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THE DISCRETE DIRICHLET PROBLEM. SOLVABILITY AND APPROXIMATION PROPERTIES

A. V. Vasilyev
National Research University "Belgorod State University"
85, Pobedy St., Belgorod 308015, Russia
alexvassel@gmail.com

V. B. Vasilyev*
National Research University "Belgorod State University"
85, Pobedy St., Belgorod 308015, Russia
vbv57@inbox.ru

A. A. Khodyreva
National Research University "Belgorod State University"
85, Pobedy St., Belgorod 308015, Russia
anastasia.kho@yandex.ru

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We consider the discrete Dirichlet boundary value problem for a discrete elliptic pseudodifferential equation in the quadrant and study its solvability in discrete counterparts of the Sobolev–Slobodetskii space. The study is based on a special factorization of the elliptic symbol. We compare the solutions to the discrete Dirichlet problem and its continuous counterpart. Bibliography: 10 *titles.*

In this paper, based on the ideas and methods of [1, 2] (cf. also [3]–[7]), we compare discrete and continuous elliptic boundary value problems in the quadrant for the simplest pseudodifferential operators. We emphasize that, in the case of a quadrant, there are principal differences from the case of a half-space, and new analytic tools are required.

1 Preliminaries

We recall the main notions and results which will be used throughout the paper (we refer to [5] for details). Let $\mathbb{Z}^2$ be an integer lattice in the plane. We denote by $\Omega = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_1 > 0, x_2 > 0\}$ the first quarter-plane, $\Omega_d = h\mathbb{Z}^2 \cap \Omega$, $h > 0$, and consider functions of discrete variables $u_d(\widetilde{x})$, $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2) \in h\mathbb{Z}^2$. We denote by $\mathbb{T}^2 = [-\pi, \pi]^2, \hbar = h^{-1}$ and

* To whom the correspondence should be addressed.

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$\zeta^2 = h^{-2}((e^{-ih\cdot\xi_1} - 1)^2 + (e^{-ih\cdot\xi_2} - 1)^2)$, $S(h\mathbb{Z}^2)$ is the discrete analog of the Schwarz space of infinitely differentiable functions rapidly decreasing at infinity.

The space $H^s(h\mathbb{Z}^2)$ consists of discrete generalized functions and is the closure of the space $S(h\mathbb{Z}^2)$ in the norm

$$||u_d||_s = \left(\int\limits_{\hbar\mathbb{T}^2} (1 + |\zeta^2|)^s |\widetilde{u}_d(\xi)|^2 d\xi \right)^{1/2}, \tag{1.1}$$

where $\widetilde{u}_d(\xi)$ denotes the discrete Fourier transform

$$(F_d u_d)(\xi) \equiv \widetilde{u}_d(\xi) = \sum_{\widetilde{x}\in h\mathbb{Z}^2} e^{-i\widetilde{x}\cdot\xi} u_d(\widetilde{x}) h^2, \quad \xi \in \hbar\mathbb{T}^2.$$

We denote by $H^s(\Omega_d)$ the space of discrete generalized functions in $H^s(h\mathbb{Z}^2)$ supported in $\overline{\Omega}_d$. The $H^s(\Omega_d)$-norm is induced by the $H^s(h\mathbb{Z}^2)$-norm. The space $H_0^s(\Omega_d)$ consists of discrete generalized functions $f_d \in S'(h\mathbb{R}^2)$ with support in $\overline{\Omega}_d$, which admit an extension to the whole space $H^s(h\mathbb{Z}^2)$. The $H_0^s(\Omega_d)$-norm is given by

$$||f_d||_s^+ = \inf ||\ell f_d||_s,$$

where the infimum is taken over all possible extensions of ℓ.

The Fourier-image of the space $H^s(\Omega_d)$ is denoted by $\widetilde{H}^s(\Omega_d)$.

Let $A_d(\xi)$ be a measurable periodic function defined on $\mathbb{R}^2$ with the main period square $\hbar\mathbb{T}^2$. By the *discrete pseudodifferential operator* A_d with symbol $A_d(\xi)$ in the discrete quadrant Ω_d we understand the operator

$$(A_d u_d)(\widetilde{x}) = \sum_{\widetilde{y}\in h\mathbb{Z}^2} h^2 \int\limits_{\hbar\mathbb{T}^2} A_d(\xi) e^{i(\widetilde{x}-\widetilde{y})\cdot\xi} \widetilde{u}_d(\xi) d\xi, \quad \widetilde{x} \in \Omega_d. \tag{1.2}$$

The operator A_d is *elliptic* if

$$\inf_{\xi\in\hbar\mathbb{T}^2} |A_d(\xi)| > 0.$$

Here, we consider the symbols satisfying the condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \leqslant |A_d(\xi)| \leqslant c_2(1 + |\zeta^2|)^{\alpha/2} \tag{1.3}$$

with positive constants c_1 and c_2 independent of h. The number $\alpha \in \mathbb{R}$ is called the *order* of the discrete pseudodifferential operator A_d.

