{HH}^1(R)$ described in Corollary 2.4 (a), satisfy the relations

$$\big(y(xy)^i, 0\big) = p_1^{i-1}u_2 \ \text{ for } 1 \le i \le k-1,$$
$$\big((xy)^k, 0\big) = d^{-1}p_1u_1,\ \big(0, y(xy)^{k-1}\big) = (p_3 + cp_4)u_1, \big(0, (xy)^k\big) = p_4u_1.$$

Hence, $\mathrm{HH}^1(R) \subset \mathcal{H}$.

Next, the basis elements of $\mathrm{HH}^2(R)$, described in Corollary 2.4 (b), satisfy the relations

$$\big(d(xy)^{i+1}, y(xy)^i + c(xy)^{i+1}\big) = p_1^i v_4 \ \text{ for } \ 0 \le i \le k-2,$$
$$\big(0, 1\big) = u_1^2,\ \big(0, (xy)^k\big) = p_4u_1^2.$$

Consequently, $\mathrm{HH}^2(R) \subset \mathcal{H}$. Here and below, we multiply the elements of $\mathcal{Y}_1$, having a positive degree, with the help of translates of these elements presented in Lemma 3.3 (see also Lemma 3.2).

We establish that the basis elements of $\mathrm{HH}^3(R)$, described in Corollary 2.4 (c), satisfy the relations

$$\begin{aligned}&\big(0, y(xy)^i\big) = p_1^i\widetilde{w} \ \text{ for } 1 \le i \le k-1,\\ &\big(dyx, xy+yx\big) = u_1v_3,\ \big(y, x(yx)^{k-1}\big) = d^{-1}u_1^3,\\ &\big(x(yx)^{k-1}, 0\big) = u_3v_1,\ \big(0, (xy)^k\big) = p_2\widetilde{w},\end{aligned}$$

whence $\mathrm{HH}^3(R) \subset \mathcal{H}$.

Now, we prove the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ by induction on n. Assume that $n \ge 4$, and let $f \in \mathrm{Hom}_\Lambda(Q_n, R)$ be a cocycle representing an element of $\mathrm{HH}^n(R)$. By Remark 2.13, we can restrict ourselves to basis elements $f = (f_1, f_2)$ where $f_1 \in \mathrm{Hom}_\Lambda(X_n, R)$ and $f_2 \in \mathrm{Hom}_\Lambda(Q_{n-4}, R)$. Moreover, using Remark 3.6, we may assume that (f_1, O) is one of the elements in (3.34) or (3.35) (depending on whether k is even or odd). Then by Proposition 3.5, (f_1, O) lies in $\mathcal{H}$. Finally, by induction hypothesis, $f_2 \in \mathcal{H}$, and then $(\mathrm{O}_2, f_2) = z \cdot f_2$ also lies in $\mathcal{H}$. □

Let $\mathcal{A}_1 = K[\mathcal{X}_1]/I_1$ be the graded K-algebra defined in Sec. 1, where $\mathcal{X}_1$ is as in (1.2) and I_1 is the corresponding ideal of relations (see (1.3)–(1.27)). The (nonzero) images of monomials in $K[\mathcal{X}_1]$ under the canonical epimorphism $K[\mathcal{X}_1] \to \mathcal{A}_1$ are also called monomials. Any element $a \in \mathcal{A}_1$ is represented as a linear combination of monomials (with coefficients in K). Propositions 3.1 and 3.7 imply that there exists a surjective homomorphism $\varphi\colon \mathcal{A}_1 \to \mathrm{HH}^*(R)$ of graded K-algebras that takes the generators in $\mathcal{X}_1$ to the corresponding generators in $\mathcal{Y}_1$ (see (3.6)); here, we use the same letter to denote elements of both sets that correspond to each other. Let $\mathcal{A}_1 = \bigoplus_{m\ge 0} \mathcal{A}_1^m$ be the direct decomposition of the algebra $\mathcal{A}_1$ into homogeneous direct summands. Now, statement (1) of Theorem 1.1 is a consequence of the following statement.

Proposition 3.8. *For any* $m \ge 0$*,*

$$\dim_K \mathcal{A}_1^m = \dim_K \mathrm{HH}^m(R).$$

Remark 3.9. It is easily verified that if $c \neq 0$, then the relation $u_1^4 = 0$ (see (3.23)) is derived from the remaining relations defining the algebra $\mathcal{A}_1$; moreover, the relation $p_2u_1^2 = 0$ is satisfied in $\mathcal{A}_1$.

Proof of Proposition 3.8. We introduce a lexicographic order on the polynomial ring $K[\mathcal{X}_1]$, such that

$$u_3 > v_1 > v_3 > v_4 > v_2 > \widetilde{w} > u_1 > u_2 > z > p_2 > p_3 > p_4 > p_1.$$

Any nonzero monomial in $\mathcal{A}_1$ is represented in the form

$$f = p_1^i p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} u_1^\ell u_2^{\beta_2} u_3^{\beta_3} v_1^r v_2^{\gamma_2} v_3^{\gamma_3} v_4^{\gamma_4} \widetilde{w}^{\varepsilon} z^s. \tag{3.36}$$

In view of the defining relations of $\mathcal{A}_1$, we have

$$\alpha_2, \alpha_3, \alpha_4, \beta_2, \beta_3, \gamma_2, \gamma_3, \gamma_4, \varepsilon \in \{0,1\}, i, \ell, r, s \in \mathbb{N}\cup\{0\}, i \le k-1, \ell \le 3.$$

Such representations of monomials in $\mathcal{A}_1$ are identified with the corresponding monomials in $K[\mathcal{X}_1]$.

Let us consider separately the cases $c \neq 0$ and $c = 0$.

1) Assume that $c \neq 0$. By definition, reduction of a monomial f in $\mathcal{A}_1$ is the process of replacement of some submonomials in f by other elements of $\mathcal{A}_1$ in accordance with the following rules ($a \mapsto b$ means the replacement of every occurrence of the monomial a by the element b):

$$\begin{array}{ll}
p_2u_1 \mapsto d^{-1}cp_1u_1 + p_1^{k-2}u_2, & u_1u_3 \mapsto cp_4v_1, \\
p_4v_1 \mapsto p_2v_2 \mapsto d^{-1}cp_1v_2, & p_3v_3 \mapsto p_2v_4 \mapsto p_1v_2 \mapsto p_4u_1^2, \\
u_1v_4 \mapsto dp_2\widetilde{w}, & u_1v_1 \mapsto cd^{-1}u_1^3, \\
u_2v_4 \mapsto p_1^2\widetilde{w}, & u_1v_2 \mapsto (p_1^{k-1} + cp_2)\widetilde{w}, \\
u_1\widetilde{w} \mapsto (p_3 + cp_4)z, & v_4^2 \mapsto p_1^2 z, \\
v_3\widetilde{w} \mapsto p_4u_1z, & v_4\widetilde{w} \mapsto u_2z, \\
\widetilde{w}^2 \mapsto cp_4u_1^2z. &
\end{array}$$

Any replacement in the above list is called an elementary step of reduction. These elementary steps correspond to (nonmonomial) defining relations of the algebra $\mathcal{A}_1$, and under such an elementary step of reduction, any nonzero monomial turns into a linear combination of monomials which are strictly smaller with respect to the lexicographic order. Hence, after finitely many steps, we obtain polynomials to which we cannot apply any elementary step of reduction. We say that a presentation of an element $a \in \mathcal{A}_1$ as a linear combination of monomials has a normal form if the reduction cannot be applied to any of these monomials. Clearly, any element in $\mathcal{A}_1$ admits at least one representation in the normal form.

Put $q_i = \dim_K \mathcal{A}_1^i$. Denote the number of monomials in $\mathcal{A}_1^i$ represented in the normal form by $\widetilde{q}_i$; it is clear that $\widetilde{q}_i \geq q_i$. Since there is an epimorphism $\mathcal{A}_1^i \to \mathrm{HH}^i(R)$, we have $q_i \geq \dim_K \mathrm{HH}^i(R)$. Consequently, it suffices to show that

$$\widetilde{q}_i = \dim_K \mathrm{HH}^i(R). \tag{3.37}$$

Assume that a monomial f in (3.36) has the normal form. If f contains the factor u_3, then f does not contain u_2, v_2, v_3, v_4, and p_i for all i (here, we use some monomial relations with factor u_3). Furthermore, f does not contain a factor u_1 (since there is a reduction $u_1u_3 \mapsto \dots$). Consequently, f coincides with one of the monomials

$$u_3v_1^rz^s, \ \ u_3\widetilde{w}v_1^rz^s.$$

Assume that u_3 is not a factor of f, but f contains v_1 as a factor. Then f does not contain the factors $v_2, v_3, v_4, u_2, p_1, p_2, p_3$ (because of the corresponding monomial relations); moreover, f does not contain u_1 (since there is a reduction $u_1v_1 \mapsto \dots$) and p_4 (there is a reduction $p_4v_1 \mapsto \dots$). Hence, f coincides with one of the monomials

$$v_1^rz^s, \ \ v_1^rz^s\widetilde{w} \ \ (r \geq 1).$$

Next, we assume that u_3, v_1 are not factors of f, but f contains v_3 as a factor. Then f does not contain the factors $v_2, v_4, u_2, p_1, p_2, p_4$ (because of the corresponding monomial relations); moreover, f does not contain $\widetilde{w}$ (since there is a reduction $v_3\widetilde{w} \mapsto \dots$) and p_3 (there is a reduction $p_3v_3 \mapsto \dots$). Hence, $f = u_1^{\beta_1}v_3z^s$ with $\beta_1 \leq 1$ (since there is the relation $u_1^2v_3 = 0$). Consequently,

$$f \in \{v_3z^s, u_1v_3z^s\}.$$

Now, assume that u_3, v_1, v_3 are not factors of f, but f contains v_4 as a factor. Then f does not contain the factors v_2, p_3, p_4; moreover, f does not contain $p_2, u_1, u_2, \widetilde{w}$ (because of the corresponding reductions). Furthermore, $i \leq k-2$ (since $p_1^{k-1}v_4 = 0$). Consequently, f coincides with one of the monomials

$$p_1^iv_4z^s, \ \ 0 \leq i \leq k-2.$$

Assume that u_3, v_1, v_3, v_4 are not factors of f, but f contains v_2 as a factor. Then f does not contain the factors $u_2, p_3, p_4, \widetilde{w}$ (because of the corresponding monomial relations); moreover, f does not contain p_1, p_2, u_1 (because of the corresponding reductions). Consequently, $f = v_2 z^s$.

Now, assume that u_3 and v_j ($j \in \{1, 2, 3, 4\}$) are not factors of f, but f contains $\widetilde{w}$ as a factor. Then f does not contain the factors u_2, p_3, p_4; moreover, f does not contain u_1 (because of the reduction $u_1\widetilde{w} \mapsto \dots$). Consequently, f coincides with one of the monomials

$$p_1^i \widetilde{w} z^s (0 \le i \le k-1), \;\; p_2 \widetilde{w} z^s.$$

Assume that u_3, v_j ($j \in \{1, 2, 3, 4\}$) and $\widetilde{w}$ are not factors of f, but f contains u_1 as a factor. Then f does not contain the factors u_2 (since $u_1 u_2 = 0$) and p_2 (there is a reduction $p_2 u_1 \mapsto \dots$). Hence, $f = p_1^i p_3^{\varepsilon_3} p_4^{\varepsilon_4} u_1^\ell t^s$. Moreover, $i \le 1$, since $p_1^2 u_1 = 0$ [if $c \ne 0$, then this relation is derived from $p_1 u_1 = dc^{-1}(p_1^{k-2} u_2 + p_2 u_1)$], and if $\ell \ge 2$, then $i = \varepsilon_3 = 0$, because $p_1 u_1^2 = p_3 u_1^2 = 0$. Consequently, f coincides with one of the monomials

$$u_1 z^s, \; p_1 u_1 z^s, \; p_3 u_1 z^s, \; p_4 u_1 z^s, \; u_1^2 z^s, \; p_4 u_1^2 z^s, \; u_1^3 z^s.$$

Now, we assume that u_1, u_3, v_j ($j \in \{1, 2, 3, 4\}$) and $\widetilde{w}$ are not factors of f, but f contains a factor u_2. Then f does not contain factors p_j, $j \in \{2, 3, 4\}$. Consequently,

$$f = p_1^i u_2 z^s, \quad 0 \le i \le k-2.$$

Finally, if does not contain factors u_j, v_m for all possible j, m, and $\widetilde{w}$, then it is clear that f coincides with one of the monomials

$$p_1^i z^s (0 \le i \le k-1), \; p_2 z^s, \; p_3 z^s, \; p_4 z^s.$$

Inspecting the degrees of the above monomials, we obtain the following list of all (nonzero) monomials that have normal form. Put $a \ge 0$.

