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# BOUNDARY-VALUE PROBLEMS FOR THE SYSTEM OF OPERATOR-DIFFERENTIAL EQUATIONS IN BANACH AND HILBERT SPACES

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We establish necessary and sufficient conditions for the existence of solutions of linear and nonlinear boundary-value problems in Hilbert and Banach spaces and present a convergent iterative procedure for finding the solutions in the nonlinear case.

## Introduction

In the present paper, we develop constructive methods for the analysis of linear and nonlinear boundary-value problems for operator-differential equations in Banach and Hilbert spaces. These problems occupy a central place in the qualitative theory of differential equations [1–18]. A specific feature of these problems is that the operator of the linear part of the equation does not have the inverse operator. This does not allow one to use the traditional methods based on the principles of contracting mappings and fixed point. For the analysis of a nonlinear system of differential equations, we develop the ideas of the Lyapunov–Schmidt method and efficient methods of the perturbation theory by using the theory of generalized inverse [19] and strongly generalized inverse operators [20].

## Statement of the Problem

Consider the following boundary-value problem:

$$\begin{cases} \varphi'(t,\varepsilon) = \varphi(t,\varepsilon) + \psi(t,\varepsilon) + \varepsilon f_1(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) + g_1(t), \\ \psi'(t,\varepsilon) = \varphi(t,\varepsilon) + \varepsilon f_2(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) + g_2(t), \quad t \in J, \end{cases} \tag{1}$$

with a boundary condition

$$l(\varphi(\cdot,\varepsilon),\psi(\cdot,\varepsilon)) = \alpha, \tag{2}$$

where $\varphi, \psi \in C^1(J,\mathcal{H})$, $C^1(J,\mathcal{H})$ is a Banach space of continuously differentiable vector functions on the interval $J \subset \mathbb{R}$ with values in the Hilbert space $\mathcal{H}$; $f_1$ and $f_2$ are strongly differentiable vector functions; $l$ is a linear and bounded operator, which translates the solutions of Eq. (1) into the Hilbert space $\mathcal{H}_1$, and $g_1(t), g_2(t) \in C(J,\mathcal{H})$ are vector functions. We find necessary and sufficient conditions for the existence of solutions $\varphi(t,\varepsilon)$, $\psi(t,\varepsilon)$

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of the boundary-value problem (1), (2), which turn, for $\varepsilon = 0$, into one of solutions of the generating linear boundary-value problem of the following form:

$$\begin{cases} \varphi_0'(t) = \varphi_0(t) + \psi_0(t) + g_1(t), \\ \psi_0'(t) = \varphi_0(t) + g_2(t), \quad t \in J, \end{cases}$$

$$l(\varphi_0(\cdot), \psi_0(\cdot)) = \alpha.$$

First, we investigate a generating linear case.

## Linear Case

Consider a linear boundary-value problem

$$\begin{cases} \varphi_0'(t) = \varphi_0(t) + \psi_0(t) + g_1(t), \\ \psi_0'(t) = \varphi_0(t) + g_2(t), \quad t \in J, \end{cases} \tag{3}$$

$$l(\varphi_0(\cdot), \psi_0(\cdot)) = \alpha. \tag{4}$$

Denote by $U(t)$ the evolution operator of the homogeneous system

$$\begin{cases} \varphi_0'(t) = \varphi_0(t) + \psi_0(t), \\ \psi_0'(t) = \varphi_0(t), \quad t \in J, \end{cases}$$

$$U'(t) = AU(t), \quad U(0) = I,$$

where the matrix operator-valued function has the form

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and the evolution operator $U(t)$ has the form

$$U(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} t^n F_{n+1} & t^n F_n \\ t^n F_n & t^n F_{n-1} \end{pmatrix},$$

where $F_n$ is a Fibonacci sequence:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n, \quad n \geq 0,$$

or

$$U(t) = \begin{pmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} e^{\frac{1+\sqrt{5}}{2}t} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{\frac{1-\sqrt{5}}{2}t} & e^{\frac{1+\sqrt{5}}{2}t} - e^{\frac{1-\sqrt{5}}{2}t} \\ e^{\frac{1+\sqrt{5}}{2}t} - e^{\frac{1-\sqrt{5}}{2}t} & \frac{2}{5+\sqrt{5}} e^{\frac{1+\sqrt{5}}{2}t} + \frac{2}{5-\sqrt{5}} e^{\frac{1-\sqrt{5}}{2}t} \end{pmatrix}. \tag{5}$$

