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INVARIANT MANIFOLDS. GLOBAL ATTRACTOR OF A GENERALIZED VERSION OF THE NONLOCAL GINZBURG–LANDAU EQUATION

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We consider a generalized nonlocal Ginzburg–Landau equation with periodic boundary conditions. For the corresponding initial-boundary value problem we prove the existence of a solution for all positive values of the evolution variable. We study the existence and properties of invariant manifolds. We extract a class of invariant manifolds the union of which forms a global attractor. We describe the structure of the attractor and find the Euclidean dimension of its components. In the metric of the space of initial conditions, we also study the Lyapunov stability and orbital stability of solutions that belong in the global attractor. Bibliography: 11 *titles.*

1 Introduction

We consider the known partial differential equation

$$u_t = u - (1 + ic)u|u|^2 + (a + ib)\Delta u, \tag{1.1}$$

where $u = u(t, x_1, \dots, x_n)$ is a complex-valued function, $n \in N$ is a natural number, Δu is the Laplace operator with respect to the variables $x_1, \dots, x_n$, $c, a, b \in R$, $a \geqslant 0$.

Equation (1.1) is known as the complex Ginzburg–Landau equation [1, 2]. This equation appears in different fields of physics [1] and chemical kinetics [2].

Equation (1.1) is called a variational Ginzburg–Landau equation [1, 3] if $c = b = 0$ and weakly dissipative or generalized nonlinear Schrödinger equation if $a = 0$ [1, 4, 5].

In connection with problems of elastic stability theory, the following version of the Ginzburg–

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Landau equation is considered:

$$u_t = u - (1+ic)u|u|^2 + (a_1+ib_1)u_{xx} - (a_2+ib_2)u_{xxxx},$$

where $a_1, b_1, a_2, b_2, c \in R$, $a_2 \geqslant 0$, and $a_1 \geqslant 0$ if $a_2 = 0$.

In connection with mathematical modeling of the ferromagnetism phenomenon (see, for example, [6, 7]), the following equation was derived:

$$u_t = u - (1+ic)uV(u) + (a+ib)u_{xx}, \tag{1.2}$$

where

$$u = u(t,x), \quad V(u) = \frac{1}{2l}\int\limits_0^{2l} |u(t,x)|^2 dx,$$

if $x \in [0, 2l]$ or the function $u(t,x)$ has period $2l$ in x. The integro-differential equation (1.2) is known as the Ginzburg–Landau equation. One of possible modifications of Equations (1.1), (1.2) is obtained if the convection phenomenon is taken into account and, consequently, additional terms appear in the corresponding equations.

In this paper, we consider a version of Equation (1.2) with periodic boundary conditions, which, after normalization of the spatial variable x, can be written as

$$u(t, x+2\pi) = u(t,x).$$

2 Statement of the Problem

We consider the following version of the nonlocal Ginzburg–Landau equation (1.2):

$$u_t = du - (1+ic)uV(u) + g_2 u_x V(u_x) + g_1 u_x + (a+ic_1)u_{xx} - (a_2+ic_2)u_{xxxx}, \tag{2.1}$$

where $c, g_2, g_1, d, a, a_2, c_1, c_2 \in R$. As was already mentioned,

$$V(u) = \frac{1}{2\pi}\int\limits_0^{2\pi} |u(t,x)|^2 dx, \text{i.e.,} V(u_x) = \frac{1}{2\pi}\int\limits_0^{2\pi} |u_x(t,x)|^2 dx.$$

We note that the third and fourth terms on the right-hand side of Equation (2.1) are interpreted in the physics as convective. Assume that $a_2 > 0$. Normalizing the variable t and function $u(t,x)$, we can assume that $a_2 = 1$ in (2.1), whereas coefficients at nonlinear terms remain unchanged. Then the remaining coefficients d, g_1, c_1, c_2, a are proportional to the original ones. The periodic boundary conditions remain unchanged, with period 2π.

We consider Equation (2.1) with boundary conditions

$$u(t, x+2\pi) = u(t,x). \tag{2.2}$$

The boundary value problem (2.1), (2.2) is completed with the initial condition

$$u(0,x) = f(x), \tag{2.3}$$

where $u(t,x)$ and $f(x)$ are complex-valued functions of variable $t > 0$ ($t \geqslant 0$), $x \in R$. Assume that $f(x) \in H_2^p$, $p = 1, 2, \ldots$, where H_2^p is the Hilbert space of functions possessing the following properties:

1) $f(x) \in H_2^p$ if $f(x) \in W_2^p[0, 2\pi]$ for $x \in [0, 2\pi]$,

2) $f(x)$ has period 2π.

For $p = 0$ we have $f(x) \in L_2(0, 2\pi)$. We denote by W_2^p the Sobolev space equipped with the traditional norm and inner product.

In this paper, we establish the existence of solutions to the initial-boundary value problem (2.1)–(2.3) for all $t > 0$ and the existence and structure of the global attractor of the boundary value problem (2.1), (2.2). Throughout the paper, we use the definition of a global attractor given in [8] (cf. also [9] and the references therein).

3 Solvability of Initial-Boundary Value Problem

Theorem 3.1. *Let $f(x) \in H_2^1$. Then the initial-boundary value problem* (2.1)–(2.3) *has a unique solution for all $t > 0$. The corresponding solution $u(t,x)$ possesses the following properties*:

(1) *for all $x \in R$, $t > 0$ the complex-valued function $u(t,x)$ belongs to the class C^∞*,

(2) *the limit equality holds*

$$\lim_{t \to 0+} ||u(t,x) - f(x)||_{H_2^1} = 0,$$

where

$$||u(t,x) - f(x)||^2_{H_2^1} = ||u(t,x) - f(x)||^2_{L_2(0,2\pi)} + ||u_x(t,x) - f'(x)||^2_{L_2(0,2\pi)},$$

$$||v(t,x) - g(x)||^2_{L_2(0,2\pi)} = \int_0^{2\pi} |v(t,x) - g(x)|^2 dx.$$

The proof of Theorem 3.1 is based on the fact that all solutions to the initial-boundary value problem (2.1)–(2.3) can be found in the form of the functional series

$$u(t,x) = \sum_{n=-\infty}^{\infty} u_n(t) \exp(inx), \tag{3.1}$$

where $u_n(t)$ admits an explicit representation.

Let $u(t,x)$ be a jointly continuous function for all x and $t > 0$. Then $u(t,x)$ can be written as the functional series (3.1) with

$$u_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(t,x) \exp(-inx) dx, \quad n = 0, \pm 1, \pm 2, \ldots.$$

We choose $u_n(t)$ such that $u(t,x)$ is a solution to the corresponding initial-boundary value problem (2.1)–(2.3). Moreover, $u_n(t)$ satisfies the countable system of ordinary differential equations

$$u_n' = (a_n + ib_n)u_n - (1 + ic)u_n V + ig_2 n V_1, \tag{3.2}$$

where $n \in Z$, $a_n = d - an^2 - n^4, b_n = -c_1n^2 - c_2n^4 + g_1n$ and, in view of the Parseval identity,

$$V = V(u) = \sum_{k=-\infty}^{\infty} |u_k|^2, \quad V_1 = V(u_x) = \sum_{k=-\infty}^{\infty} k^2|u_k|^2.$$

We complete the system (3.2) by the initial condition

$$u_n(0) = f_n, \tag{3.3}$$

where f_n denotes the Fourier coefficient of $f(x)$:

$$f_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-inx)dx.$$

Remark 3.1. From the condition $f_m = 0$ for some $m \in Z$ it follows that $u_m(t) = 0$ (the linear subspace $f_m = 0$ is invariant with respect to solutions to the system (3.2)). We can regard the case $f_n \neq 0$ for all $n \in Z$ as the main one. If $f_m = 0$ for some $m \in Z_* \subset Z$, then we can restrict ourselves to equations with $k \in Z \backslash Z_*$.

We consider the case $f_n \neq 0$ for all $n \in Z$. We set

$$u_n(t) = \rho_n(t) \exp(i\varphi_n(t)), \quad f_n = r_n \exp(i\psi_n). \tag{3.4}$$

Making the change (3.4) for $\rho_n(t)$ and $\varphi_n(t)$, we obtain the Cauchy problem

$$\rho_n' = a_n\rho_n - \rho_n V(\rho), \tag{3.5}$$

$$\rho_n(0) = r_n, \tag{3.6}$$

$$\varphi_n' = b_n - cV(\rho) + g_2V_1(\rho)n, \tag{3.7}$$

$$\varphi_n(0) = \psi_n, \tag{3.8}$$

where, in this case,

$$V(\rho) = \sum_{k=-\infty}^{\infty} \rho_k^2, \quad V_1(\rho) = \sum_{k=-\infty}^{\infty} k^2\rho_k^2.$$

The Cauchy problem (3.5), (3.6) can be studied autonomously without addressing to the Cauchy problem (3.7), (3.8).