The monomials of degree $4a$:

$$\{u_3 \widetilde{w} v_1^{2(a-m)-2} z^m\}_{m=0}^{a-1}, \{v_1^{2(a-m)} z^m\}_{m=0}^{a-1},$$
$$\{p_1^i z^a\}_{i=0}^{k-1}, p_2 z^a, p_3 z^a, p_4 z^a$$

(the number of them equals $2a + k + 3$).

The monomials of degree $4a + 1$:

$$\{u_3 v_1^{2(a-m)} z^m\}_{m=0}^{a}, \{\widetilde{w} v_1^{2(a-m)-1} z^m\}_{m=0}^{a-1},$$
$$\{p_1^i u_2 z^a\}_{i=0}^{k-2}, u_1 z^a, p_1 u_1 z^a, p_3 u_1 z^a, p_4 u_1 z^a$$

(the number of them equals $2a + k + 4$).

The monomials of degree $4a + 2$:

$$\{u_3 \widetilde{w} v_1^{2(a-m)-1} z^m\}_{m=0}^{a-1}, \{v_1^{2(a-m)+1} z^m\}_{m=0}^{a},$$
$$\{p_1^i v_4 z^a\}_{i=0}^{k-2}, v_2 z^a, v_3 z^a, u_1^2 z^a, p_4 u_1^2 z^a$$

(the number of them equals $2a + k + 4$).

The monomials of degree $4a + 3$:

$$\{u_3 v_1^{2(a-m)+1} z^m\}_{m=0}^{a}, \{\widetilde{w} v_1^{2(a-m)} z^m\}_{m=0}^{a-1},$$
$$\{p_1^i \widetilde{w} z^a\}_{i=0}^{k-1}, u_1 v_3 z^a, p_2 \widetilde{w} z^a, u_1^3 z^a$$

(the number of them equals $2a + k + 4$). It is easily seen that all monomials in this list have the normal form. Using Corollary 2.11, we obtain equality (3.37).

2) If $c = 0$, then we need minor modifications in the above arguments. Now we choose the following elementary steps of reduction:

$$\begin{array}{ll} p_2u_1 \mapsto p_1^{k-2}u_2, & p_3v_3 \mapsto p_2v_4 \mapsto p_1v_2 \mapsto p_4u_1^2, \\ u_1v_4 \mapsto dp_2\widetilde{w}, & u_1v_1 \mapsto cd^{-1}u_1^3, \\ u_1v_2 \mapsto p_1^{k-1}\widetilde{w}, & u_1\widetilde{w} \mapsto (p_3+cp_4)z, \\ v_4^2 \mapsto p_1^2 z, & v_3\widetilde{w} \mapsto p_4u_1z, \\ v_4\widetilde{w} \mapsto u_2z, & \widetilde{w}^2 \mapsto cp_4u_1^2z. \end{array}$$

We remark that new monomial relations appear in the algebra $\mathcal{A}_1$ (in comparison with the previous case), namely,

$$u_1u_3 = p_2v_2 = p_4v_1 = u_1v_1 = \widetilde{w}^2 = 0.$$

Now, successively analyzing several possibilities for occurrence of the elements from the set $\mathcal{X}_1$ in the representation of monomials in (3.36), we obtain the same list of monomials having the normal form as in the case $c \neq 0$. □

Case 2. Assume that k is even, $k > 2$, and $d \neq 0$.
We pick the following homogeneous elements of $\mathrm{HH}^*(R)$:

$$\begin{array}{ll} \text{— of degree 0 :} & p_1, p_2, p_3, p_4 \text{ in (3.1);} \\ \text{— of degree 1 :} & \begin{cases} u_3 \text{ in (3.2), and} \\ u_1' := (d(xy)^{k-1}, 1+cx), u_2' := (y, 0); \end{cases} \\ \text{— of degree 2 :} & v_1, v_2, v_3, v_4 \text{ in (3.4);} \\ \text{— of degree 3 :} & w := (0, y); \\ \text{— of degree 4 :} & z \text{ in (3.5).} \end{array} \tag{3.38}$$

Proposition 3.10. *Assume that k is even, $k > 2$, and $d \neq 0$. In the algebra* $\mathrm{HH}^*(R)$, *the elements of the set*

$$\mathcal{Y}_2 = \{p_1, p_2, p_3, p_4, u_1', u_2', u_3, v_1, v_2, v_3, v_4, w, z\} \tag{3.39}$$

satisfy the relations (3.7), (3.10), (3.12), (3.16), (3.17), (3.22), (3.23), (3.26), *and the following relations*:

$$p_1u_1' = (dp_2 + cp_1)u_2',\ p_2u_1' = p_1^{k-1}u_2',$$
$$p_1^{k-1}v_4 = dp_4v_1 + cp_3v_3,\ p_4(u_1')^2 = p_1v_2 = p_3v_3 = p_2v_4,$$
$$p_1(u_1')^2 = p_3(u_1')^2 = 0,$$
$$u_1'u_2' = \theta_{k-1}(cdp_4v_1 + c^2p_3v_3),$$
$$(u_1')^3 = du_2'v_1,\ u_1'v_2 = p_1^{k-1}w,\ u_2'v_2 = p_2w,$$
$$u_1'v_4 = u_2'(dv_2 + cv_4),\ u_2'v_4 = p_1w,$$
$$u_1'v_1 = u_2'v_3 = 0,$$
$$p_3w = p_4w = 0,$$
$$(u_1')^2v_j = 0 \ \text{for}\ 2 \leq j \leq 4,$$
$$u_1'w = u_2'w = 0,$$
$$v_2w = p_2u_2'z,\ v_4w = p_1u_2'z,\ v_3w = 0,$$
$$w^2 = (d\theta_{k+1}p_4v_1 + c\theta_{k-1}p_3v_3)z.$$

Proof. The proof of the above relations is similar to the proof of Proposition 3.1. For this, we need to know the translates those elements in (3.39) for which they have not been calculated earlier (see Remark 3.4). These translates are described in the following lemma.

Lemma 3.11. *As the translates of the elements u_1', u_2', v_3, and w, one can take the homomorphisms determined by the following matrices:*

$$\mathrm{T}^0(u_1') = \big(d(xy)^{k-1}\otimes 1, 1\otimes 1 + cx\otimes 1\big);$$

$\mathrm{T}^1(u_1')$ *is represented by a* 2×2*-matrix in which*

$$\begin{aligned}
(\mathrm{T}^1(u_1'))_{11} &= d\sum_{i=1}^{k-2}(xy)^i\otimes(xy)^{k-1-i} + d\sum_{i=0}^{k-1}(yx)^i\otimes(xy)^{k-1-i}\\
&\quad + cd\sum_{i=1}^{k-1} ix(yx)^{i-1}\otimes(xy)^{k-i};\\
(\mathrm{T}^1(u_1'))_{12} &= \sum_{i=0}^{k-2} y(xy)^i\otimes(xy)^{k-2-i}\\
&\quad + c\sum_{i=0}^{k-1}(xy)^i\otimes(xy)^{k-1-i} + c\sum_{i=2}^{k-2}(i+1)(yx)^i\otimes(xy)^{k-1-i}\\
&\quad + c^2\sum_{i=0}^{k-2}(i+1)x(yx)^i\otimes(xy)^{k-1-i}\\
&\quad + dx(yx)^{k-1}\otimes(xy)^{k-2},\\
(\mathrm{T}^1(u_1'))_{21} &= d\sum_{i=0}^{k-2} y(xy)^i\otimes y(xy)^{k-2-i} + cd\sum_{i=1}^{k-1} i(xy)^i\otimes y(xy)^{k-1-i},\\
(\mathrm{T}^1(u_1'))_{22} &= 1\otimes 1 + cx\otimes 1 + c\sum_{i=0}^{k-2}(i+1)y(xy)^i\otimes y(xy)^{k-2-i}\\
&\quad + c^2\sum_{i=1}^{k-1} i(xy)^i\otimes y(xy)^{k-1-i} + d(xy)^{k-1}\otimes y(xy)^{k-2};\\
\mathrm{T}^0(u_2') &= \big(y\otimes 1, 0\big),\\
\mathrm{T}^1(u_2') &= \begin{pmatrix} * & * \\ d\sum\limits_{i=1}^{k-1} i(xy)^i\otimes y(xy)^{k-1-i} & * \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
(\mathrm{T}^1(u_2'))_{11} &= y\otimes 1 + d\sum_{i=1}^{k-1} ix(yx)^i\otimes(xy)^{k-1-i},\\
(\mathrm{T}^1(u_2'))_{12} &= \sum_{i=1}^{k-1} i(yx)^i\otimes(xy)^{k-1-i} + c\sum_{i=1}^{k-1} ix(yx)^i\otimes(xy)^{k-1-i},\\
(\mathrm{T}^1(u_2'))_{22} &= \sum_{i=0}^{k-2}(i+1)y(xy)^i\otimes y(xy)^{k-2-i} + c\sum_{i=1}^{k-1} i(xy)^i\otimes y(xy)^{k-1-i};
\end{aligned}$$

$\mathrm{T}^0(v_3)$ *and* $\mathrm{T}^1(v_3)$ *can be given by the same formulas as for add* k (*see Lemma* 3.3);

$\mathrm{T}^2(v_3)$ *is a* 2×3*-matrix in which*

$$\begin{aligned}
(\mathrm{T}^2(v_3))_{11} &= dx^2\otimes 1 + d(yx)^{k-1}\otimes y + cdx(yx)^{k-1}\otimes y\\
&\quad + d^2x(yx)^{k-2}\otimes (xy)^k,\\
(\mathrm{T}^2(v_3))_{12} &= dx(yx)^{k-2}\otimes (xy)^k + dx(yx)^{k-1}\otimes (xy)^{k-1}\\
&\quad + cdx(yx)^{k-1}\otimes x(yx)^{k-1},\\
(\mathrm{T}^2(v_3))_{13} &= x(yx)^{k-2}\otimes (xy)^k + x(yx)^{k-1}\otimes (xy)^{k-1}\\
&\quad + (yx)^{k-1}\otimes x(yx)^{k-1} + cx(yx)^{k-1}\otimes x(yx)^{k-1},\\
(\mathrm{T}^2(v_3))_{21} &= c^2d^2y(xy)^{k-1}\otimes (xy)^k + d^2\sum_{i=0}^{k-1}(xy)^i\otimes (xy)^{k-i}\\
&\quad + cd^2x(yx)^{k-1}\otimes xy,\\
(\mathrm{T}^2(v_3))_{22} &= d\sum_{i=1}^{k}(xy)^i\otimes (xy)^{k-i} + d\sum_{i=1}^{k-1}y(xy)^{i-1}\otimes x(yx)^{k-i}\\
&\quad +dx\otimes y(xy)^{k-1} + c^2d(xy)^k\otimes y(xy)^{k-1} + cdy(xy)^{k-1}\otimes y(xy)^{k-1},\\
(\mathrm{T}^2(v_3))_{23} &= \sum_{i=1}^{k}y(xy)^{i-1}\otimes x(yx)^{k-i} + \sum_{i=1}^{k-1}(xy)^i\otimes (xy)^{k-i}\\
&\quad + c^2(xy)^k\otimes y(xy)^{k-1} + cy(xy)^{k-1}\otimes y(xy)^{k-1};
\end{aligned}$$

$$\mathrm{T}^0(w) = \big(0, y\otimes 1\big),$$

$\mathrm{T}^1(w)$ *is a* 2×3*-matrix in which*

$$\begin{aligned}
(\mathrm{T}^1(w))_{11} &= dy\otimes 1 + d^2\sum_{i=1}^{k-1}ix(yx)^i\otimes (xy)^{k-1-i},\\
(\mathrm{T}^1(w))_{12} &= y\otimes 1 + d\sum_{i=1}^{k-1}ix(yx)^i\otimes (xy)^{k-1-i}\\
&\quad + d\sum_{i=2}^{k-2}(i-1)(yx)^i\otimes x(yx)^{k-1-i} + cdy(xy)^{k-1}\otimes (xy)^{k-1},\\
(\mathrm{T}^1(w))_{13} &= \sum_{i=1}^{k-1}ix(yx)^i\otimes (xy)^{k-1-i} + \sum_{i=1}^{k-1}i(yx)^i\otimes x(yx)^{k-1-i}+\\
&\quad + cy(xy)^{k-1}\otimes (xy)^{k-1},\\
(\mathrm{T}^1(w))_{21} &= d^2\sum_{i=1}^{k-1}i(xy)^i\otimes y(xy)^{k-1-i},\\
(\mathrm{T}^1(w))_{22} &= d\sum_{i=1}^{k-1}i(xy)^i\otimes y(xy)^{k-1-i} + d\sum_{i=1}^{k-1}iy(xy)^i\otimes (yx)^{k-1-i},\\
(\mathrm{T}^1(w))_{23} &= \sum_{i=1}^{k-1}i(xy)^i\otimes y(xy)^{k-1-i} + \sum_{i=0}^{k-2}(i+1)y(xy)^i\otimes (yx)^{k-1-i};
\end{aligned}$$