In this case, the set of solutions of Eqs. (3) has the form

$$\begin{pmatrix} \varphi_0(t,c) \\ \psi_0(t,c) \end{pmatrix} = e^{tA}c + \int\limits_0^t e^{(t-\tau)A} g(\tau)d\tau = \sum_{n=0}^{+\infty} \frac{1}{n!} \begin{pmatrix} t^n F_{n+1} c_1 + t^n F_n c_2 \\ t^n F_n c_1 + t^n F_{n-1} c_2 \end{pmatrix}$$

$$+ \sum_{n=0}^{+\infty} \frac{1}{n!} \int\limits_0^t \begin{pmatrix} (t-\tau)^n F_{n+1} g_1(\tau) + (t-\tau)^n F_n g_2(\tau) \\ (t-\tau)^n F_n g_1(\tau) + (t-\tau)^n F_{n-1} g_2(\tau) \end{pmatrix} d\tau,$$

where $c = (c_1, c_2)^T$, $c_1, c_2 \in \mathcal{H}$, and $g(t) = (g_1(t), g_2(t))^T$ [or by using representation (5)]. Substituting in the boundary condition (4) we obtain the following operator equation:

$$Qc = \alpha - l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau, \quad Q = lU(\cdot)\colon \mathcal{H} \to \mathcal{H}_1.$$

By using the theory of strong generalized solutions [21], we get the following result:

$$Qc = \alpha - l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau, \qquad Q = lU(\cdot)\colon \mathcal{H} \times \mathcal{H} \to \mathcal{H}_1.$$

**Theorem 1.** *(i) (a) The boundary-value problem (3), (4) has strongly generalized solutions if and only if the following condition holds:*

$$\mathcal{P}_{N(\overline{Q}^*)} \left\{ \alpha - l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau \right\} = 0; \tag{6}$$

*if*

$$\alpha - l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)f(\tau)d\tau \in R(Q),$$

*then the generalized solutions is classical;*

(b) *under condition (6), the set of solutions has the form*

$$\begin{pmatrix} \varphi_0(t,\overline{c}) \\ \psi_0(t,\overline{c}) \end{pmatrix} = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c} + \overline{(G[g,\alpha])(t)} \quad \forall\, \overline{c} \in \mathcal{H}, \tag{7}$$

*where* $\mathcal{P}_{N(\overline{Q})}$, $\mathcal{P}_{N(\overline{Q}^*)}$ *are the orthoprojectors onto the kernel and cokernel of the operator* $\overline{Q}$, *respectively* ($\overline{Q}$ *is the extension of the operator* $Q$ *[20]),*

$$\overline{(G[g,\alpha])(t)} = \int\limits_0^t U(t)U^{-1}(\tau)g(\tau)d\tau + \overline{Q}^+ \left\{ \alpha - l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)g(\tau)d\tau \right\}$$

*is a generalized Green's operator, and* $\overline{Q}^+$ *is a strongly Moore–Penrose pseudoinvertible operator [20];*

(ii) (a) *the boundary-value problem (3), (4) has strong pseudosolutions if and only if the following condition holds:*

$$\mathcal{P}_{N(\overline{Q}^*)} \left\{ \alpha - l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)f(\tau)d\tau \right\} \neq 0; \tag{8}$$

(b) *under condition (8) the set of strong pseudosolutions has the form*

$$\begin{pmatrix} \varphi_0(t,\overline{c}) \\ \psi_0(t,\overline{c}) \end{pmatrix} = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c} + \overline{(G[g,\alpha])(t)} \quad \forall\, \overline{c} \in \mathcal{H}.$$

## Nonlinear Case

The following statement is true:

**Theorem 2.** *Suppose that the boundary-value problem (1), (2) has a solution which turns into one of solutions of the generating boundary-value problem (3), (4) in the form (7)* ($\varepsilon = 0$) *with an element* $\overline{c} = c_0$. *Then the element* $c_0$ *satisfies the following operator equation for generating elements:*

$$F(c) = \mathcal{P}_{N(\overline{Q}^*)} l \int\limits_0^{\cdot} U(\cdot)U^{-1}(\tau)f(\tau,\varphi_0(\tau,c),\psi_0(\tau,c),0)d\tau = 0.$$

Here,

$$f(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) = \begin{pmatrix} f_1(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) \\ f_2(t,\varphi(t,\varepsilon),\psi(t,\varepsilon),\varepsilon) \end{pmatrix}.$$

***Proof.*** If the boundary-value problem (1), (2) has a solution, then it follows from Theorem 1 that the following condition is true:

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{\alpha - l\int_0^{\cdot} U(\cdot)U^{-1}(\tau)\,(g(\tau)+\varepsilon f(\tau,\varphi(\tau,\varepsilon),\psi(\tau,\varepsilon),\varepsilon))\,d\tau\right\} = 0.$$

