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ON THE CLASSIFICATION OF SYMMETRY REDUCTIONS FOR THE (1+3)-DIMENSIONAL MONGE–AMPÈRE EQUATION

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We propose a classification of the symmetry reductions for the Monge–Ampère equation in the space $M(1,3)\times R(u)$. We present some results obtained by using the classification of three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$.

Keywords: classification of symmetry reductions, Monge–Ampère equation, classification of the Lie algebras, nonconjugate subalgebras of the Lie algebras, Poincaré group $P(1,4)$.

From Newton's time, differential equations serve as one of the main tools for the construction of mathematical models of the processes running in nature. In numerous cases, the differential equations of these models have nontrivial symmetries. For the investigation of these equations, we can use, in particular, the classical Lie–Ovsyannikov method. The application of this approach, in particular, enables one to perform symmetry reduction and construct the classes of invariant solutions of the analyzed equations (see [1, 21, 23] and the references therein).

In the course of symmetry reduction of some differential equations important for the theoretical and mathematical physics, it was discovered that, in some cases, the reduced equations obtained with the help of nonconjugate subalgebras of given ranks of the Lie algebras of symmetry groups for these equations are of different types (see, e.g., [3, 9, 12, 14, 22] and the references therein). Note that the investigations of this type of reduction were originated as earlier as in 1984 in the work by A. M. Grundland, J. Harnad, and P. Winternitz [14].

According to the classical group analysis (see, e.g., [1, 23]), the invariant solutions of differential equations should be classified according to their ranks (the ranks of the corresponding nonconjugate subalgebras). However, in this approach, it is impossible to explain the appearance of different types of reduced equations (invariant solutions) in the case of application of nonconjugate subalgebras of given ranks of the Lie algebras of symmetry groups for these equations.

In [11], for the classification of symmetry reductions (invariant solutions) of the indicated differential equations, it was proposed to use the structural properties of low-dimensional nonconjugate subalgebras of the same rank for the Lie algebras of symmetry groups for the investigated equations.

At present, we performed the classification of symmetry reductions and invariant solutions for the eikonal equation and the Euler–Lagrange–Born–Infeld equation in the space $M(1,3)\times R(u)$ with the use of the classification of low-dimensional $(\dim L \leq 3)$ nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ (for details, see [10–13] and the references therein). Here and in what follows, $M(1,3)$ is the (1+3)-dimensional Minkowski space and $R(u)$ is the real axis of the dependent variable u.

The solution of numerous problems of geometry, geometric analysis, string theory, cosmology, geometric

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optics, optimal transfer, one-dimensional gas dynamics, meteorology, and oceanography is connected with the investigation of the Monge–Ampère equations in spaces of various dimensions and different types. At present, in the available literature, there is a great number of works devoted to the investigation of equations of this kind (see, in particular, [2, 7, 8, 15–20, 24–29] and the references therein).

The present paper is devoted to the classification of the symmetry reductions and invariant solutions for the Monge–Ampère equation in the space $M(1,3)\times R(u)$. Here, we present only some results obtained by using the classification of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$. For this purpose, we first consider some results obtained for the Lie algebra of the group $P(1,4)$ and its nonconjugate subalgebras.

1. Lie Algebra of the Group $P(1,4)$ and its Nonconjugate Subalgebras

The Poincaré group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. Among the groups most important for the theoretical and mathematical physics, the group $P(1,4)$ occupies a special place. This is the least group that contains both the symmetry groups of relativistic physics (Poincaré group $P(1,3)$) and the symmetry groups of nonrelativistic physics (extended Galileo group $\tilde{G}(1,3)$ as subgroups [5].

The Lie algebra of the group $P(1,4)$ is given by 15 basis elements $M_{\mu\nu}=-M_{\nu\mu}$, $\mu,\nu=0,1,2,3,4$, and P_μ, $\mu=0,1,2,3,4$, satisfying the commutation relations

$$[P_\mu,P_\nu]=0,$$

$$[M_{\mu\nu},P_\sigma]=g_{\nu\sigma}P_\mu-g_{\mu\sigma}P_\nu,$$

$$[M_{\mu\nu},M_{\rho\sigma}]=g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma}-g_{\mu\rho}M_{\nu\sigma}-g_{\nu\sigma}M_{\mu\rho},$$

where $g_{\mu\nu}$, $\mu,\nu=0,1,2,3,4$, is a metric tensor with the components $g_{00}=-g_{11}=-g_{22}=-g_{33}=-g_{44}=1$ and $g_{\mu\nu}=0$ for $\mu\neq\nu$.