$$\mathrm{T}^2(w) = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & 0 \end{pmatrix}$$

with

$$\begin{aligned}
(\mathrm{T}^2(w))_{12} &= \widetilde{y} \otimes 1 + c^3 d(xy)^k \otimes x(yx)^{k-1},\\
(\mathrm{T}^2(w))_{13} &= 1 \otimes x(yx)^{k-1} + c^2x^2 \otimes x(yx)^{k-1},\\
(\mathrm{T}^2(w)))_{14} &= (yx)^{k-1} \otimes (xy)^{k-1} + c(yx)^{k-1} \otimes x(yx)^{k-1},\\
(\mathrm{T}^2(w)))_{22} &= dyx \otimes 1 + dy \otimes x + cdyx \otimes x + d^2x(yx)^{k-1} \otimes x,\\
(\mathrm{T}^2(w)))_{23} &= 1 \otimes xy + x \otimes y + cx \otimes xy + cx^2 \otimes y + dy(xy)^{k-1} \otimes (yx)^{k-1};
\end{aligned}$$

$$\mathrm{T}^3(w) = \begin{pmatrix} * & * & * & 0 \\ 0 & 0 & * & * \end{pmatrix}$$

with

$$\begin{aligned}
(\mathrm{T}^3(w))_{11} &= dy \otimes 1 + d^2x(yx)^{k-1} \otimes 1 + c^3d^2x(yx)^{k-1} \otimes (xy)^k,\\
(\mathrm{T}^3(w))_{12} &= y \otimes 1 + dx(yx)^{k-1} \otimes 1 + c^3dx(yx)^{k-1} \otimes (xy)^k,\\
(\mathrm{T}^3(w))_{13} &= x(yx)^{k-1} \otimes 1 + cy(xy)^{k-1} \otimes (xy)^{k-1}\\
&\quad + c(yx)^{k-1} \otimes y(xy)^{k-1} + c^2x^2 \otimes x(yx)^{k-1};\\
(\mathrm{T}^3(w)_{23} &= 1 \otimes y + d\sum_{i=1}^{k-3} ix(yx)^i \otimes (xy)^{k-1-i}\\
&\quad + d\sum_{i=1}^{k-1}(yx)^i \otimes x(yx)^{k-1-i},\\
(\mathrm{T}^3(w)_{24} &= \sum_{i=1}^{k-1} i(yx)^{i-1} \otimes (xy)^{k-i} + c\sum_{i=1}^{k-1} ix(yx)^i \otimes (xy)^{k-1-i}\\
&\quad + c\sum_{i=0}^{k-1}(yx)^i \otimes x(yx)^{k-1-i} + c^2y(xy)^{k-1} \otimes (xy)^{k-1}.
\end{aligned}$$

The proof of this lemma is similar to the proof of Lemma 3.3.

Now the proof of Proposition 3.10 is completed similarly to the proof of Proposition 3.1 with the help of direct computations with matrices presented in Lemma 3.11. We leave to the reader the corresponding detailed computations. □

Proposition 3.12. *Assume that k is even, $k > 2$, and $d \neq 0$. The set $\mathcal{Y}_2$ in* (3.39) *generates* $\mathrm{HH}^*(R)$ *as a K-algebra.*

We need the following auxiliary statement.

Lemma 3.13. *For any $\ell \in \mathbb{N} \setminus \{1\}$, we have*

$$\begin{gathered}
v_1^{\ell-2}(v_1w + u_3z) = (0, \widetilde{y}, \mathrm{O}) \in \mathrm{HH}^{2\ell+1}(R),\\
v_1^{\ell-1}u_3w = \big(0, (xy)^k, \mathrm{O}\big) \in \mathrm{HH}^{2\ell+2}(R).
\end{gathered}$$

Proof. It is directly verified that $v_1w + u_3z = (0, \widetilde{y}, \mathrm{O}_2)$. Then the proof of the first equality is carried out by induction on ℓ similarly to the proof of Proposition 3.5. We also note that the translates $\mathrm{T}^i(f)$, $i \in \{1,2\}$, of the element $f = (0, \widetilde{y}, \mathrm{O})$ can be taken in the same form as in the proof of that proposition.

The second equality is proved similarly. □

Proof of Proposition 3.12. Let $\mathcal{H}$ be a K-subalgebra of $\mathrm{HH}^*(R)$, generated by the set $\mathcal{Y}_2 \cup \{1\}$. First, we prove that $\bigcup_{i=0}^{3} \mathrm{HH}^i(R) \subset \mathcal{H}$, and then prove the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ by induction on n.

It is clear that $\mathrm{HH}^0(R) \subset \mathcal{H}$ (see the proof of Proposition 3.7). The basis elements of $\mathrm{HH}^1(R)$ described in Corollary 2.6 (a), satisfy the relations

$$\begin{aligned}
&\big(y(xy)^i, 0\big) = p_1^i u_2' \ \text{ for } 1 \le i \le k-1, \ \big((xy)^k, 0\big) = p_2 u_2',\\
&\big(0, y(xy)^{k-1}\big) = (p_3 + c p_4) u_1', \big(0, (xy)^k\big) = p_4 u_1'.
\end{aligned}$$

Hence, $\mathrm{HH}^1(R) \subset \mathcal{H}$.

Next, the basis elements of $\mathrm{HH}^2(R)$ described in Corollary 2.6 (b), satisfy the relations

$$\begin{aligned}
&\big(d(xy)^{i+1}, y(xy)^i + c(xy)^{i+1}\big) = p_1^i v_4 \ \text{ for } \ 0 \le i \le k-2,\\
&\big(0, 1\big) = (u_1')^2 + c^2 d\theta_{k-1} p_4 v_1 + c^3 \theta_{k+1} p_3 v_3,\\
&\big((xy)^k, 0\big) = p_4 v_1, \ \big(0, (xy)^k\big) = p_3 v_3.
\end{aligned}$$

Consequently, $\mathrm{HH}^2(R) \subset \mathcal{H}$.

Then the basis elements of $\mathrm{HH}^3(R)$ described in Corollary 2.6 (c), satisfy the relations

$$\begin{aligned}
&\big(0, y(xy)^i\big) = p_1^i w \ \text{ for } 1 \le i \le k-1,\\
&\big(dyx, xy + yx\big) = u_1' v_3, \ \big(y, x(yx)^{k-1}\big) = u_2' v_1,\\
&\big(x(yx)^{k-1}, 0\big) = u_3 v_1, \ \big(0, (xy)^k\big) = u_2' v_2,
\end{aligned}$$

whence $\mathrm{HH}^3(R) \subset \mathcal{H}$.

Now the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ follows by induction on n similarly to the proof of Proposition 3.7. Here, it is worth to note that relations (3.32) are valid for even k too, and one needs to use relations in Lemma 3.13 instead of relations (3.33). □

Let $\mathcal{A}_2 = K[\mathcal{X}_2]/I_2$ be the graded K-algebra defined in Sec. 1 with $\mathcal{X}_2$ as in (1.28), and I_2 the corresponding ideal of relations. Propositions 3.10 and 3.12 imply that there exists a surjective homomorphism $\varphi\colon \mathcal{A}_2 \to \mathrm{HH}^*(R)$ of graded K-algebras, that takes the generators in $\mathcal{X}_2$ to the corresponding generators in $\mathcal{Y}_2$ (see (3.39)). Let $\mathcal{A}_2 = \bigoplus_{m \ge 0} \mathcal{A}_2^m$ be the direct decomposition of the algebra $\mathcal{A}_2$ into homogeneous direct summands. Now, statement (2) of Theorem 1.1 is a consequence of the following statement.

Proposition 3.14. *For any $m \ge 0$,*

$$\dim_K \mathcal{A}_2^m = \dim_K \mathrm{HH}^m(R).$$

First, we state the following auxiliary assertion.

Lemma 3.15. *The following relations are satisfied in the algebra $\mathcal{A}_2$:*

$$\begin{aligned}
&p_2 u_2' v_2 = p_2 u_2' v_4 = p_1 u_2' v_2 = p_3 u_1' v_3 = p_4 u_2' v_1 = (u_1')^4 = (u_2')^3 = p_1^2 v_2 = p_4 v_1^2 = 0,\\
&p_1^{k-2} u_2' v_4 = u_1' v_2.
\end{aligned}$$

All relations in the lemma follow directly from the defining relations of the algebra $\mathcal{A}_2$.

Proof of Proposition 3.14. We introduce a lexicographic order on the polynomial ring $K[\mathcal{X}_2]$, such that

$$w > u_1' > u_2' > u_3 > v_4 > v_3 > v_2 > v_1 > z > p_2 > p_3 > p_4 > p_1.$$

Any nonzero monomial in $\mathcal{A}_2$ is represented in the form

$$f = p_1^i p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} (u_1')^{\beta_1} (u_2')^{\beta_2} u_3^{\beta_3} v_1^r v_2^{\gamma_2} v_3^{\gamma_3} v_4^{\gamma_4} \widetilde{w}^{\varepsilon} z^s; \tag{3.40}$$

here, by the defining relations of the algebra $\mathcal{A}_2$, we have

$$\alpha_2, \alpha_3, \alpha_4, \beta_3, \gamma_2, \gamma_3, \gamma_4, \varepsilon \in \{0,1\}, i, \ell, r, s \in \mathbb{N} \cup \{0\}, i \leq k-1, \beta_1 \leq 3, \beta_2 \leq 2.$$

As in the proof of Proposition 3.8, we introduce the following list of elementary steps of reduction and then study the normal forms of monomials (with respect to such steps of reductions):

$$\begin{array}{ll}
p_1 u_1' \mapsto (dp_2 + cp_1)u_2', & p_2 u_1' \mapsto p_1^{k-1} u_2', \\
p_2 v_2 \mapsto p_4 v_1, & p_4 (u_1')^2 \mapsto p_2 v_4 \mapsto p_3 v_3 \mapsto p_1 v_2, \\
p_1^{k-1} v_4 \mapsto dp_4 v_1 + cp_1 v_2, & u_1' u_2' \mapsto \theta_{k-1} c(dp_4 v_1 + cp_1 v_2), \\
(u_2')^2 \mapsto d\theta_{k+1} p_4 v_1 + c\theta_{k-1} p_3 v_3, & u_2' u_3 \mapsto p_4 v_1, \\
(u_1')^3 \mapsto du_2' v_1, & p_1^{k-1} w \mapsto u_1' v_2, \\
p_1 w \mapsto u_2' v_4, & p_2 w \mapsto u_2' v_2, \\
p_1^{k-2} u_2' v_4 \mapsto u_1' v_2, & p_2 w \mapsto u_2' v_2, \\
u_1' v_4 \mapsto (dv_2 + cv_4) u_2', & v_4^2 \mapsto p_1^2 z, \\
v_2 w \mapsto p_2 u_2' z, & v_4 w \mapsto p_1 u_2' z, \\
w^2 \mapsto (d\theta_{k+1} p_4 v_1 + c\theta_{k-1} p_3 v_3) z. &
\end{array}$$

Put $q_i = \dim_K \mathcal{A}_2^i$. Denote the number of monomials in $\mathcal{A}_2^i$ occurring in the normal form by $\widetilde{q}_i$. Since there is an epimorphism $\mathcal{A}_2^i \to \mathrm{HH}^i(R)$, we have $q_i \geq \dim_K \mathrm{HH}^i(R)$. Consequently, it suffices to show that

$$\widetilde{q}_i = \dim_K \mathrm{HH}^i(R). \tag{3.41}$$

Assume that a monomial f in (3.40) has the normal form. If f contains the factor w, then f does not contain $p_3, p_4, u_1', u_2', v_3$ (here, we use some monomial relations with factor w); moreover, f does not contain p_1, p_2 (since there are reductions $p_1 w \mapsto \dots, p_2 w \mapsto \dots$), and v_2, v_4 (there are reductions $v_2 w \mapsto \dots, v_4 w \mapsto \dots$). Hence, f coincides with one of the monomials

$$w v_1^r z^s, \ \ u_3 w v_1^r z^s.$$

Assume that w is not a factor of f, but f contains u_1' as a factor. Then f does not contain the factors v_1, u_3. Moreover, f does not contain p_1, p_2, u_2' and v_4 since there are reductions $p_1 u_1' \mapsto \dots$, $p_2 u_1' \mapsto \dots$, and $u_1' u_2' \mapsto \dots$, $u_1' v_4 \mapsto \dots$ respectively. Note that $\beta_1 \leq 2$ in (3.40) (since $(u_1')^3 \mapsto \dots$). Hence, f coincides with one of the monomials

$$u_1' z^m, p_3 u_1' z^m, p_4 u_1' z^m, u_1' v_2 z^m, u_1' v_3 z^m, (u_1')^2 z^m.$$

Now, assume that w, u_1' are not factors of f, but f contains u_2' as a factor. Then f does not contain the factors p_3, p_4, v_3 (because of the corresponding monomial relations), and u_3 (since there is a reduction $u_2' u_3 \mapsto \dots$). Moreover, $i \leq k-3$ in (3.40), since there is a reduction $p_1^{k-2} u_2' v_4 \mapsto \dots$ (see Lemma 3.15). It easily follows that f coincides with one of the monomials

$$p_1^i u_2' z^m \ (0 \leq i \leq k-1), p_2 u_2' z^m, u_2' v_1 z^m, u_2' v_2 z^m, p_1^i u_2' v_4 z^m \ (0 \leq i \leq k-3).$$

Note that, in the above argument, we use the relations $p_2 u_2' v_4 = 0$ (see Lemma 3.15) and $u_2' v_1^2 = 0$.

