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LIMIT BEHAVIOR OF SOLUTIONS TO THE RADIATIVE TRANSFER EQUATION AS COEFFICIENTS OF ABSORPTION AND SCATTERING TEND TO INFINITY

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We consider the boundary value problem for the radiative transfer equation with conditions of internal diffusive reflection of radiation. Under the assumption that the absorption and scattering coefficients tend to infinity, we study the limit behavior of solutions. Bibliography: 11 *titles.*

1 Introduction

When studying complex heat transfer from the mathematical point of view, it is of interest to find out the limit connection between the solutions to the problem of radiative-conductive heat transfer in semitransparent materials with large absorption and scattering coefficients and the solutions to the corresponding problems in opaque materials. This topic is important because, in practice, it is assumed that semitransparent materials with a large absorption coefficient can be considered as opaque and the energy radiation and absorption occur only on the boundaries. This assumption was confirmed by the mathematical results [1], [2] in the case where the radiation scattering is negligible in comparison with absorption or it is absent at all.

The propagation of the monochromatic radiation in a semitransparent body G is described by the radiative transfer integro-differential equations

$$\omega \cdot \nabla I + \beta I = s\mathscr{S}(I) + \varkappa k^2 F, \quad (\omega, x) \in D. \tag{1.1}$$

The sought function $I(\omega, x)$ is defined on the set $D = \Omega \times G$, where $\Omega = \{\omega \in \mathbb{R}^3 \mid |\omega| = 1\}$, and is the intensity at $x \in G$ of radiation propagating in the direction $\omega \in \Omega$.

In Equation (1.1), $\omega \cdot \nabla I = \sum\limits_{i=1}^{3} \omega_i \dfrac{\partial}{\partial x_i} I$ is the derivative of I along a direction ω. Here, $\mathscr{S}$

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denotes the isotropic scattering operator

$$\mathscr{S}(I)(\omega, x) = \frac{1}{4\pi} \int\limits_{\Omega} I(\omega', x)\, d\omega', \quad (\omega, x) \in D.$$

Furthermore, $\beta = s + \varkappa$ is the extinction coefficient, $0 \leqslant s$ is the scattering coefficient, $0 < \varkappa$ is the absorption coefficient, $F = F(x)$ is the density of isotropic volume sources of radiation, and $1 \leqslant k$ is the refractive index.

Equation (1.1) has a simple form, but is difficult to solve. Therefore, the radiative transfer equation is often replaced by its rough approximate counterpart, the so-called diffusion approximation (P_1-approximation), which is a diffusion type equation for the volume density of the energy of radiation

$$u(x) = \frac{1}{4\pi} \int\limits_{\Omega} I(\omega', x)\, d\omega'.$$

In the case where the medium occupying the body G is optically dense, such an approximation sufficiently well describe the behavior of radiation inside G. However, since there are no information about the boundary value of the function u and the function I possesses a sharp boundary layer, the diffusion approximation is not suitable to describe the behavior of radiation near ∂G. Therefore, it is used in practice together with formulation of approximate boundary conditions or construction of special approximations of u near the boundary.

The famous works [3]–[5] were the first ones devoted to justification of diffusion approximation in the case where one can neglect the radiation scattering and then the asymptotics of the solution I near the boundary becomes simple; in this case, it is possible to justify the diffusion approximation (cf., for example, [2]). To the knowledge of the author, in the general case, in the presence of scattering, there are no mathematically correct justification of this approximation.

In this paper, we assume that $\beta \to \infty$, i.e., $\beta = \dfrac{1}{\varepsilon}$, where $\varepsilon \to 0$. We introduce the *albedo* $\varpi = s/\beta$ and assume that it is constant. In this case, the absorption and scattering coefficients have the form $\varkappa = \dfrac{1 - \varpi}{\varepsilon}$ and $s = \dfrac{\varpi}{\varepsilon}$ and tend to infinity as $\varepsilon \to 0$, whereas Equation (1.1) takes the form

$$\omega \cdot \nabla I_\varepsilon + \frac{1}{\varepsilon} I_\varepsilon = \frac{\varpi}{\varepsilon} \mathscr{S}(I_\varepsilon) + \frac{1 - \varpi}{\varepsilon} k^2 F, \quad (\omega, x) \in D. \tag{1.2}$$

We assume that the body G is a bounded domain in $\mathbb{R}^3$ with smooth boundary ∂G of class $C^{1+\lambda}$, $0 < \lambda < 1$. For $x \in \partial G$ we denote by $n(x)$ the outward normal to the boundary at the point x. We put

$$\Omega^-(x) = \{\omega \in \Omega \mid \omega \cdot n(x) < 0\}, \quad \Omega^+(x) = \{\omega \in \Omega \mid \omega \cdot n(x) > 0\}$$

and introduce the sets

$$\Gamma = \Omega \times \partial G, \quad \Gamma^- = \{(\omega, x) \in \Gamma \mid \omega \cdot n(x) < 0\}, \quad \Gamma^+ = \{(\omega, x) \in \Gamma \mid \omega \cdot n(x) > 0\}.$$

Denote by $I_\varepsilon|_{\Gamma^-}$ and $I_\varepsilon|_{\Gamma^+}$ the values (traces) of the solution to Equation (1.2) on Γ^- and Γ^+.

Together with Equations (1.2), we consider the boundary internal diffusive reflection and diffusive refraction of radiation coming from outside

$$I_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(I_\varepsilon|_{\Gamma^+}) + \mathscr{P}^-(J_*), \quad (\omega, x) \in \Gamma^-. \tag{1.3}$$

Here, $\mathscr{R}^-$ is the diffusive reflection operator and $\mathscr{P}^-$ is the diffusive refraction operator:

$$\mathscr{R}^-(I_\varepsilon|_{\Gamma^+})(\omega,x) = \theta\mathscr{M}^+(I_\varepsilon|_{\Gamma^+})(x), \quad (\omega,x)\in\Gamma^-,$$
$$\mathscr{P}^-(J_*)(\omega,x) = (1-\theta)k^2\mathscr{M}^-(J_*)(\omega,x), \quad (\omega,x)\in\Gamma^-,$$

where

$$\mathscr{M}^+(I_\varepsilon|_{\Gamma^+})(x) = \frac{1}{\pi}\int\limits_{\Omega^+(x)} I_\varepsilon|_{\Gamma^+}(\omega',x)\,\omega'\cdot n(x)\,d\omega', \quad x\in\partial G, \tag{1.4}$$

$$\mathscr{M}^-(J_*)(x) = \frac{1}{\pi}\int\limits_{\Omega^-(x)} J_*(\omega',x)\,|\omega'\cdot n(x)|\,d\omega', \quad x\in\partial G.$$

Here, $0\leqslant\theta<1$ is the internal reflection coefficient, $J_*(\omega,x)$ is the intensity of radiation incoming from outside and falling at a point $x\in\partial G$ in a direction $\omega\in\Omega^-(x)$.

The goal of the paper is to study the limit behavior of solutions I_ε to the problem (1.2), (1.3) and their traces $I_\varepsilon|_{\Gamma^+}$ and $I_\varepsilon|_{\Gamma^-}$ as $\varepsilon\to 0$.

The paper is organized as follows. In Section 2, we introduce the notation and some function spaces. In Section 3, we consider an auxiliary problem that is a spatially one-dimension counterpart of the problem under consideration. The main result of the paper, the theorem on the limit behavior of the solutions to the problem (1.2), (1.3) as $\varepsilon\to 0$, is formulated and proved in Section 4.

2 Notation and Function Spaces

We denote $\mathbb{R}^+ = (0,+\infty)$ and $\overline{\mathbb{R}}^+ = [0,+\infty)$. Let $C(\overline{\mathbb{R}}^+)$ be the Banach space of continuous bounded functions on $\overline{\mathbb{R}}^+$ equipped with the norm

$$\|f\|_{C(\overline{\mathbb{R}}^+)} = \sup_{\tau\in\overline{\mathbb{R}}^+}|f(\tau)|.$$

We denote by $B_r(x_0)$ the ball in $\mathbb{R}^3$ with center x_0 and radius r. Let Z be a set equipped with a given measure $d\mu$, and let Z_1 be its subset, measurable with respect to the measure $d\mu$. We denote by $L^1(Z_1;d\mu)$ the Lebesgue space of functions f that are defined on Z_1, measurable with respect to the measure $d\mu$, and possessing the finite norm

$$\|f\|_{L^1(Z_1;d\mu)} = \int\limits_Z |f(z)|\,d\mu(z).$$

We denote by $d\sigma(x)$ and $d\omega$ the measures on ∂G and Ω induced by the Lebesgue measure in $\mathbb{R}^3$. We set $L^1(\partial G) = L^1(\partial G;d\sigma)$ and $L^1(D) = L^1(D;d\omega dx)$. On Γ^- and Γ^+, we introduce the measures

$$\widehat{d}\Gamma^-(\omega,x) = |\omega\cdot n(x)|\,d\omega d\sigma(x), \quad (\omega,x)\in\Gamma^-,$$

$$\widehat{d}\Gamma^+(\omega,x) = \omega\cdot n(x)\,d\omega d\sigma(x), \quad (\omega,x)\in\Gamma^+.$$