We are interested in the solvability of the discrete equation

$$(A_d u_d)(\widetilde{x}) = v_d(\widetilde{x}), \quad \widetilde{x} \in \Omega_d, \tag{1.4}$$

in the space $H^s(\Omega_d)$ under the assumption that $v_d \in H_0^{s-\alpha}(\Omega_d)$.

We need special domains in the two-dimensional complex space $\mathbb{C}^2$. A domain of the form $\mathscr{T}_h(\Omega) = \hbar\mathbb{T}^2 + i\Omega$ is called a *tubular domain* over the quadrant Ω. We will operate with analytic functions $f(x + i\tau)$ in the domain $\mathscr{T}_h(\Omega) = \hbar\mathbb{T}^2 + i\Omega$.

We define a periodic analog of the Bochner kernel (cf. [7])

$$B_h(z) = \sum_{\widetilde{x}\in\Omega_d} e^{i\widetilde{x}\cdot(\xi+i\tau)} h^2, \quad \xi \in \hbar\mathbb{T}^2, \quad \tau \in \Omega,$$

and the corresponding integral operator

$$(B_h\widetilde{u}_d)(\xi) = \lim_{\tau\to 0,\tau\in\Omega} \frac{1}{4\pi^2} \int\limits_{\hbar\mathbb{T}^2} B_h(\xi + i\tau - \eta)\widetilde{u}_d(\eta)d\eta.$$

To describe the solvability conditions for the discrete equation (1.4), we need a special representation of the symbol of a discrete operator.

Definition 1.1. By the *periodic wave factorization* of the elliptic symbol $A_d(\xi) \in E_\alpha$ we mean its representation

$$A_d(\xi) = A_{d,\neq}(\xi)A_{d,=}(\xi),$$

where $A_{d,\neq}(\xi)$ and $A_{d,=}(\xi)$ can be analytically extended to the tubular domains $\mathscr{T}_h(\Omega)$ and $\mathscr{T}_h(-\Omega)$ respectively and

$$c_1(1 + |\widehat{\zeta}^2|)^{\frac{æ}{2}} \leqslant |A_{d,\neq}(\xi + i\tau)| \leqslant c_1'(1 + |\widehat{\zeta}^2|)^{\frac{æ}{2}},$$

$$c_2(1 + |\widehat{\zeta}^2|)^{\frac{\alpha-æ}{2}} \leqslant |A_{d,=}(\xi - i\tau)| \leqslant c_2'(1 + |\widehat{\zeta}^2|)^{\frac{\alpha-æ}{2}}$$

where c_1, c_1', c_2, c_2' are positive constants independent of h;

$$\widehat{\zeta}^2 \equiv \hbar^2((e^{-ih(\xi_1+i\tau_1)} - 1)^2 + (e^{-ih(\xi_2+i\tau_2)} - 1)^2), \quad \xi = (\xi_1, \xi_2) \in \hbar\mathbb{T}^2, \quad \tau = (\tau_1, \tau_2) \in \Omega.$$

The number $æ \in \mathbb{R}$ is called the *index* of periodic wave factorization.

We will assume that we have a periodic wave factorization of the symbol $A_d(\xi)$ with index æ.

The theorem on the general form of solutions to Equation (1.4) is proved by methods of [5].

Theorem 1.1. *If $æ - s = n + \delta$, $n \in \mathbb{N}$, $|\delta| < 1/2$, then the general solution to the discrete equation* (1.4) *has the form*

$$\widetilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi)Q_n(\xi)B_h(Q_n^{-1}(\xi)A_{d,=}^{-1}(\xi)\widetilde{(\ell v_d)}(\xi)) + A_{d,\neq}^{-1}(\xi)\left(\sum_{k=0}^{n-1} \widetilde{c}_k(\xi_1)\widehat{\zeta}_2^k + \widetilde{d}_k(\xi_2)\widehat{\zeta}_1^k,\right),$$

where $Q_n(\xi)$ is an arbitrary polynomial of degree n in variables $\zeta_k = \hbar(e^{-ih\xi_k} - 1)$, $k = 1, 2$, satisfying the condition (1.3) *for $\alpha = n$, $\widetilde{c}_k(\xi_1)$, $\widetilde{d}_k(\xi_2)$, $k = 0, 1, \cdots, n-1$, are arbitrary functions in $H^{s_k}(h\mathbb{T})$, $s_k = s - æ + k + 1/2$, ℓv_d is an extension of $v_d \in H_0^{s-\alpha}(D_d)$ to the whole space $H^{s-\alpha}(h\mathbb{Z}^2)$. Furthermore, the following a priori estimate holds*:

$$||u_d||_s \leqslant \text{const}\left(||v_d||_{s-\alpha}^+ + \sum_{k=0}^{n-1}([c_k]_{s_k} + [d_k]_{s_k})\right),$$

where $[\cdot]_{s_k}$ is the $H^{s_k}(h\mathbb{Z})$-norm and the constant const *is independent of h.*