Now, assume that w, u_1', u_2' are not factors of f, but f contains u_3 as a factor. Then f does not contain the factors p_i for all i, and v_2, v_3, v_4 (because of the corresponding monomial relations). Consequently, f coincides with one of the monomials

$$f = u_3 v_1^r z^m \text{ for } r \geq 0.$$

Assume that w, u_1', u_2', u_3 are not factors of f, but f contains v_4 as a factor. Then f does not contain the factors p_3, p_4, v_1, v_2, v_3, and also does not contain p_2 (since there is a reduction $p_2 v_4 \mapsto \dots$). Moreover, $i \leq k-2$ (because of $p_1^{k-1} v_4 \mapsto \dots$). Consequently,

$$f = p_1^i v_4 z^m \text{ for } 0 \leq i \leq k-2.$$

Now, assume that w, u_1', u_2', u_3, v_4 are not factors of f, but f contains v_3 as a factor. It is easily seen that f does not contain p_i for all i as factors, and hence, $f = v_3 z^m$.

Now we assume that $w, u_1', u_2', u_3, v_4, v_3$ are not factors of f, but f contains v_2 as a factor. It is easily seen that f does not contain p_2, p_3, p_4. Moreover, $i \leq 1$ in (3.40) (since $p_1^2 v_2 = 0$). Consequently, f coincides with one of the monomials

$$v_2 z^m, \ p_1 v_2 z^m.$$

If f does not contain $w, u_1', u_2', u_3, v_4, v_3, v_2$ as factors, but contains v_1, then f does not contain p_1, p_2, p_3. Clearly, f coincides with one of the monomials

$$v_1^r z^m, p_4 v_1^r z^m.$$

Finally, if $w, u_1', u_2', u_3, v_4, v_3, v_2, v_1$ are not factors of f, then f coincides with one of the monomials

$$p_1^i \ (0 \leq i \leq k-1) z^m, \ p_2 z^m, \ p_3 z^m, \ p_4 z^m.$$

Inspecting the degrees of these monomials, we obtain the following list of all (nonzero) monomials that have the normal form. Put $a \geq 0$.

The monomials of degree $4a$:

$$\{u_3 w v_1^{2(a-m)-2} z^m\}_{m=0}^{a-1}, \{v_1^{2(a-m)} z^m\}_{m=0}^{a-1},$$
$$\{p_1^i z^a\}_{i=0}^{k-1}, p_2 z^a, p_3 z^a, p_4 z^a$$

(the number of them equals $2a + k + 3$).

The monomials of degree $4a + 1$:

$$\{w v_1^{2(a-m)-1} z^m\}_{m=0}^{a-1}, \{u_3 v_1^{2(a-m)} z^m\}_{m=0}^{a},$$
$$\{p_1^i u_2' z^a\}_{i=0}^{k-1}, u_1' z^a, p_3 u_1' z^a, p_4 u_1' z^a, p_2 u_2' z^a$$

(the number of them equals $2a + k + 5$).

The monomials of degree $4a + 2$:

$$\{u_3 w v_1^{2(a-m)-1} z^m\}_{m=0}^{a-1}, \{v_1^{2(a-m)+1} z^m\}_{m=0}^{a},$$
$$\{p_1^i v_4 z^a\}_{i=0}^{k-2}, p_4 v_1 z^a, v_2 z^a, v_3 z^a, p_1 v_2 z^a, (u_1')^2 z^a$$

(the number of them equals $2a + k + 5$).

The monomials of degree $4a + 3$:

$$\{u_3 v_1^{2(a-m)+1} z^m\}_{m=0}^{a}, \{w v_1^{2(a-m)} z^m\}_{m=0}^{a},$$
$$\{p_1^i u_2' v_4 z^a\}_{i=0}^{k-3}, u_1' v_2 z^a, u_1' v_3 z^a, u_2' v_1 z^a, u_2' v_2 z^a$$

(the number of them equals $2a + k + 4$). It is easily seen that all monomials in this list have normal form. Using Corollary 2.11, we derive from this the equality (3.41). □

Case 3. Assume that k is odd and $d = 0$.
We consider the following homogeneous elements of $\mathrm{HH}^*(R)$:

$$\begin{aligned}
&\text{— of degree 0 :} && p_1, p_2, p_3, p_4 \text{ in (3.1);} \\
&\text{— of degree 1 :} && \begin{cases} u_1, u_2 \text{ in (3.2), and} \\ u_0 := (1, y(xy)^{k-2} + c(xy)^{k-1}); \end{cases} && (3.42) \\
&\text{— of degree 2 :} && v_2, v_3, v_4 \text{ in (3.3);} \\
&\text{— of degree 3 :} && w_0 := (0, 1),\ w_1 := (0, y); && (3.43) \\
&\text{— of degree 4 :} && z \text{ in (3.5).}
\end{aligned}$$

Proposition 3.16. *Assume that k is odd and $d = 0$. In the algebra* $\mathrm{HH}^*(R)$, *the elements of the set*

$$\mathcal{Y}_3 = \{p_1, p_2, p_3, p_4, u_0, u_1, u_2, v_2, v_3, v_4, w_0, w_1, z\} \tag{3.44}$$

satisfy the relations (3.7), (3.9), (3.11), (3.16), (3.23), (3.26), *and the following relations*:

$$\begin{gathered}
p_1u_0 = p_3u_1,\ p_3u_0 = p_1^{k-2}u_2, \\
p_2u_1 = (p_3 + cp_4)u_0,\ p_1u_1 = 0; \\
u_0u_2 = p_4u_1^2 = p_4u_0u_1 = 0, \\
p_1v_2 = p_2v_4 = 0,\ p_jv_3 = 0 \ \textit{for}\ \ 1 \le j \le 4; \\
u_0v_3 = p_3w_0 = p_1^{k-1}w_1,\ p_2w_1 = p_4w_0, \\
p_3w_1 = p_4w_1 = 0, \\
u_1u_0^2 = cu_0v_2 + cp_2w_0,\ u_0u_1^2 = u_1^3 = 0, \\
u_0v_4 = u_1v_3 = p_1w_0,\ u_2v_4 = p_1^2w_1, \\
u_1v_2 = (p_1^{k-1} + cp_2)w_1, \\
u_1v_4 = u_2v_2 = u_2v_3 = 0, \\
u_1w_1 = (p_3 + cp_4)z,\ u_1w_0 = cu_0w_1 + cp_2z + p_1^{k-1}z; \\
v_2w_0 = u_0^2w_1,\ v_3w_0 = p_1^{k-2}u_2z, \\
v_4w_0 = p_3u_1z,\ v_3w_1 = p_4u_1z,\ v_4w_1 = u_2z,\ v_2w_1 = 0; \\
w_0^2 = (1 + c^3p_4)u_0^2z, \\
w_0w_1 = v_2z,\ w_1^2 = 0.
\end{gathered}$$

Proof. The proof of the above relations is similar to the proof of Proposition 3.1. But we need to know the translates of those elements in (3.44) for which they have not been calculated earlier. These translates are described in the following lemma.

Lemma 3.17. *As the translates of u_0, w_0, one can take the homomorphisms determined by the following matrices*:

$$\begin{aligned}
\mathrm{T}^0(u_0) &= \big(1 \otimes 1, y(xy)^{k-2} \otimes 1 + c(xy)^{k-1} \otimes 1\big), \\
\mathrm{T}^1(u_0) &= \begin{pmatrix} 1 \otimes 1 & * \\ 0 & * \end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}(\mathrm{T}^1(u_0))_{12} = &\sum_{i=0}^{k-2} x(yx)^i \otimes (xy)^{k-2-i} + cy(xy)^{k-2} \otimes (xy)^{k-1}\\ &+ c^2(xy)^{k-1} \otimes (xy)^{k-1},\\ (\mathrm{T}^1(u_0))_{22} = &\sum_{i=0}^{k-2}(xy)^i \otimes y(xy)^{k-2-i} + \sum_{i=1}^{k-2}(yx)^i \otimes y(xy)^{k-2-i}\\ &+ y(xy)^{k-2} \otimes 1 + c\sum_{i=0}^{k-2} x(yx)^i \otimes y(xy)^{k-2-i} + c(xy)^{k-1} \otimes 1;\end{aligned}$$

$$\mathrm{T}^0(w_0) = \big(0, 1 \otimes 1\big), \quad \mathrm{T}^1(w_0) = \begin{pmatrix} 0 & 1\otimes 1 & * \\ 0 & 0 & * \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^1(w_0))_{13} = &\sum_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} + c(yx)^{k-2} \otimes (xy)^k,\\ (\mathrm{T}^1(w_0))_{23} = &\sum_{i=0}^{k-1}(xy)^i \otimes (yx)^{k-1-i} + \sum_{i=0}^{k-2}(xy)^i \otimes (xy)^{k-1-i}\\ &+ \sum_{i=1}^{k-1}(yx)^i \otimes (yx)^{k-1-i} + cy(xy)^{k-2} \otimes y(xy)^{k-1};\end{aligned}$$

$$\mathrm{T}^2(w_0) = \begin{pmatrix} 0 & 1\otimes 1 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^2(w_0))_{13} = &c(xy)^k \otimes (xy)^{k-2},\\ (\mathrm{T}^2(w_0))_{14} = &(yx)^{k-1} \otimes x(yx)^{k-2} + x(yx)^{k-2} \otimes (xy)^{k-1}\\ &+ x(yx)^{k-1} \otimes (xy)^{k-2} + cx(yx)^{k-1} \otimes x(yx)^{k-2},\\ (\mathrm{T}^2(w_0))_{23} = &x \otimes 1 + 1 \otimes x + \sum_{i=0}^{k-2} y(xy)^i \otimes y(xy)^{k-2-i}\\ &+ cx \otimes x + cy(xy)^{k-1} \otimes 1 + c^2y(xy)^{k-1} \otimes x,\\ (\mathrm{T}^2(w_0)))_{24} = &\sum_{i=0}^{k-1}(xy)^i \otimes (yx)^{k-1-i} + x \otimes y(xy)^{k-2}\\ &+ (yx)^{k-1} \otimes 1 + cy(xy)^{k-2} \otimes y(xy)^{k-1};\end{aligned}$$

$$\mathrm{T}^3(w_0) = \begin{pmatrix} 0 & 1\otimes 1 & * & 0 \\ 0 & 0 & * & * \end{pmatrix}$$

with

$$
\begin{aligned}
(\mathrm{T}^3(w_0))_{13} &= c^2(yx)^{k-1}\otimes x(yx)^{k-1} + c^3x(yx)^{k-1}\otimes x(yx)^{k-1},\\
(\mathrm{T}^3(w_0))_{23} &= 1\otimes 1 + c^2y(xy)^{k-1}\otimes 1 + c^3(xy)^k\otimes 1,\\
(\mathrm{T}^3(w_0))_{24} &= \sum_{i=0}^{k-2} x(yx)^i\otimes (xy)^{k-2-i} + c(yx)^{k-1}\otimes y(xy)^{k-2}\\
&\qquad + cy(xy)^{k-2}\otimes (xy)^{k-1} + c^2x(yx)^{k-1}\otimes y(xy)^{k-2} + c^2(yx)^{k-1}\otimes (xy)^{k-1}.
\end{aligned}
$$

The proof of this lemma is similar to the proof of Lemma 3.3.

Now the proof of Proposition 3.16 is completed with the help of direct calculation. We leave to the reader the corresponding details. □

Remark 3.18. Note that formulas for the translates of w_1 are obtained from the corresponding translates of $\widetilde{w}$ (see Lemma 3.3) for $d = 0$. Recall that formulas for the translates of v_2, v_3, v_4 described in that lemma are valid also for $d = 0$.

