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PROBLEMS OF OPTIMAL RESOURCE HARVESTING FOR INFINITE TIME HORIZON

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We consider a population such that, in absence of exploitation, its dynamics is described by a system of differential equations. It is assumed that, at certain times $\tau_k = kd$, $d > 0$, *resource shares* $u(k)$, $k = 0, 1, 2, \ldots$, *are extracted from the population. Regarding* $\overline{u} = (u(0), u(1), \ldots, u(k), \ldots)$ *as a control for reaching a desired harvesting result, we construct* $\overline{u}$ *at which the resource harvesting characteristics (the time-average harvesting profit and the harvesting efficiency) attain given values, in particular, the case where the harvesting efficiency becomes infinite is included. We consider the problems of constructing stationary controls delivering the maximum value for one of the characteristics provided that the other is fixed and demonstrate the solution of these problems by considering examples of homogeneous and two-species populations. Bibliography*: 9 *titles. Illustrations*: 2 *figures.*

In this paper, we consider models of homogeneous populations consisting of one species or one age class and models of structured populations as well. We assume that it is possible to extract some resource shares and control the process to achieve given values of the time-average profit or the harvesting efficiency. We obtain conditions under which to reach the best resource harvesting result it is necessary to extract all or some species of the structured population.

We note that controls providing the maximal time-average profit were constructed in [1]. In this paper, we propose a new approach owing to which we obtain such results for not only the time-average profit, but also for the harvesting efficiency. In particular, we show how to construct control actions to achieve the infinite value of the harvesting efficiency.

In addition, we consider two new problems. One problem is to find stationary controls guaranteeing the maximal harvesting efficiency providing that the time-average profit is fixed. The

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second one is to reach the maximal time-average profit provided that the harvesting efficiency is fixed. The results obtained are illustrated by examples of a homogeneous population and a two-species population.

1 Characteristics of Resource Harvesting in Models of Exploited Populations

Let $\mathbb{R}^n_+ \doteq \{x \in \mathbb{R}^n : x_1 \geqslant 0, \ldots, x_n \geqslant 0\}$. We consider the model of a population whose dynamics is described, in the absence of exploitation, by the system of differential equations

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n_+. \tag{1.1}$$

In the case $n = 1$, the population consists of one species and is called *homogeneous*. If $n \geqslant 2$, the population is *structured*, i.e., consists of different species $x_1, \ldots, x_n$ or n age groups.

We assume that biological resource shares $u(k) = (u_1(k), \ldots, u_n(k)) \in [0,1]^n$, $k = 0, 1, 2, \ldots$, are extracted from the population at times $\tau_k = kd$, $d > 0$, $k = 0, 1, 2, \ldots$. We set $U \doteq \{\overline{u} : \overline{u} = (u(0), u(1), \ldots, u(k), \ldots)\}$ and interpret $\overline{u} \in U$ as a control which can be changed to achieve a certain harvesting result. Thus, we consider the exploited population with dynamics described by the control system with impulse action

$$\begin{aligned} &\dot{x}_i = f_i(x), \quad t \neq kd, \quad k = 0, 1, 2, \ldots, \\ &x_i(kd) = (1 - u_i(k)) \cdot x_i(kd - 0), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{1.2}$$

where $x_i(kd - 0)$ and $x_i(kd)$ denote the resource amounts of the ith species before and after harvesting at time kd, $k = 1, 2, \ldots$ respectively. Furthermore, for the ith species, $i = 1, \ldots, n$, we denote by $x_i(0-0)$ the initial resource amount and by $x_i(0)$ the resource amount after harvesting at the initial time. It is assumed that the solutions to the system (1.2) are continuous from the right and the functions $f_1(x), \ldots, f_n(x)$ are defined and continuously differentiable for all $x \in \mathbb{R}^n_+$.

Since the systems (1.1) and (1.2) describe the biological population dynamics, the solutions must be nonnegative for any nonnegative initial data. The system (1.1) possesses this condition if and only if the functions $f_1(x), \ldots, f_n(x)$ satisfies the *quasipositivity* condition [4]

$$f_i(x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \geqslant 0, \quad i = 1, \ldots, n.$$

From the equalities $x_i(kd) = (1 - u_i(k)) \cdot x_i(kd - 0)$ in (1.2) it follows that the solutions to the system (1.2) are nonnegative for any nonnegative initial data if the quasipositivity condition holds.

For the ith species we denote by $X_i(k) = x_i(kd - 0)$ the resource amount before harvesting at time kd, $k = 1, 2, \ldots$, and by $X_i(0)$ the initial resource amount¿ We set $X(0) = (X_1(0), \ldots, X_n(0))$. We denote by $C_i \geqslant 0$ the resource cost of the ith species (assuming that $C_1, \ldots, C_n$ do not vanish simultaneously). Then the total cost of the harvested resource at time $\tau_k = kd$ is equal to $\sum\limits_{i=1}^{n} C_i X_i(k) u_i(k)$.