Since the boundary-value problem (1), (2) has the solution, by using condition (6), we finally obtain ($\varepsilon \to 0$)

$$\mathcal{P}_{N(\overline{Q}^*)}\left\{l\int_0^{\cdot} U(\cdot)U^{-1}(\tau)f(\tau,\varphi_0(\tau,\overline{c}),\psi_0(\tau,\overline{c}),0)d\tau\right\} = 0.$$

In order to obtain the sufficient condition for the existence of solutions, we use the following change of variables:

$$\varphi(t,\varepsilon) = \overline{\varphi}(t,\varepsilon) + \varphi_0(t,c_0),$$

$$\psi(t,\varepsilon) = \overline{\psi}(t,\varepsilon) + \psi_0(t,c_0),$$

where the element $c_0$ satisfies the equation for generating elements. We arrive at the boundary-value problem

$$\begin{aligned}\overline{\varphi}'(t,\varepsilon) &= \overline{\varphi}(t,\varepsilon) + \overline{\psi}(t,\varepsilon) + \varepsilon f_1(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon),\\ \overline{\psi}'(t,\varepsilon) &= \overline{\varphi}(t,\varepsilon) + \varepsilon f_2(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon),\end{aligned} \tag{9}$$

$$l(\overline{\varphi}(\cdot,\varepsilon),\overline{\psi}(\cdot,\varepsilon)) = 0. \tag{10}$$

Suppose that the vector functions $f_1$ and $f_2$ are strongly differentiable in the neighborhood of the generating solution

$$f_1, f_2 \in C^1(\|\varphi-\varphi_0\| \le q_1, \|\psi-\psi_0\| \le q_2),$$

where $q_1$ and $q_2$ are positive constants.

We use the following expansions:

$$\begin{aligned}&f_1(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon)\\ &\qquad = f_1(t,\varphi_0(t,c_0),\psi_0(t,c_0),0) + f'_{1\varphi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\varphi}(t,\varepsilon)\\ &\qquad\quad + f'_{1\psi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\psi}(t,\varepsilon) + \mathcal{R}_1(t,\overline{\varphi}(t,\varepsilon),\overline{\psi}(t,\varepsilon),\varepsilon),\\ &f_2(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon)\\ &\qquad = f_2(t,\varphi_0(t,c_0),\psi_0(t,c_0),0) + f'_{2\varphi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\varphi}(t,\varepsilon)\\ &\qquad\quad + f'_{2\psi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\psi}(t,\varepsilon) + \mathcal{R}_2(t,\overline{\varphi}(t,\varepsilon),\overline{\psi}(t,\varepsilon),\varepsilon),\end{aligned}$$

where

$$\mathcal{R}_1(t,0,0,0) = \mathcal{R}'_{1\varphi}(t,0,0,0) = \mathcal{R}'_{1\psi}(t,0,0,0) = 0,$$

$$\mathcal{R}_2(t,0,0,0) = \mathcal{R}'_{2\varphi}(t,0,0,0) = \mathcal{R}'_{2\psi}(t,0,0,0) = 0.$$

Thus, we can rewrite the boundary-value problem (9)–(10) in the following form:

$$\begin{aligned} \overline{\varphi}' &= \overline{\varphi} + \overline{\psi} + \varepsilon \left\{ f_1 + f'_{1\varphi}\overline{\varphi} + f'_{1\psi}\overline{\psi} + \mathcal{R}_1 \right\}, \\ \overline{\psi}' &= \overline{\varphi} + \varepsilon \left\{ f_2 + f'_{2\varphi}\overline{\varphi} + f'_{2\psi}\overline{\psi} + \mathcal{R}_2 \right\}, \end{aligned} \tag{11}$$

$$l\big(\overline{\varphi}(\cdot,\varepsilon), \overline{\psi}(\cdot,\varepsilon)\big) = 0. \tag{12}$$

Let

$$F(t,\varepsilon) = \begin{pmatrix} f_1 + f'_{1\varphi}\overline{\varphi} + f'_{1\psi}\overline{\psi} + \mathcal{R}_1 \\ f_2 + f'_{2\varphi}\overline{\varphi} + f'_{2\psi}\overline{\psi} + \mathcal{R}_2 \end{pmatrix}.$$

Under condition of solvability [19, 20]

$$\mathcal{P}_{N(\overline{Q}^*)} \left\{ l \int_0^{\cdot} U(\cdot)U^{-1}(\tau)F(\tau,\varepsilon)\,d\tau \right\} = 0, \tag{13}$$

the set of solutions of the boundary-value problem (11), (12) has the following form:

$$\begin{pmatrix} \overline{\varphi}(t,\overline{c}) \\ \overline{\psi}(t,\overline{c}) \end{pmatrix} = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c} + \varepsilon\overline{(G[F,0])(t)} \quad \forall\, \overline{c} \in \mathcal{H}.$$