In the present work, we consider the following representation [6] for the Lie algebra of the group $P(1,4)$:

$$P_0=\frac{\partial}{\partial x_0},\qquad P_1=-\frac{\partial}{\partial x_1},\qquad P_2=-\frac{\partial}{\partial x_2},$$

$$P_3=-\frac{\partial}{\partial x_3},\qquad P_4=-\frac{\partial}{\partial u},\qquad M_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu,\qquad x_4\equiv u.$$

Further, we pass from $M_{\mu\nu}$ and P_μ to the following linear combinations:

$$G=M_{04},\qquad L_1=M_{23},\qquad L_2=-M_{13},\qquad L_3=M_{12},$$

$$P_a=M_{a4}-M_{0a},\qquad C_a=M_{a4}+M_{0a},\qquad a=1,2,3,$$

$$X_0 = \frac{P_0 - P_4}{2}, \quad X_k = P_k, \quad k = 1,2,3, \quad X_4 = \frac{P_0 + P_4}{2}.$$

In [4], one can find the classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ (whose dimensions do not exceed 3) into the classes of isomorphic subalgebras.

2. On the Classification of Symmetry Reductions of the (1+3)-Dimensional Monge–Ampère Equation

In the present paper, we consider a Monge–Ampère equation of the form

$$\det(u_{\mu\nu}) = 0, \tag{1}$$

where

$$u = u(x), \quad x = (x_0, x_1, x_2, x_3) \in M(1,3), \quad u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu,\nu = 0,1,2,3.$$

In [6], the authors studied the symmetry and constructed multiparameter families of the exact solutions to the multidimensional Monge–Ampère equation. In particular, it follows from the cited work that the Lie algebra of the symmetry group of the investigated equation (1) contains the Lie algebra of the Poincaré group $P(1,4)$ as a subalgebra.

To perform the classification of symmetry reductions of the (1+3)-dimensional Monge–Ampère equation, we use the classification of three-dimensional nonconjugate subalgebras [4] of the Lie algebra of the group $P(1,4)$. As a result of the performed classification, it was established that there exist the following types of three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$:

$$3A_1, \quad A_2 \oplus A_1, \quad A_{3,1}, \quad A_{3,2}, \quad A_{3,3}, \quad A_{3,4}, \quad A_{3,6}, \quad A_{3,7}^a, \quad A_{3,8}, \text{ and } A_{3,9}.$$

As a result of the symmetry reduction of the new (1+3)-dimensional Monge–Ampère equation, we obtained the following reduced equations:

- identities,
- linear ordinary differential equations,
- nonlinear ordinary differential equations,
- partial differential equations.

We now present a short survey of the accumulated results.

2.1. Reductions to Identities. Reductions of this type were obtained for some nonconjugate subalgebras of the following types:

$$3A_1, \quad A_2 \oplus A_1, \quad A_{3,1}, \quad A_{3,2}, \quad A_{3,3}, \text{ and } A_{3,6}.$$

Examples

Subalgebras of the Type $3A_1$.

1. $\langle P_1 - \gamma X_3, \gamma > 0\rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0\rangle \oplus \langle X_4\rangle$:
Ansatz

$$x_3(x_0+u)^2 - (\gamma x_1 + \delta x_2 - x_3)(x_0+u) - \gamma x_1 = \varphi(\omega), \qquad \omega = x_0 + u.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$x_3(x_0+u)^2 - (\gamma x_1 + \delta x_2 - x_3)(x_0+u) - \gamma x_1 = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

2. $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_3\rangle$:
Ansatz

$$\frac{x_0^2 - x_1^2 - u^2}{x_0+u} - \frac{x_2^2}{x_0+u+1} = \varphi(\omega), \qquad \omega = x_0 + u.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$\frac{x_0^2 - x_1^2 - u^2}{x_0+u} - \frac{x_2^2}{x_0+u+1} = \varphi(x_0+u),$$

where φ is an arbitrary smooth function.