2 The Discrete Dirichlet Problem

We consider the Dirichlet boundary condition. In this section, we assume that $æ - s = 1 + \delta$, $|\delta| < 1/2$, $v_d \equiv 0$. By Theorem 1.1, the general solution to Equation (1.4) has the form

$$\widetilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi)(\widetilde{c}_0(\xi_1) + \widetilde{d}_0(\xi_2)), \tag{2.1}$$

where $c_0, d_0 \in H^{s-æ+1/2}(\hbar\mathbb{Z})$ are arbitrary functions. To determine uniquely these functions, we add the discrete Dirichlet condition on the angle sides

$$u_{d|_{\tilde{x}_1=0}} = f_d(\tilde{x}_2), \quad u_{d|_{\tilde{x}_2=0}} = g_d(\tilde{x}_1). \tag{2.2}$$

Thus, (1.4), (2.2) is the discrete Dirichlet problem.

We set

$$\int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_1 \equiv \widetilde{a}_0(\xi_2), \quad \int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_2 \equiv \widetilde{b}_0(\xi_1)$$

and, under the assumption that $\widetilde{a}_0(\xi_2), \widetilde{b}_0(\xi_1) \neq 0$ for all $\xi_1 \neq 0$, $\xi_2 \neq 0$,

$$\widetilde{F}_d(\xi_2) = \widetilde{f}_d(\xi_2)\widetilde{a}_0^{-1}(\xi_2), \quad \widetilde{G}_d(\xi_1) = \widetilde{g}_d(\xi_1)\widetilde{b}_0^{-1}(\xi_1),$$

$$k_1(\xi) = A^{-1}_{d,\neq}(\xi)\widetilde{a}_0^{-1}(\xi_2), \quad k_2(\xi) = A^{-1}_{d,\neq}(\xi)\widetilde{b}_0^{-1}(\xi_1).$$

Using the new notation, we write the following system of two linear integral equations for two unknown functions $\widetilde{c}_0(\xi_1)$, $\widetilde{d}_0(\xi_2)$:

$$\begin{aligned} &\int_{-\hbar\pi}^{\hbar\pi} k_1(\xi)\widetilde{c}_0(\xi_1)d\xi_1 + \widetilde{d}_0(\xi_2) = \widetilde{F}_d(\xi_2), \\ &\widetilde{c}_0(\xi_1) + \int_{-\hbar\pi}^{\hbar\pi} k_2(\xi)\widetilde{d}_0(\xi_2)d\xi_2 = \widetilde{G}_d(\xi_1). \end{aligned} \tag{2.3}$$

Theorem 2.1. *Assume that* $f_d, g_d \in H^{s-1/2}(\mathbb{R}_+)$, $s > 1/2$, $v_d \equiv 0$, $\inf|\widetilde{a}_0(\xi_2)| \neq 0$, $\inf|\widetilde{b}_0(\xi_1)| \neq 0$. *Then the discrete Dirichlet problem* (1.4), (2.2) *is equivalent to the system* (2.3) *of linear integral equations.*

Proof. Applying the discrete Fourier transform to the discrete conditions (2.2), we obtain the Fourier-images

$$\begin{aligned} &\int_{-\hbar\pi}^{\hbar\pi} \widetilde{u}_d(\xi_1,\xi_2)d\xi_1 = \widetilde{f}_d(\xi_2), \\ &\int_{-\hbar\pi}^{\hbar\pi} \widetilde{u}_d(\xi_1,\xi_2)d\xi_2 = \widetilde{g}_d(\xi_1). \end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.1), we get

$$\int_{-\hbar\pi}^{\hbar\pi} \widetilde{u}_d(\xi)d\xi_1 = \int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)\widetilde{c}_0(\xi_1)d\xi_1 + \widetilde{d}_0(\xi_2)\int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_1,$$

$$\int_{-\hbar\pi}^{\hbar\pi} \widetilde{u}_d(\xi)d\xi_2 = \widetilde{c}_0(\xi_1)\int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_2 + \int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)\widetilde{d}_0(\xi_2)d\xi_2$$

Taking into account the above notation, we arrive at the system (2.3). □

We consider the continuous case. We describe a similar scheme for the continuous analog of the discrete Dirichlet problem (1.4), (2.2). It is reduced to the system of integral equations by the same method with the help of the Fourier transform.