Lemma 3.1. *Let $f_n \neq 0, a_n \neq 0$ for all integers n. Then*

$$\rho_n(t) = \frac{r_n \exp(a_n t)}{\sqrt{1 + Q(t)}}, \quad n \in Z.$$

Proof. In the case under consideration, we have $r_n \neq 0$ for all n. We first show that

$$\rho_n(t) = \frac{r_n}{r_0}\rho_0(t) \exp(-\alpha_n t), \tag{3.9}$$

where $\alpha_n = an^2 + n^4$, $n \in Z \backslash \{0\}$. To prove (3.9), we consider the equation of the system (3.5) with some $n \neq 0$ and the equation with $n = 0$

$$\rho_n' = a_n\rho_n - \rho_n V(\rho),$$

$$\rho_0' = a_0\rho_0 - \rho_0 V(\rho), \quad a_0 = d.$$

We multiply the first equation by ρ_0 and the second one by ρ_n and subtract term-by-term one from the other. As a result, we find

$$\rho_n'\rho_0 - \rho_0'\rho_n = (a_n - a_0)\rho_0\rho_n = -\alpha_n\rho_0\rho_n(\alpha_n = a_0 - a_n).$$

Consequently,

$$\left(\frac{\rho_n}{\rho_0}\right)' = -\alpha_n\frac{\rho_n}{\rho_0}, \quad \frac{\rho_n(0)}{\rho_0(0)} = \frac{r_n}{r_0}.$$

Hence

$$\frac{\rho_n(t)}{\rho_0(t)} = \frac{r_n}{r_0}\exp(-\alpha_n t).$$

Substituting (3.9) into the equation for $\rho_0 = \rho_0(t)$, we obtain the Bernoulli equation for $\rho_0(t)$

$$\rho_0' = a_0\rho_0 - \frac{\rho_0^3}{r_0^2}S(t), \tag{3.10}$$

where

$$S(t) = \sum_{n=-\infty}^{\infty} r_n^2 \exp(-2\alpha_n t);$$

moreover, the series uniformly converges for $t \geqslant 0$ since $f(x) \in H_2^1$ by assumption. Consequently, the series

$$\sum_{n=-\infty}^{\infty} |f_n|^2, \quad \sum_{n=-\infty}^{\infty} n^2|f_n|^2, \quad |f_n| = r_n,$$

converge. Therefore, $S(t) \in C[0,\infty)$. Furthermore, the function $S(t)$, $t \in (0,\infty)$, has continuous derivatives of any order k. Indeed, we differentiate the corresponding series term-by-term

$$S^{(k)}(t) = \sum_{n=-\infty}^{\infty} r_n^2(-2\alpha_n)^k \exp(-2\alpha_n t).$$

The last series uniformly converges for $t \in [t_0,\infty)$, where t_0 is an arbitrarily small positive constant. Moreover, one should take into account that $r_n^2 \leqslant M$, where $M > 0, \alpha_n > 0$ for all $n^2 \geqslant n_0^2$. Thus, $S^{(k)}(t)$ is continuous for any natural number k and $t \in (0,\infty)$, i.e., $S(t) \in C^\infty(0,\infty)$.

It remains to find $\rho_0(t)$. Then we find all $\rho_k(t)$ with the help of formula (3.9). Replacing $1/\rho_0^2(t) = y_0(t)$, we reduce the Bernoulli equation (3.10) to the linear inhomogeneous equation

$$y_0' = -2a_0y_0 + \frac{2}{r_0^2}S(t), \tag{3.11}$$

where

$$y_0(0) = \frac{1}{r_0^2}. \tag{3.12}$$

We consider Equation (3.11). The function $y_0 = C\exp(-2a_0t)$ is a solution to the homogeneous equation $y_0' = -2a_0y_0$. For a partial solution to the inhomogeneous equation (3.11) we can take

$$y_p = \sum_{n=-\infty}^{\infty} \frac{r_n^2}{a_n r_0^2}\exp(-2\alpha_n t).$$

We emphasize that the series $S(t)$ contains no resonance terms since $a_n \neq 0$. Thus, the general solution to Equation (3.11) is the function

$$y_0(t) = C\exp(-2a_0 t) + \sum_{n=-\infty}^{\infty} \frac{r_n^2}{a_n r_0^2} \exp(-2\alpha_n t).$$

The choice of a suitable C is provided by the initial condition (3.12). As a result, a solution to the problem Cauchy (3.11), (3.12) has the form

$$y_0(t) = \frac{\exp(-2a_0 t)}{r_0^2}\Big(1 + \sum_{k=-\infty}^{\infty} \frac{r_k^2}{a_k}(\exp(2a_k t) - 1)\Big).$$

Finally, we have

$$\rho_0(t) = \frac{r_0 \exp(a_0 t)}{\sqrt{1 + Q(t)}},$$

where

$$Q(t) = \sum_{k=-\infty}^{\infty} \frac{r_k^2}{a_k}(\exp(2a_k t) - 1).$$

We note that $Q(t) > 0$ for $t > 0$ and $r_m \neq 0$, at least for one number m. Finally, (3.9) implies

$$\rho_n(t) = \frac{r_n \exp(a_n t)}{\sqrt{1 + Q(t)}}, \quad n \in Z. \tag{3.13}$$

Lemma 3.1 is proved. □

Remark 3.2. Formula (3.13) is obtained under the conditions $r_n \neq 0$ and $a_n \neq 0$ for all $n \in Z$. We show that formula (3.13) remains valid if the first condition is removed. If the second condition is removed, then the formula should be corrected since, in this case, resonance terms appear in the nonlinearity of Equation (3.11).

Let $r_m = 0$ for some $m \in Z_*$. Then it is obvious that $\rho_m(t) = 0$ for the corresponding number m. In this case,

$$Q(t) = Q_1(t) = \sum_{k \neq Z \setminus Z_*} \frac{r_k^2}{a_k}(\exp(2a_k t) - 1),$$

where $r_k \neq 0$. The proof of the modified formula (3.13) with $Q(t)$ replaced by $Q_1(t)$ is based on a version of the identity (3.9). In this case,

$$\rho_n(t) = \frac{r_n}{r_s}\rho_s(t)\exp(-\alpha_{n,s} t),$$

where $\alpha_{n,s} = a(n^2 - s^2) + (n^4 - s^4)$, i.e., $\alpha_{n,0} = \alpha_n$ and s is chosen in such a way that $r_s \neq 0$. The proof of the last formula is word-by-word repeats the proof of formula (3.9).

Let $a_p = 0$ for some integer p. Then $a_{-p} = 0$. We consider the case $a_m \neq a_p$ ($a_m \neq 0$) for all remaining $m \neq \pm p$. Then

$$\rho_{\pm p}(t) = \frac{r_{\pm p}}{\sqrt{1 + Q_2(t)}}, \quad \rho_m(t) = \frac{r_m \exp(a_m t)}{\sqrt{1 + Q_2(t)}},$$

where

$$Q_2(t) = \sum_{m \neq \pm p} \frac{r_m^2}{a_m}(\exp(2a_m t) - 1) + 2(r_p^2 + r_{-p}^2)t.$$

Now, formula for $Q_2(t)$ takes into account resonance terms.

Assume that $a_p = a_{-p} = 0$, $a_q = a_{-q} = 0$, $a_m \neq 0$ for the remaining m. Then

$$\rho_{\pm p}(t) = \frac{r_{\pm p}}{\sqrt{1 + Q_3(t)}}, \quad \rho_{\pm q}(t) = \frac{r_{\pm q}}{\sqrt{1 + Q_3(t)}}, \quad \rho_m(t) = \frac{r_m \exp(a_m t)}{\sqrt{1 + Q_3(t)}},$$

where

$$Q_3(t) = \sum_{m \neq \pm p, \pm q} \frac{r_m^2}{a_m}(\exp(2a_m t) - 1) + 2(r_p^2 + r_{-p}^2 + r_q^2 + r_{-q}^2)t.$$

Assume that $a_p = 0$ and $a_m \neq 0$ for $m \neq 0$. Then

$$\rho_0 = \frac{r_0}{\sqrt{1 + Q_4(t)}},$$

$$\rho_m(t) = \frac{r_m \exp(a_m t)}{\sqrt{1 + Q_4(t)}}, \quad m \neq 0,$$

$$Q_4(t) = \sum_{m \neq 0} \frac{r_m^2}{a_m}(\exp(2a_m t) - 1) + 2r_0^2 t.$$

We assume that $a_0 = 0$ and $a_{\pm q} = 0$ for some $q, -q \neq 0$. Then

$$\rho_0(t) = \frac{r_0}{\sqrt{1 + Q_5(t)}},$$

$$\rho_{\pm q}(t) = \frac{r_{\pm q}}{\sqrt{1 + Q_5(t)}},$$

$$\rho_m(t) = \frac{r_m \exp(a_m t)}{\sqrt{1 + Q_5(t)}},$$

where

$$Q_5(t) = \sum_{m \neq 0, \pm q} \frac{r_m^2}{a_m}(\exp(2a_m t) - 1) + 2(r_0^2 + r_q^2 + r_{-q}^2)t.$$

It is obvious that there are no other variants of the choice of n provided that $a_n = 0$ because the corresponding numbers for which $a_n = 0$ are found as roots (if exist) of the biquadratic equation $d - an^2 - n^4 = 0$.