Subalgebra of the Type $A_2 \oplus A_1$

$\langle -(G + \alpha X_3), X_4, \alpha > 0\rangle \oplus \langle L_3 + \beta X_3, \beta > 0\rangle$:
Ansatz

$$x_3 - \alpha \ln(x_0+u) + \beta \arctan\frac{x_1}{x_2} = \varphi(\omega), \qquad \omega = (x_1^2 + x_2^2)^{1/2}.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$x_3 - \alpha \ln(x_0+u) + \beta \arctan\frac{x_1}{x_2} = \varphi(x_1^2 + x_2^2),$$

where φ is an arbitrary smooth function.

Subalgebra of the Type $A_{3,1}$

$\langle -2\beta X_4, L_3 + \beta X_3, P_3 - 2X_0, \beta > 0\rangle$:

Ansatz

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3 = \varphi(\omega), \qquad \omega = (x_1^2 + x_2^2)^{1/2}.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}(x_0 + u)^2 + x_3 = \varphi(x_1^2 + x_2^2),$$

where φ is an arbitrary smooth function.

Subalgebra of the Type $A_{3,2}$

$\langle 2\beta X_4, P_3, G + \alpha X_1 + \beta X_3, \alpha > 0, \beta > 0 \rangle$:
Ansatz

$$x_1 - \alpha \ln(x_0 + u) = \varphi(\omega), \qquad \omega = x_2.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$x_1 - \alpha \ln(x_0 + u) = \varphi(x_2),$$

where φ is an arbitrary smooth function.

Subalgebra of the Type $A_{3,3}$

$\left\langle P_3, X_4, \frac{1}{\lambda} L_3 + G, \lambda > 0 \right\rangle$:
Ansatz

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(\omega), \qquad \omega = (x_1^2 + x_2^2)^{1/2}.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$\ln(x_0 + u) + \lambda \arctan \frac{x_1}{x_2} = \varphi(x_1^2 + x_2^2),$$

where φ is an arbitrary smooth function.

Subalgebra of the Type $A_{3,6}$

$\langle X_1, -X_2, P_3 - L_3 - 2\alpha X_0, \alpha > 0 \rangle$:
Ansatz

$$(x_0+u)^3+6\alpha x_3(x_0+u)+6\alpha^2(x_0-u)=\varphi(\omega), \qquad \omega=(x_0+u)^2+4\alpha x_3.$$

The solution of the (1+3)-dimensional Monge–Ampère equation

$$(x_0+u)^3+6\alpha x_3(x_0+u)+6\alpha^2(x_0-u)=\varphi((x_0+u)^2+4\alpha x_3),$$

where φ is an arbitrary smooth function.

It is worth noting that, for this type of symmetry reductions, the nonsingular manifolds in the space $M(1,3)\times R(u)$ invariant under the corresponding nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ are themselves the solutions of the (1+3)-dimensional Monge–Ampère equation.

2.2. Reductions to Ordinary Linear Differential Equations. Reductions of this type were obtained for some nonconjugate subalgebras of the types $3A_1$ and $A_{3,6}$.

Examples

Subalgebras of the Type $3A_1$.

1. $\langle P_1\rangle\oplus\langle P_2\rangle\oplus\langle P_3\rangle$:
Ansatz

$$x_0^2-x_1^2-x_2^2-x_3^2-u^2=\varphi(\omega), \qquad \omega=x_0+u.$$

Reduced equation

$$\omega^2\varphi''-2\omega\varphi'+2\varphi=0.$$

The solution of the reduced equation

$$\varphi(\omega)=c_1\omega^2+c_2\omega,$$

where c_1 and c_2 are arbitrary constants.

The solution of the (1+3)-dimensional Monge–Ampère equation

$$x_0^2-x_1^2-x_2^2-x_3^2-u^2=c_1(x_0+u)^2+c_2(x_0+u).$$

2. $\langle P_1\rangle\oplus\langle P_2-\alpha X_2,\alpha>0\rangle\oplus\langle P_3-\gamma X_3,\gamma\neq 0\rangle$:
Ansatz

$$2u+\frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+\alpha}+\frac{x_3^2}{x_0+u+\gamma}=\varphi(\omega), \qquad \omega=x_0+u.$$

Reduced equation

$$\omega(\omega+\alpha)(\omega+\gamma)\varphi'' = 0.$$

The solutions of the reduced equation

$$\varphi(\omega) = c_1\omega + c_2, \quad \omega = 0, \quad \omega+\alpha = 0, \quad \omega+\gamma = 0,$$

where c_1 and c_2 are arbitrary constants.