If we consider the pseudodifferential equation

$$(Au)(x) = 0, \quad x \in \Omega, \tag{2.5}$$

where the symbol $A(\xi)$ satisfies the condition

$$c_1(1+|\xi|)^\alpha \leqslant |A(\xi)| \leqslant c_2(1+|\xi|)^\alpha \tag{2.6}$$

and admits the wave factorization with respect to D

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

with index æ such that $æ - s = 1 + \delta, |\delta| < 1/2$, then the general solution to the equation can be written as

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(\widetilde{C}_0(\xi_1) + \widetilde{D}_0(\xi_2))$$

where $\widetilde{C}_0(\xi_1),\ \widetilde{D}_0(\xi_2) \in \widetilde{H}^{s-æ-1/2}(\mathbb{R})$ are found from the system of integral equations (if it is uniquely solvable)

$$\begin{aligned} &\int_{-\infty}^{\infty} K_1(\xi)\widetilde{C}_0(\xi_1)d\xi_1 + \widetilde{D}_0(\xi_2) = \widetilde{F}(\xi_2), \\ &\widetilde{C}_0(\xi_1) + \int_{-\infty}^{\infty} K_2(\xi)\widetilde{D}_0(\xi_2)d\xi_2 = \widetilde{G}(\xi_1), \end{aligned} \tag{2.7}$$

with boundary conditions

$$u_{|x_1=0} = f(x_2), \quad u_{|x_2=0} = g(x_1) \tag{2.8}$$

and the condition that $\inf|\widetilde{A}_0(\xi_2)| \neq 0$, $\inf|\widetilde{B}_0(\xi_1)| \neq 0$,

$$\int_{-\infty}^{\infty} A_{\neq}^{-1}(\xi)d\xi_1 \equiv \widetilde{A}_0(\xi_2), \quad \int_{-\infty}^{\infty} A_{\neq}^{-1}(\xi)d\xi_2 \equiv \widetilde{B}_0(\xi_1),$$

$$\widetilde{F}(\xi_2) = \widetilde{f}(\xi_2)\widetilde{A}_0^{-1}(\xi_2), \quad \widetilde{G}(\xi_1) = \widetilde{g}(\xi_1)\widetilde{B}_0^{-1}(\xi_1),$$

$$K_1(\xi) = A_{\neq}^{-1}(\xi)\widetilde{A}_0^{-1}(\xi_2), \quad K_2(\xi) = A_{\neq}^{-1}(\xi)\widetilde{B}_0^{-1}(\xi_1).$$

Theorem 2.2 ([1]). *Assume that $s > 1/2$ and the symbol $A(\xi)$ satisfying the condition* (2.6) *admit the wave factorization with respect to Ω with index æ such that $æ - s = 1 + \delta$, $|\delta| < 1/2$. If $\inf|\widetilde{A}_0(\xi_2)| \neq 0$ and $\inf|\widetilde{B}_0(\xi_1)| \neq 0$, then the Dirichlet problem* (2.5), (2.8) *with $f, g \in H^{s-1/2}(\mathbb{R}_+)$ is equivalent to the system of integral equations* (2.7) *with unknown functions $\widetilde{C}_0, \widetilde{D}_0 \in \widetilde{H}^{s_0}(\mathbb{R})$ and right-hand sides $\widetilde{F}, \widetilde{G} \in \widetilde{H}^{s_0}(\mathbb{R})$.*

Remark 2.1. Equation (2.5) with nonzero right-hand side and symbol $A(\xi) = (\xi_1^2 + \xi_2^2 + k^2)^{1/2}$, $k \in \mathbb{C}$, appears in the theory of electromagnetic wave diffraction by a flat screen [8] and also in the problem of pressing a wedge-shaped stamp into an elastic half-space [9].

The Dirichlet problem (2.5), (2.8) for the Laplacian was studied by the second author in [1], where conditions for the unique solvability were obtained for the system of integral equations (2.7), but no constructions of the solution were proposed there. Hence it becomes necessary to develop the discrete theory and constructions which could be used as approximation elements.

3 Comparison of Discrete and Continuous Solutions

Here and below, we consider the case $æ - s = 1 + \delta$, $s > 1/2$, $|\delta| < 1/2$. In this situation, $æ - 1 \geqslant s - 1/2$.

The main difficulty in comparison of solutions to the systems (2.3) and (2.7) is that the solutions belong to different function spaces and the integral operators of these systems act in different spaces. Therefore, we divide the comparison process into two steps. We first consider the truncated integral operator of the system (2.7) on $\hbar\mathbb{T}$ and compare it with the original operator. Then we compare the truncated operator with the discrete operator of the system (2.3). We use the general results concerning projection methods (cf. [10]).