Proposition 3.19. *For any $\ell \in \mathbb{N}$, we have*

$$u_0^\ell = \big(1, \mathrm{O}\big) \in \mathrm{HH}^\ell(R), \quad u_0^\ell v_2 = \big(y, \mathrm{O}\big) \in \mathrm{HH}^{\ell+2}(R), \tag{3.45}$$

$$u_0^\ell w_0 = \big(0, 1, \mathrm{O}\big) \in \mathrm{HH}^{\ell+3}(R), \quad u_0^\ell w_1 = \big(0, y, \mathrm{O}\big) \in \mathrm{HH}^{\ell+3}(R). \tag{3.46}$$

Remark 3.20. Owing to the description of bases for the groups $\mathrm{H}^{2\ell}(\mathcal{X}^\bullet)$ $(\ell \geq 4)$ given in [2, Lemma 4.11], the elements

$$\big(1, \mathrm{O}_{r+1}\big), \big(y, \mathrm{O}_{r+1}\big), \big(0, 1, \mathrm{O}_r\big), \big(0, y, \mathrm{O}_r\big) \tag{3.47}$$

(for suitable r) are included in a basis of the image of $\mathrm{H}^\ell(\mathcal{X}^\bullet)$ in the group $\mathrm{HH}^\ell(R)$. Moreover, the remaining basis elements of the image are expressed in terms of the above elements,

$$
\begin{aligned}
&\big(x(yx)^{k-1}, \mathrm{O}_{r+1}\big) = p_2\big(1, \mathrm{O}_{r+1}\big),\ \big((xy)^k, \mathrm{O}_{r+1}\big) = p_4\big(1, \mathrm{O}_{r+1}\big),\\
&\big(0, x(yx)^{k-1}, \mathrm{O}_r\big) = p_2\big(0, 1, \mathrm{O}_r\big),\ \big(0, (xy)^k, \mathrm{O}_r\big) = p_4\big(0, 1, \mathrm{O}_r\big).
\end{aligned}
$$

Proof of Proposition 3.19. The base of induction for all mentioned elements is established directly (using Lemma 3.17).

1) Now, assuming $u_0^\ell = \big(1, \mathrm{O}\big)$ $(\ell \geq 2)$, we find the following translates of this element:

$$
\begin{aligned}
\mathrm{T}^0(u_0^\ell) &= \big(1\otimes 1, \mathrm{O}\big),\\
\mathrm{T}^1(u_0^\ell) &= \begin{pmatrix} 1\otimes 1 & \sum_{i=0}^{k-2} x(yx)^i\otimes x(yx)^{k-2-i} & \mathrm{O}\\ 0 & \sum_{i=0}^{k-1}(xy)^i\otimes (yx)^{k-1-i} + c^3(xy)^{k-1}\otimes (xy)^k & \mathrm{O}\end{pmatrix}.
\end{aligned}
$$

Consequently,

$$u_0^{\ell+1} = \mu\mathrm{T}^0(u_0)\mathrm{T}^1(u_0^\ell) = \big(1, \mathrm{O}\big).$$

2) Assuming $u_0^\ell v_2 = \big(y, \mathrm{O}\big)$, we find the translates

$$
\begin{aligned}
\mathrm{T}^0(u_0^\ell v_2) &= \big(y\otimes 1, \mathrm{O}\big),\\
\mathrm{T}^1(u_0^\ell v_2) &= \begin{pmatrix} y\otimes 1 & \sum_{i=1}^{k-1}(yx)^i\otimes x(yx)^{k-1-i} + c^3x(yx)^{k-1}\otimes (xy)^k & \mathrm{O}\\ 0 & \sum_{i=0}^{k-1} y(xy)^i\otimes (yx)^{k-1-i} & \mathrm{O}\end{pmatrix}.
\end{aligned}
$$

Then

$$u_0^{\ell+1} v_2 = \mu \mathrm{T}^0(u_0)\mathrm{T}^1(v_1^\ell v_2) = \big(y, \mathrm{O}\,\big).$$

3) Assuming $u_0^\ell w_0 = \big(0, 1, \mathrm{O}\,\big)$, we find

$$\mathrm{T}^0(u_0^\ell w_0) = \big(0, 1\otimes 1, \mathrm{O}\,\big),$$

$$\mathrm{T}^1(u_0^\ell w_0) = \begin{pmatrix} 0 & 1\otimes 1 & \sum\limits_{i=0}^{k-2} x(yx)^i \otimes x(yx)^{k-2-i} & \mathrm{O} \\ 0 & 0 & \sum\limits_{i=0}^{k-1} (xy)^i \otimes (yx)^{k-1-i} & \mathrm{O} \end{pmatrix}.$$

Then

$$u_0^{\ell+1} w_0 = \mu \mathrm{T}^0(u_0)\mathrm{T}^1(u_0^\ell w_0) = \big(0, 1, \mathrm{O}\,\big).$$

4) Assuming $u_0^\ell w_1 = \big(0, y, \mathrm{O}\,\big)$ ($\ell \in \mathbb{N}$), we find the translates

$$\mathrm{T}^0(u_0^\ell w_1) = \big(0, y\otimes 1, \mathrm{O}\,\big),$$

$$\mathrm{T}^1(u_0^\ell w_1) = \begin{pmatrix} 0 & y\otimes 1 & \sum\limits_{i=1}^{k-1} (yx)^i \otimes x(yx)^{k-1-i} & \mathrm{O} \\ 0 & 0 & \sum\limits_{i=0}^{k-1} y(xy)^i \otimes (yx)^{k-1-i} & \mathrm{O} \end{pmatrix}.$$

Then

$$u_0^{\ell+1} w_1 = \mu \mathrm{T}^0(u_0)\mathrm{T}^1(u_0^\ell w_1) = \big(0, y, \mathrm{O}\,\big).$$ □

Proposition 3.21. *Assume that k is odd and $d = 0$. The set $\mathcal{Y}_3$ in* (3.44) *generates* $\mathrm{HH}^*(R)$ *as a K-algebra.*

Proof. Let $\mathcal{H}$ denote a K-subalgebra of $\mathrm{HH}^*(R)$, generated by the set $\mathcal{Y}_3 \cup \{1\}$. First, we prove that $\bigcup\limits_{i=0}^{3} \mathrm{HH}^i(R) \subset \mathcal{H}$.

It is clear that $\mathrm{HH}^0(R) \subset \mathcal{H}$ (see the proof of Proposition 3.7). The basis elements of $\mathrm{HH}^1(R)$, described in Corollary 2.8 (a), satisfy the relations

$$\begin{aligned} &\big(y(xy)^i, 0\big) = p_1^{i-1} u_2 \ \text{ for } 1 \le i \le k-1, \\ &\big(x(yx)^{k-1}, 0\big) = p_2 u_0,\ \big((xy)^k, 0\big) = p_4 u_0, \\ &\big(0, y(xy)^{k-1}\big) = (p_3 + c p_4) u_1,\ \big(0, (xy)^k\big) = p_4 u_1. \end{aligned}$$

Hence, $\mathrm{HH}^1(R) \subset \mathcal{H}$.

Next, the basis elements of $\mathrm{HH}^2(R)$, described in Corollary 2.8 (b), satisfy the relations

$$\begin{aligned} &\big(0, y(xy)^i + c(xy)^{i+1}\big) = p_1^i v_4 \ \text{ for } \ 0 \le i \le k-3, \\ &\big(1, 0\big) = u_0^2,\ \big(x(yx)^{k-1}, 0\big) = p_2 u_0^2,\ \big((xy)^k, 0\big) = p_4 u_0^2, \\ &\big(0, 1\big) = u_1^2,\ \big(0, y(xy)^{k-2}\big) = u_0 u_1 + c v_2, \\ &\big(0, (xy)^{k-1}\big) = c^{-1}(p_1^{k-2} v_4 + u_0 u_1) + v_2. \end{aligned}$$

Consequently, $\mathrm{HH}^2(R) \subset \mathcal{H}$.

Then the basic elements of $\mathrm{HH}^3(R)$ described in Corollary 2.8 (c), satisfy the relations

$$\begin{aligned}
&\big(0, y(xy)^i\big) = p_1^i \widetilde{w} \ \text{ for } 1 \le i \le k-1,\\
&\big(1,0\big) = u_0^3,\ \big(y,0\big) = u_0 v_2,\\
&\big(x(yx)^{k-1},0\big) = p_2 u_0^3,\ \big((xy)^k,0\big) = p_4 u_0^3,\\
&\big(0, xy+yx\big) = u_1 v_3,\ \big(0, x(yx)^{k-1}\big) = p_2 w_0,\\
&\big(0,(xy)^k\big) = p_2 w_1,
\end{aligned}$$

whence $\mathrm{HH}^3(R) \subset \mathcal{H}$.

Now the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ is verified by induction on n. Assume that $n \ge 4$. Let $f = (f_1, f_2) \in \mathrm{Hom}_\Lambda(Q_n, R)$ be a cocycle representing an element of $\mathrm{HH}^n(R)$, where $f_1 \in \mathrm{Hom}_\Lambda(X_n, R)$ and $f_2 \in \mathrm{Hom}_\Lambda(Q_{n-4}, R)$. By Proposition 3.19, (f_1, O) lies in $\mathcal{H}$. Finally, by induction hypothesis, $f_2 \in \mathcal{H}$, and hence $(\mathrm{O}_2, f_2) = z \cdot f_2$ also lies in $\mathcal{H}$. □

Let $\mathcal{A}_3 = K[\mathcal{X}_3]/I_3$ be the graded K-algebra defined in Sec. 1 with $\mathcal{X}_3$ as in (1.31) and I_3 the corresponding ideal of relations. Propositions 3.16 and 3.21 imply that there exists a surjective homomorphism $\varphi\colon \mathcal{A}_3 \to \mathrm{HH}^*(R)$ of graded K-algebras that takes the generators in $\mathcal{X}_3$ to the corresponding generators in $\mathcal{Y}_3$ (see (3.44)). Let $\mathcal{A}_3 = \bigoplus_{m \ge 0} \mathcal{A}_3^m$ be the direct decomposition of $\mathcal{A}_3$ into homogeneous direct summands. Now, statement (3) of Theorem 1.1 is a consequence of the following statement.

Proposition 3.22. *For any $m \ge 0$,*

$$\dim_K \mathcal{A}_3^m = \dim_K \mathrm{HH}^m(R).$$

First, we state an auxiliary assertion.

Lemma 3.23. *The following relations are satisfied in the algebra $\mathcal{A}_3$:*

$$p_1^2 w_0 = p_1 u_0 w_0 = p_1 u_0 w_1 = p_4 u_0^2 u_1 = p_3 u_0^2 = p_1 u_0^2 = p_1^2 u_0 = 0.$$

All relations in lemma 3.23 follow directly from the defining relations of the algebra $\mathcal{A}_3$.

Proof of Proposition 3.22. We introduce a lexicographic order on the polynomial ring $K[\mathcal{X}_3]$, such that

$$v_3 > v_4 > w_0 > v_2 > w_1 > u_1 > u_2 > u_0 > z > p_2 > p_3 > p_4 > p_1.$$

Any nonzero monomial in $\mathcal{A}_3$ is represented in the form

$$f = p_1^i p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} u_0^\ell u_1^{\beta_1} u_2^{\beta_2} v_2^{\gamma_2} v_3^{\gamma_3} v_4^{\gamma_4} w_0^{\varepsilon_0} w_1^{\varepsilon_1} z^s; \tag{3.48}$$

here, by the defining relations of the algebra $\mathcal{A}_3$, we have

$$\alpha_2, \alpha_3, \alpha_4, \beta_2, \gamma_2, \gamma_3, \gamma_4, \varepsilon_0, \varepsilon_1 \in \{0,1\},\ \ i, \ell, \beta_1, s \in \mathbb{N} \cup \{0\}, i \le k-1, \beta_1 \le 2.$$

As in the proof of Proposition 3.8, we introduce the following list of elementary steps of reduction, and study the normal forms of monomials (with respect to these steps of reductions):

$$\begin{array}{ll}
p_3u_1 \mapsto p_1u_0, & p_2u_1 \mapsto (p_3+cp_4)u_0,\\
p_1^{k-2}u_2 \mapsto p_3u_0, & p_2v_2 \mapsto p_4u_0^2,\\
u_1v_3 \mapsto u_0v_4 \mapsto p_1w_0, & u_0v_3 \mapsto p_3w_0 \mapsto p_1^{k-1}w_1,\\
p_2w_0 \mapsto u_0v_2 + c^{-1}u_0^2u_1, & p_4w_0 \mapsto p_2w_1,\\
u_2v_4 \mapsto p_1^2w_1, & u_1v_2 \mapsto (p_1^{k-1}+cp_2)w_1,\\
u_1w_1 \mapsto (p_3+cp_4)z, & u_1w_0 \mapsto cu_0w_1 + (cp_2+p_1^{k-1})z,\\
v_4w_0 \mapsto p_3u_1z, & v_4^2 \mapsto p_1^2z,\\
v_3w_1 \mapsto p_4u_1z, & v_4w_1 \mapsto u_2z,\\
v_2w_0 \mapsto u_0^2w_1, & v_3w_0 \mapsto p_1^{k-2}u_2z,\\
w_0^2 \mapsto (1+c^3p_4)u_0^2z, & w_0w_1 \mapsto v_2z.
\end{array}$$

Put $q_i = \dim_K \mathcal{A}_3^i$. Denote the number of monomials in $\mathcal{A}_3^i$, represented in the normal form by $\widetilde{q}_i$. It is clear that $\widetilde{q}_i \geq q_i$. Since there is an epimorphism $\mathcal{A}_3^i \to \mathrm{HH}^i(R)$, we have $q_i \geq \dim_K \mathrm{HH}^i(R)$. Consequently, it suffices to show that

$$\widetilde{q}_i = \dim_K \mathrm{HH}^i(R). \tag{3.49}$$

Finally, successively analyzing several cases (cf. the proof of Proposition 3.8), we prove that all (nonzero) monomials having the normal form are contained in the following list (where $a \geq 0$).