The solutions of the (1+3)-dimensional Monge–Ampère equation

$$2u + \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\alpha} + \frac{x_3^2}{x_0+u+\gamma} = c_1(x_0+u)+c_2,$$

$$x_0+u = 0, \quad x_0+u+\alpha = 0, \quad x_0+u+\gamma = 0.$$

Subalgebras of the Type $A_{3,6}$.

1. $\langle P_1 - X_1, P_2 - X_2, -P_3 + L_3\rangle$:

Ansatz

$$\frac{x_1^2+x_2^2}{x_0+u+1} + \frac{x_3^2}{x_0+u} + 2u = \varphi(\omega), \quad \omega = x_0+u.$$

Reduced equation

$$\omega(\omega+1)\varphi'' = 0.$$

The solutions of the reduced equation

$$\varphi(\omega) = c_1\omega + c_2, \quad \omega = 0, \quad \omega+1 = 0,$$

where c_1 and c_2 are arbitrary constants.

The solutions of the (1+3)-dimensional Monge–Ampère equation

$$\frac{x_1^2+x_2^2}{x_0+u+1} + \frac{x_3^2}{x_0+u} + 2u = c_1(x_0+u)+c_2, \quad u = -x_0-1, \quad u = -x_0.$$

2. $\langle P_1, P_2, L_3 - P_3\rangle$:

Ansatz

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega), \quad \omega = x_0+u.$$

Reduced equation

$$\omega^2\varphi'' - 2\omega\varphi' + 2\varphi = 0.$$

The solution of the reduced equation

$$\varphi(\omega) = c_2\omega^2 + c_1\omega,$$

where c_1 and c_2 are arbitrary constants.

The solution of the (1+3)-dimensional Monge–Ampère equation

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_2(x_0 + u)^2 + c_1(x_0 + u).$$

2.3. Reductions to Nonlinear Ordinary Differential Equations. We obtain reductions of this kind for some nonconjugate subalgebras of the types $A_2 \oplus A_1$, $A_{3,3}$, $A_{3,7}^a$, and $A_{3,9}$.

Examples

Subalgebra of the Type $A_2 \oplus A_1$

$\langle -G, P_3 \rangle \oplus \langle L_3 \rangle$:

Ansatz

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega), \qquad \omega = (x_1^2 + x_2^2)^{1/2}.$$

Reduced equation

$$\varphi\varphi'\varphi'' = 0.$$

The solutions of the reduced equation

$$\varphi(\omega) = c_1\omega + c_2, \qquad \varphi(\omega) = c,$$

where c_1, c_2, and c are arbitrary constants.

The solutions of the (1+3)-dimensional Monge–Ampère equation

$$(x_0^2 - x_3^2 - u^2)^{1/2} = c_1(x_1^2 + x_2^2)^{1/2} + c_2, \qquad (x_0^2 - x_3^2 - u^2)^{1/2} = c.$$

Subalgebra of the Type $A_{3,3}$

$\langle P_1, P_2, G + \alpha X_3, \alpha > 0 \rangle$:

Ansatz

$$x_3 - \alpha \ln(x_0 + u) = \varphi(\omega), \qquad \omega = x_0^2 - x_1^2 - x_2^2 - u^2.$$

Reduced equation

$$(2\omega\varphi'\varphi'' + \alpha\varphi'' + (\varphi')^2)(\varphi')^2 = 0.$$

Subalgebra of the Type $A_{3,7}^a$

$\langle P_1, P_2, L_3 + \lambda G, \lambda > 0 \rangle$:
Ansatz

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \qquad \omega = x_3.$$

Reduced equation

$$\varphi\varphi'' = 0.$$

The solution of the reduced equation

$$\varphi(\omega) = c_1\omega + c_2,$$

where c_1 and c_2 are arbitrary constants.
The solution of the (1+3)-dimensional Monge–Ampère equation

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = c_1 x_3 + c_2.$$

Subalgebra of the Type $A_{3,9}$

$\left\langle -\frac{1}{2}\left(L_3 + \frac{1}{2}(P_3 + C_3)\right), \frac{1}{2}\left(L_2 + \frac{1}{2}(P_2 + C_2)\right), \frac{1}{2}\left(L_1 + \frac{1}{2}(P_1 + C_1)\right)\right\rangle$:
Ansatz

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varphi(\omega), \qquad \omega = x_0.$$

Reduced equation

$$\varphi\varphi'' = 0.$$

The solution of the reduced equation

$$\varphi(\omega) = c_1\omega + c_2,$$

where c_1 and c_2 are arbitrary constants.