We set $\widehat{L}^1(\Gamma^\pm) = L^1(\Gamma^\pm;\widehat{d}\Gamma^\pm)$ and note that the operator $\mathscr{M}^+$ given by formula (1.4) possesses the following properties:

$\mathscr{M}^+ : \widehat{L}^1(\Gamma^+) \to L^1(\partial G)$; moreover, $\mathscr{M}^+(1) = 1$ and

$$\|\mathscr{M}^+(\varphi)\|_{L^1(\partial G)} \leqslant \frac{1}{\pi} \int\limits_{\partial G} \Big[\int\limits_{\Omega^+(x)} |\varphi(\omega, x)|\, \omega \cdot n(x)\, d\omega \Big] d\sigma(x) = \frac{1}{\pi} \|\varphi\|_{\widehat{L}^1(\Gamma^+)} \quad \forall\, \varphi \in \widehat{L}^1(\Gamma^+).$$

As a consequence, $\mathscr{R}^- = \theta \mathscr{M}^+ : \widehat{L}^1(\Gamma^+) \to \widehat{L}^1(\Gamma^-)$; moreover,

$$\|\mathscr{R}^-\|_{\widehat{L}^1(\Gamma^+) \to \widehat{L}^1(\Gamma^-)} \leqslant \theta. \tag{2.1}$$

Indeed,

$$\|\mathscr{R}^-(\varphi)\|_{\widehat{L}^1(\Gamma^-)} = \theta \int\limits_{\partial G} \Big[\int\limits_{\Omega^-(x)} |\mathscr{M}^+(\varphi)(x)|\, |\omega \cdot n(x)|\, d\omega \Big] d\sigma(x)$$

$$= \theta\pi \int\limits_{\partial G} |\mathscr{M}^+(\varphi)(x)|\, d\sigma(x) = \theta\pi \|\mathscr{M}^+(\varphi)\|_{L^1(\partial G)} \leqslant \theta \|\varphi\|_{\widehat{L}^1(\Gamma^+)}.$$

By a *weak derivative along a direction* ω of a function $f \in L^1(D)$ we understand a function $w \in L^1(D)$, denoted by $w = \omega \cdot \nabla f$, that satisfies the identity

$$\int\limits_D \big[f(\omega, x)\, \omega \cdot \nabla\varphi(x) + w(\omega, x)\varphi(x) \big] \psi(\omega)\, d\omega dx = 0 \quad \forall\, \varphi \in C_0^\infty(G), \quad \forall\, \psi \in L^\infty(\Omega).$$

We denote by $\mathscr{W}^1(D)$ the Banach space of functions $f \in L^1(D)$ possessing the weak derivative $\omega \cdot \nabla f \in L^1(D)$ and equipped with the norm

$$\|f\|_{\mathscr{W}^1(D)} = \|f\|_{L^1(D)} + \|\, \omega \cdot \nabla f\|_{L^1(D)}.$$

We denote by $f|_{\Gamma^-}$ and $f|_{\Gamma^+}$ the traces of a function $f \in \mathscr{W}^1(D)$ on Γ^- and Γ^+ respectively.

In $\mathscr{W}^1(D)$, we introduce $\widehat{\mathscr{W}}^1(D) = \{f \in \mathscr{W}^1(D) \mid f|_{\Gamma^+} \in \widehat{L}^1(\Gamma^+)\}$. We remark the following properties of functions $f \in \widehat{\mathscr{W}}^1(D)$:

1) If $f \in \widehat{\mathscr{W}}^1(D)$, then $|f| \in \widehat{\mathscr{W}}^1(D)$; moreover, $(\omega \cdot \nabla f) \cdot \operatorname{sgn} f = \omega \cdot \nabla |f|$.

2) If $f \in \widehat{\mathscr{W}}^1(D)$, then $f|_{\Gamma^-} \in \widehat{L}^1(\Gamma^-)$ and

$$\int\limits_D \omega \cdot \nabla |f|\, d\omega dx = \|f|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} - \|f|_{\Gamma^-}\|_{\widehat{L}^1(\Gamma^-)}. \tag{2.2}$$

For a more detailed information about properties of functions $f \in \widehat{\mathscr{W}}^1(D)$ and their traces $f|_{\Gamma^\pm}$ we refer, for example, to [10, 11].

3 Spatially One-Dimensional Problem

We consider the following spatially one-dimensional counterpart of the problem (1.2), (1.3):

$$-\,\mu \frac{d\psi(\mu, \tau)}{d\tau} + \psi(\mu, \tau) = \frac{\varpi}{2} \int\limits_{-1}^{1} \psi(\mu', \tau)\, d\mu', \quad \mu \in [-1, 0) \cup (0, 1], \quad \tau \in \overline{\mathbb{R}}^+, \tag{3.1}$$

$$\psi(\mu, 0) = 2\theta \int\limits_0^1 \psi(\mu', 0)\mu'\, d\mu' + 1 - \theta, \quad \mu \in [-1, 0), \tag{3.2}$$

$$\psi(\mu, +\infty) = 0, \quad \mu \in (0, 1]. \tag{3.3}$$

Equation (3.1) is widely used in astrophysics to describe the propagation of light radiation in the atmospheres of stars and planets in the case of plane symmetry [6]–[9]. The problem coincides, up to a boundary condition of the form (3.2), with the inhomogeneous Milne problem Here, $\psi(\mu, \tau)$ is interpreted as the intensity of radiation propagating in the half-space $\tau > 0$, μ is the cosine of the angle of radiation propagation, and τ is the optical depth.

The condition (3.2) describes the internal diffusive reflection of radiation. We note that, in view of this condition, the intensity of the reflected radiation $\psi_0 = \psi(\mu, 0)$ is independent of $\mu \in [-1, 0)$. In the degenerate case $\theta = 0$, where reflection is absent, we have $\psi_0 = 1$.

We indicate some properties of the problem (3.1)–(3.3) which will be used below. Let

$$\Psi(\tau) = \frac{1}{2}\int\limits_{-1}^{1} \psi(\mu, \tau)\, d\mu.$$

If $\Psi \in C(\overline{\mathbb{R}}^+)$, then a solution to Equation (3.1) has the form

$$\psi(\mu, \tau) = \begin{cases} \psi_-(\mu, \tau) = \psi_0 e^{-\tau/|\mu|} + \varpi \int\limits_0^\tau \dfrac{1}{|\mu|} e^{-(\tau-\tau')/|\mu|} \Psi(\tau')\, d\tau', & \mu \in [-1, 0), \\ \psi_+(\mu, \tau) = \varpi \int\limits_\tau^\infty \dfrac{1}{\mu} e^{-(\tau'-\tau)/\mu} \Psi(\tau')\, d\tau', & \mu \in (0, 1]. \end{cases} \tag{3.4}$$

Hence

$$\Psi(\tau) = \frac{\psi_0}{2}\int\limits_{-1}^{0} e^{-\tau/|\mu|}\, d\mu + \frac{\varpi}{2}\int\limits_0^\tau \int\limits_{-1}^{0} \frac{1}{|\mu|} e^{-(\tau-\tau')/|\mu|}\, d\mu\, \Psi(\tau')\, d\tau'$$

$$+ \frac{\varpi}{2}\int\limits_\tau^\infty \int_0^1 \frac{1}{\mu} e^{-(\tau'-\tau)/\mu}\, d\mu\, \Psi(\tau')\, d\tau' = \frac{\psi_0}{2} E_2(\tau) + \frac{\varpi}{2}\int\limits_0^\infty E_1(|\tau - \tau'|)\, \Psi(\tau')\, d\tau'.$$

Here,

$$E_1(\tau) = \int\limits_0^1 \frac{1}{\mu} e^{-\tau/\mu}\, d\mu, \quad E_2(\tau) = \int\limits_0^1 e^{-\tau/\mu}\, d\mu$$

are integro-exponential functions of order 1 and 2. Thus, $\Psi = \psi_0 \Phi$, where Φ is a solution to the integral equation

$$\Phi(\tau) = \varpi \Lambda[\Phi](\tau) + \frac{1}{2} E_2(\tau) \tag{3.5}$$

with the operator Λ defined by

$$\Lambda[\Phi](\tau) = \frac{1}{2}\int\limits_0^\infty E_1(|\tau - \tau'|)\Phi(\tau')\, d\tau'.$$

We note that

$$\frac{1}{2}\int\limits_0^\infty E_1(|\tau - \tau'|)\, d\tau = 1 - \frac{1}{2} E_2(\tau') < 1, \quad \tau' \in \overline{\mathbb{R}}^+.$$