We proceed by integrating the Cauchy problem (3.7), (3.8). It is obvious that

$$\varphi_n(t) = \varphi_n(t, \psi_n) = \psi_n + b_n t - c\chi_1(t) + g_2 \chi_2(t) n,$$

$$\chi_1(t) = \int_0^t V(\rho) ds, \quad \chi_2(t) = \int_0^t V_1(\rho) ds,$$

$$V(\rho) = \frac{1}{1 + Q(s)} \sum_{k=-\infty}^{\infty} \rho_k^2(s), \quad V_1(\rho) = \frac{1}{1 + Q(s)} \sum_{m=-\infty}^{\infty} m^2 \rho_m^2(s).$$

Moreover,

$$\chi_1(t) = \frac{1}{2}\ln(1+Q(s))\Big|_0^t = \frac{1}{2}\ln(1+Q(t)), \quad Q(0)=0.$$

The second integral with variable upper limit has the form

$$\chi_2(t) = \int_0^t V_1(s)ds = \int_0^t \frac{1}{1+Q(s)} \sum_{k=-\infty}^{\infty} r_k^2 k^2 \exp(2a_k s)ds.$$

Unlike the first integral, we cannot compute this integral in an explicit form. Nevertheless, the function $\chi_2(t)$ possesses the following properties:

(1) $\chi_2(t)$ is defined for all $t \geqslant 0$,

(2) $\chi_2(0) = 0$,

(3) $\chi_2(t) > 0$ if $t > 0$,

(4) $\chi_2(t)$ has derivatives of any order if $t > 0$.

Property (4) of $\chi_2(t)$ follows from properties of $Q(t)$ ($Q(t) > 0$ for $t > 0$ and $Q(t) \in C^\infty(0,\infty)$) and the inclusion

$$F(t) = \sum_{k=-\infty}^{\infty} r_k^2 k^2 \exp(2a_k t) \in C^\infty(0,\infty).$$

We can verify the last inclusion in the same way as $S(t) \in C^\infty(0,\infty)$. In particular, we use the convergence of the series $\sum\limits_{k=-\infty}^{\infty} r_k^2 k^2$.

The above constructions lead to the following assertion.

Lemma 3.2. *The solution to the Cauchy problem* (3.2), (3.3) *satisfies the equality*

$$u_n(t) = \frac{f_n}{\sqrt{1+Q(t)}} \exp(a_n t + i\chi(t,n)), \quad n \in Z, \tag{3.14}$$

where $\chi(t,n) = b_n t - c\ln(1+Q(t))/2 + g_2\chi_2(t)n$ *and* $f_n = r_n \exp(i\psi_n)$. *Furthermore,*

$$u(t,x) = \frac{1}{\sqrt{1+Q(t)}} \sum_{n=-\infty}^{\infty} f_n \exp(a_n t + i\chi(t,n) + inx). \tag{3.15}$$

The function defined by (3.15) *for* $t > 0$ *has continuous partial derivatives of any order.*

We note that the series (3.15) is obtained by substituting the right-hand side of (3.14) into the functional series (3.1) with (3.4) is taken into account. The proof of the convergence of the series (3.15), together with all its derivatives for $t > 0$, repeats the constructions used to justify the Fourier method applied to the analysis of the first boundary value problem for the heat equation.

4 Invariant Manifolds of Boundary Value Problem

The analysis of invariant manifolds is based on the study of the systems of differential equations (3.2) and (3.5). Both systems contain a coefficient a_n such that $a_{-n} = a_n$. Therefore, we assume that $n \geqslant 0$. We denote $Z_+ = \{0\} \cup N$, where N is the set of natural numbers. We divide the set Z_+ into three subsets

$Z_{+,1} = \{m \in Z_+ \mid a_m > 0\}$, $\quad Z_{+,0} = \{p \in Z_+ \mid a_p = 0,$ $\quad Z_{+,-1} = \{k \in Z_+ \mid a_k < 0\}$.

It is obvious that $Z_{+,-1} \neq \varnothing$ and $Z_{+,0}$, $Z_{+,1}$ contain at most finitely many elements and can be empty. The corresponding p are found as roots of the biquadratic equation $p^4 + ap^2 - d = 0$, which belong to Z_+ (Z).

Let $\rho_k(t)$ be a component of the system of differential equations (3.5) with number $k \in Z_{+,-1}$. Then $\lim\limits_{t\to\infty} \rho_k(t) = 0$. Indeed, in this case, $a_k < 0$. Hence

$$\lim_{t\to 0} \frac{r_k \exp(a_k t)}{\sqrt{1 + Q(t)}} = 0 \quad \Big(\rho_k = \frac{r_k \exp(a_k t)}{\sqrt{1 + Q(t)}}\Big).$$

A similar limit equality holds if $p \in Z_{+,0}$. In this case, $\rho_p' = -\rho_p V(\rho) < 0$. Consequently, the function $\rho_p(t)$ monotonically decreases and $\lim\limits_{t\to\infty} \rho_p(t) = 0$.

For $\eta = (\ldots, \eta_{-1}, \eta_0, \eta_1, \ldots)$ we denote by $S(\eta)$ the equilibrium state of the system (3.5). By the above, the equilibrium state can have nonzero coordinates η_m, η_{-m}, $m \in Z_{+,1}$. The remaining coordinates η_k, $k \notin Z_{+,1}$, vanish.

The set $Z_{+,1}$ can be divided into two subsets $\Lambda_1 = \{m \in Z_{+,1} \mid a_m \neq a_k$ for the remaining $k \in Z_{+,1}\}$ and $\Lambda_2 = \{m_1, m_2 \in Z_{+,1} \mid a_{m_1} = a_{m_2}, m_1 \neq m_2\}$. In this case, there are either four elements a_{m_1}, a_{-m_1}, a_{m_2}, a_{-m_2} such that $a_{m_1} = a_{-m_1} = a_{m_2} = a_{-m_2}$ or three elements a_0, a_m, a_{-m}. We emphasize that we do not exclude the case where Λ_1 or Λ_2 is empty. Finally, Λ_2, if not empty, contains two elements (m_1, m_2) or $(0, m)$. Owing to this fact, we can formulate the following assertion.

Lemma 4.1. 1. *If $\Lambda_1 \neq \varnothing$ and $m \in \Lambda_1$, then the system* (3.5) *has the family $S(m)$ of equilibrium states $\rho_m = \eta_m$, $\rho_{-m} = \eta_{-m}$, $\rho_n = 0$ if $n \neq \pm m$. Moreover, $\eta_m^2 + \eta_{-m}^2 = a_m$, $m \neq 0$. If $m = 0$, then $\rho_0^2 = a_0$, and we obtain the equilibrium state $S(0)$.*

2. *Let $\Lambda_2 \neq \varnothing$. The following variants can occur:*

(a) *if $m_1, m_2 \neq 0$ $(a_{-m_1} = a_{m_1}$, $a_{-m_2} = a_{m_2})$, then we have the following family of equilibrium states $S(m_1, m_2)$: $\rho_{m_1}(t) = \eta_{m_1}$, $\rho_{-m_1}(t) = \eta_{-m_1}$, $\rho_{m_2}(t) = \eta_{m_2}$, $\rho_{-m_2}(t) = \eta_{-m_2}$, $\rho_n = 0$ for $n \neq \pm m_1$, $\pm m_2$; moreover, $\eta_{m_1}^2 + \eta_{-m_1}^2 + \eta_{m_2}^2 + \eta_{-m_2}^2 = a_{m_1}$ $(= a_{m_2})$;*

(b) *if $m_1 = 0$, then we have the family of equilibrium states $S(0, m)$: $\rho_m = \eta_m$, $\rho_{-m} = \eta_{-m}$, $\rho_0(t) = \eta_0$, $m_2 = m$, for $k \neq 0$, $\pm m$; moreover, $\eta_m^2 + \eta_{-m}^2 + \eta_0^2 = a_0$ $(= a_m)$.*

To prove Lemma 4.1, we need to analyze the system of algebraic equations

$$\rho_k(a_k - V(\rho)) = 0, \quad k \in Z_{+,1},$$

$$V(\rho) = \sum_{k\in Z_{+,1}} \rho_k^2.$$

Consequently, for some $k \in Z_{+,1}$ two cases are possible: $\rho_k = 0$ and $a_k = V(\rho)$.

Let $\rho_k \neq 0$. Then the following variants are possible.

(i) $k = m$, where $m \in \Lambda_1$. Then $\rho_m = \eta_m$ and $\rho_{-m} = \eta_{-m}$, where $\eta_m^2 + \eta_{-m}^2 = a_m$.

(ii) $k = m_1$ or $k = m_2$ $(m_1 \neq m_2)$, where $m_1, m_2 \in \Lambda_2$; moreover, $m_1, m_2 \neq 0$. Then $\rho_{m_1} = \eta_{m_1}$, $\rho_{-m_1} = \eta_{-m_1}$, $\rho_{m_2} = \eta_{m_2}$, $\rho_{-m_2} = \eta_{-m_2}$, where $\eta_{m_1}^2 + \eta_{-m_1}^2 + \eta_{m_2}^2 + \eta_{-m_2}^2 = a_{m_1}$ $(a_{m_1} = a_{m_2})$.

(iii) $k = m$ $(m = m_2)$ or $k = 0$, where $m, 0 \in \Lambda_2$. Then $\rho_m = \eta_m$, $\rho_{-m} = \eta_{-m}$, $\rho_0 = \eta_0$, where $\eta_m^2 + \eta_{-m}^2 + \eta_0^2 = a_{m_1}$ $(a_0 = a_{m_1})$ and η_j are positive constants.

From Lemma 4.1 we obtain the following assertion.