The solution of the (1+3)-dimensional Monge–Ampère equation

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = c_1 x_0 + c_2 .$$

2.4. Reductions to Partial Differential Equations. We obtain reductions of this kind for some nonconjugate subalgebras of the types $A_{3,6}$, $A_{3,8}$, and $A_{3,9}$.

Examples

Subalgebra of the Type $A_{3,6}$

$\langle P_1, P_2, L_3 \rangle$:

Ansatz

$$x_3 = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_0 + u, \quad \omega_2 = x_0^2 - x_1^2 - x_2^2 - u^2 .$$

Reduced equation

$$(\omega_1^2 \varphi_{11} \varphi_{22} - 2\omega_1 \varphi_2 \varphi_{12} - \omega_1^2 \varphi_{12}^2 - 2\omega_2 \varphi_2 \varphi_{22} - \varphi_2^2) \varphi_2 = 0 .$$

The solution of the reduced equation

$$\varphi(\omega_1, \omega_2) = f(\omega_1),$$

where f is an arbitrary smooth function.

The solution of the (1+3)-dimensional Monge–Ampère equation

$$x_3 = f(x_0 + u) .$$

Subalgebra of the Type $A_{3,8}$

$\langle P_3, G, -C_3 \rangle$:

Ansatz

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varphi(\omega_1, \omega_2), \quad \omega_1 = x_1, \quad \omega_2 = x_2 .$$

Reduced equation

$$(\varphi_{11} \varphi_{22} - \varphi_{12}^2) \varphi = 0 .$$

The solution of the reduced equation

$$\varphi(\omega_1,\omega_2) = f(c_1\omega_1 + c_2\omega_2 + c_3),$$

where f is an arbitrary smooth function and c_1, c_2, and c_3 are arbitrary constants.

The solution of the (1+3)-dimensional Monge–Ampère equation

$$(x_0^2 - x_3^2 - u^2)^{1/2} = f(c_1x_1 + c_2x_2 + c_3).$$

Subalgebra of the type $A_{3,9}$

$\langle -L_3, -L_2, -L_1 \rangle$:

Ansatz

$$u = \varphi(\omega_1,\omega_2), \quad \omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

Reduced equation

$$(\varphi_{11}\varphi_{22} - \varphi_{12}^2)\varphi_2 = 0.$$

The solution of the reduced equation

$$\varphi(\omega_1,\omega_2) = f(c_1\omega_1 + c_2\omega_2 + c_3),$$

where f is an arbitrary smooth function and c_1, c_2, and c_3 are arbitrary constants.

The solution of the (1+3)-dimensional Monge–Ampère equation

$$u = f(c_1x_0 + c_2(x_1^2 + x_2^2 + x_3^2)^{1/2} + c_3).$$

Reductions of this kind are caused by the fact that the rank of the corresponding subalgebras is equal to 2.

The set of nonconjugate subalgebras of the types $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, and $A_{3,6}$ contains subalgebras such that the ansatzes reducing the (1+3)-dimensional Monge–Ampère equation cannot be constructed for their invariants. It was discovered that it is impossible to construct ansatzes reducing the (1+3)-dimensional Monge–Ampère equation from the invariants of all four nonconjugate subalgebras of the type $A_{3,4}$.

We now present basis elements of one of these subalgebras and its invariants.

Subalgebra of the Type $A_{3,4}$

$\left\langle -X_0, X_4, -\frac{L_3}{\lambda} - G - \frac{\alpha}{\lambda}X_3, \alpha > 0, \lambda > 0 \right\rangle$:

Invariants

$$(x_1^2 + x_2^2)^{1/2}, \quad x_3 + \alpha \arctan\frac{x_1}{x_2}.$$

Note that the subalgebras for the invariants of which it is impossible to construct ansatzes do not satisfy necessary conditions for the existence of invariant solutions (for details, see [1]).