Consequently, $\Lambda : L^1(\mathbb{R}^+) \to L^1(\mathbb{R}^+)$; moreover, $\|\Lambda\|_{L^1(\mathbb{R}^+)\to L^1(\mathbb{R}^+)} \leqslant 1$. Since $E_2 \in L^1(\mathbb{R}^+)$, and $\|\varpi\Lambda\|_{L^1(\mathbb{R}^+)\to L^1(\mathbb{R}^+)} \leqslant \varpi < 1$, Equation (3.5) has a unique solution $\Phi \in L^1(\mathbb{R}^+)$, expressed by the Neumann series converging in $L^1(\mathbb{R}^+)$:

$$\Phi = \frac{1}{2}\sum_{k=0}^{\infty} \varpi^k \Lambda^k [E_2] \tag{3.6}$$

Since $\|E_2\|_{L^1(\mathbb{R}^+)} = 1/2$, from the inequalities

$$\|\Phi\|_{L^1(\mathbb{R}^+)} \leqslant \varpi \|\Lambda[\Phi]\|_{L^1(\mathbb{R}^+)} + \frac{1}{2}\|E_2\|_{L^1(\mathbb{R}^+)} \leqslant \varpi\|\Phi\|_{L^1(\mathbb{R}^+)} + \frac{1}{4}$$

we obtain the estimate

$$\|\Phi\|_{L^1(\mathbb{R}^+)} \leqslant \frac{1}{4(1-\varpi)}. \tag{3.7}$$

Lemma 3.1. *The operator Λ acts from $C(\overline{\mathbb{R}}^+)$ to $C(\overline{\mathbb{R}}^+)$; moreover, $\|\Lambda\|_{C(\overline{\mathbb{R}}^+)\to C(\overline{\mathbb{R}}^+)} \leqslant 1$.*

Proof. Let $\Phi \in C(\overline{\mathbb{R}}^+)$. Then

$$\sup_{\tau\in\overline{\mathbb{R}}^+} |\Lambda[\Phi](\tau)| \leqslant \sup_{\tau\in\overline{\mathbb{R}}^+} \frac{1}{2}\int_0^\infty E_1(|\tau-\tau'|)\,d\tau' \|\Phi\|_{C(\overline{\mathbb{R}}^+)} \leqslant \|\Phi\|_{C(\overline{\mathbb{R}}^+)}. \tag{3.8}$$

Furthermore, for all $\tau \geqslant 0$ and $\Delta\tau > 0$ we have

$$\Big|\Lambda[\Phi](\tau+\Delta\tau) - \Lambda[\Phi](\tau)\Big| = \frac{1}{2}\Big|\int_0^\infty E_1(|\tau+\Delta\tau-\tau'|)\Phi(\tau')\,d\tau' - \int_0^\infty E_1(|\tau-\tau'|)\Phi(\tau')\,d\tau'\Big|$$

$$\leqslant \frac{1}{2}\int_0^{\Delta\tau} E_1(\tau+\Delta\tau-\tau')\,d\tau'\|\Phi\|_{C(\overline{\mathbb{R}}^+)} + \frac{1}{2}\int_0^\infty E_1(|\tau-\tau'|)|\Phi(\tau'+\Delta\tau)-\Phi(\tau')|\,d\tau' \to 0, \quad \Delta\tau\to 0$$

in view of the Lebesgue majorized convergence theorem. If $\tau > 0$ and $0 < \Delta\tau < \tau$, then, setting $\Phi(\tau') = \Phi(-\tau')$ for $\tau' < 0$, we have

$$|\Lambda[\Phi](\tau-\Delta\tau) - \Lambda[\Phi](\tau)|$$

$$\leqslant \frac{1}{2}\int_0^\infty E_1(|\tau-\tau'|)|\Phi(\tau'-\Delta\tau)-\Phi(\tau')|\,d\tau' + \frac{1}{2}\int_0^{\Delta\tau} E_1(\tau-\tau')\,d\tau'\|\Phi\|_{C(\mathbb{R}^+)} \to 0$$

as $\Delta\tau \to 0$. Thus, $\Lambda[\Phi] \in C(\mathbb{R}^+)$. The estimate $\|\Lambda\|_{C(\overline{\mathbb{R}}^+)\to C(\overline{\mathbb{R}}^+)} \leqslant 1$ follows from the inequality (3.8). The lemma is proved. □

Corollary 3.1. *The solution Φ to Equation* (3.5) *belongs to the space $C(\overline{\mathbb{R}}^+)$ and is expressed as the Neumann series* (3.6) *converging in $C(\overline{\mathbb{R}}^+)$.*

Lemma 3.2. *The solution to Equation* (3.5) *satisfies the estimates*

$$0 < \frac{1}{2}E_2(\tau) \leqslant \Phi(\tau) \leqslant \frac{\varpi}{2-\varpi} + \frac{1-\varpi}{2-\varpi}E_2(\tau) \leqslant \frac{1}{2-\varpi} < 1.$$

Proof. Since the kernel of the operator Λ is positive, from (3.6) it follows that $\Phi(\tau) \geqslant \frac{1}{2}E_2(\tau)$. Taking into account that

$$\Lambda[1](\tau) = \frac{1}{2}\int\limits_0^\infty E_1(|\tau-\tau'|)\,d\tau' = 1 - \frac{1}{2}E_2(\tau),$$

we note that the function $\Phi_1(\tau) = \dfrac{1}{2-\varpi} - \Phi(\tau)$ satisfies the equation

$$\Phi_1(\tau) = \varpi\Lambda[\Phi_1](\tau) + \frac{1-\varpi}{2-\varpi}\left(1 - E_2(\tau)\right).$$

As a consequence,

$$\frac{1}{2-\varpi} - \Phi(\tau) = \Phi_1(\tau) \geqslant \frac{1-\varpi}{2-\varpi}\left(1 - E_2(\tau)\right).$$

Thus,

$$\Phi(\tau) \leqslant \frac{\varpi}{2-\varpi} + \frac{1-\varpi}{2-\varpi}E_2(\tau) \leqslant \frac{1}{2-\varpi}.$$

The lemma is proved. □

Corollary 3.2. *For the solution to Equation* (3.5) *the following estimate holds*:

$$\|\Phi\|_{C(\overline{\mathbb{R}}^+)} \leqslant \frac{1}{2-\varpi} < 1. \tag{3.9}$$

Lemma 3.3. *The solution to Equation* (3.5) *for all* $\Delta\tau > 0$ *satisfies the estimate*

$$\|\Phi(\cdot+\Delta\tau) - \Phi(\cdot)\|_{C(\overline{\mathbb{R}}^+)} \leqslant \frac{1}{(1-\varpi)(2-\varpi)}(1 - E_2(\Delta\tau)). \tag{3.10}$$

Proof. We note that

$$\Phi(\tau+\Delta\tau) - \Phi(\tau) = \frac{1}{2}(E_2(\tau+\Delta\tau) - E_2(\tau))$$
$$+ \frac{\varpi}{2}\int\limits_0^{\Delta\tau} E_1(\tau+\Delta\tau-\tau')\Phi(\tau')\,d\tau' + \frac{\varpi}{2}\int\limits_0^{\infty} E_1(|\tau-\tau'|)[\Phi(\tau'+\Delta\tau) - \Phi(\tau')]\,d\tau'.$$

Taking into account that

$$E_2(\tau) - E_2(\tau+\Delta\tau) = \int\limits_0^1 e^{-\tau/\mu}[1 - e^{-\Delta\tau/\mu}]\,d\mu \leqslant \int\limits_0^1 [1 - e^{-\Delta\tau/\mu}]\,d\mu = 1 - E_2(\Delta\tau),$$

$$\int\limits_0^{\Delta\tau} E_1(\tau+\Delta\tau-\tau')\,d\tau' = \int\limits_0^{\Delta\tau} E_1(\tau+\tau')\,d\tau' \leqslant \int\limits_0^{\Delta\tau} E_1(\tau')\,d\tau' = 1 - E_2(\Delta\tau)$$

and using the estimate (3.9), we get

$$\|\Phi(\cdot+\Delta\tau) - \Phi(\cdot)\|_{C(\overline{\mathbb{R}}^+)} \leqslant \frac{1}{2}(1 - E_2(\Delta\tau)) + \frac{\varpi}{2}(1 - E_2(\Delta\tau))\frac{1}{2-\varpi}$$
$$+ \varpi\|\Phi(\cdot+\Delta\tau) - \Phi(\cdot)\|_{C(\overline{\mathbb{R}}^+)},$$

which implies the estimate (3.10). The lemma is proved. □

We return to the study of the problem (3.1)–(3.3). In the degenerate case $\theta = 0$, when the internal reflection is absent, we have $\psi_0 = 1$.