Theorem 4.1. *Let the system of differential equations* (3.5) *have the equilibrium state* $S(0)$ $(\rho_0 = \eta_0 = \sqrt{a_0} > 0,\ \rho_j = 0, j \neq 0)$. *Then* $S(0)$ *corresponds to the limit cycle*

$$M_1(0): \ u_0(t) = w_0(t, \varphi_0) = \eta_0 \exp(i\omega_0 t + i\varphi_0),\ u_n(t) = 0,\ n \neq 0$$

of the system (3.2) *and also the cycle*

$$W_1(0): \ u_0(t, x) = w_0(t, \varphi_0)$$

of the boundary value problem (2.1), (2.2). *Here,* $\varphi_0 \in R$ *and* $\omega_0 = -ca_0$. *The one-parameter family of solutions* $W_1(0)$ *contains functions independent of* x.

Theorem 4.2. *Let the system* (3.5) *have the family of equilibrium states* $S(m)$. *Then this family is associated with the* 3*-dimensional invariant manifolds* $M_3(m)$, $W_3(m)$, $m \in Z_{+,1}$, *of the system of differential equations* (3.2) *and the boundary value problem* (2.1), (2.2)

$$\begin{aligned} M_3(m): \quad & u_m(t) = w_m(t, \varphi_m) = \eta_m \exp(i\omega_m t + i\varphi_m), \\ & u_{-m}(t) = w_{-m}(t, \varphi_{-m}) = \eta_{-m} \exp(i\omega_{-m} t + i\varphi_{-m}), \quad u_n(t) = 0, \quad n \neq \pm m, \\ W_3(m): \quad & u_m(t) = w_m(t, \varphi_m) \exp(imx) + w_{-m}(t, \varphi_{-m}) \exp(-imx). \end{aligned}$$

Here, $\varphi_m, \varphi_{-m} \in R$ *are arbitrary real constants,* $\omega_m = b_m - ca_m + g_2 m^3 a_m$, $\omega_{-m} = b_{-m} - ca_{-m} - g_2 m^3 a_{-m}$, $b_m = -c_1 m^2 - c_2 m^4 + g_1 m$, $b_{-m} = -c_1 m^2 - c_2 m^4 - g_1 m$, $a_m = a_{-m} = d - am^2 - m^4$, $\eta_m^2 + \eta_{-m}^2 = a_m$.

Theorem 4.3. *Let the system of differential equations* (3.5) *have the family of equilibrium states* $S(m_1, m_2)$. *Then this family is associated with the* 7*-dimensional invariant manifolds* $M_7(m_1, m_2)$ *and* $W_7(m_1, m_2)$ *of the system of differential equations* (3.2) *and the boundary value problem* (2.1), (2.2)

$$\begin{aligned} M_7(m_1, m_2): \quad & u_{m_1}(t) = w_{m_1}(t, \varphi_{m_1}) = \eta_{m_1} \exp(i\omega_{m_1} t + i\varphi_{m_1}), \\ & u_{-m_1}(t) = w_{-m_1}(t, \varphi_{-m_1}) = \eta_{-m_1} \exp(i\omega_{-m_1} t + i\varphi_{-m_1}), \\ & u_{m_2}(t) = w_{m_2}(t, \varphi_{m_2}) = \eta_{m_2} \exp(i\omega_{m_2} t + i\varphi_{m_2}), \\ & u_{-m_2}(t) = w_{-m_2}(t, \varphi_{-m_2}) = \eta_{-m_2} \exp(i\omega_{-m_2} t + i\varphi_{-m_2}), \\ W_7(m_1, m_2): \quad & u(t, x) = w_{m_1}(t, \varphi_{m_1}) \exp(im_1 x) + w_{-m_1}(t, \varphi_{-m_1}) \exp(-m_1 x) \\ & \qquad + w_{m_2}(t, \varphi_{m_2}) \exp(im_2 x) + w_{-m_2}(t, \varphi_{-m_2}) \exp(-im_2 x). \end{aligned}$$

Here, φ_{m_1}, φ_{-m_1}, φ_{m_2}, φ_{-m_2} *are arbitrary real constants,* $\omega_{m_1} = b_{m_1} - ca_{m_1} + g_2 m_1 \Omega$, $\omega_{-m_1} = b_{-m_1} - ca_{-m_1} - g_2 m_1 \Omega$, $\omega_{m_2} = b_{m_2} - ca_{m_2} + g_2 m_2 \Omega$, $\omega_{-m_2} = b_{-m_2} - ca_{-m_2} - g_2 m_2 \Omega$, $\Omega = m_1^2(\eta_{m_1}^2 + \eta_{-m_1}^2) + m_2^2(\eta_{m_2}^2 + \eta_{-m_2}^2)$, $a_{m_1} = a_{-m_1} = a_{m_2} = a_{-m_2}$, $\eta_{m_1}^2 + \eta_{-m_1}^2 + \eta_{m_2}^2 + \eta_{-m_2}^2 = a_{m_1} = a_{m_2}$.

Theorem 4.4. *Let the system* (3.5) *have the family of equilibrium states* $S(0, m)$. *Then this family is associated with the* 5*-dimensional invariant manifolds* $M_5(0, m)$ *and* $W_5(0, m)$ *of the system of differential equations* (3.2) *and the boundary value problem* (2.1), (2.2)

$$M_5(0,m): \quad u_0(t) = w_0(t, \varphi_0) = \eta_0 \exp(i\omega_0 t + i\varphi_0), \ u_m(t) = w_m(t, \varphi_m) = \eta_m \exp(i\omega_m t + i\varphi_m),$$

$$u_{-m}(t) = w_{-m}(t, \varphi_{-m}) = \eta_{-m} \exp(i\omega_{-m} t + i\varphi_{-m}),$$

$$W_5(0,m): \quad u(t,x) = w_0(t, \varphi_0) + w_m(t, \varphi_m)\exp(imx) + w_{-m}(t, \varphi_{-m})\exp(-imx).$$

Here, φ_0, φ_m, φ_{-m} *are arbitrary real constants,* $\omega_0 = -ca_0$, $\omega_m = b_m - ca_m + g_2 m\Omega_0$, $\omega_{-m} = b_{-m} - ca_{-m} - g_2 m\Omega_0$, $\Omega_0 = m^2(\eta_m^2 + \eta_{-m}^2)$, $\eta_0^2 + \eta_m^2 + \eta_{-m}^2 = a_0 = a_m$.

Now, we consider invariant manifolds of some other form. We set

$$Z_1 = \{n \in Z \mid a_n > 0\}, \quad Z_0 = \{n \in Z \mid a_n = 0\}, \quad Z_{-1} = \{n \in Z \mid a_n < 0\}$$

and denote by $H^1_{2,+}$ the subspace of H^1_2 consisting of functions $f(x)$ such that

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp(-inx) dx = 0, \quad n \in Z_0 \cup Z_{-1}, \tag{4.1}$$

and by $H^1_{2,0}$ the subspace of H^1_2 consisting of functions $f(x) \in H^1_2$ such that the equality (4.1) holds for $n \in Z_1 \cup Z_{-1}$. Finally, $f(x) \in H^1_{2,-}$ if f satisfies (4.1) for $n \in Z_1 \cup Z_0$.

We emphasize that the subspace $H^1_{2,0}$ can be empty or not. Thus, $H^1_{2,0} \neq \varnothing$, if $d = -m^2k^2$, $a = -(m^2 + k^2)$, i.e., the equation $d - a\xi^2 - \xi^4 = 0$ has the roots $\xi = \pm m$, $\xi = \pm k$. It is natural that $H^1_{2,+}$ can be empty if $a_m \leqslant 0$ for all m.

Let $W_{in} = H^1_{2,+} \cup H^1_{2,0}$. This is a linear subspace of H^1_2. The linear subspaces $H^1_{2,+}$, $H^1_{2,0}$, $H^1_{2,-}$, W_{in} possess the following properties:

(1) $H^1_{2,+}$, $H^1_{2,0}$, $H^1_{2,-}$, W_{in} are invariant with respect to solutions to the boundary value problem (2.1), (2.2),

(2) $H^1_{2,+}$, $H^1_{2,0}$, $H^1_{2,-}$ are orthogonal and W_{in} is orthogonal to $H^1_{2,-}$ in the sense of the inner product in the complex Hilbert space $L_2(0, 2\pi)$:

$$(f, g) = \int_0^{2\pi} f(x)\overline{g}(x) dx, \quad f(x), g(x) \in L_2(0, 2\pi),$$

(3) $H^1_{2,+}$, $H^1_{2,0}$, W_{in} are finite-dimensional, and $H^1_{2,-}$ is infinite-dimensional.

Theorem 4.5. *The linear subspace* W_{in} *is an inertial manifold of the boundary value problem* (2.1), (2.2).

Proof. Indeed, W_{in} is invariant with respect to solutions to the boundary value problem (2.1), (2.2). Let $f(x) \in W_{in} \cap H^1_2$. Then

$$f(x) = \sum_{n \in Z_1 \cup Z_0} f_n \exp(inx)$$

and the corresponding solution is represented as (cf. (3.1))

$$u(t,x) = \sum_{n\in Z_1\cup Z_0} u_n(t)\exp(inx).$$

It is obvious that W_{in} is finite-dimensional since the set $Z_1 \cup Z_0$ can contain only finitely many elements with $a_n \geqslant 0$.