CONCLUSIONS

We have established the relationship between the structural properties of three-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$ and the types of obtained reduced equations for the (1+3)-dimensional Monge–Ampère equation. We now present some invariant solutions of the considered equation.

There exist three-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of the following types [4]: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, $A_{3,6}$, $A^a_{3,7}$, $A_{3,8}$, and $A_{3,9}$.

- Reductions to identities are obtained for some nonconjugate subalgebras of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,6}$.
- Reductions to linear ordinary differential equations are obtained for some nonconjugate subalgebras of the following types: $3A_1$ and $A_{3,6}$.
- Reductions to nonlinear ordinary differential equations are obtained for some nonconjugate subalgebras of the following types: $A_2 \oplus A_1$, $A_{3,3}$, $A^a_{3,7}$, and $A_{3,9}$.
- Reductions to partial differential equations are obtained for some nonconjugate subalgebras of the following types: $A_{3,6}$, $A_{3,8}$, and $A_{3,9}$.
- Among nonconjugate subalgebras of the types $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, and $A_{3,6}$, there are subalgebras from the invariants of which it is impossible to construct ansatzes reducing the (1+3)-dimensional Monge–Ampère equation.
- From the invariants of all four nonconjugate subalgebras of the type $A_{3,4}$, it is impossible to construct ansatzes, which reduce the (1+3)-dimensional Monge–Ampère equation.

These subalgebras do not satisfy the necessary conditions for the existence of invariant solutions (for details, see [1]).

REFERENCES

1. L. V. Ovsiannikov, *Group Analysis of Differential Equations* [in Russian], Nauka, Moscow (1978); ***English translation:*** Academic Press, New York (1982)
2. A. V. Pogorelov, *Multidimensional Minkowski Problem* [in Russian], Nauka, Moscow (1975).
3. V. M. Fedorchuk, "Symmetry reduction and some exact solutions of a nonlinear five-dimensional wave equation," *Ukr. Mat. Zh.*, **48**, No. 4, 573–577 (1996); ***English translation:*** *Ukr. Math. J.*, **48**, No. 4, 636–640 (1996); https://doi.org/10.1007/BF02390625.
4. V. M. Fedorchuk and V. I. Fedorchuk, "On the classification of low-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$," in: *Symmetry and Integrability of the Equations of Mathematical Physics: Collection of Works of the Institute of Mathematics, Nats. Akad. Nauk Ukr.* [in Ukrainian], Vol. 3, No. 2 (2006), pp. 301–307.
5. V. I. Fushchich and A. G. Nikitin, *Symmetry of the Equations of Quantum Mechanics* [in Russian], Nauka, Moscow (1990).