The monomials of degree $4a$:

$$\begin{gathered}
\{w_0u_0^{4(a-m)-3}z^m\}_{m=0}^{a-1}, \{v_2u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1},\\
\{w_1u_0^{4(a-m)-3}z^m\}_{m=0}^{a-1}, \{p_2w_1u_0^{4(a-m)-3}z^m\}_{m=0}^{a-1},\\
\{u_1u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \{u_0^{4(a-m)}z^m\}_{m=0}^{a-1},\\
\{p_2u_0^{4(a-m)}z^m\}_{m=0}^{a-1}, \{p_4u_0^{4(a-m)}z^m\}_{m=0}^{a-1},\\
\{p_1^iz^a\}_{i=0}^{k-1}, p_2z^a, p_3z^a, p_4z^a
\end{gathered}$$

(the number of them equals $8a+k+3$).

The monomials of degree $4a+1$:

$$\begin{gathered}
\{w_0u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1}, \{v_2u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1},\\
\{w_1u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1}, \{p_2w_1u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1},\\
\{u_1u_0^{4(a-m)}z^m\}_{m=0}^{a}, \{u_0^{4(a-m)+1}z^m\}_{m=0}^{a},\\
\{p_2u_0^{4(a-m)+1}z^m\}_{m=0}^{a}, \{p_4u_0^{4(a-m)+1}z^m\}_{m=0}^{a},\\
\{p_1^iu_2z^a\}_{i=0}^{k-3}, p_4u_1z^a, p_1u_0z^a, p_3u_0z^a
\end{gathered}$$

(the number of them equals $8a+k+5$).

The monomials of degree $4a+2$:

$$\begin{gathered}
\{w_0u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \{v_2u_0^{4(a-m)}z^m\}_{m=0}^{a},\\
\{w_1u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \{p_2w_1u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1},\\
\{u_1u_0^{4(a-m)+1}z^m\}_{m=0}^{a}, \{u_0^{4(a-m)+2}z^m\}_{m=0}^{a},\\
\{p_2u_0^{4(a-m)+2}z^m\}_{m=0}^{a}, \{p_4u_0^{4(a-m)+2}z^m\}_{m=0}^{a},
\end{gathered}$$

$$\{p_1^i v_4 z^a\}_{i=0}^{k-2}, u_1^2 z^a, v_3 z^a$$

(the number of them equals $8a + k + 6$).

The monomials of degree $4a + 3$:

$$\{w_0 u_0^{4(a-m)} z^m\}_{m=0}^a, \{v_2 u_0^{4(a-m)+1} z^m\}_{m=0}^a,$$
$$\{w_1 u_0^{4(a-m)} z^m\}_{m=0}^a, \{p_2 w_1 u_0^{4(a-m)} z^m\}_{m=0}^a,$$
$$\{u_1 u_0^{4(a-m)+2} z^m\}_{m=0}^a, \{u_0^{4(a-m)+3} z^m\}_{m=0}^a,$$
$$\{p_2 u_0^{4(a-m)+3} z^m\}_{m=0}^a, \{p_4 u_0^{4(a-m)+3} z^m\}_{m=0}^a,$$
$$\{p_1^i w_1 z^a\}_{i=1}^{k-1}, p_1 w_0 z^a$$

(the number of them equals $8a + k + 8$).

Using Corollary 2.11, we derive from this the equality (3.49). □

Case 4. Now assume that k is even, $k > 2$, and $d = 0$.
We consider the following homogeneous elements of $\mathrm{HH}^*(R)$:

— of degree 0 : p_1, p_2, p_3, p_4 in (3.1);

— of degree 1 : u_0 in (3.42), u_2' in (3.38), and $u_1' := (0, 1 + cx)$;

— of degree 2 : v_2, v_3, v_4 in (3.3);

— of degree 3 : w_0, w_1 in (3.43);

— of degree 4 : z in (3.5).

Proposition 3.24. *Assume that k is even, $k > 2$, and $d = 0$. In the algebra* $\mathrm{HH}^*(R)$*, the elements of the set*

$$\mathcal{Y}_4 = \{p_1, p_2, p_3, p_4, u_0, u_1', u_2', v_2, v_3, v_4, w_0, w_1, z\} \tag{3.50}$$

satisfy the relations (3.7), (3.16), (3.23), (3.26), *and the following relations*:

$$p_3 u_0 = p_2 u_1',\ p_3 u_0 = p_1^{k-1} u_2',\ p_1 u_1' = c p_1 u_2'$$
$$p_3 u_1' = p_1 u_0,\ p_2 u_2' = p_4 u_0,\ p_3 u_2' = p_4 u_2' = 0;$$
$$u_1' u_2' = c^2 \theta_{k-1} p_3 v_3,$$
$$p_1 v_2 = p_3 v_3 = p_2 v_4 = p_4 (u_1')^2,$$
$$u_0 u_1' = 0,\ p_2 v_2 = p_4 u_0^2,\ (u_2')^2 = c\theta_{k-1} p_3 v_3,$$
$$p_1^{k-1} v_4 = c p_3 v_3,$$
$$p_1 v_3 = p_2 v_3 = p_4 v_3 = 0,$$
$$u_1' v_4 = u_2' v_3 = p_1 w_1,\ u_0 v_4 = p_1 w_0,$$
$$u_0 v_3 = p_1^{k-2} u_2' v_3 = p_3 w_0 = u_1' v_2,\ (u_1')^3 = 0,$$
$$u_0 v_2 = u_0^2 u_2' + p_2 w_0,\ p_2 w_1 = p_4 w_0 = u_2' v_2,$$
$$u_2' v_4 = u_1' v_3 = p_3 w_1 = p_4 w_1 = 0,$$
$$u_2' w_0 = u_0 w_1,\ u_1' w_0 = u_1' w_1 = u_2' w_1 = 0,$$
$$v_2 w_0 = u_0^2 w_1 + p_2 u_0 z,\ v_3 w_0 = p_1^{k-1} u_2' z,$$
$$v_2 w_1 = p_4 u_0 z,\ v_4 w_0 = (c p_1^{k-1} u_2' + p_1 u_0) z,$$
$$v_4 w_1 = p_1 u_2' z,\ v_3 w_1 = 0;$$

$$w_0^2=(1+c^3p_4)u_0^2z,\ w_0w_1=u_0u_2'z,\ w_1^2=c\theta_{k-1}p_3v_3z.$$

Proof. The proof of the above relations is similar to the proof of Proposition 3.1. But we need to know the translates of the element u_1' in $\mathcal{Y}_4$; the translates for the other elements have been calculated earlier.

Lemma 3.25. *As the translates of the element u_1', one can take the homomorphisms determined by the following matrices*:

$$\mathrm{T}^0(u_1')=\big(0,1\otimes 1+cx\otimes 1\big),\quad \mathrm{T}^1(u_1')=\begin{pmatrix}0 & *\\ 0 & *\end{pmatrix}$$

with

$$\begin{aligned}(\mathrm{T}^1(u_1'))_{12}&=\sum_{i=0}^{k-2}y(xy)^i\otimes(xy)^{k-2-i}+c\sum_{i=0}^{k-2}(i+1)(yx)^i\otimes(xy)^{k-1-i}\\&\quad+c\sum_{i=1}^{k-1}(xy)^i\otimes(xy)^{k-1-i}+c^2\sum_{i=1}^{k-1}ix(yx)^{i-1}\otimes(xy)^{k-i},\\(\mathrm{T}^1(u_1'))_{22}&=1\otimes 1+cx\otimes 1+c\sum_{i=0}^{k-2}(i+1)y(xy)^i\otimes y(xy)^{k-2-i}\\&\quad+c^2\sum_{i=1}^{k-1}i(xy)^i\otimes y(xy)^{k-1-i}.\end{aligned}$$

The proof of this lemma is similar to that of Lemma 3.3.

Now the proof of Proposition 3.24 is completed with the help of direct calculation. We leave to the reader the corresponding details. □

Remark 3.26. Note that formulas for the translates of w_1 are obtained from the corresponding translates of w, that have been calculated earlier (see Lemma 3.11) for $d\neq 0$ (and even k). Recall that formulas for the translates of v_2,v_3,v_4 in Lemmas 3.3, 3.11 are valid also for even k (and $d=0$).

Proposition 3.27. *For any $\ell\in\mathbb{N}$,*

$$u_0^\ell=\big(1,\mathrm{O}\big)\in\mathrm{HH}^\ell(R),\quad u_0^\ell v_2=\big(y,\mathrm{O}\big)\in\mathrm{HH}^{\ell+2}(R),$$
$$u_0^\ell w_0=\big(0,1,\mathrm{O}\big)\in\mathrm{HH}^{\ell+3}(R),\quad u_0^\ell w_1+p_2u_0^{\ell-1}z=\big(0,y,\mathrm{O}\big)\in\mathrm{HH}^{\ell+3}(R).$$

The proof of this statement is completely similar to the proof of Proposition 3.19.

Proposition 3.28. *Assume that k is even, $k>2$, and $d=0$. The set $\mathcal{Y}_4$ in* (3.50) *generates $\mathrm{HH}^*(R)$ as a K-algebra.*

Proof. Let $\mathcal{H}$ denote a K-subalgebra of $\mathrm{HH}^*(R)$, generated by the set $\mathcal{Y}_4\cup\{1\}$. First, we prove that $\bigcup_{i=0}^{3}\mathrm{HH}^i(R)\subset\mathcal{H}$.

It is clear that $\mathrm{HH}^0(R)\subset\mathcal{H}$ (see the proof of Proposition 3.7). The basis elements of $\mathrm{HH}^1(R)$, described in Corollary 2.10 (a), satisfy the relations

$$\begin{aligned}&\big(y(xy)^i,0\big)=p_1^iu_2'\ \text{ for } 1\le i\le k-1,\\&\big(x(yx)^{k-1},0\big)=p_2u_0,\ \big((xy)^k,0\big)=p_4u_0,\\&\big(0,y(xy)^{k-1}\big)=(p_3+cp_4)u_1',\ \big(0,(xy)^k\big)=p_4u_1'.\end{aligned}$$

Hence, $\mathrm{HH}^1(R) \subset \mathcal{H}$.

Next, the basis elements of $\mathrm{HH}^2(R)$, described in Corollary 2.10 (b), satisfy the relations

$$\begin{aligned}
&\big(0, y(xy)^i + c(xy)^{i+1}\big) = p_1^i v_4 \ \text{ for } \ 0 \le i \le k-3,\\
&\big(1,0\big) = u_0^2,\ \big(x(yx)^{k-1},0\big) = p_2 u_0^2,\\
&\big(0,1\big) = (u_1')^2 + c^3(1+\theta_{k-1})p_3 v_3,\ \big((xy)^k,0\big) = p_4 u_0^2,\\
&\big(0, y(xy)^{k-2}\big) = p_1^{k-1} v_4 + c(u_0 u_2' + v_2),\\
&\big(0,(xy)^{k-1}\big) = u_0 u_2' + v_2.
\end{aligned}$$

Consequently, $\mathrm{HH}^2(R) \subset \mathcal{H}$.

Then the basis elements of $\mathrm{HH}^3(R)$ described in Corollary 2.10 (c), satisfy the relations

$$\begin{aligned}
&\big(0, y(xy)^i\big) = p_1^i w_1 \ \text{ for } 1 \le i \le k-1,\\
&\big(1,0\big) = u_0^3,\ \big(y,0\big) = u_0 v_2,\\
&\big(x(yx)^{k-1},0\big) = p_2 u_0^3,\ \big((xy)^k,0\big) = p_4 u_0^3,\\
&\big(0, xy+yx\big) = u_0 v_4,\ \big(0, x(yx)^{k-1}\big) = p_2 w_0,\\
&\big(0,(xy)^k\big) = p_4 w_0,
\end{aligned}$$

whence $\mathrm{HH}^3(R) \subset \mathcal{H}$.