Let $u(t,x)$ be a solution to the initial-boundary value problem (2.1)–(2.3) for arbitrary $f(x) \in H_2^1$. Then $u(t,x)$ is defined by (3.1). Since W_{in} and $H^1_{2,-}$ are orthogonal, we note that the squared distance from a point $u(t,x)$ of the phase space H_2^1 to W_{in} is defined by

$$d^2(t) = \left\| \sum_{k\in Z_{-1}} u_k(t)\exp(ikx)\right\|^2_{H_2^1} = 2\pi \sum_{k\in Z_{-1}} |u_k(t)|^2(1+k^2)$$

$$= 2\pi \sum_{k\in Z_-} \frac{r_k^2\exp(2a_kt)}{1+Q(t)}(1+k^2) \leqslant 2\pi\exp(-2\nu t)\sum_{k\in Z_{-1}} r_k^2(1+k^2) \leqslant 2\pi||f||^2_{H_2^1}\exp(-2\nu t).$$

To prove the last inequality, we used the equality $|u_k(t)| = r_k\exp(a_kt)/\sqrt{1+Q(t)}$, the inequality $Q(t) \geqslant 0$, and the equality

$$||f(x)||^2_{H_2^1} = \sum_{k=-\infty}^{\infty} (1+k^2)|f_k|^2.$$

Thus,

$$d(t) \leqslant M\exp(-\nu t), M = \sqrt{2\pi}||f||_{H_2^1}. \tag{4.2}$$

Consequently, the distance from the solution $u(t,x)$ to W_{in} is decreasing at exponential rate. This means that W_{in} is an inertial manifold in the sense of the definition in [10]. □

We make the following remark. The linear subspace $H^1_{2,+}$ is also invariant and finite-dimensional. The solutions which does not belong to this subspace approach $H^1_{2,+}$, but the approximation rate is not necessarily exponential. Let us discuss this fact. Let $u(t,x)$ be a solution to the initial-boundary value problem (2.1)–(2.3). By the orthogonality of $H^1_{2,+}$, $H^1_{2,0}$, $H^1_{2,-}$ the squared distance from points of the phase spaces $u(t,x)$ to $H^1_{2,+}$ is equal to $d^2(t) = d_1^2(t) + d_2^2(t)$, where

$$d_1^2(t) = 2\pi\sum_{k\in Z_-} |u_k(t)|^2(1+k^2),$$

$$d_2^2(t) = 2\pi\sum_{p\in Z_0} |u_p(t)|^2(1+p^2).$$

For $d_1^2(t)$ the estimate (4.2) holds. A different result holds for $d_2(t)$. For the sake of definiteness we assume that Z_0 contains four elements p_1, $-p_1$, p_2, $-p_2$. Then

$$d_2^2(t) = 2\pi\left((1+p_1^2)\frac{r^2_{p_1}+r^2_{-p_1}}{1+Q_3(t)} + (1+p_2^2)\frac{r^2_{p_2}+r^2_{-p_2}}{1+Q_3(t)}\right)$$

(recall that $a_{p_1} = a_{-p_1} = a_{p_2} = a_{-p_2} = 0$). Moreover,

$$Q_3(t) \geqslant 2(r^2_{p_1} + r^2_{-p_1} + r^2_{p_2} + r^2_{-p_2})t.$$

Therefore,

$$d_2^2(t) \leqslant 2\pi \frac{(1+p_1^2)(r_{p_1}^2 + r_{-p_1}^2) + (1+p_2^2)(r_{p_2}^2 + r_{-p_2}^2)}{1 + 2(r_{p_1}^2 + r_{-p_1}^2 + r_{p_2}^2 + r_{-p_2}^2)t}$$

$$\leqslant M_2 \frac{r_{p_1}^2 + r_{-p_1}^2 + r_{p_2}^2 + r_{-p_2}^2}{1 + 2(r_{p_1}^2 + r_{-p_1}^2 + r_{p_2}^2 + r_{-p_2}^2)t}, M_2 = 2\pi \max(1+p_1^2, 1+p_2^2).$$

Consequently, $\lim\limits_{t\to\infty} d_2(t) = 0$, i.e., $\lim\limits_{t\to\infty} d(t) = 0$, where $d(t) = \sqrt{d_1^2(t) + d_2^2(t)}$. Thus, all solutions to the boundary value problem (2.1), (2.2) approach $H^1_{2,+}$ with time. At the same time,

$$d^2(t) \leqslant M^2 \exp(-2\nu t) + M_2 \frac{R}{1+2Rt}, R = r_{p_1}^2 + r_{-p_1}^2 + r_{p_2}^2 + r_{-p_2}^2.$$

We assume that $t \geqslant 1$ and $R \neq 0$. Then

$$d(t) \leqslant \frac{M_0}{\sqrt{t}}, \tag{4.3}$$

where M_0 is a positive constant. To prove the inequality (4.3), we used the inequalities

$$\frac{R}{1+2Rt} \leqslant \frac{1}{2t}, \quad \exp(-2\nu t) \leqslant \frac{M_3}{t}, \quad M_3 > 0.$$

The estimate (4.3) shows that, in the case of general position, the solutions to the boundary value problem (2.1), (2.2) converge to $H^1_{2,+}$, if this linear subspace exists, at the rate $1/\sqrt{t}$, but not exponentially, i.e., $H^1_{2,+}$ is not inertial in the sense of the definition in [10].

5 Global Attractor

We consider the invariant manifold $A = \{0\} \cup A_1 \cup A_2 \cup A_3 \cup A_4$, where $\{0\}$ is the zero equilibrium state, $A_1 = W_1(0)$, $A_2 = \bigcup\limits_{m \in Z_{+,1}} W_3(m)$ is the union of all three-dimensional invariant manifolds of the boundary value problem (2.1), (2.2) mentioned in Theorem 4.2. Finally, $A_3 = W_5(0, m)$, $A_4 = W_5(m_1, m_2)$ (cf. Theorems 4.3 and 4.4). In the case of general position, the invariant manifolds A_3 and A_4 are absent. In an arbitrary case, for the existence of these manifolds it is necessary to choose the coefficients a and d; otherwise, the component A_3 (A_4 or $A_3 \cup A_4$) is empty. The case of absence of nontrivial components is also included, and then $A = \{0\}$. This case occurs if $a_k = d - ak^2 - k^4 \leqslant 0$ for all $k \in Z$.

Theorem 5.1. *Assume that $u(t,x)$ is a solution to the initial-boundary value problem* (2.1)–(2.3) *and $u(0,x) = f(x) \in H^1_2$. Then $u(t,x)$ approaches A with time. Moreover, every solution approaches one of the components of A. The choice of the component depends on the choice of $f(x)$ from the initial condition* (2.3).

Before to prove Theorem 5.1, we make some remarks. Let $Z_{+,1} \neq \varnothing$. We first consider the case of general position, i.e., assume that there exists a set of nonnegative numbers m such that $m_1, \dots, m_s \in Z_{+,1}$ ($a_{m_j} > 0$) and $a_{m_j} \neq a_{m_p}$ for any m_j, m_p ($m_j \neq m_p$). In this case, there exists a family of invariant sets $W_3(m_j)$ ($\dim W_3(m_j) = 3$ if $m_j \neq 0$) and $W_1(0)$ ($\dim W_1(0) = 1$ if $m_j = 0$).

Let the sequence $\{a_m\}$, $m \in Z_{+,1}$, be enumerated in such a way that $a_{m_1} > a_{m_2} > \ldots > a_{m_s}$, and let $f(x) \in H_2^1$ be such that $|f_{m_1}|^2 + |f_{-m_1}|^2 \neq 0$ ($m_1 = 0$ implies $f_0 \neq 0$). Then the solution to the boundary value problem (2.1), (2.2) satisfying the initial condition $u(0, x) = f(x)$ converges to $W_3(m_1)$ (or $W_1(0)$ if $m_1 = 0$) in the H_2^1-norm. If $f_{m_1} = f_{-m_1} = 0$ ($f_0 = 0$), but $|f_{m_2}|^2 + |f_{-m_2}|^2 \neq 0$ ($f_0 \neq 0, m_2 = 0$), then the solution $u(t, x)$ approaches $W_3(m_2)$ and so on. Finally, assume that $f_p = 0$ for all $p = Z_{+,1}$ and $f_{-p} = 0$. Then the solution approaches the zero equilibrium state.

If $a_{m_1} = a_{m_2}$, then the above should be corrected as follows: $u(t, x)$ approaches $W_7(m_1, m_2)$ ($W_5(0, m)$, $m = m_2$, if $m_1 = 0$).

The proof of Theorem 5.1 is based on the study of the behavior of solutions to the system of ordinary differential equations (3.5). In the case of general position ($\Lambda_2 = 0$), we divide (3.5) into two groups

$$\rho'_{m_j} = a_{m_j}\rho_{m_j} - \rho_{m_j} V(\rho), \quad j = 1, \ldots, s, \tag{5.1}$$

$$\rho'_k = a_k \rho_k - \rho_k V(\rho), \tag{5.2}$$

where $m_j \in Z_{+,1}$, $k \in Z_{+,0} \cup Z_{+,-1}$. Moreover, we can enumerate in (5.1) in such a way that $a_{m_1} > a_{m_2} > \ldots > a_{m_s}$. In this case, the following assertion holds.

Lemma 5.1. *Let $r_{m_1}^2 + r_{-m_1}^2 \neq 0$. Then the solutions to the system* (3.5) *converge to* $S(m_1)$. *If $r_{m_1}^2 + r_{-m_1}^2 = 0$, but $r_{m_2}^2 + r_{-m_2}^2 \neq 0$, then the solutions approach $S(m_2)$ and so on.*

In the case $m_j = 0$, we replace $S(m_j)$ with $S(0)$.