6. V. I. Fushchich and N. I. Serov, "Symmetry and some exact solutions of the multidimensional Monge–Ampère equation," *Dokl. Akad. Nauk SSSR,* **273**, No. 3, 543–546 (1983).
7. S. V. Khabirov, "Application of contact transformations of the inhomogeneous Monge–Ampère equation in one-dimensional gas dynamics," *Dokl. Akad. Nauk SSSR,* **310**, No. 2, 333–336 (1990); ***English translation:*** *Sov. Phys. Dokl.,* **35**, No. 1, 29–30 (1990).
8. M. J. P. Cullen and R. J. Douglas, "Applications of the Monge–Ampère equation and Monge transport problem to meteorology and oceanography," in: *Proc. Conf. Monge–Ampère Equation: Applications to Geometry and Optimization (Deerfield Beach, FL),* 1997; Contemp. Math., Vol. 226, Amer. Math. Soc., Providence, RI (1999), pp. 33–53.
9. V. Fedorchuk, "Symmetry reduction and exact solutions of the Euler–Lagrange–Born–Infeld multidimensional Monge–Ampère and eikonal equations," *J. Nonlin. Math. Phys.,* **2**, No. 3-4, 329–333 (1995); https://doi.org/10.2991/jnmp.1995.2.3-4.13.
10. V. Fedorchuk and V. Fedorchuk, *Classification of Symmetry Reductions for the Eikonal Equation,* Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv (2018).
11. V. Fedorchuk and V. Fedorchuk, "On classification of symmetry reductions for partial differential equations," in: Nonclassical Problems of the Theory of Differential Equations: Collection of Sci. Works Dedicated to the 80th Birthday of B. I. Ptashnyk, Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv (2017).
12. V. Fedorchuk and V. Fedorchuk, "On classification of symmetry reductions for the eikonal equation," *Symmetry,* **8**, No. 6, Art. 51, 1–32 (2016); https://doi.org/10.3390/sym8060051.
13. V. M. Fedorchuk and V. I. Fedorchuk, "On the classification of symmetry reductions and invariant solutions for the Euler–Lagrange–Born–Infeld equation," *Ukr. J. Phys.,* **64**, No. 12, 1103–1107 (2019); https://doi.org/10.15407/ujpe64.12.1103.
14. A. M. Grundland, J. Harnad, and P. Winternitz, "Symmetry reduction for nonlinear relativistically invariant equations," *J. Math. Phys.,* **25**, No. 4, 791–806 (1984); https://doi.org/10.1063/1.526224.
15. C. E. Gutiérrez and T. van Nguyen, "On Monge–Ampère type equations arising in optimal transportation problems," *Calcul. Var. Partial Differ. Equat.,* **28**, No. 3, 275–316 (2007); https://doi.org/10.1007/s00526-006-0045-x.
16. F. Jiang and N. S. Trudinger, "On the second boundary value problem for Monge–Ampère type equations and geometric optics," *Arch. Ration. Mech. Anal.,* **229**, No. 2, 547–567 (2018); https://doi.org/10.1007/s00205-018-1222-8.
17. X. Jia, D. Li, and Zh. Li, "Asymptotic behavior at infinity of solutions of Monge–Ampère equations in half spaces," *J. Different. Equat.,* **269**, No. 1, 326–348 (2020); https://doi.org/10.1016/j.jde.2019.12.007.
18. A. Kushner., V. V. Lychagin, and J. Slovák, "Lectures on geometry of Monge–Ampère equations with Maple," in: R. A. Kycia, M. Ulan, and E. Schneider (editors), *Nonlinear PDEs, Their Geometry, and Applications,* Birkhäuser, Basel (2019), pp. 53–94.
19. Q. Le Nam, "Global Hölder estimates for 2D linearized Monge–Ampère equations with right-hand side in divergence form," *J. Math. Anal. Appl.,* **485**, No. 2, Art. 123865, 1–13 (2020); https://doi.org/10.1016/j.jmaa.2020.123865.
20. D. Li, Zh. Li, and Yu. Yuan, "A Bernstein problem for special Lagrangian equations in exterior domains," *Adv. Math.,* **361,** Art. 106927, 1–29 (2020); https://doi.org/10.1016/j.aim.2019.106927.
21. S. Lie, "Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung," *Berichte Sächs. Ges.,* **47,** 53–128, Leipzig (1895).
22. A. G. Nikitin and O. Kuriksha, "Invariant solutions for equations of axion electrodynamics," *Comm. Nonlin. Sci. Numer. Simulat.,* **17**, No. 12, 4585–4601 (2012); https://doi.org/10.1016/j.cnsns.2012.04.009.
23. P. J. Olver, *Applications of Lie Groups to Differential Equations,* Springer, New York (1986).
24. V. P. Pingali, "A vector bundle version of the Monge–Ampère equation," *Adv. Math.,* **360,** Art. 106921, 1–40 (2020); https://doi.org/10.1016/j.aim.2019.106921.
25. M. Sroka, "The C^0 estimate for the quaternionic Calabi conjecture," *Adv. Math.,* **370,** Art. 107237, 1–15 (2020); https://doi.org/10.1016/j.aim.2020.107237.
26. Ł. T. Stępień, "On some exact solutions of heavenly equations in four dimensions," *AIP Advances.,* **10,** Art. 065105 (2020); https://doi.org/10.1063/1.5144327.
27. C. Udrişte and N. Bîlă, "Symmetry group of Ţiţeica surfaces PDE," *Balkan J. Geom. Appl.,* **4**, No. 2, 123–140 (1999).
28. E. Witten, "Superstring perturbation theory via super Riemann surfaces: an overview," *Pure Appl. Math. Quart.,* **15**, No. 1, 517–607 (2019); https://doi.org/10.4310/PAMQ.2019.v15.n1.a4.
29. Sh.-T. Yau and S. Nadis, *The Shape of a Life. One Mathematician's Search for the Universe's Hidden Geometry,* Yale Univ. Press, New Haven (2019).