Now the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ is verified by induction on n similarly to the proof of Proposition 3.21. □

Let $\mathcal{A}_4 = K[\mathcal{X}_4]/I_4$ be the graded K-algebra defined in Sec. 1 with $\mathcal{X}_4$ as in (1.32) and I_4 the corresponding ideal of relations. Propositions 3.24 and 3.28 imply that there exists a surjective homomorphism $\varphi\colon \mathcal{A}_4 \to \mathrm{HH}^*(R)$ of graded K-algebras that takes the generators in $\mathcal{X}_4$ to the corresponding generators in $\mathcal{Y}_4$. Let $\mathcal{A}_4 = \bigoplus_{m\ge 0} \mathcal{A}_4^m$ be the direct decomposition of $\mathcal{A}_4$ into homogeneous direct summands. Now, statement (4) of Theorem 1.1 is a consequence of the following statement.

Proposition 3.29. *For any* $m \ge 0$,

$$\dim_K \mathcal{A}_4^m = \dim_K \mathrm{HH}^m(R).$$

First, we need the following auxiliary assertion.

Lemma 3.30. *The following relations are satisfied in the algebra* $\mathcal{A}_4$:

$$p_1^2 u_0 = p_1 u_0^2 = u_0^2 v_3 = u_0^2 v_4 = p_3 u_0^2 = (u_2')^3 = p_1 u_0 v_4 = 0.$$

All relations in lemma 3.30 follow directly from the defining relations of the algebra $\mathcal{A}_4$.

Proof of Proposition 3.29. We introduce a lexicographic order on the polynomial ring $K[\mathcal{X}_4]$, such that

$$w_0 > w_1 > u_1' > u_2' > v_2 > v_3 > v_4 > u_0 > z > p_2 > p_3 > p_4 > p_1.$$

Any nonzero monomial in $\mathcal{A}_4$ is represented in the form

$$f = p_1^i p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} u_0^{\ell} (u_1')^{\beta_1} (u_2')^{\beta_2} v_2^{\gamma_2} v_3^{\gamma_3} v_4^{\gamma_4} w_0^{\varepsilon_0} w_1^{\varepsilon_1} z^s; \tag{3.51}$$

here, by the defining relations of the algebra $\mathcal{A}_4$ (see also Lemma 3.30), we have

$$\alpha_2, \alpha_3, \alpha_4, \beta_4, \gamma_2, \gamma_3, \gamma_4, \varepsilon_0, \varepsilon_1 \in \{0,1\},\ \ i, \ell, \beta_1, s \in \mathbb{N} \cup \{0\}, i \le k-1, \beta_1 \le 2,\ \beta_2 \le 2.$$

We introduce the following list of elementary steps of reduction and study the normal forms of monomials (with respect to these steps of reductions):

$$
\begin{array}{ll}
p_2u_1' \mapsto p_1^{k-1}u_2' \mapsto p_3u_0, & p_1u_1' \mapsto cp_1u_2', \\
p_3u_1' \mapsto p_1u_0, & p_2u_2' \mapsto p_4u_0, \\
p_4(u_1')^2 \mapsto p_1v_2 \mapsto p_3v_3 \mapsto p_2v_4, & p_2v_4 \mapsto c^{-1}p_1^{k-1}v_4, \\
u_1'u_2' \mapsto c^2\theta_{k-1}p_3v_3 \text{ (ifff } c\theta_{k-1} \neq 0), & (u_2')^2 \mapsto c\theta_{k-1}p_3v_3 \text{ (ifff } c\theta_{k-1} \neq 0), \\
p_2v_2 \mapsto p_4u_0^2, & p_1w_1 \mapsto u_1'v_4 \mapsto u_2'v_3, \\
p_3w_0 \mapsto u_1'v_2 \mapsto p_1^{k-2}u_2'v_3 \mapsto u_0v_3, & p_2w_0 \mapsto u_0^2u_2' + u_0v_2, \\
p_4w_0 \mapsto p_2w_1 \mapsto u_2'v_2, & p_1w_0 \mapsto u_0v_4, \\
u_2'w_0 \mapsto u_0w_1, & v_4^2 \mapsto p_1^2z, \\
v_4w_1 \mapsto p_1u_2'z, & v_3w_0 \mapsto p_1^{k-1}u_2'z, \\
v_2w_0 \mapsto u_0^2w_1 + p_2u_0z, & v_4w_0 \mapsto (cp_1^{k-1}u_2' + p_1u_0)z, \\
v_2w_1 \mapsto p_4u_0z, & w_1^2 \mapsto c\theta_{k-1}p_3v_3z \text{ (ifff } c\theta_{k-1} \neq 0), \\
w_0^2 \mapsto (1+c^3p_4)u_0^2z, & w_0w_1 \mapsto u_0u_2'z.
\end{array}
$$

Put $q_i = \dim_K \mathcal{A}_4^i$. Denote the number of monomials in $\mathcal{A}_4^i$ represented in the normal form by $\widetilde{q}_i$. It is clear that $\widetilde{q}_i \geq q_i$. Consequently, it suffices to show that

$$\widetilde{q}_i = \dim_K \mathrm{HH}^i(R). \tag{3.52}$$

Now, successively analyzing several cases (cf. the proof of Proposition 3.8), we prove that all (nonzero) monomials having the normal form are contained in the following list (where $a \geq 0$).

The monomials of degree $4a$:

$$
\begin{gathered}
\{w_0u_0^{4(a-m)-3}z^m\}_{m=0}^{a-1}, \{v_2u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1}, \\
\{w_1u_0^{4(a-m)-3}z^m\}_{m=0}^{a-1}, \{u_2'v_2u_0^{4(a-m)-3}z^m\}_{m=0}^{a-1}, \\
\{u_2'u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \{u_0^{4(a-m)}z^m\}_{m=0}^{a-1}, \\
\{p_2u_0^{4(a-m)}z^m\}_{m=0}^{a-1}, \{p_4u_0^{4(a-m)}z^m\}_{m=0}^{a-1}, \\
\{p_1^iz^a\}_{i=0}^{k-1}, p_2z^a, p_3z^a, p_4z^a
\end{gathered}
$$

(the number of them equals $8a+k+3$).

The monomials of degree $4a+1$:

$$
\begin{gathered}
\{w_0u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1}, \{v_2u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \\
\{w_1u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1}, \{u_2'v_2u_0^{4(a-m)-2}z^m\}_{m=0}^{a-1}, \\
\{u_2'u_0^{4(a-m)}z^m\}_{m=0}^{a}, \{u_0^{4(a-m)+1}z^m\}_{m=0}^{a}, \\
\{p_2u_0^{4(a-m)+1}z^m\}_{m=0}^{a}, \{p_4u_0^{4(a-m)+1}z^m\}_{m=0}^{a}, \\
\{p_1^iu_2'z^a\}_{i=1}^{k-2}, u_1'z^a, p_4u_1'z^a, p_1u_0z^a, p_3u_0z^a
\end{gathered}
$$

(the number of them equals $8a+k+6$).

The monomials of degree $4a+2$:

$$
\begin{gathered}
\{w_0u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \{v_2u_0^{4(a-m)}z^m\}_{m=0}^{a}, \\
\{w_1u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \{u_2'v_2u_0^{4(a-m)-1}z^m\}_{m=0}^{a-1}, \\
\{u_2'u_0^{4(a-m)+1}z^m\}_{m=0}^{a}, \{u_0^{4(a-m)+2}z^m\}_{m=0}^{a},
\end{gathered}
$$

$$\{p_2u_0^{4(a-m)+2}z^m\}_{m=0}^a, \{p_4u_0^{4(a-m)+2}z^m\}_{m=0}^a,$$
$$\{p_1^iv_4z^a\}_{i=0}^{k-1}, (u_1')^2z^a, v_3z^a$$

(the number of them equals $8a+k+7$).

The monomials of degree $4a+3$:

$$\{w_0u_0^{4(a-m)}z^m\}_{m=0}^a, \{v_2u_0^{4(a-m)+1}z^m\}_{m=0}^a,$$
$$\{w_1u_0^{4(a-m)}z^m\}_{m=0}^a, \{u_2'v_2u_0^{4(a-m)}z^m\}_{m=0}^a,$$
$$\{u_2'u_0^{4(a-m)+2}z^m\}_{m=0}^a, \{u_0^{4(a-m)+3}z^m\}_{m=0}^a,$$
$$\{p_2u_0^{4(a-m)+3}z^m\}_{m=0}^a, \{p_4u_0^{4(a-m)+3}z^m\}_{m=0}^a,$$
$$\{p_1^iu_2'v_3z^a\}_{i=0}^{k-3}, v_3u_0z^a, v_4u_0z^a$$

(the number of them equals $8a+k+8$).

Using Corollary 2.11, we derive from this the equality (3.52). □

Case 5. Assume that $k=2$ and $d\neq 0$.
We consider the homogeneous elements

$$p_1, p_2, p_3, p_4, u_1', u_2', u_3, v_1, v_2, v_3, v_4, w, z \tag{3.53}$$

of $\mathrm{HH}^*(R)$, defined by the same formulas as in the Case 2 (for $k=2$).

Proposition 3.31. *Let $\mathcal{Y}_5$ be the set formed by the elements in* (3.53)*. In the algebra* $\mathrm{HH}^*(R)$*, these elements satisfy the following relations*:

$$p_ip_j=0 \ \ \textit{for all}\, i,j\in\{1,2,3,4\};$$
$$p_1u_1'=(dp_2+cp_1)u_2',\ p_2u_1'=p_1u_2',$$
$$p_ju_3=0 \ \ \textit{for}\ \ 1\le j\le 4,\ p_3u_2'=p_4u_2'=0,$$
$$p_1v_4=dp_4v_1+cp_3v_3,\ u_3^2=0,$$
$$p_3v_1=p_1v_2=p_3v_3=p_2v_4=u_1'u_3=p_4(u_1')^2,$$
$$p_2v_2=p_4v_1=u_2'u_3,\ p_3(u_1')^2=0,$$
$$u_1'u_2'=cdp_4v_1+c^2p_3v_1,\ (u_2')^2=cp_3v_1,$$
$$u_1'v_4=u_2'(dv_2+cv_4),\ u_2'v_4=u_1'v_2=p_1w,$$
$$u_1'v_1=u_1'v_3,\ p_2w=u_2'v_2$$
$$(u_1')^3=du_2'v_1,\ p_3w=p_4w=0,$$
$$v_2^2=v_3^2=v_4^2=0;\ v_iv_j=0 \ \ \textit{for}\, i<j,$$
$$u_1'w=u_2'w=0;$$
$$v_2w=p_2u_2'z,\ v_4w=p_1u_2'z,\ v_3w=0;$$
$$w^2=cp_3v_3z.$$

Proof. The proof of the above relations is similar to the proof of Proposition 3.1. Note that the formulas for the translates of the elements in $\mathcal{Y}_5$, that have been calculated earlier are valid in the case under consideration. □

Proposition 3.32. *Assume that $k=2$ and $d\neq 0$. The set $\mathcal{Y}_5$ generates* $\mathrm{HH}^*(R)$ *as a K-algebra.*

We need the following auxiliary statement.

Lemma 3.33. *For any* $\ell \in \mathbb{N} \setminus \{1\}$,

$$v_1^{\ell-2}(v_1 w + u_3 z) = (0, \widetilde{y}, \mathrm{O}) \in \mathrm{HH}^{2\ell+1}(R),$$
$$v_1^{\ell-1} u_3 w = \big(0, (xy)^2, \mathrm{O}\,\big) \in \mathrm{HH}^{2\ell+2}(R).$$

Proof. It is immediately verified that $v_1 w + u_3 z = (0, \widetilde{y}, \mathrm{O}_2)$. Then the proof of the first equality is carried out by induction on ℓ similarly to the proof of Proposition 3.5. Note that the translates $\mathrm{T}^i(f)$, $i \in \{1, 2\}$, of the element $f = (0, \widetilde{y}, \mathrm{O})$ can be taken in the form obtained in the proof of this proposition.

The second equality is proved similarly. □

Proof of Proposition 3.32. Let $\mathcal{H}$ denote a K-subalgebra of $\mathrm{HH}^*(R)$, generated by the set $\mathcal{Y}_5 \cup \{1\}$. First, we prove that $\bigcup_{i=0}^3 \mathrm{HH}^i(R) \subset \mathcal{H}$.

It is clear that $\mathrm{HH}^0(R) \subset \mathcal{H}$ (see the proof of Proposition 3.7). The basis elements of $\mathrm{HH}^1(R)$, described in Corollary 2.6 (a), satisfy the relations

$$\big(yxy, 0\big) = p_1^i u_2', \ \big((xy)^2, 0\big) = p_2 u_2',$$
$$\big(0, yxy\big) = (p_3 + cp_4)u_1', \big(0, (xy)^2\big) = p_4 u_1'.$$

Hence, $\mathrm{HH}^1(R) \subset \mathcal{H}$.