We introduce the space $\mathbf{H}^s(\mathbb{R})$ of two-component vector-valued functions $f = (f_1, f_2)$, $f_j \in H^s(\mathbb{R})$, $j = 1, 2$, $||f||_s \equiv ||f_1||_s + ||f_2||_s$, and matrix operators

$$K = \begin{pmatrix} K_1 & I \\ I & K_2 \end{pmatrix}, \quad k = \begin{pmatrix} k_1 & I_h \\ I_h & k_2 \end{pmatrix},$$

acting in the spaces $\mathbf{H}^{s-æ+1/2}(\mathbb{R})$ and $\mathbf{H}^{s-æ+1/2}(\hbar\mathbb{T})$ respectively.

We recall that $s_0 = s - æ + 1/2$. Using the Cauchy–Schwarz inequality, it is easy to verify the following assertion.

Lemma 3.1. *For $æ > 1$ the operator $K : \mathbf{H}^{s_0}(\mathbb{R}) \to \mathbf{H}^{s_0}(\mathbb{R})$ is boundedly acts in the space $\mathbf{H}^{s_0}(\mathbb{R})$.*

3.1. Estimate for the norm of the integral operator of the system (2.7)**.** We denote by $\chi_h : H^s(\mathbb{R}) \to H^s(\hbar\mathbb{T})$ the projection on the segment $\hbar\mathbb{T}$. The projection on $\hbar\mathbb{T}$ in the space $\mathbf{H}^s(\mathbb{R})$ will be denoted by Ξ_h, so that for $f = (f_1, f_2) \in \mathbf{H}^s(\mathbb{R})$

$$\Xi_h f = (\chi_h f_1, \chi_h f_2).$$

In all estimates below, we assume that h is sufficiently small, $0 < h < 1$,

Lemma 3.2. *For $æ > 1$*

$$||\Xi_h K - K\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbb{R}) \to \widetilde{\mathbf{H}}^{s_0}(\mathbb{R})} \leqslant \text{const}\, h^{s-1/2}.$$

Proof. It is easy to see that

$$\Xi_h K - K\Xi_h = \begin{pmatrix} \chi_h K_1 - K_1\chi_h & 0 \\ 0 & \chi_h K_2 - K_2\chi_h \end{pmatrix}.$$

We have

$$((\chi_h K_1 - K_1\chi_h)f)(\xi_2) = \begin{cases} \left(\int\limits_{-\infty}^{-\hbar\pi} + \int\limits_{\hbar\pi}^{+\infty} \right) K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1, & \xi_2 \in \hbar\mathbb{T}, \\ -\int\limits_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1, \xi_2) f(\xi_1) d\xi_1, & \xi_2 \notin \hbar\mathbb{T}. \end{cases}$$

In the first case, we estimate only one of the integrals since the other is estimated in the same way:

$$\left| \int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right| \leqslant \int\limits_{\hbar\pi}^{+\infty} |K_1(\xi)||f(\xi_1)| d\xi_1 \leqslant \text{const} \int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-æ} |f(\xi_1)| d\xi_1$$

$$\leqslant \text{const} \left(\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2æ} (1+|\xi_1|)^{-2s_0} d\xi_1 \right)^{1/2} \left(\int\limits_{\hbar\pi}^{+\infty} |f(\xi_1)|^2 (1+|\xi_1|)^{2s_0} d\xi_1 \right)^{1/2}$$

$$\leqslant \text{const} \left(\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2(æ+s_0)} d\xi_1 \right)^{1/2} ||f||_{s_0},$$

where the Cauchy–Schwarz inequality was used. We have

$$\int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-2(æ+s_0)} d\xi_1 \sim (1+|\xi_2|+\hbar\pi)^{-2(æ+s_0)+1}$$

since $-2(æ+s_0)+1 = -2(s+1/2)+1 = -2s < 0$. Thus,

$$\left| \int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right| \leqslant \text{const}\, ||f||_{s_0} (1+|\xi_2|+\hbar\pi)^{-(æ+s_0)+1/2}.$$

Squaring the last inequality, multiplying by $(1+|\xi_2|)^{2s_0}$, and integrating over $\hbar\mathbb{T}$, we get

$$\int\limits_{\hbar\mathbb{T}} (1+|\xi_2|)^{2s_0} \left| \int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right|^2 d\xi_2$$

$$\leqslant \text{const}\, ||f||_{s_0}^2 \int\limits_{\hbar\mathbb{T}} (1+|\xi_2|+\hbar\pi)^{-2(æ+s_0)+1} (1+|\xi_2|)^{2s_0} d\xi_2$$

$$\leqslant \text{const}\, ||f||_{s_0}^2 (1+\hbar)^{-2s} \int\limits_{\hbar\mathbb{T}} (1+|\xi_2|)^{s_0} d\xi_2 \leqslant \text{const}\, ||f||_{s_0}^2 (1+\hbar)^{-2s+1}$$

since $1+|\xi_2|+\hbar\pi \geqslant 1+\hbar$, $-2(æ+s_0)+1 = -2s < 0$. Then

$$\int\limits_{\hbar\mathbb{T}} (1+|\xi_2|)^{2s_0} \left| \int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right|^2 d\xi_2 \leqslant \text{const}\, ||f||_{s_0}^2 h^{2(s-1/2)}.$$