Lemma 3.4. *For* $0 < \theta < 1$

$$\psi_0 < 1. \tag{3.11}$$

Proof. Using the condition (3.2) and taking into account that (cf. (3.4))

$$\psi_+(\mu, 0) = \varpi \int_0^\infty e^{-\tau'/\mu}\, \Psi(\tau')\, d\tau',$$

we have

$$\psi_0 = 2\theta\varpi \int_0^\infty \int_0^1 e^{-\tau'/\mu}\, d\mu\, \Psi(\tau')\, d\tau' + 1 - \theta = 2\theta\varpi\psi_0 \int_0^\infty E_2(\tau')\, \Phi(\tau')\, d\tau' + 1 - \theta.$$

Hence

$$\psi_0 = \frac{1-\theta}{1 - 2\theta\varpi \int\limits_0^\infty E_2(\tau')\Phi(\tau')\, d\tau'}.$$

By the estimate (3.9), we obtain the inequality

$$\int_0^\infty E_2(\tau')\Phi(\tau')\, d\tau' \leqslant \frac{1}{2-\varpi} \int_0^\infty E_2(\tau')\, d\tau' = \frac{1}{2-\varpi}\frac{1}{2}$$

which implies

$$\psi_0 \leqslant \frac{1-\theta}{1 - \dfrac{\theta\varpi}{2-\varpi}} < 1.$$

The lemma is proved. □

We write the solution (3.4) to the problem (3.1)–(3.3) in the form

$$\psi(\mu,\tau) = \begin{cases} \psi_-(\mu,\tau) = \psi_0 \left[e^{-\tau/|\mu|} + \varpi \int\limits_0^{\tau/|\mu|} e^{-s}\Phi(\tau - |\mu| s)\, ds \right], & \mu \in [-1, 0), \\ \psi_+(\mu,\tau) = \psi_0\varpi \int\limits_0^\infty e^{-s}\Phi(\tau + \mu s)\, ds, & \mu \in (0, 1]. \end{cases} \tag{3.12}$$

From this formula and the inequality (3.9) we obviously derive the estimates

$$\psi_0 e^{-\tau/|\mu|} \leqslant \psi_-(\mu,\tau) \leqslant \psi_0, \quad 0 \leqslant \psi_+(\mu,\tau) \leqslant \frac{\varpi}{2-\varpi}\psi_0. \tag{3.13}$$

Lemma 3.5. *For all* $0 < \mu \leqslant 1$, $\tau > 0$ *the following estimate holds*:

$$|\psi_+(\mu, 0) - \psi_+(\mu,\tau)| \leqslant \frac{\varpi\psi_0}{(1-\varpi)(2-\varpi)}(1 - E_2(\tau)). \tag{3.14}$$

Proof. Using formula (3.12) and the estimate (3.10), we find

$$|\psi_+(\mu,0)-\psi_+(\mu,\tau)| \leqslant \psi_0\varpi\int\limits_0^\infty e^{-s}|\Phi(\mu s)-\Phi(\tau+\mu s)|\,ds \leqslant \frac{\varpi\psi_0}{(1-\varpi)(2-\varpi)}(1-E_2(\tau)).$$

The lemma is proved. □

Lemma 3.6. *For all* $0<\mu_1<\mu_2\leqslant 1$ *the following estimate holds*:

$$|\psi_+(\mu_1,0)-\psi_+(\mu_2,0)| \leqslant \frac{2\varpi\psi_0}{2-\varpi}\frac{\mu_2-\mu_1}{\mu_2}. \tag{3.15}$$

Proof. Using formula (3.4) and the estimate (3.10), we find

$$|\psi_+(\mu_1,0)-\psi_+(\mu_2,0)| = \varpi\psi_0\left|\int\limits_0^\infty\left(\frac{1}{\mu_1}e^{-\tau'/\mu_1}-\frac{1}{\mu_2}e^{-\tau'/\mu_2}\right)\Phi(\tau')\,d\tau'\right|$$

$$\leqslant \frac{\varpi\psi_0}{2-\varpi}\left[\int\limits_0^\infty\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)e^{-\tau'/\mu_1}\,d\tau'+\int\limits_0^\infty\frac{1}{\mu_2}\left(e^{-\tau'/\mu_2}-e^{-\tau'/\mu_1}\right)d\tau'\right]$$

$$\leqslant \frac{2\varpi\psi_0}{2-\varpi}\left[\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)\mu_1+\frac{1}{\mu_2}(\mu_2-\mu_1)\right] = \frac{2\varpi\psi_0}{2-\varpi}\frac{\mu_2-\mu_1}{\mu_2}.$$

The lemma is proved. □

Lemma 3.7. *The following estimate holds*:

$$\|\psi\|_{L^1((-1,1)\times\mathbb{R}^+)} \leqslant \frac{\psi_0}{2(1-\varpi)}. \tag{3.16}$$

Proof. It suffices to note that

$$\|\psi\|_{L^1((-1,1)\times\mathbb{R}^+)} = 2\|\Psi\|_{L^1(\mathbb{R}^+)} = 2\psi_0\|\Phi\|_{L^1(\mathbb{R}^+)}$$

and apply the estimate (3.7). The lemma is proved. □

4 The Limit Behavior of Solution

We proceed by studying the limit behavior of the solutions I_ε to the problem (1.2), (1.3) as $\varepsilon\to 0$. We write the solution in the form

$$\omega\cdot\nabla I_\varepsilon+\frac{1}{\varepsilon}I_\varepsilon = \frac{\varpi}{\varepsilon}\mathscr{S}(I_\varepsilon)+\frac{1-\varpi}{\varepsilon}k^2F,\quad (\omega,x)\in D, \tag{4.1}$$

$$I_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(I_\varepsilon|_{\Gamma^+})+(1-\theta)g_*,\quad (\omega,x)\in\Gamma^-, \tag{4.2}$$

where $g_*=k^2\mathscr{M}^-(J_*)$. Assume that $g_*\in L^1(\partial G)$ and $F\in W^{1,1}(G)$.

By a *solution* to the problem (4.1), (4.2) we mean a function $I_\varepsilon\in\widehat{\mathscr{W}}^1(D)$ satisfying Equation (4.1) almost everywhere in D and the boundary condition (4.2) almost everywhere on Γ^-.

We note that, in view of the boundary condition (4.2), the value $I_\varepsilon|_{\Gamma^-}(\omega, x)$ is independent of $\omega \in \Omega^-(x)$.

According to [10, 11] it follows that a solution to the problem (4.1), (4.2) exists and is unique.

Let ψ be a solution to the problem (3.1)– (3.3). We define $\widehat{\psi}_+$ on Γ^+ by

$$\widehat{\psi}_+(\omega, x) = \psi_+(\omega \cdot n(x), 0), \quad (\omega, x) \in \Gamma^+ \tag{4.3}$$

and denote by $F|_{\partial G}$ the trace of F on ∂G.

The goal of this section is to prove the following theorem.

Theorem 4.1. *The solutions I_ε to the problem* (4.1), (4.2) *have the following limit property as $\varepsilon \to 0$:*

$$I_\varepsilon \to k^2 F \quad \text{in} \quad L^1(D), \tag{4.4}$$

$$I_\varepsilon|_{\Gamma^+} \to \widehat{\psi}_+ g_* + (1 - \widehat{\psi}_+) k^2 F|_{\partial G} \quad \text{in} \quad \widehat{L}^1(\Gamma^+), \tag{4.5}$$

$$I_\varepsilon|_{\Gamma^-} \to \psi_0 g_* + (1 - \psi_0) k^2 F|_{\partial G} \quad \text{in } L^1(\partial G). \tag{4.6}$$

4.1. Proof of (4.4).

Lemma 4.1. *Let the functions $\widehat{I}_\varepsilon \in \widehat{\mathscr{W}}^1(D)$ satisfy the equation*

$$\omega \cdot \nabla \widehat{I}_\varepsilon + \frac{1}{\varepsilon} \widehat{I}_\varepsilon = \frac{\varpi}{\varepsilon} \mathscr{S}(\widehat{I}_\varepsilon) + f, \quad (\omega, x) \in D, \tag{4.7}$$

almost everywhere in D and the boundary condition

$$\widehat{I}_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(\widehat{I}_\varepsilon|_{\Gamma^+}) + \zeta, \quad (\omega, x) \in \Gamma^-, \tag{4.8}$$

almost everywhere on Γ^-, where $f \in L^1(D)$, $\zeta \in \widehat{L}^1(\Gamma^-)$. Then the following estimate holds:

$$(1 - \theta)\|\widehat{I}_\varepsilon|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} + \frac{1 - \varpi}{\varepsilon}\|\widehat{I}_\varepsilon\|_{L^1(D)} \leqslant \|f\|_{L^1(D)} + \|\zeta\|_{\widehat{L}^1(\Gamma^-)}. \tag{4.9}$$

Proof. Multiplying (4.7) by sgn $\widehat{I}_\varepsilon$, we get

$$\omega \cdot \nabla |\widehat{I}_\varepsilon| + \frac{1}{\varepsilon} |\widehat{I}_\varepsilon| \leqslant \frac{\varpi}{\varepsilon} \mathscr{S}(|\widehat{I}_\varepsilon|) + |f|. \tag{4.10}$$

Integrating (4.10) over D and taking into account that

$$\int\limits_D \mathscr{S}(|\widehat{I}_\varepsilon|)\, d\omega dx = \|\widehat{I}_\varepsilon\|_{L^1(D)},$$

we arrive at the inequality

$$\|\widehat{I}_\varepsilon|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} + \frac{1 - \varpi}{\varepsilon}\|\widehat{I}_\varepsilon\|_{L^1(D)} \leqslant \|\widehat{I}_\varepsilon|_{\Gamma^-}\|_{\widehat{L}^1(\Gamma^-)} + \|f\|_{L^1(D)}. \tag{4.11}$$