Substituting the representation of solutions in condition (13), we obtain the operator equation

$$B_0\overline{c} = b, \tag{14}$$

where the operator $B_0$ has the form

$$B_0 = \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} f'_{1\varphi} & f'_{1\psi} \\ f'_{2\varphi} & f'_{2\psi} \end{pmatrix} U(\tau)\mathcal{P}_{N(\overline{Q})}\,d\tau,$$

$$b = -\mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} d\tau$$

$$- \varepsilon \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} f'_{1\varphi} & f'_{1\psi} \\ f'_{2\varphi} & f'_{2\psi} \end{pmatrix} \overline{G[F,0]}(\tau) d\tau.$$

Suppose that the following condition is satisfied:

$$\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0.$$

Then the equation (14) is solvable. One of its solutions has the form

$$c = \overline{B}_0^+ b.$$

In this way, we obtain the following theorem:

**Theorem 3.** *Suppose that the following condition is satisfied:*

$$\mathcal{P}_{N(\overline{B}_0^*)} \mathcal{P}_{N(\overline{Q}^*)} = 0.$$

*Then, for any element $c = c_0 \in \mathcal{H}$ satisfying the equation for generating elements, there exists a solution of the boundary-value problem (1), (2). This solution can be found by using the iterative procedure:*

$$\begin{pmatrix} \overline{\varphi}_{k+1}(t, \overline{c}_k) \\ \overline{\psi}_{k+1}(t, \overline{c}_k) \end{pmatrix} = U(t)\mathcal{P}_{N(\overline{Q})}\overline{c}_k + \overline{h}_{k+1}(t, \varepsilon),$$

$$\overline{c}_k = -\overline{B}_0^+ \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} \mathcal{R}_1(\tau, \overline{\varphi}_k, \overline{\psi}_k, \varepsilon) \\ \mathcal{R}_2(\tau, \overline{\varphi}_k, \overline{\psi}_k, \varepsilon) \end{pmatrix} d\tau$$

$$- \overline{B}_0^+ \mathcal{P}_{N(\overline{Q}^*)} l \int_0^{\cdot} U(\cdot)U^{-1}(\tau) \begin{pmatrix} f'_{1\varphi} & f'_{1\psi} \\ f'_{2\varphi} & f'_{2\psi} \end{pmatrix} \overline{h}_k(\tau, \varepsilon) d\tau,$$

$$\overline{h}_{k+1}(t, \varepsilon) = \varepsilon G[f(\cdot, \overline{\varphi}_k + \varphi_0, \overline{\psi}_k + \psi_0, \varepsilon), 0](t),$$

$$\mathcal{R}_1(t, \overline{\varphi}(t,\varepsilon), \overline{\psi}(t,\varepsilon), \varepsilon) = f_1(t, \overline{\varphi}(t,\varepsilon) + \varphi_0(t, c_0), \overline{\psi}(t,\varepsilon) + \psi_0(t, c_0), \varepsilon)$$

$$- f_1(t, \varphi_0(t, c_0), \psi_0(t, c_0), 0) - f'_{1\varphi}(t, \varphi_0(t, c_0), \psi_0(t, c_0), 0)\overline{\varphi}(t, \varepsilon)$$

$$- f'_{1\psi}(t, \varphi_0(t, c_0), \psi_0(t, c_0), 0)\overline{\psi}(t, \varepsilon),$$

$$\mathcal{R}_2(t,\overline{\varphi}(t,\varepsilon),\overline{\psi}(t,\varepsilon),\varepsilon) = f_2(t,\overline{\varphi}(t,\varepsilon)+\varphi_0(t,c_0),\overline{\psi}(t,\varepsilon)+\psi_0(t,c_0),\varepsilon)$$

$$- f_2(t,\varphi_0(t,c_0),\psi_0(t,c_0),0) - f'_{2\varphi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\varphi}(t,\varepsilon)$$

$$- f'_{2\psi}(t,\varphi_0(t,c_0),\psi_0(t,c_0),0)\overline{\psi}_k(t,\varepsilon),$$

$$\varphi(t,\varepsilon) = \varphi_0(t,c_0) + \lim_{k\to\infty}\overline{\varphi}_k(t,\varepsilon),$$

$$\psi(t,\varepsilon) = \psi_0(t,c_0) + \lim_{k\to\infty}\overline{\psi}_k(t,\varepsilon).$$

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