Proof of Lemma 5.1. From (3.13) it follows that $\lim\limits_{t\to\infty} \rho_k(t) = 0$ for $k \in Z_{+,0} \cup Z_{+,-1}$ since $a_k \leqslant 0$. Thus, all components of (5.2) converge to zero. It remains to study the behavior of $\rho_{m_j}(t)$, where ρ_{m_j} are components of (5.1). By (3.13),

$$\rho_{m_j}(t) = \frac{r_{m_j} \exp(a_{m_j} t)}{\sqrt{1 + Q_+(t)}},$$

$$Q_+(t) = \sum_{m_j \in Z_{+,1}} \frac{r_{m_j}^2 + r_{-m_j}^2}{a_{m_j}} (\exp(2a_{m_j} t) - 1) + \frac{r_0^2}{a_0} (\exp(2a_0 t) - 1).$$

Here, the last term is absent if $a_0 \leqslant 0$ ($0 \notin Z_{+,1}$).

Let $m_1 \neq 0$. Then it is easy to verify that $\lim\limits_{t\to\infty} \rho_n(t) = 0$ if $n \neq \pm m_1$. At the same time, for $n = \pm m_1$

$$\lim_{t\to\infty} \rho_{m_1}(t) = \frac{r_{m_1}\sqrt{a_{m_1}}}{\sqrt{r_{m_1}^2 + r_{-m_1}^2}},$$

$$\lim_{t\to\infty} \rho_{-m_1}(t) = \frac{r_{-m_1}\sqrt{a_{-m_1}}}{\sqrt{r_{m_1}^2 + r_{-m_1}^2}}.$$

Since $a_{m_1} = a_{-m_1}$, it is easy to obtain the identities

$$\Big(\frac{r_{m_1}\sqrt{a_{m_1}}}{\sqrt{r_{m_1}^2 + r_{-m_1}^2}}\Big)^2 + \Big(\frac{r_{-m_1}\sqrt{a_{-m_1}}}{\sqrt{r_{m_1}^2 + r_{-m_1}^2}}\Big)^2 = a_{m_1}.$$

Consequently, if $r_{m_1}^2 + r_{-m_1}^2 \neq 0$, then the solutions to the system (3.5) converge to $S(m_1)$. If it turns out that $m_1 = 0$, then

$$\lim_{t\to\infty} \frac{r_0 \exp(a_0 t)}{\sqrt{1 + Q_+(t)}} = a_0,$$

i.e., the solutions converge to $S(0)$.

Let $r_{m_1}^2 + r_{-m_1}^2 = 0$ (or $r_0 = 0$ if $m_1 = 0$). In this case, we repeat the above procedure starting with m_2. Then the solutions to the system (3.5) approach $S(m_2)$ and so on. □

A special case appears if $a_{m_1} = a_{m_2}$ ($m_2 = 0$ implies $a_{m_1} = a_0$), $\Lambda_2 \neq \varnothing$.

Lemma 5.2. *Let $r_{m_1}^2 + r_{-m_1}^2 + r_{m_2}^2 + r_{-m_2}^2 \neq 0$. Then the solutions to the system* (3.5) *converge to $S(m_1, m_2)$. If $m_1 = 0$, then the solutions to the system* (3.5) *approach $S(0, m)$, $m = m_2$.*

Proof. The arguments partially repeat some fragments of the proof of Lemma 5.1:

$$\lim_{t\to\infty} (\rho_{m_1}^2 + \rho_{-m_1}^2 + \rho_{m_2}^2 + \rho_{-m_2}^2) = a_{m_1} (= a_{m_2})$$

or for $m_1 = 0$ ($r_{m_2}^2 + r_{-m_2}^2 + r_0^2 \neq 0$)

$$\lim_{t\to\infty} (\rho_{m_2}^2 + \rho_{-m_2}^2 + \rho_0^2) = a_{m_1} \quad (a_0 = a_{m_2})$$

and $\lim\limits_{t\to\infty} \rho_n(t) = 0$ for the remaining n ($n \neq \pm m_1, \pm m_2$). □

Now, we can show that the solution $u(t,x)$ to the boundary value problem (2.1), (2.2) satisfying the initial condition $u(0,x) = f(x)$, approaches one of the invariant manifolds $W_3(m_j)$ ($W_1(0)$, $W_7(m_j, m_{j+1})$, $W_5(0, m_j)$) corresponding to the family of equilibrium states of the system (3.5) to which the solution to this system with the initial condition $\rho_k(0) = r_k = |f_k|$ approaches; here, $\{f_k\}$ are the Fourier coefficients of $f(x)$:

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(x) \exp(-ikx) dx.$$

The verification of this assertion is reduced to calculating the limits. Similar calculations were made in [11], where a similar problem was considered for the more traditional version of the nonlocal Ginzburg–Landau equation

$$u_t = u - (1 + ic) u V(u) + (a + ib) u_{xx}, V(u) = \frac{1}{2\pi} \int_0^{2\pi} |u|^2 dx.$$

We proceed by analyzing the dynamics of solutions in the global attractor A. In the trivial case, where A consists only of the zero equilibrium state, it is obvious that the solution is stable and even asymptotically stable.

Assume that the attractor A has at least one nonzero component $W_1(0)$, $W_3(m_j)$, $W_5(0, m_j)$, $W_7(m_j, m_{j+1})$. The remaining nontrivial components of A, if exist, are formed, in the case of general position, by almost t-periodic solutions. Thus, $W_3(m_j)$, $m_j \neq 0$, are formed by solutions possessing two basis frequencies ω_{m_j} and ω_{-m_j}. The solutions in $W_5(0, m_j)$ have three basis frequencies ω_0, ω_{m_j}, ω_{-m_j}, whereas the solutions in $W_7(m_j, m_{j+1})$ have four basis frequencies ω_{m_j}, ω_{-m_j}, $\omega_{m_{j+1}}$, $\omega_{-m_{j+1}}$. We emphasize that for $W_3(m_j)$ we mean that m_j does not have to be equal to just m_1 and the same is true for the manifolds $W_7(m_j, m_{j+1})$ and $W_5(0, m_j)$.

We consider, for example, the solutions constituting $W_3(m_j)$. Then ω_{m_j} and ω_{-m_j} can be calculated by

$$\omega_{m_j} = b_{m_j} - ca_{m_j} + g_2 m_j^3 a_{m_j},$$

$$\omega_{-m_j} = b_{-m_j} - ca_{-m_j} - g_2 m_j^3 a_{-m_j}.$$

in the case of general position, the numbers ω_{m_j} and ω_{-m_j} are rationally incommensurable and, consequently, the solutions in $W_3(m_j)$ are not necessarily periodic in t.

We consider a more meaningful question about invariant manifolds entering the global attractor A as its components. We say that a component is *dominating* if it corresponds to the number k or numbers k_1, k_2, $k_1 \neq k_2$, such that $a_k > a_n$, $a_{k_1} = a_{k_2} > a_n$, where $n \neq k$, $n \neq k_1, k_2$. In the case of general position $k = m_1$ and the above-introduced enumeration, we have $a_{m_1} > a_{m_2} > \ldots > a_{m_s} > \ldots$. If $a_{m_1} = a_{m_2}$, $m_1 \neq m_2$, then $k_1 = m_1$ and $k_2 = m_2$. In what follows, we will take into account singularity of the new enumeration. Finally, if the attractor A has only one nontrivial component, then this component is regarded as dominating.

Theorem 5.2. *All solutions in nondominating components are Lyapunov unstable. These solutions cannot be orbitally stable.*

Proof. Let $W_3(m_2)$ be a nondominating component, where $m_2 \neq m_1$, and let $W_3(m_1)$ be a dominating component. Consider the solution $u_{m_2}(t, x) \in W_3(m_2)$. Then

$$f_{m_2}(x) = u_{m_2}(0, x) = \eta_{m_2} \exp(im_2 x) + \eta_{-m_2} \exp(-im_2 x),$$

$$\eta_{m_2}^2 + \eta_{-m_2}^2 = a_{m_2}.$$

Let $f_{m_2,\delta}(x) = f_{m_2}(x) + \delta \exp(im_1 x)$. If δ is small, then the functions $f_{m_2,\delta}(x)$ and $f_{m_2}(x)$ are close in the metric of the phase space H_2^1. At the same time, $f_{m_2,\delta}(x)$ has nonzero Fourier coefficient $f_{m_1,\delta}$, i.e., $r_{m_1} = \delta \neq 0$ $(r_{m_1}^2 + r_{-m_1}^2 \neq 0)$. Thus, the solution $u_{m_2,\delta}(t, x)$ satisfying the equality $u_{m_2,\delta}(0, x) = f_2(x, \delta)$ leaves a neighborhood of $W_3(m_2)$ and approaches $W_3(m_1)$. Moreover, the distance between $W_3(m_1)$ and $W_3(m_2)$ is positive. This means that any solution in $W_3(m_2)$ is unstable. The invariant manifold $W_3(m_2)$ is unstable either. The remaining variants of the choice of dominating and nondominating sets are handled in a similar way. □

It remains to discuss the stability of solutions belonging to a dominating component.

Theorem 5.3. *Let $W_3(m_1)$ be a dominating component. Then all solutions in $W_3(m_1)$ are stable. Let $W_7(m_1, m_2)$ be a dominating component* (*or $W_5(0, m)$, $m = m_2$, if $m_1 = 0$*), *and let $g_2 \neq 0$. Then all solutions to the boundary value problem* (2.1), (2.2) *lying in $W_7(m_1, m_2)$* ($W_5(0, m)$) *are unstable* (*in the case $g_2 = 0$, they are stable*).