Taking into account (2.1), from (4.8) we get

$$\|\widehat{I}_\varepsilon|_{\Gamma^-}\|_{\widehat{L}^1(\Gamma^-)} \leqslant \|\mathscr{R}^-(\widehat{I}_\varepsilon|_{\Gamma^+})\|_{\widehat{L}^1(\Gamma^-)} + \|\zeta\|_{\widehat{L}^1(\Gamma^-)} = \theta\|\widehat{I}_\varepsilon|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} + \|\zeta\|_{\widehat{L}^1(\Gamma^-)}. \tag{4.12}$$

Substituting (4.12) into (4.11), we obtain the estimate (4.10). The lemma is proved. □

Lemma 4.2. *For the functions* $\widetilde{I}_\varepsilon = I_\varepsilon - k^2 F$ *the following estimate holds*:

$$\|\widetilde{I}_\varepsilon\|_{L^1(D)} \leqslant \frac{\varepsilon}{1-\varpi}\left[4\pi k^2 \|\nabla F\|_{L^1(G)} + (1-\theta)\pi\|g\|_{L^1(\partial G)}\right], \tag{4.13}$$

where $g = g_* - k^2 F|_{\partial G}$.

Proof. We note that $\widetilde{I}_\varepsilon \in \widehat{\mathscr{W}}^1(D)$, and this function is a solution to the problem

$$\omega \cdot \nabla \widetilde{I}_\varepsilon + \frac{1}{\varepsilon}\widetilde{I}_\varepsilon = \frac{\varpi}{\varepsilon}\mathscr{S}(\widetilde{I}_\varepsilon) - k^2 \omega \cdot \nabla F, \quad (\omega, x) \in D, \tag{4.14}$$

$$\widetilde{I}_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(\widetilde{I}_\varepsilon|_{\Gamma^+}) + (1-\theta)g. \tag{4.15}$$

Therefore, in view of Lemma 4.1, the following estimate holds:

$$\begin{aligned}&(1-\theta)\|\widetilde{I}_\varepsilon|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} + \frac{1-\varpi}{\varepsilon}\|\widetilde{I}_\varepsilon\|_{L^1(D)}\\ &\leqslant k^2\|\omega \cdot \nabla F\|_{L^1(D)} + (1-\theta)\|g\|_{\widehat{L}^1(\Gamma^-)} \leqslant 4\pi k^2\|\nabla F\|_{L^1(G)} + (1-\theta)\pi\|g\|_{L^1(\partial G)}.\end{aligned}$$

Roughening the obtained estimate, we arrive at the inequality (4.13). The lemma is proved. □

Corollary 4.1. *The property (4.4) holds.*

Proof. From the estimate (4.13) it follows that $I_\varepsilon - k^2 F \to 0$ in $L^1(D)$, i.e., $I_\varepsilon \to k^2 F$ in $L^1(D)$. □

4.2. Proof of (4.5) and (4.6). We consider the disc $V_r = \{y' = (y_1, y_2) \in \mathbb{R}^2 \mid |y'| \leqslant r\}$. in $\mathbb{R}^2$ with center at zero and radius r. By the assumption that $\partial G \in C^{1+\lambda}$, $0 < \lambda < 1$, for each point $x_0 \in \partial G$ there exists A Cartesian coordinate system with origin at the point x_0 and basis e_1, e_2, $e_3 = n(x_0)$, as well as the cylinder

$$\mathscr{C}(x_0) = \{x = x_0 + y_1 e_1 + y_2 e_2 + y_3 n(x_0) \mid |y'| \leqslant r_0,\ |y_3| < r_0\}$$

(where $r_0 > 0$ is independent of x_0) and a function $\gamma \in C^{1+\lambda}(V_{r_0})$ depending on x_0 is such that

$$G \cap \mathscr{C}(x_0) = \{x = x_0 + y_1 e_1 + y_2 e_2 + y_3 n(x_0) \mid y' \in V_{r_0},\ -r_0 < y_3 < \gamma(y')\},$$

$$\partial G \cap \mathscr{C}(x_0) = \{x = x_0 + y_1 e_1 + y_2 e_2 + y_3 n(x_0) \mid y' \in V_{r_0},\ y_3 = \gamma(y')\}.$$

Furthermore, $\gamma(0,0) = 0$, $\nabla_{y'}\gamma(0,0) = (0,0)$ and the following estimates hold:

$$|\gamma(y')| \leqslant C_1 |y'|^{1+\lambda} \quad \forall\, y' \in V_r, \tag{4.16}$$

$$|n(x) - n(x_0)| \leqslant C_2 |x - x_0|^{\lambda} \quad \forall\, x \in \partial G \cap \mathscr{C}(x_0) \tag{4.17}$$

with constants C_1 and C_2 independent of x_0.

We set $r_\varepsilon = \varepsilon^{1-\lambda/2}$ and $h_\varepsilon = C_1 \varepsilon^{-1}(2r_\varepsilon)^{1+\lambda}$. Further, we assume that $2r_\varepsilon \leqslant r_0$. We note that

$$\frac{\varepsilon}{r_\varepsilon} \to 0, \quad h_\varepsilon \to 0, \quad \varepsilon \to 0 \tag{4.18}$$

Furthermore,

$$|\gamma(y')| \leqslant \varepsilon h_\varepsilon \quad \forall\, y' \in V_{2r_\varepsilon} \tag{4.19}$$

and, consequently,

$$0 \leqslant h_\varepsilon - \varepsilon^{-1}\gamma(y') \leqslant 2h_\varepsilon \quad \forall\, y' \in V_{2r_\varepsilon}. \tag{4.20}$$

Let $x_0 \in \partial G$. We set

$$U_{2r_\varepsilon}(x_0) = G \cap B_{2r_\varepsilon}(x_0), \quad D_{2r_\varepsilon}(x_0) = \Omega \times U_{2r_\varepsilon}(x_0),$$
$$\Sigma_{r_\varepsilon}(x_0) = \partial G \cap B_{r_\varepsilon}(x_0), \quad \Sigma_{2r_\varepsilon}(x_0) = \partial G \cap B_{2r_\varepsilon}(x_0), \quad S_{2r_\varepsilon}(x_0) = \operatorname{mes}(\Sigma_{2r_\varepsilon}(x_0); d\sigma),$$
$$\Gamma^\pm_{r_\varepsilon}(x_0) = \{(\omega, x) \in \Gamma^\pm \mid x \in \Sigma_{r_\varepsilon}(x_0)\}, \quad \Gamma^\pm_{2r_\varepsilon}(x_0) = \{(\omega, x) \in \Gamma^\pm \mid x \in \Sigma_{2r_\varepsilon}(x_0)\}.$$

We use the spaces

$$L^1(U_{2r_\varepsilon}(x_0)) = L^1(U_{2r_\varepsilon}(x_0); dx), \quad L^1(\Sigma_{2r_\varepsilon}(x_0)) = L^1(\Sigma_{2r_\varepsilon}(x_0); d\sigma),$$
$$L^1(D_{2r_\varepsilon}(x_0)) = L^1(D_{2r_\varepsilon}(x_0); d\omega dx),$$
$$\widehat{L}^1(\Gamma^\pm_{r_\varepsilon}(x_0)) = \widehat{L}^1(\Gamma^\pm_{r_\varepsilon}(x_0), \widehat{d}\Gamma^\pm), \quad \widehat{L}^1(\Gamma^\pm_{2r_\varepsilon}(x_0)) = \widehat{L}^1(\Gamma^\pm_{2r_\varepsilon}(x_0), \widehat{d}\Gamma^\pm).$$

Let ψ be a solution to the problem (3.1)–(3.3), and let I_ε be a solution to the problem (1.2), (1.3) with $g \in C(\partial G)$.