In the second case, $(|\xi_2| > \hbar\pi)$, we have

$$\left| \int\limits_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1 \right| \leqslant \text{const} \int\limits_{-\hbar\pi}^{+\hbar\pi} (1+|\xi|)^{-æ} |f(\xi_1)| d\xi_1$$

$$\leqslant \text{const} \left(\int\limits_{-\hbar\pi}^{\hbar\pi} (1+|\xi|)^{-2æ} (1+|\xi_1|)^{-2s_0} d\xi_1 \right)^{1/2} \left(\int\limits_{-\hbar\pi}^{\hbar\pi} |f(\xi_1)|^2 (1+|\xi_1|)^{2s_0} d\xi_1 \right)^{1/2}.$$

Here, we again used the Cauchy–Schwarz inequality. Taking into account that inequality $1+|\xi| \geqslant 1+|\xi_1|$, we derive the estimate

$$\int\limits_{-\hbar\pi}^{\hbar\pi} (1+|\xi|)^{-2æ}(1+|\xi_1|)^{-2s_0} d\xi_1 \leqslant 2 \int\limits_{0}^{|\xi_2|} (1+\xi_1)^{-2(s_0+æ)} d\xi_1 \leqslant \mathrm{const}(1+|\xi_2|)^{-2s}$$

since $-2(s_0+æ) = -2(s+1/2) = -2s-1$. Thus, we have

$$\left| \int\limits_{-\hbar\pi}^{+\hbar\pi} K_1(\xi_1,\xi_2) f(\xi_1) d\xi_1 \right| \leqslant \mathrm{const}\, ||f||_{s_0} (1+|\xi_2|)^{-s}.$$

Multiplying both sides of the last inequality by $(1+|\xi_2|)^{s_0}$, squaring the result, and integrating over $\mathbb{R}\setminus\hbar\mathbb{T}$, we get

$$\int\limits_{\mathbb{R}\setminus\mathbb{T}} (1+|\xi_2|)^{2s_0} \left| \int\limits_{\hbar\pi}^{+\infty} K_1(\xi) f(\xi_1) d\xi_1 \right|^2 d\xi_2$$

$$\leqslant \mathrm{const}\, ||f||^2_{s_0} \int\limits_{\hbar\pi}^{+\infty} (1+\xi_2)^{2(s_0-s)} d\xi_2 \leqslant \mathrm{const}\, ||f||^2_{s_0} h^{2(æ-1)}.$$

The last integral can be easily calculated ($s_0 - s = -æ + 1/2$). Similar estimates are valid for the operator K_2. □

It is easy to prove the following assertion.

Corollary 3.1. *Let the assumptions of Lemma* 3.2 *hold. If the operator K possesses the bounded inverse, then the operator K^{-1} satisfies the estimate*

$$||\Xi_h K^{-1} - K^{-1}\Xi_h||_{\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})\to\widetilde{\mathbf{H}}^{s_0}(\mathbb{R})} \leqslant \mathrm{const}\, h^{s-1/2}.$$

To prove the following assertion, we need to choose elements of the periodic wave factorization generating a special periodic symbol.

We construct the symbol $A_d(\xi)$ of the discrete operator A_d as follows. If there is the wave factorization of the symbol $A(\xi)$, $A(\xi) = A_{\neq}(\xi)\cdot A_{=}(\xi)$, then we take the restrictions of $A_{\neq}(\xi)$ and $A_{=}(\xi)$ on $\hbar\mathbb{T}^2$ and periodically extend them to the whole space $\mathbb{R}^2$. The obtained elements are denoted by $A_{d,\neq}(\xi)$, $A_{d,=}(\xi)$. For these elements we construct the periodic symbol $A_d(\xi)$ which admits a periodic wave factorization with respect to Ω

$$A_d(\xi) = A_{d,\neq}(\xi)\cdot A_{d,=}(\xi)$$

with the same index æ. Below, comparing the discrete and continuous solutions, we consider the discrete pseudodifferential equation with this symbol.