Next, the basis elements of $\mathrm{HH}^2(R)$, described in Corollary 2.6 (b), satisfy the relations

$$\big(0, 1\big) = (u_1')^{\,2} + c^2 d p_4 v_1 + cd p_3 v_3,$$
$$\big((xy)^2, 0\big) = p_4 v_1, \ \big(0, (xy)^2\big) = p_3 v_3.$$

Consequently, $\mathrm{HH}^2(R) \subset \mathcal{H}$.

Then the basis elements of $\mathrm{HH}^3(R)$, described in Corollary 2.6 (c), satisfy the relations

$$\big(0, yxy\big) = p_1 w,$$
$$\big(dyx, xy + yx\big) = u_1' v_3, \ \big(y, xyx\big) = u_2' v_1,$$
$$\big(xyx, 0\big) = u_3 v_1, \ \big(0, (xy)^2\big) = u_2' v_2,$$

whence $\mathrm{HH}^3(R) \subset \mathcal{H}$.

Now the inclusion $\mathrm{HH}^n(R) \subset \mathcal{H}$ is proved by induction on n similarly to the proof of Proposition 3.7. Note that relations (3.32) are valid also for even k, and the equalities in Lemma 3.33 should be used instead of relations (3.33). □

Let $\mathcal{A}_5 = K[\mathcal{X}_5]/I_5$ be the graded K-algebra defined in Sec. 1, where $\mathcal{X}_5$ coincides with the set $\mathcal{X}_2$ in (1.28), and I_5 the corresponding ideal of relations. Since there exists a surjective homomorphism $\varphi\colon \mathcal{A}_5 \to \mathrm{HH}^*(R)$ of graded K-algebras that takes the generators in $\mathcal{X}_5$ to the corresponding generators in $\mathcal{Y}_5$, statement (5) of Theorem 1.1 is a consequence of the following statement.

Proposition 3.34. *For any* $m \geq 0$,

$$\dim_K \mathcal{A}_5^m = \dim_K \mathrm{HH}^m(R).$$

First, we need the following auxiliary assertion.

Lemma 3.35. *The following relations are satisfied in the algebra* $\mathcal{A}_5$:

$$p_2 u_2' v_2 = p_1 u_2' v_2 = p_3 u_1' v_1 = p_4 u_1' v_1 = (u_1')^4 = (u_2')^3 = p_4 v_1^2 = u_1' v_1^2 = u_2' v_1^2 = 0.$$

All relations in Lemma 3.35 follow directly from the defining relations of the algebra $\mathcal{A}_5$.

Proof of Proposition 3.34. We introduce a lexicographic order on the polynomial ring $K[\mathcal{X}_5]$, such that

$$w > u_1' > u_2' > u_3 > v_4 > v_3 > v_2 > v_1 > z > p_2 > p_3 > p_4 > p_1.$$

Then we introduce the following list of elementary steps of reduction, and study the normal forms of monomials (with respect to these steps):

$$\begin{array}{ll}
p_1u_1' \mapsto (dp_2 + cp_1)u_2', & p_2u_1' \mapsto p_1u_2', \\
u_2'u_3 \mapsto p_2v_2 \mapsto p_4v_1, & p_4(u_1')^2 \mapsto p_2v_4 \mapsto p_3v_3 \mapsto p_1v_2 \mapsto p_3v_1, \\
u_1'u_3 \mapsto p_3v_1, & p_1v_4 \mapsto dp_4v_1 + cp_3v_3, \\
u_1'u_2' \mapsto c(dp_4 + cp_3)v_1, & (u_2')^2 \mapsto cp_3v_1, \\
(u_1')^3 \mapsto du_2'v_1, & u_1'v_3 \mapsto u_1'v_1, \\
u_1'v_4 \mapsto u_2'(dv_2 + cv_4), & p_1w \mapsto u_2'v_4 \mapsto u_1'v_2, \\
p_2w \mapsto u_2'v_2, & v_2w \mapsto p_2u_2'z, \\
v_4w \mapsto p_1u_2'z, & w^2 \mapsto cp_3v_3z.
\end{array}$$

And then, as in the proof of Proposition 3.14, we obtain the list of monomials having the normal form, which almost coincides with such list in the Case 2: the monomial $p_1v_2z^a$ is replaced by $p_3v_1z^a$, and the monomial $u_1'v_3z^a$ is replaced by $u_1'v_1z^a$. This completes the proof of Proposition 3.34. □

Case 6. Now, we assume that $k = 2$ and $d = 0$.
We consider the homogeneous elements

$$p_1, p_2, p_3, p_4, u_0, u_1', u_2', v_2, v_3, v_4, w_0, w_1, z \tag{3.54}$$

of $\mathrm{HH}^*(R)$, defined by the same formulas as in the Case 4 (for $k = 2$).

Proposition 3.36. *Let $\mathcal{Y}_6$ be the set formed by the elements in* (3.54). *In the algebra* $\mathrm{HH}^*(R)$, *these elements satisfy the following relations*:

$$\begin{gathered}
p_ip_j = 0 \quad \textit{for all}\, i, j \in \{1, 2, 3, 4\}; \\
p_1u_1' = cp_3u_0,\ p_2u_1' = p_3u_0 = p_1u_2', \\
p_3u_1' = p_1u_0,\ p_2u_2' = p_4u_0,\ p_3u_2' = p_4u_2' = 0; \\
p_1v_2 = p_3v_3 = p_2v_4 = c^{-1}p_1v_4 = c^{-1}(u_2')^2 = c^{-1}u_1'u_2' = p_4(u_1')^2, \\
p_2v_2 = p_4u_0^2,\ p_3v_2 = p_4v_2 = 0, \\
p_1v_3 = p_2v_3 = p_4v_3 = p_3v_4 = p_4v_4 = 0,\ u_0u_1' = 0; \\
u_0v_3 = u_1'v_2 = c^{-1}u_1'v_4 = u_2'v_4 = p_3w_0 = p_1w_1, \\
u_0v_4 = u_1'v_3 = p_1w_0,\ (u_1')^3 = u_2'v_3 = 0, \\
p_2w_0 = u_0v_2 + u_0^2u_2', \\
p_2w_1 = p_4w_0 = u_2'v_2,\ p_3w_1 = p_4w_1 = 0; \\
v_iv_j = 0 \quad \textit{for all} \ \ i, j \in \{2, 3, 4\}, \\
u_2'w_0 = u_0w_1,\ u_1'w_0 = u_1'w_1 = u_2'w_1 = 0, \\
v_2w_0 = u_0^2w_1 + p_2u_0z,\ v_3w_0 = v_4w_1 = p_1u_2'z, \\
v_2w_1 = p_4u_0z,\ v_4w_0 = p_1u_0z,\ v_3w_1 = 0, \\
w_0^2 = (1 + c^3p_4)u_0^2z,\ w_0w_1 = u_0u_2'z,\ w_1^2 = cp_3v_3z.
\end{gathered}$$

The proof of the above relations is similar to the proof of Proposition 3.1 (cf. Proposition 3.24). But we need to know the translates $\mathrm{T}^i(w_0)$ for $i = 2, 3$, whose description differs from that in the Case 3 (and, respectively, in Case 4) (see Lemma 3.17).

Lemma 3.37. *The translates* $\mathrm{T}^0(w_0)$, $\mathrm{T}^1(w_0)$ *can be defined by the formulas in Lemma* 3.17. *Furthermore, the translates* $\mathrm{T}^i(w_0)$, $i = 2, 3$, *can be described by the following matrices*:

$$\mathrm{T}^2(w_0) = \begin{pmatrix} 0 & 1\otimes 1 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

with

$$\begin{aligned}
(\mathrm{T}^2(w_0))_{13} =& c(xy)^2\otimes 1 + c\cdot 1\otimes (xy)^2,\\
(\mathrm{T}^2(w_0))_{14} =& yx\otimes x + x\otimes xy + xyx\otimes 1\\
&+ cxyx\otimes x + c^2(xy)^2\otimes xy,\\
(\mathrm{T}^2(w_0))_{23} =& x\otimes 1 + 1\otimes x + y\otimes y + cx\otimes x\\
&+ cyxy\otimes 1 + c^2yxy\otimes x,\\
(\mathrm{T}^2(w_0)))_{24} =& 1\otimes yx + xy\otimes 1 + x\otimes y\\
&+ yx\otimes 1 + cy\otimes yxy;
\end{aligned}$$

$$\mathrm{T}^3(w_0) = \begin{pmatrix} 0 & 1\otimes 1 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

with

$$\begin{aligned}
(\mathrm{T}^3(w_0))_{13} &= c(xy)^2\otimes 1 + c\cdot 1\otimes (xy)^2\\
&\quad + c^2yx\otimes xyx + c^3xyx\otimes xyx,\\
(\mathrm{T}^3(w_0))_{14} &= yx\otimes xy + (xy)^2\otimes 1\\
&\quad + cyx\otimes xyx + cxyx\otimes xy,\\
(\mathrm{T}^3(w_0))_{23} &= 1\otimes 1 + c^2yxy\otimes 1 + c^3(xy)^2\otimes 1,\\
(\mathrm{T}^3(w_0))_{24} &= x\otimes 1 + c\cdot 1\otimes yxy + cyxy\otimes 1\\
&\quad + c^2yx\otimes xy + c^2(yx)^2\otimes 1.
\end{aligned}$$

Now the proof of Proposition 3.36 is completed with the help of direct calculation. We leave to the reader the corresponding details.

Proposition 3.38. *Assume that* $k = 2$ *and* $d = 0$. *The set* $\mathcal{Y}_6$ *generates* $\mathrm{HH}^*(R)$ *as a* K*-algebra.*

Proof. The proof is carried out similarly to the proof of Proposition 3.28. The formulas in Proposition 3.27, that are used in the proof, are all valid with one exception: we have $u_0^\ell = \big(1, \mathrm{O}\big) \in \mathrm{HH}^\ell(R)$ only for $\ell \geq 4$. □

Let $\mathcal{A}_6 = K[\mathcal{X}_6]/I_6$ be the graded K-algebra defined in Sec. 1, where $\mathcal{X}_6$ coincides with the set $\mathcal{X}_4$ in (1.32), and I_6 is the corresponding ideal of relations. Since there exists a surjective homomorphism $\varphi\colon \mathcal{A}_6 \to \mathrm{HH}^*(R)$ of graded K-algebras that takes the generators in $\mathcal{X}_6$ to the corresponding generators in $\mathcal{Y}_6$, statement (6) of Theorem 1.1 is a consequence of the following statement.

Proposition 3.39. *For any* $m \geq 0$,

$$\dim_K \mathcal{A}_6^m = \dim_K \mathrm{HH}^m(R).$$

Proof. The proof of this statement is similar to the proof of Proposition 3.29. We introduce a lexicographic order on the polynomial ring $K[\mathcal{X}_6]$, such that

$$w_0 > w_1 > u_1' > u_2' > v_2 > v_3 > v_4 > u_0 > z > p_2 > p_3 > p_4 > p_1.$$

Then we introduce the following list of elementary steps of reduction, and study the normal forms of monomials (with respect to these steps):

$p_2u_1' \mapsto p_1u_2' \mapsto p_3u_0,$ $\quad p_1u_1' \mapsto cp_3u_0,$

$p_3u_1' \mapsto p_1u_0,$ $\quad p_2u_2' \mapsto p_4u_0,$

$p_4(u_1')^2 \mapsto p_1v_2 \mapsto p_3v_3 \mapsto p_2v_4 \mapsto c^{-1}p_1v_4,$

$u_1'u_2' \mapsto (u_2')^2 \mapsto p_1v_4,$ $\quad p_2v_2 \mapsto p_4u_0^2,$

$u_1'v_2 \mapsto u_2'v_4 \mapsto u_0v_3,$ $\quad p_3w_0 \mapsto p_1w_1 \mapsto u_0v_3,$

$u_1'v_4 \mapsto cu_0v_3,$ $\quad p_2w_0 \mapsto u_0^2u_2' + u_0v_2,$

$p_4w_0 \mapsto p_2w_1 \mapsto u_2'v_2,$ $\quad p_1w_0 \mapsto u_1'v_3 \mapsto u_0v_4,$

$u_2'w_0 \mapsto u_0w_1,$ $\quad v_3w_0 \mapsto v_4w_1 \mapsto p_1u_2'z,$

$v_2w_0 \mapsto u_0^2w_1 + p_2u_0z,$ $\quad v_4w_0 \mapsto p_1u_0z,$

$v_2w_1 \mapsto p_4u_0z,$ $\quad w_1^2 \mapsto cp_3v_3z,$

$w_0^2 \mapsto (1 + c^3p_4)u_0^2z,$ $\quad w_0w_1 \mapsto u_0u_2'z.$

Then, as in the proof of Proposition 3.29, we obtain the list of monomials having the normal form that coincides with the list in Case 4. This completes the proof of Proposition 3.39. □

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