We use the modulus of continuity g

$$w(g, \varepsilon) = \sup_{x', x'' \in \partial G,\ |x' - x''| < 2r_\varepsilon} |g(x') - g(x'')|$$

and introduce the following functions on $D_{2r_\varepsilon}(x_0)$:

$$\psi_\varepsilon(\omega, x) = \psi(\mu, \tau), \quad \mu = \omega \cdot n(x_0), \quad \tau = h_\varepsilon - \varepsilon^{-1} y_3 = h_\varepsilon - \varepsilon^{-1}(x - x_0) \cdot n(x_0),$$
$$z_\varepsilon(\omega, x) = g(x_0)\psi_\varepsilon(\omega, x) - \widetilde{I}_\varepsilon(\omega, x), \quad \widetilde{I}_\varepsilon(\omega, x) = I_\varepsilon(\omega, x) - k^2 F(x).$$

We note that

$$\psi_\varepsilon(\omega, x) = \psi(\omega \cdot n(x_0), h_\varepsilon - \varepsilon^{-1}\gamma(y')), \quad (\omega, x) \in \Gamma^\pm_{2r_\varepsilon}(x_0),$$
$$\psi_\varepsilon(\omega, x_0) = \psi(\omega \cdot n(x_0), h_\varepsilon).$$

It is easy to see that

$$\omega \cdot \nabla \psi_\varepsilon(\omega, x) = -\varepsilon^{-1}\mu \frac{d}{d\tau}\psi(\mu, \tau), \quad (\omega, x) \in D_{2r_\varepsilon}(x_0),$$
$$\mathscr{S}(\psi_\varepsilon)(x) = \frac{1}{4\pi}\int_\Omega \psi_\varepsilon(\omega', x)\, d\omega' = \frac{1}{2}\int_{-1}^{1} \psi(\mu', \tau)\, d\mu', \quad x \in U_{2r_\varepsilon}(x_0).$$

Therefore, ψ_ε satisfies the equation

$$\omega \cdot \nabla \psi_\varepsilon + \frac{1}{\varepsilon}\psi_\varepsilon = \frac{\varpi}{\varepsilon}\mathscr{S}(\psi_\varepsilon), \quad (\omega, x) \in D_{2r_\varepsilon}(x_0).$$

As a consequence, z_ε satisfies the equation

$$\omega\cdot\nabla z_\varepsilon+\frac{1}{\varepsilon}z_\varepsilon=\frac{\varpi}{\varepsilon}\mathscr{S}(z_\varepsilon)+k^2\omega\cdot\nabla F,\quad (\omega,x)\in D_{2r_\varepsilon}(x_0).$$

We also note that the function $\widehat{\psi}_+$ given by formula (4.3) is such that

$$\mathscr{R}^-(\widehat{\psi}_+)(\omega,x)=\frac{\theta}{\pi}\int\limits_{\Omega^+(x)}\psi_+(\omega'\cdot n(x),0)\,\omega'\cdot n(x)\,d\omega'=2\theta\int\limits_0^1\psi_+(\mu',0)\mu'\,d\mu',\quad (\omega,x)\in\Gamma^-.$$

Therefore, in view of (3.2),

$$\psi_0=\mathscr{R}^-(\widehat{\psi}_+)+1-\theta,\quad (\omega,x)\in\Gamma^-. \tag{4.21}$$

As a consequence, we have

$$z_\varepsilon|_{\Gamma^-}=\mathscr{R}^-(z_\varepsilon|_{\Gamma^+})+\delta_\varepsilon,\quad (\omega,x)\in\Gamma^-_{2,\varepsilon}(x_0), \tag{4.22}$$

where the function

$$\delta_\varepsilon(\omega,x)=g(x_0)[\psi_\varepsilon(\omega,x)-\psi_0+\mathscr{R}^-(\widehat{\psi}_+-\psi_\varepsilon)(x)]+(1-\theta)(g(x_0)-g(x))$$

plays the role of a residual in the boundary condition.

Now, we estimate the residual. Below, $\mu_0\in(0,1)$, μ_0 is a parameter.

Lemma 4.3. *For all $x\in\Sigma_{2r_\varepsilon}(x_0)$ and $0<\varepsilon<\varepsilon_0(\mu_0)$ the following estimates hold:*

$$\int\limits_{\Omega^-(x)}|\psi_\varepsilon(\omega,x)-\psi_0|\,|\omega\cdot n(x)|\,d\omega\leqslant\pi\psi_0\big[\zeta_1(\varepsilon,\mu_0)+\mu_0^2\big], \tag{4.23}$$

$$\int\limits_{\Omega^+(x)}|\psi_\varepsilon(\omega,x)-\widehat{\psi}_+(\omega,x)|\,\omega\cdot n(x)\,d\omega\leqslant\pi\psi_0\Big[\zeta_2(\varepsilon,\mu_0)+\mu_0^2\Big], \tag{4.24}$$

$$\int\limits_{\Omega^-(x)}|\delta_\varepsilon(\omega,x)|\,|\omega\cdot n(x)|\,d\omega\leqslant\pi\psi_0[\zeta(\varepsilon,\mu_0)+2\mu_0^2]\,\|g\|_{C(\partial G)}+(1-\theta)\pi w(g,\varepsilon), \tag{4.25}$$

where

$$\zeta_1(\varepsilon,\mu_0)=1-e^{-4h_\varepsilon/\mu_0},\quad \zeta_2(\varepsilon,\mu_0)=\frac{\varpi}{2-\varpi}\Big[\frac{1}{1-\varpi}(1-E_2(2h_\varepsilon))+\frac{4C_2(2r_\varepsilon)^\lambda}{\mu_0}\Big],$$

$$\zeta(\varepsilon,\mu_0)=\zeta_1(\varepsilon,\mu_0)+\zeta_2(\varepsilon,\mu_0).$$

Proof. We set $\mu'=\omega\cdot n(x)$, $\mu=\omega\cdot n(x_0)$, and

$$\Omega^{-,\mu_0}(x)=\{\omega\in\Omega^-(x)\mid \mu'<-\mu_0\},\quad \Omega^{+,\mu_0}(x)=\{\omega\in\Omega^+(x)\mid \mu'>\mu_0\}.$$

By (4.17), there exists $\varepsilon_0(\mu_0)>0$ such that $|n(x)-n(x_0)|<\mu_0/2$ for all $x\in\Sigma_{2r_\varepsilon}(x_0)$.

Let $\omega\in\Omega^{-,\mu_0}(x)$. Then from the estimate

$$|\mu-\mu'|\leqslant|n(x_0)-n(x)|<\mu_0/2 \tag{4.26}$$

it follows that $\mu < -\mu_0/2$ and $\omega \in \Omega^{-,\mu_0/2}(x_0)$. By the first inequality in (3.13), we have

$$|\psi_\varepsilon(\omega,x) - \psi_0| = |\psi_-(\mu, h_\varepsilon - \varepsilon^{-1}\gamma(y')) - \psi_0| \leqslant \psi_0(1 - e^{-(h_\varepsilon - \varepsilon^{-1}\gamma(y'))/|\mu|}) \leqslant \psi_0\zeta_1(\varepsilon,\mu_0).$$

If $\omega \in \Omega^-(x) \setminus \Omega^{-,\mu_0}(x)$, then we have the rough estimate

$$|\psi_\varepsilon(\omega,x) - \psi_0| \leqslant \psi_0.$$

Thus,

$$\int\limits_{\Omega^-(x)} |\psi_\varepsilon(\omega,x) - \psi_0|\, |\mu'|\, d\omega \leqslant \int\limits_{\Omega^{-,\mu_0}(x)} \psi_0\zeta_1(\varepsilon,\mu_0)|\mu'|\, d\omega + \int\limits_{\Omega^-(x)\setminus\Omega^{-,\mu_0}(x)} \psi_0\, |\mu'|\, d\omega$$

$$= 2\pi \int\limits_{-1}^{-\mu_0} \psi_0\zeta_1(\varepsilon,\mu_0)|\mu'|\, d\mu' + 2\pi \int\limits_{-\mu_0}^{0} \psi_0|\mu'|\, d\mu' \leqslant \pi\psi_0\big[\zeta_1(\varepsilon,\mu_0) + \mu_0^2\big].$$

The estimate (4.23) is proved. Let us prove the estimate (4.24).

Let $\omega \in \Omega^{+,\mu_0}(x)$. Then from (4.26) it follows that $\mu > \mu_0/2$ and $\omega \in \Omega^{+,\mu_0/2}(x_0)$. Using the estimates (3.14), (3.15), and (4.17), we find

$$\begin{aligned} |\psi_\varepsilon(\omega,x) - \widehat{\psi}_+(\omega,x)| &= |\psi_+(\mu, h_\varepsilon - \varepsilon^{-1}\gamma(y')) - \psi_+(\mu',0)| \\ &\leqslant |\psi_+(\mu, h_\varepsilon - \varepsilon^{-1}\gamma(y')) - \psi_+(\mu,0)| + |\psi_+(\mu,0) - \psi_+(\mu',0)| \\ &\leqslant \frac{\psi_0\varpi}{2-\varpi}\Big[\frac{1}{1-\varpi}(1 - E_2(h_\varepsilon - \varepsilon^{-1}\gamma(y'))) + 2\frac{|n(x) - n(x_0)|}{\mu_0/2}\Big] \\ &\leqslant \frac{\psi_0\varpi}{2-\varpi}\Big[\frac{1}{1-\varpi}(1 - E_2(2h_\varepsilon)) + \frac{4C_2(2r_\varepsilon)^\lambda}{\mu_0}\Big] = \psi_0\zeta_2(\varepsilon,\mu_0). \end{aligned}$$