Lemma 3.3. *For $æ > 1$*

$$|K_1(\xi) - k_1(\xi)| \leqslant \mathrm{const}(1+|\xi|)^{-æ} h^{æ-1}, \quad \xi \in \hbar\mathbb{T}^2.$$

Proof. Indeed, by the choice of $A^{-1}_{d,\neq}(\xi)$,

$$|K_1(\xi) - k_1(\xi)| = |A^{-1}_{\neq}(\xi)\widetilde{A}_0^{-1}(\xi_2) - A^{-1}_{d,\neq}(\xi)\widetilde{a}_0^{-1}(\xi_2)| \leqslant \mathrm{const}(1+|\xi|)^{-æ}|\widetilde{A}_0(\xi_2) - \widetilde{a}_0(\xi_2)|.$$

Let us estimate $|\widetilde{A}_0(\xi_2) - \widetilde{a}_0(\xi_2)|$. We have

$$|\widetilde{A}_0(\xi_2) - \widetilde{a}_0(\xi_2)| = \left| \int\limits_{-\infty}^{\infty} A^{-1}_{\neq}(\xi)d\xi_1 - \int\limits_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,\neq}(\xi)d\xi_1 \right|$$

$$\leqslant \mathrm{const} \int\limits_{\hbar\pi}^{+\infty} (1+|\xi|)^{-æ}d\xi_1 \leqslant \mathrm{const}(1+|\xi_2|+\hbar)^{-æ+1} \leqslant \mathrm{const}\, h^{æ-1}$$

for sufficiently small h. Hence $\inf|\widetilde{B}_0(\xi_1)| \neq 0$ implies $\inf|\widetilde{b}_0(\xi_1)| \neq 0$ for sufficiently small h. Collecting the obtained estimates, we complete the proof of the lemma. □

We introduce the operator $\Xi_h K \Xi_h$. By Lemma 3.1, for sufficiently small h the invertibility of $\Xi_h K \Xi_h$ in $\widetilde{\mathbf{H}}^{s-æ-1/2}(\hbar\mathbb{T})$ is a consequence of the invertibility of K in $\widetilde{\mathbf{H}}^{s-æ-1/2}(\mathbb{R})$ (cf. [10]). Furthermore, for sufficiently small h

$$||(\Xi_h K \Xi_h)^{-1}||_{\widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T}) \to \widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T})} \leqslant \mathrm{const}\,.$$

Lemma 3.4. *For $æ > 1$ the comparison of the norms of the operators $\Xi_h K \Xi_h$ and k is given by the estimate*

$$||\Xi_h K \Xi_h - k||_{\widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T}) \to \widetilde{\mathbf{H}}^{s_0}(\hbar\mathbb{T})} \leqslant \mathrm{const}\, h^{æ-1}.$$

The proof mainly repeats that of Lemma 3.2, and we omit it.

3.2. Discrete and continuous solutions. We compare the discrete and continuous solutions. Since Theorems 2.1 and 2.2 establish the equivalence between the boundary value problems (1.4), (2.2) and (2.5), (2.8) for the systems of integral equations (2.3) and (2.7) respectively, we assume that the original (continuous) boundary value problem is uniquely solvable for any right-hand side $v \in H_0^{s-\alpha}(K)$ and any boundary functions $f, g \in H^{s-1/2}(\mathbb{R}_+)$. In other words, the bounded inverse operator K^{-1} exists, i.e., the system (2.7) of integral equations has a unique solution for any $(\widetilde{F}, \widetilde{G})^T$.

Taking into account the choice of the discrete operator A_d made in Subsection 3.1, we construct the boundary functions f_d and g_d in a similar way. Namely, we restrict the Fourier transforms $\widetilde{f}$ and $\widetilde{g}$ on $\hbar\mathbb{T}$, extend periodically to the whole line $\mathbb{R}$, and then apply the inverse discrete Fourier transform. Thus, we obtain the corresponding discrete boundary value problem (1.4), (2.2) with $v_d \equiv 0$. We compare the obtained problem with the problem (2.5), (2.8).

Theorem 3.1. *Let all the assumptions of Theorem* 2.2 *hold, and let $æ > 1$. The comparison of solutions to the problems* (1.4), (2.2) *with $v_d \equiv 0$ and* (2.5), (2.8) *for sufficiently small h is given by the estimate*

$$||u - u_d||_{H^s(\hbar\mathbb{T}^2)} \leqslant \mathrm{const}\, h^{s-1/2}(||f||_{s-1/2} + ||g||_{s-1/2})$$

where const *is independent of h.*

Proof. We first compare the Fourier-images of solutions to the systems (2.3) and (2.7). We have the continuous

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(\widetilde{C}_0(\xi_1) + \widetilde{D}_0(\xi_2))$$

and discrete

$$\widetilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi)(\widetilde{c}_0(\xi_1) + \widetilde{d}_0(\xi_2))$$

solutions. We denote by $\widetilde{\Phi}_d$ and $\widetilde{\Phi}$ vectors with components $(\widetilde{F}_d, \widetilde{G}_d)^T$ and $(\widetilde{F}, \widetilde{G})^T$ and by $\widetilde{C}$ and $\widetilde{c}$ vectors with components $(\widetilde{C}_0, \widetilde{D}_0)^T$ and $(\widetilde{c}_0, \widetilde{d}_0)^T$ respectively. Then

$$\widetilde{C} = K^{-1}\widetilde{\Phi}, \quad \widetilde{c} = k^{-1}\widetilde{\Phi}_d.$$