Proof. We first prove the assertion about the stability of solutions in $W_3(m_1)$. In this part, we write m instead of m_1; for example, $W_3(m)$ instead of $W_3(m_1)$. We choose a solution $w(t, x) \in W_3(m)$ which, as was mentioned in Section 4, can be written as

$$w(t, x) = w_m(t) \exp(imx) + w_{-m}(t) \exp(-imx), \tag{5.3}$$

where $w_{\pm m}(t) = g_{\pm m} \exp(i\omega_{\pm m} t)$, $g_{\pm m}$ is the Fourier coefficient of $g(x) = w(0, x)$ and $\omega_{\pm m}$ are given in Section 4.

At the first step of the proof of the stability of $w(t, x)$, we consider solutions to the boundary value problem (2.1), (2.2) lying in the manifold $W_3(m)$,

$$v(t, x) = v_m(t) \exp(imx) + v_{-m} \exp(-imx), \tag{5.4}$$

where $v_{\pm m}(t) = h_{\pm m} \exp(i\omega_{\pm m} t)$ and $h_{\pm m}$ is the Fourier coefficient of $h(x) = v(0, x)$.

We recall that $g(x), h(x) \in H_2^1$ and the closeness of these functions means that $||h(x) - g(x)||_{H_2^1} < \delta$ or $|h_m - g_m|^2 + |h_{-m} - g_{-m}|^2 < \delta^2/(2\pi(1 + m^2))$. At the same time,

$$|v_{\pm m}(t) - w_{\pm m}(t)| = |(h_{\pm m} - g_{\pm m}) \exp(i\omega_{\pm m} t)| = |h_{\pm m} - g_{\pm m}|.$$

Consequently,

$$||v(t, x) - w(t, x)||^2_{H_2^1} = 2\pi(1 + m^2)(|v_m(t) - w_m(t)|^2 + |v_{-m}(t) - w_{-m}(t)|^2)$$
$$= ||h(x) - g(x)||^2_{H_2^1} \leqslant \delta^2.$$

Let $u(t, x)$ be a solution to the boundary value problem (2.1), (2.3) that does not belong to $W_3(m)$. Then

$$u(t, x) = u_+(t, x) + u_-(t, x), \quad u(0, x) = f(x) = f_+(x) + f_-(x),$$
$$u_+(t, x) = u_m(t) \exp(imx) + u_{-m}(t) \exp(-imx), \quad f_+(x) = f_m \exp(imx) + f_{-m} \exp(-imx),$$
$$u_-(t, x) = \sum_{k \neq \pm m} u_k(t) \exp(ikx), \quad f_-(x) = \sum_{k \neq \pm m} f_k \exp(ikx).$$

We emphasize that for all $t \geqslant 0$ the solutions $u_+(t, x)$ and $u_-(t, x)$ belong to the subspaces $H^1_{2,+}$ and $H^1_{2,-}$ of the space H_2^1, which are mutually orthogonal in the sense of the inner product in the Hilbert space $L_2(0, 2\pi)$ and also $W_2^1(0, 2\pi)$. Therefore,

$$||f(x) - g(x)||^2_{H_2^1} = \Delta_+^2 + \Delta_-^2,$$

where

$$\Delta_+^2 = 2\pi(1 + m^2)(|f_m - g_m|^2 + |f_{-m} - g_{-m}|^2),$$
$$\Delta_-^2 = \sum_{k \neq \pm m} (1 + k^2)|f_k|^2.$$

Let $||f(x) - g(x)||_{H_2^1} < \delta$. Then $\Delta_+, \Delta_- < \delta$.

From the formula for $u_k(t)$ in Section 3 we find

$$|u_k(t)| \leqslant M_- \exp(-\nu_0 t)|f_k|, \quad \nu_0 = \max_{k \neq \pm m} (a_m - a_k), \quad M_- \leqslant 2.$$

Consequently, $||u_-(t, x)||_{H_2^1} \leqslant 2 \exp(-\nu_0 t)\delta$, if $t \geqslant 0$. We note that $\nu_0 > 0$.

For an auxiliary solution $v(t, x)$ we take the following solution to the boundary value problem (2.1), (2.2):

$$v(t, x) = v_m(t) \exp(imx) + v_{-m}(t) \exp(-imx),$$

where

$$v_{\pm m}(t) = \widetilde{f}_{\pm m} \exp(i\omega_{\pm m} t), \quad \widetilde{f}_{\pm m} = \frac{f_{\pm m}\sqrt{a_{\pm m}}}{\sqrt{|f_m|^2 + |f_{-m}|^2}}.$$

By assumption, $||f(x) - g(x)||_{H_2^1} < \delta$. Consequently, $||v(t,x) - w(t,x)||_{H_2^1} < \delta$ for all $t \geqslant 0$.

Using the explicit formulas for $u_n(t)$ in Section 3, one can verify that $||u_+(t,x) - v(t,x)|| \leqslant M_+||f_+(x) - g(x)||_{H_2^1}$ or $||u_+(t,x) - v(t,x)||_{H_2^1} \leqslant M_+\delta$, where M_+ is a positive constant independent of δ. In this part of the proof, we use the inequality $||f(x) - g(x)||_{H_2^1} < \delta$ and the closeness of $\widetilde{f}_m$ to $f_{\pm m}$ (cf. Lemma 5.1).

Thus, we obtain the inequalities

$$\begin{aligned} ||u(t,x) - w(t,x)||_{H_2^1} &\leqslant ||v(t,x) - w(t,x)||_{H_2^1} + ||u_+(t,x) - v(t,x)||_{H_2^1} + ||u_-(t,x)||_{H_2^1} \\ &< (\delta + M_+\delta + 2\delta) \leqslant \varepsilon, \end{aligned}$$

where $\delta \leqslant \varepsilon/(3 + M_+)$, which is required.

We note that the stability is established in the H_2^1-norm. It is essential that, in the case of general position, all solutions in $W_3(m)(W_3(m_1))$ are almost periodic functions of t with the same basis frequencies ω_m and ω_{-m} for all solutions in $W_3(m)(W_3(m_1))$.

Now, we pass to the second part of Theorem 5.3 dealing with solutions in $W_7(m_1, m_2)$ or $W_5(0, m)$. We begin with the case where $W_7(m_1, m_2)$ is a dominating invariant manifold.

Let $g_2 \neq 0$. We choose two solutions to the boundary value problem (2.1), (2.2) in $W_7(m_1, m_2)$ (cf. Section 4)

$$\begin{aligned} u(t,x,p) &= w_{m_1,p}(t,p)\exp(im_1x) + w_{-m_1,p}(t,p)\exp(-im_1x) \\ &\quad + w_{m_2,p}(t,p)\exp(im_2x) + w_{-m_2,p}(t,p)\exp(-im_2x), \end{aligned}$$

where $p = 1, 2$. Finally,

$$\begin{aligned} w_{m_1}(t,p) &= f_{m_1}(p)\exp(i\omega_{m_1}(p)t), \quad f_{m_1}(p) = \eta_{m_1}(p)\exp(i\psi_{m_1}(p)), \\ w_{-m_1}(t,p) &= f_{-m_1}(p)\exp(i\omega_{-m_1}(p)t), \quad f_{-m_1}(p) = \eta_{-m_1}(p)\exp(i\psi_{-m_1}(p)), \\ w_{m_2}(t,p) &= f_{m_2}(p)\exp(i\omega_{m_2}(p)t), \quad f_{m_2}(p) = \eta_{m_2}(p)\exp(i\psi_{m_2}(p)), \\ w_{-m_2}(t,p) &= f_{-m_2}(p)\exp(i\omega_{-m_2}(p)t), \quad f_{-m_2}(p) = \eta_{-m_2}(p)\exp(i\psi_{-m_2}(p)). \end{aligned}$$

The closeness of the initial conditions for the solutions labeled by 1 and 2 means the smallness of the sum

$$\begin{aligned} &2\pi[(1 + m_1^2)|f_{m_1}(1) - f_{m_1}(2)|^2 + (1 + m_1^2)|f_{-m_1}(1) - f_{-m_1}(2)|^2 \\ &+ (1 + m_2^2)|f_{m_2}(1) - f_{m_2}(2)|^2 + (1 + m_2^2)|f_{-m_2}(1) - f_{-m_2}(2)|^2] < \delta^2, \end{aligned}$$

where δ is an arbitrarily small positive constant.

Let us establish the existence of $\{t_n\}$ such that

1) $\lim\limits_{n\to\infty} t_n = +\infty$,

2) the following inequality holds:

$$2\pi((1+m_1^2)(|w_{m_1}(t_n,1)-w_{m_1}(t_n,2)|^2+|w_{-m_1}(t_n,1)-w_{-m_1}(t_n,2)|^2)$$
$$+(1+m_2^2)(|w_{m_2}(t_n,1)-w_{m_2}(t_n,2)|^2+|w_{-m_2}(t_n,1)-w_{-m_2}(t_n,2)|^2))\geqslant \varepsilon_0^2, \qquad (5.5)$$

where ε_0 is independent of the choice of δ.

Thereby we prove the Lyapunov unstability of the solutions under consideration.