In the case $\omega \in \Omega^+(x) \setminus \Omega^{+,\mu_0}(x)$, we use the rough estimate

$$|\psi_\varepsilon(\omega,x) - \widehat{\psi}_+(\omega,x)| \leqslant \psi_0.$$

Thus,

$$\int\limits_{\Omega^+(x)} |\psi_\varepsilon(\omega,x) - \widehat{\psi}_+(\omega,x)|\, \mu'\, d\omega \leqslant \int\limits_{\Omega^{+,\mu_0}(x)} \psi_0\zeta_2(\varepsilon,\mu_0)\mu'\, d\omega + \int\limits_{\Omega^+(x)\setminus\Omega^{+,\mu_0}(x)} \psi_0\, \mu'\, d\omega$$

$$= 2\pi \int_{\mu_0}^{1} \psi_0\zeta_2(\varepsilon,\mu_0)\mu'\, d\mu' + 2\pi \int\limits_{0}^{\mu_0} \psi_0\mu'\, d\mu' \leqslant \pi\psi_0[\zeta_2(\varepsilon,\mu_0) + \mu_0^2].$$

From the estimates (4.23) and (4.24) it follows that

$$\int\limits_{\Omega^-(x)} |\delta_\varepsilon(\omega,x)||\omega\cdot n(x)|\, d\omega$$

$$\leqslant \|g\|_{C(\partial G)}\Bigg[\int\limits_{\Omega^-(x)} |\psi_\varepsilon(\omega,x) - \psi_0||\omega\cdot n(x)|\, d\omega + \theta \int\limits_{\Omega^+(x)} |\widehat{\psi}_+(\omega,x) - \psi_\varepsilon(\omega,x)|\omega\cdot n(x)\, d\omega\Bigg]$$

$$+ (1-\theta)\pi w(g,\varepsilon) \leqslant \pi\psi_0[\zeta(\varepsilon,\mu_0) + 2\mu_0^2]\, \|g\|_{C(\partial G)} + (1-\theta)\pi w(g,\varepsilon).$$

The lemma is proved. □

Corollary 4.2. *The following estimates hold*:

$$\|\psi_\varepsilon - \psi_0\|_{\widehat{L}^1(\Gamma^-_{2r_\varepsilon}(x_0))} \leqslant \pi\psi_0\big[\zeta_1(\varepsilon,\mu_0) + \mu_0^2\big] S_{2r_\varepsilon}(x_0), \tag{4.27}$$

$$\|\psi_\varepsilon - \widehat{\psi}_+\|_{\widehat{L}^1(\Gamma^+_{2r_\varepsilon}(x_0))} \leqslant \pi\psi_0\big[\zeta_2(\varepsilon,\mu_0) + \mu_0^2\big] S_{2r_\varepsilon}(x_0), \tag{4.28}$$

$$\|\delta_\varepsilon\|_{\widehat{L}^1(\Gamma^-_{2r_\varepsilon}(x_0))} \leqslant \pi\Big\{\psi_0\big[\zeta(\varepsilon,\mu_0) + 2\mu_0^2\big]\|g\|_{C(\partial G)} + (1-\theta)w(g,\varepsilon)\Big\} S_{2r_\varepsilon}(x_0). \tag{4.29}$$

Lemma 4.4. *Let $g \in C(\partial G)$. Then the following estimate holds*:

$$\|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} \leqslant \frac{1}{1-\theta}\left[\frac{1}{r_\varepsilon}\|\widetilde{I}_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_0))} + 4\pi k^2\|\nabla F\|_{L^1(U_{2r_\varepsilon}(x_0))}\right]$$
$$+ \left\{\frac{2\pi\psi_0}{1-\theta}\Big[\zeta(\varepsilon,\mu_0) + 2\mu_0^2 + \frac{\varepsilon}{r_\varepsilon}\frac{1}{1-\varpi}\Big]\|g\|_{C(\partial G)} + \pi(1+\psi_0)w(g,\varepsilon)\right\} S_{2r_\varepsilon}(x_0). \tag{4.30}$$

Proof. We recall that the function $z_\varepsilon = g(x_0)\psi_\varepsilon - \widetilde{I}_\varepsilon$ is such that

$$\omega\cdot\nabla z_\varepsilon + \frac{1}{\varepsilon}z_\varepsilon = \frac{\varpi}{\varepsilon}\mathscr{S}(z_\varepsilon) + k^2\omega\cdot\nabla F, \quad (\omega,x)\in D_{2r_\varepsilon}(x_0), \tag{4.31}$$

$$z_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(z_\varepsilon|_{\Gamma^+}) + \delta_\varepsilon, \quad (\omega,x)\in\Gamma^-_{2r_\varepsilon}(x_0). \tag{4.32}$$

We multiply z_ε by the cut-off function

$$\eta_\varepsilon(x) = \begin{cases} 1, & |x-x_0|\leqslant r_\varepsilon, \\ 2 - |x-x_0|/r_\varepsilon, & r_\varepsilon < |x-x_0| < 2r_\varepsilon, \\ 0, & 2r_\varepsilon \leqslant |x-x_0| \end{cases}$$

and extend $z_\varepsilon\eta_\varepsilon$ by zero on $D\setminus D_{2r_\varepsilon}(x_0)$. We note that the function $z_\varepsilon\eta_\varepsilon$ is a solution to the problem

$$\omega\cdot\nabla(z_\varepsilon\eta_\varepsilon) + \frac{1}{\varepsilon}(z_\varepsilon\eta_\varepsilon) = \frac{\varpi}{\varepsilon}\mathscr{S}(z_\varepsilon\eta_\varepsilon) - (\omega\cdot\nabla\eta_\varepsilon)z_\varepsilon + k^2(\omega\cdot\nabla F)\eta_\varepsilon, \quad (\omega,x)\in D, \tag{4.33}$$

$$(z_\varepsilon\eta_\varepsilon)|_{\Gamma^-} = \mathscr{R}^-((z_\varepsilon\eta_\varepsilon)|_{\Gamma^+}) + \delta_\varepsilon\eta_\varepsilon, \quad (\omega,x)\in\Gamma^-. \tag{4.34}$$

Using Lemma 4.1, we arrive at the estimate

$$(1-\theta)\|(z_\varepsilon\eta_\varepsilon)|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} + \frac{1-\varpi}{\varepsilon}\|z_\varepsilon\eta_\varepsilon\|_{L^1(D)}$$
$$\leqslant \|(\nabla\eta_\varepsilon)z_\varepsilon\|_{L^1(D)} + k^2\|(\nabla F)\eta_\varepsilon\|_{L^1(D)} + \|\delta_\varepsilon\eta_\varepsilon\|_{\widehat{L}^1(\Gamma^-)} \tag{4.35}$$

which, together with the inequality (4.20), implies

$$(1-\theta)\|z_\varepsilon|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} \leqslant \frac{1}{r_\varepsilon}\|z_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_0))} + 4\pi k^2\|\nabla F\|_{L^1(U_{2r_\varepsilon}(x_0))} + \|\delta_\varepsilon\|_{\widehat{L}^1(\Gamma^-_{2r_\varepsilon}(x_0))}$$
$$\leqslant \frac{1}{r_\varepsilon}\Big[\|g(x_0)\psi_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_0))} + \|\widetilde{I}_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_0))}\Big] + 4\pi k^2\|\nabla F\|_{L^1(U_{2r_\varepsilon}(x_0))}$$
$$+ \pi\Big\{\psi_0\big[\zeta(\varepsilon,\mu_0) + 2\mu_0^2\big]\|g\|_{C(\partial G)} + (1-\theta)w(g,\varepsilon)\Big\} S_{2r_\varepsilon}(x_0).$$

Taking into account the equality

$$\int\limits_{\Omega} \psi_\varepsilon(\omega,\tau)\,d\omega = 2\pi \int\limits_{-1}^{1} \psi(\mu,\tau)\,d\mu = 4\pi\psi_0\Phi(\tau)$$

and the estimate (3.7), we find

$$\begin{aligned}\|\psi_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_0))} &= \int\limits_{V_{2r_\varepsilon}} \Bigg[\int\limits_{-r_0}^{\gamma(y')} 4\pi\psi_0\Phi(h_\varepsilon - \varepsilon^{-1}y_3)\,dy_3\Bigg]dy' \\ &\leqslant 4\pi\psi_0 S_{2r_\varepsilon}(x_0)\|\Phi\|_{L^1(\mathbb{R}^+)}\varepsilon \leqslant \frac{\pi\psi_0}{1-\varpi}S_{2r_\varepsilon}(x_0)\varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\|z_\varepsilon|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} &\leqslant \frac{1}{1-\theta}\left[\frac{1}{r_\varepsilon}\|\widetilde{I}_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_0))} + 4\pi k^2\|\nabla F\|_{L^1(U_{2r_\varepsilon}(x_0))}\right] \\ &+ \left\{\frac{\pi\psi_0}{1-\theta}\Big[\zeta(\varepsilon,\mu_0) + 2\mu_0^2 + \frac{\varepsilon}{r_\varepsilon}\frac{1}{1-\varpi}\Big]\|g\|_{C(\partial G)} + \pi w(g,\varepsilon)\right\} S_{2r_\varepsilon}(x_0). \quad (4.36)\end{aligned}$$