We denote by C_1, C_2 and c_1, c_2 the jth coordinates of the vectors $\widetilde{C}$ and $\widetilde{c}$, $j = 1, 2$. Then

$$\begin{aligned}(\chi_h\widetilde{u})(\xi) - \widetilde{u}_d(\xi) &= \chi_h A_{\neq}^{-1}(\xi)((\widetilde{C}_0(\xi_1) - \widetilde{c}_0(\xi_1)) + (\widetilde{D}_0(\xi_2) - \widetilde{d}_0(\xi_2))\\ &= \chi_h A_{\neq}^{-1}(\xi)((K^{-1}\widetilde{\Phi})_1(\xi_1) - (k^{-1}\widetilde{\Phi}_d)_1(\xi_1) + (K^{-1}\widetilde{\Phi})_2(\xi_2) - (k^{-1}\widetilde{\Phi}_d)_2(\xi_2)).\end{aligned}$$

This means that it suffices to estimate the norm $||\Xi_h K^{-1}\Phi - k^{-1}\Phi_d||_{\mathbf{H}^{s_0}(\hbar\mathbb{T})}$. We write

$$\Xi_h K^{-1}\Phi - k^{-1}\Phi_d = (\Xi_h K^{-1}\Phi - K^{-1}\Xi_h\Phi) + (K^{-1}\Xi_h\Phi - k^{-1}\Phi_d).$$

To estimate the first term, we use Corollary 3.1. We have

$$||\Xi_h K^{-1}\Phi - K^{-1}\Xi_h\Phi||_{s_0} \leqslant \operatorname{const} h^{s-1/2}||\Phi||_{s_0} \leqslant \operatorname{const} h^{s-1/2}(||f||_{s_0} + ||g||_{s_0}).$$

We represent the second term as the sum

$$K^{-1}\Xi_h\Phi - k^{-1}\Phi_d = (K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi) + (k^{-1}\Xi_h\Phi - k^{-1}\Phi_d)$$

and estimate each term separately.

We consider $k^{-1}\Xi_h\Phi - k^{-1}\Phi_d$. Since the norm of k^{-1} is bounded by a constant independent of h, we find

$$||k^{-1}\Xi_h\Phi - k^{-1}\Phi_d||_{s_0} \leqslant \operatorname{const} ||\Xi_h\Phi - \Phi_d||_{s_0} \leqslant \operatorname{const}(||\chi_h F - F_d||_{s_0} + ||\chi_h G - G_d||_{s_0}).$$

It remains to estimate, for example, $||\chi_h F - F_d||_{s_0}$. We have

$$\begin{aligned}||\chi_h F - F_d||_{s_0}^2 &= \int_{-\hbar\pi}^{\hbar\pi} |\widetilde{f}(\xi_2)A_0^{-1}(\xi_2) - \widetilde{f}_d(\xi_2)a_0^{-1}(\xi_2)|^2(1 + |\xi_2|)^{2s_0}d\xi_2\\ &\leqslant \operatorname{const} h^{2æ-2}\int_{-\hbar\pi}^{\hbar\pi} |\widetilde{f}(\xi_2)|^2(1 + |\xi_2|)^{2s_0}d\xi_2 \leqslant \operatorname{const} h^{2æ-2}||f||_{s_0}^2\end{aligned}$$

taking into account that f_d and f are identical on $\hbar\mathbb{T}$ and using the estimate from Lemma 3.3.

The remaining term is estimated by the following inequality which can be easily verified: $K^{-1} - k^{-1} = K^{-1}(k - K)k^{-1}$. We recall that the invertibility of the operator k follows from the invertibility of the operator K. Comparing on $\hbar\mathbb{T}$

$$K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi = \Xi_h(K^{-1} - k^{-1})\Xi_h\Phi = \Xi_h K^{-1}(k - K)k^{-1}\Xi_h\Phi$$

and taking into account Lemma 3.4, we obtain the estimate

$$||K^{-1}\Xi_h\Phi - k^{-1}\Xi_h\Phi||_{s_0} \leqslant \text{const}\, h^{æ-1}||\Phi||_{s_0} \leqslant \text{const}\, h^{æ-1}(||f||_{s_0} + ||g||_{s_0}).$$

Collecting the obtained estimates, we obtain the assertion of Theorem 3.1. Here, we took into account the properties of pseudodifferential operators [5] owing to which we can pass to the H^s-norm. □

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