Making some transformations, we find

$$\omega_{m_1}(p)=\sigma_{m_1}+g_2m_1(m_2^2-m_1^2)\Theta_1(p),$$
$$\sigma_{m_1}=-c_1m_1^2-c_2m_1^4-ca_{m_1}+g_1m_1+g_2m_1^3a_{m_1},$$
$$\omega_{-m_1}(p)=\sigma_{-m_1}-g_2m_1(m_2^2-m_1^2)\Theta_1(p),$$
$$\sigma_{-m_1}=-c_1m_1^2-c_2m_1^4-ca_{-m_1}-g_1m_1-g_2m_1^3a_{m_1},$$
$$\omega_{m_2}(p)=\sigma_{m_2}+g_2m_2(m_1^2-m_2^2)\Theta_2(p),$$
$$\sigma_{m_2}=-c_1m_2^2-c_2m_2^4-ca_{m_2}+g_1m_2+g_2m_2^3a_{m_2},$$
$$\omega_{-m_2}(p)=\sigma_{-m_2}-g_2m_2(m_1^2-m_2^2)\Theta_2(p),$$
$$\sigma_{-m_2}=-c_1m_2^2-c_2m_2^4-ca_{-m_2}-g_1m_2-g_2m_2^3a_{m_2},$$
$$\Theta_1(p)=\eta_{m_2}^2+\eta_{-m_2}^2, \quad \Theta_2(p)=\eta_{m_1}^2+\eta_{-m_1}^2,$$
$$\Theta_1(p)+\Theta_2(p)=a_{m_1}=a_{m_2}, \quad p=1,2.$$

The quantities $\Theta_1(p)$ and $\Theta_2(p)$ depend on the choice of $\eta_{m_1}=|f_{m_1}|$, $\eta_{-m_1}=|f_{-m_1}|$, $\eta_{m_2}=|f_{m_2}|$, $\eta_{-m_2}=|f_{-m_2}|$ and, consequently, in the case of general position, the differences $\omega_{m_1}(1)-\omega_{m_1}(2)=\Delta_{m_1}$, $\omega_{m_2}(1)-\omega_{m_2}(2)=\Delta_{m_2}$, $\omega_{-m_1}(1)-\omega_{-m_1}(2)=\Delta_{-m_1}$, $\omega_{-m_2}(1)-\omega_{-m_2}(2)=\Delta_{-m_2}$ do not vanish.

Let a positive constant δ be sufficiently small. Then at least for $k=\pm m_1$ or $k=\pm m_2$

$$\max_{k,p}|f_k(p)|\geqslant \frac{\sqrt{a_{m_1}}}{2}, \quad k=\pm m_1,\pm m_2, \quad p=1,2.$$

For the sake of definiteness we assume that $|f_{m_1}(1)|\geqslant \sqrt{a_{m_1}}/2$. Then there exists a sequence $t_n\to\infty$ as $n\to\infty$ such that

$$|f_{m_1}(1)\exp(i\omega_{m_1}(1)t_n)-f_{m_1}(2)\exp(i\omega_{m_1}(2)t_n)|\geqslant \frac{\sqrt{a_{m_1}}}{2}. \qquad (5.6)$$

The inequality (5.6) can be written as

$$|\eta_{m_1}(1)\exp(i\psi_{m_1}(1))\exp(i\omega_{m_1}(1)t_n)-\eta_{m_1}(2)\exp(i\psi_{m_1}(2))\exp(i\omega_{m_1}(2)t_n)|\geqslant \frac{\sqrt{a_{m_1}}}{2}$$

or

$$|\eta_{m_1}(1)\exp(i\Delta\psi_{m_1})\exp(i\Delta_{m_1}t_n)-\eta_{m_1}(2)|\geqslant \frac{\sqrt{a_{m_1}}}{2},$$

where $\Delta\psi_{m_1}=\psi_{m_1}(1)-\psi_{m_1}(2)$. Finally, owing to the choice of t_n, we can conclude that the inequality (5.6) holds if $\eta_{m_1}(1)+\eta_{m_1}(2)\geqslant\sqrt{a_{m_1}}/2$, $\eta_{m_1}(1)=|f_{m_1}(1)|$, $\eta_{m_1}(2)=|f_{m_1}(2)|$. If the inequality (5.6) holds, then the inequality (5.5) holds for the corresponding t_n with $\varepsilon_0 = \sqrt{2\pi(1+m_1^2)a_{m_1}}/2$, where ε_0 is independent of δ. The possibility of the choice of t_n justifying the inequality (5.5) follows from the simple assertion.

Lemma 5.3. *Let $\xi_1, \xi_2 \in C$ and $\alpha_1, \alpha_2 \in R$ be such that $\Delta\alpha = \alpha_2 - \alpha_1 \neq 0$, and let $\xi(t) = |\xi_2 \exp(i\alpha_2 t) - \xi_1 \exp(i\alpha_1 t)|$. Then there exists a sequence t_n such that $\lim\limits_{n\to\infty} t_n = \infty$ and $\xi(t_n) = |\xi_1| + |\xi_2|$.*

Proof. The following equality holds: $\xi(t) = ||\xi_2|\exp(i\alpha_2 t + i\varphi_2) - |\xi_1|\exp(i\alpha_1 t + i\varphi_1)|$, where $\varphi_j = \arg\,\xi_j$, $j = 1, 2$. Taking into account that $|\exp(i\beta)| = 1$ for $\beta \in R$, we find

$$\xi(t) = ||\xi_2|\exp(i(\Delta\alpha)t_n + i\Delta\varphi) - |\xi_1||.$$

Let t_n be taken such that $t_n \to \infty$ and $(\Delta\alpha)t_n + \Delta\varphi = (2n-1)\pi$. If $(\Delta\alpha) > 0$, then we can assume that $n = n_0, n_0+1, \ldots$, where $n_0 \in N$ and $n = -n_0, -(n_0+1), \ldots$ if $(\Delta\alpha) < 0$. □

In a similar way, we can verify that the solutions in $W_5(0,m)$ are unstable if the manifold $W_5(0,m)$ turns out to be dominating.

A different situation occurs if $g_2 = 0$, but the role of a dominating component is played by the manifold $W_7(m_1,m_2)$ ($W_5(0,m)$). Then, as in the case of $W_3(m_1)$, the frequencies of solutions in $W_7(m_1,m_2)$ ($W_5(0,m)$) turn out to be the same and independent of the choice of the solution.

Therefore, the solutions in $W_7(m_1,m_2)$ or $W_5(0,m)$, as in the main variant where $W_3(m_1)$ is dominating, are stable. The proof of the stability of solutions in $W_7(m_1,m_2)$ or $W_5(0,m)$ mainly repeats the first part of the proof. □

Remark 5.1. As was already mentioned, for $g_2 \neq 0$ the solutions to the boundary value problem (2.1), (2.2) in $W_7(m_1,m_2)$ are Lyapunov unstable, but orbitally stable. The trajectories of the solutions $u(t,x,1)$ and $u(t,x,2)$ are close if their initial conditions are close in the following sense: the solution $u(t,x,2)$ is located as $t \to \infty$ in a sufficiently small neighborhood of the trajectory of the solution $u(t,x,1)$.

Let $u(t,x,p) \in W_7(m_1,m_2)$, $p = 1,2$. As was noted in the proof of Theorem 5.3, we have

$$||u(t,x,1) - u(t,x,2)||^2_{H_2^1} = 2\pi(1+m_1^2)Q(m_1) + 2\pi(1+m_2^2)Q(m_2),$$

where

$$\begin{aligned} Q(t,m_1) &= |w_{m_1}(t,1) - w_{m_1}(t,2)|^2 + |w_{-m_1}(t,1) - w_{-m_1}(t,2)|^2 \\ &= |f_{m_1}(1)\exp(i\Delta_{m_1}t) - f_{m_1}(2)|^2 + |f_{-m_1}(1)\exp(i\Delta_{-m_1}t) - f_{-m_1}(2)|^2, \\ Q(t,m_2) &= |w_{m_2}(t,1) - w_{m_2}(t,2)|^2 + |w_{-m_2}(t,1) - w_{-m_2}(t,2)|^2 \\ &= |f_{m_2}(1)\exp(i\Delta_{m_2}t) - f_{m_2}(2)|^2 + |f_{-m_2}(1)\exp(i\Delta_{-m_2}t) - f_{-m_2}(2)|^2, \end{aligned}$$

whereas Δ_{m_1}, Δ_{-m_1}, Δ_{m_2}, Δ_{-m_2} are not equal to zero in the case of general position. The function $Q(t) = 2\pi(1+m_1^2)Q(t,m_1) + 2\pi(1+m_2^2)Q(t,m_2)$ is almost periodic in the sense of Bohr. Moreover, $\mu = Q(0) \ll 1$. By the almost periodicity, there exists a sequence τ_k such that $\lim\limits_{k\to\infty} \tau_k = \infty$ for any sufficiently small ε, i.e., $Q(\tau_k) < 2\varepsilon$ if $\mu < \varepsilon$. Consequently, the solution $u(t,x,2)$ is located in a small neighborhood of the trajectory of the solution $u(t,x,1)$.

If $u(t,x,1) \in W_7(m_1,m_2)$ and $u(t,x,2) \notin W_7(m_1,m_2)$, then the verification of the orbital stability of the trajectory of the solution $u(t,x,1)$ mainly repeats the constructions of the corresponding fragment of the proof of the stability of solutions in $W_3(m_1)$ in the case where one of the solutions belongs to $W_3(m_1)$, unlike the other.

If $W_7(m_1,m_2)$ is changed with $W_5(0,m)$, then the orbital stability of solutions in $W_5(0,m)$ is proved in a similar way.

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