Noting that

$$\begin{aligned}&\|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} \\ &\leqslant \|\widetilde{I}_\varepsilon|_{\Gamma^+} - g(x_0)\psi_\varepsilon\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} + \|(g(x_0)-g)\psi_\varepsilon\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} + \|(\psi_\varepsilon - \widehat{\psi}_+)g\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} \\ &\leqslant \|z_\varepsilon\|_{\widehat{L}^1(\Gamma^+_{r_\varepsilon}(x_0))} + \big\{\pi\psi_0 w(g,\varepsilon) + \pi\psi_0[\zeta_2(\varepsilon,\mu_0) + \mu_0^2]\,\|g\|_{C(\partial G)}\big\} S_{2r_\varepsilon}(x_0),\end{aligned}$$

we pass from (4.36) to the estimate (4.30). The lemma is proved. □

Theorem 4.2. *The properties* (4.5) *and* (4.6) *hold.*

Proof. We cover $\mathbb{R}^3$ by the system of cubes

$$\Pi^{r_\varepsilon}_{i,j,k} = \left[\frac{(i-1/2)r_\varepsilon}{2}, \frac{(i+1/2)r_\varepsilon}{2}\right] \times \left[\frac{(j-1/2)r_\varepsilon}{2}, \frac{(j+1/2)r_\varepsilon}{2}\right] \times \left[\frac{(k-1/2)r_\varepsilon}{2}, \frac{(k+1/2)r_\varepsilon}{2}\right]$$

with edges of length $r_\varepsilon/2$, where i,j,k are integers. We choose a finite subsystem of cubes intersecting ∂G. Let $\{z_\ell\}_{\ell=1}^N$ be the set of centers of the cubes in this subsystem. With a cube with center z_ℓ we associate a point x_ℓ in the intersection of this cube and ∂G. It is clear that the system of open balls $\{B_{r_\varepsilon}(x_\ell)\}_{\ell=1}^N$ with centers x_ℓ and radii r_ε covers ∂G.

We note that for every ℓ the ball $B_{2r_\varepsilon}(x_\ell)$ can intersect at most M other balls $B_{r_\varepsilon}(x_k)$, where M is a constant independent of ℓ and r_ε.

Let $g \in C(\partial G)$. Applying the estimate (4.30) with x_ℓ instead of x_0 and summarizing the obtained inequalities, we find

$$\begin{aligned}\sum_{\ell=1}^N \|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+_{1,\varepsilon}(x_\ell))} &\leqslant \frac{1}{1-\theta}\frac{1}{r_\varepsilon}\sum_{\ell=1}^N \|\widetilde{I}_\varepsilon\|_{L^1(D_{2r_\varepsilon}(x_\ell))} + \frac{4\pi k^2}{1-\theta}\sum_{\ell=1}^N \|\nabla F\|_{L^1(U_{2r_\varepsilon}(x_\ell))} \\ &+ \left\{\frac{2\pi\psi_0}{1-\theta}\Big[\zeta(\varepsilon,\mu_0) + 2\mu_0^2 + \frac{\varepsilon}{r_\varepsilon}\frac{1}{1-\varpi}\Big]\|g\|_{C(\partial G)} + \pi(1+\psi_0)w(g,\varepsilon)\right\}\sum_{\ell=1}^N S_{2_\varepsilon}(x_\ell). \quad (4.37)\end{aligned}$$

Since $\Gamma^+ \subset \bigcup\limits_{\ell=1}^{N} \Gamma^+_{r_\varepsilon}(x_\ell)$ and each ball $B_{2r_\varepsilon}(x_\ell)$ intersects at most M other balls $B_{2r_\varepsilon}(x_k)$, from (4.37) we obtain the estimate

$$\begin{aligned}\|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+)} &\leqslant \frac{M}{1-\theta}\frac{1}{r_\varepsilon}\|\widetilde{I}_\varepsilon\|_{L^1(D)} + \frac{M}{1-\theta}4\pi k^2\|\nabla F\|_{L^1(G_\varepsilon)} \\ &\leqslant M\left\{\frac{2\pi\psi_0}{1-\theta}\Big[\zeta(\varepsilon,\mu_0) + 2\mu_0^2 + \frac{\varepsilon}{r_\varepsilon}\frac{1}{1-\varpi}\Big]\|g\|_{C(\partial G)} + \pi(1+\psi_0)w(g,\varepsilon)\right\}\cdot \operatorname{mes}(\partial G; d\sigma), \quad (4.38)\end{aligned}$$

where $G_\varepsilon = \{x \in G \mid \rho(x, \partial G) < 2r_\varepsilon\}$. Passing to the limit as $\varepsilon \to 0$ in this inequality and using the fact that, in view of the inequality (4.13), the first term on the right-hand side of (4.38) converges to zero, we find

$$\overline{\lim_{\varepsilon\to\infty}}\ \|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+)} \leqslant M\frac{4\pi\psi_0}{1-\theta}\mu_0^2\|g\|_{C(\partial G)}\cdot \operatorname{mes}(\partial G; d\sigma).$$

By the arbitrariness in the choice of $\mu_0 \in (0,1)$, we have

$$\overline{\lim_{\varepsilon\to\infty}}\ \|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+)} = 0.$$

Let $g \in L^1(\partial G)$. We construct a sequence $\{g_n\}_{n=1}^\infty \subset C(\partial G)$ such that $g_n \to g$ in $L^1(\partial G)$ as $n \to \infty$. We denote by $\widetilde{I}_{n,\varepsilon}$ a solution to the problem (4.14), (4.15) with g_n instead of g. The function $\Delta\widetilde{I}_{n,\varepsilon} = \widetilde{I}_{n,\varepsilon} - \widetilde{I}_\varepsilon$ is a solution to the problem

$$\begin{aligned}&\omega\cdot\nabla\Delta\widetilde{I}_{n,\varepsilon} + \frac{1}{\varepsilon}\Delta\widetilde{I}_{n,\varepsilon} = \frac{\varpi}{\varepsilon}\mathscr{S}(\Delta\widetilde{I}_{n,\varepsilon}), \quad (\omega, x) \in D,\\ &\Delta\widetilde{I}_{n,\varepsilon}|_{\Gamma^-} = \mathscr{R}^-(\Delta\widetilde{I}_{n,\varepsilon}|_{\Gamma^+}) + (1-\theta)(g_n - g).\end{aligned}$$

Hence, by Lemma 4.1, the following estimate holds:

$$\|\Delta\widetilde{I}_{n,\varepsilon}|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} \leqslant \pi\|g - g_n\|_{L^1(\partial G)}.$$

Consequently,

$$\begin{aligned}\|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+)} &\leqslant \|\Delta\widetilde{I}_{n,\varepsilon}|_{\Gamma^+}\|_{\widehat{L}^1(\Gamma^+)} + \|\widetilde{I}_{n,\varepsilon}|_{\Gamma^+} - \widehat{\psi}_+ g_n\|_{\widehat{L}^1(\Gamma^+)} + \|\widehat{\psi}_+(g_n - g)\|_{\widehat{L}^1(\Gamma^+)}\\ &\leqslant \|\widetilde{I}_{n,\varepsilon}|_{\Gamma^+} - \widehat{\psi}_+ g_n\|_{\widehat{L}^1(\Gamma^+)} + \pi(1+\psi_0)\|g_n - g\|_{L^1(\partial G)}.\end{aligned}$$

By the above, for fixed n the first term on the right-hand side of this inequality converges to zero as $\varepsilon \to 0$. Therefore,

$$\overline{\lim_{\varepsilon\to 0}}\ \|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+)} \leqslant \pi(1+\psi_0)\|g_n - g\|_{L^1(\partial G)}.$$

Passing to the limit as $n \to \infty$, we find

$$\overline{\lim_{\varepsilon\to 0}}\ \|\widetilde{I}_\varepsilon|_{\Gamma^+} - \widehat{\psi}_+ g\|_{\widehat{L}^1(\Gamma^+)} = 0.$$

Thus,

$$\widetilde{I}_\varepsilon|_{\Gamma^+} = I_\varepsilon|_{\Gamma^+} - k^2F|_{\partial G} \to \widehat{\psi}_+ g = \widehat{\psi}_+ g_* - k^2\widehat{\psi}_+ F|_{\partial G} \quad \text{in} \quad \widehat{L}^1(\Gamma^+),$$

i.e., (4.5) holds.

To complete the proof, it remains to note that from (4.5) and (4.21) it follows that

$$I_\varepsilon|_{\Gamma^-} = \mathscr{R}^-(I_\varepsilon|_{\Gamma^+}) + (1-\theta)g_* \to \mathscr{R}^-\big(\widehat{\psi}_+ g_* + (1-\widehat{\psi}_+)k^2 F|_{\partial G}\big) + (1-\theta)g_*$$
$$= [\mathscr{R}^-(\widehat{\psi}_+) + (1-\theta)]g_* + \mathscr{R}^-(1-\widehat{\psi}_+)k^2 F|_{\partial G} = \psi_0 g_* + (1-\psi_0)k^2 F|_{\partial G} \quad \text{in} \quad L^1(\partial G).$$

The theorem is proved. □

Thus, Theorem 4.1 is completely proved.

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