Definition 1.1 ([5]). The *time-average profit* from the resource extraction is defined by

$$H_*(\overline{u}, X(0)) \doteq \varliminf_{k \to +\infty} \frac{1}{k} \sum_{j=0}^{k-1} \sum_{i=1}^{n} C_i X_i(j) u_i(j).$$

Replacing the lower limit with the upper one in (1.1), we defined the function $H^*(\overline{u}, X(0))$. If $H_*(\overline{u}, X(0)) = H^*(\overline{u}, X(0))$, we set

$$H(\overline{u}, X(0)) \doteq \lim_{k\to+\infty} \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=1}^{n} C_i X_i(j) u_i(j).$$

Definition 1.2. The *harvesting efficiency* is the lower limit as $k \to +\infty$ of the ratio of the resource amount after k harvestings to the sum of the corresponding controls (the harvesting efforts):

$$E_*(\overline{u}, X(0)) \doteq \varliminf_{k\to+\infty} \frac{\sum_{j=0}^{k-1}\sum_{i=1}^{n} C_i X_i(j) u_i(j)}{\sum_{j=0}^{k-1}\sum_{i=1}^{n} u_i(j)}, \tag{1.3}$$

where $\sum_{i=1}^{n} u_i(0) > 0$. The harvesting efficiency will be denoted by $E(\overline{u}, X(0))$ if the limit in (1.3) exists.

We note that a similar characteristic was studied in [6] in the case of a periodic harvesting of a renewable resource distributed in a domain of the arithmetic space and obeying the logistic growth law. As shown in [6], there exists a stationary control providing the maximal harvesting efficiency at infinite horizon.

2 Construction of Controls to Achieve Given Values of Time-Average Profit

We denote by $\varphi(t, x) = (\varphi_1(t, x), \ldots, \varphi_n(t, x))$ the solution to the system (1.1) satisfying the initial condition $\varphi(0, x) = x$, where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n_+$. Assume that for any $x \in \mathbb{R}_n^+$ solutions $\varphi(t, x)$ to the system exist for $t \in [0, d]$, $d > 0$. By the *stationary exploitation regime* for a population described by the system (1.2) we understand a mode where the harvesting is provided by $u(k) \equiv u = (u_1, \ldots, u_n) \in [0, 1]^n$ for all $k = 0, 1, 2, \ldots$. In this case, the dynamics of an exploited population is described by the system of difference equations

$$X(k + 1) = \varphi(d, (1 - u)X(k)), \quad k = 0, 1, 2, \ldots, \tag{2.1}$$

where $X(k) = (X_1(k), \ldots, X_n(k))$ is the species composition of the population before harvesting at time $\tau_k = kd$ and $(1-u)X(k) = ((1-u_1)X_1(k), \ldots, (1-u_n)X_n(k))$. We denote by $X(k, u, X_0)$, $k = 0, 1, 2, \ldots$ the solution to the system (2.1) satisfying the initial condition $X(0) = X_0 \in \mathbb{R}^n_+$. If the system (2.1) has a fixed point (an equilibrium state) $\xi(u) = (\xi_1(u), \ldots, \xi_n(u)) \in \mathbb{R}^n_+$, then $\xi(u) = \varphi(d, (1 - u)\xi(u))$. Let $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$. We recall (see [7]) that a fixed point $\xi(u)$ of the system (2.1) is *asymptotically stable* if for any initial data $X(0) = X_0$ in some neighborhood of the point $\xi(u)$ we have

$$\lim_{k\to\infty} \|X(k, u, X_0) - \xi(u)\| = 0. \tag{2.2}$$

The set of initial points $X(0)$ satisfying (2.2) is called the *attraction set* of the point $\xi(u)$ and denoted by $A(\xi(u))$.

Proposition 2.1. *Let the stationary exploitation regime $u(k) \equiv u$, $k = 0, 1, \ldots$, hold, and let the system* (2.1) *have a fixed point $\xi(u)$. Then for any initial point $X(0) \in A(\xi(u))$*

$$H(\overline{u}, X(0)) = \sum_{i=1}^{n} C_i \xi_i(u) u_i, \tag{2.3}$$

$$E(\overline{u}, X(0)) = \sum_{i=1}^{n} C_i \xi_i(u) u_i \Big(\sum_{i=1}^{n} u_i\Big)^{-1}. \tag{2.4}$$

The proof of Proposition 2.1 is similar to the proof of Assertion 1 in [8].

We introduce the set $\mathscr{D}_+ \doteq \{x \in \mathbb{R}_+^n : x_i \leqslant \varphi_i(d, x) \neq 0, i = 1, \ldots, n\}$ and the function

$$D(x) \doteq \sum_{i=1}^{n} C_i(\varphi_i(d, x) - x_i). \tag{2.5}$$

Theorem 2.1. *Let $D(x)$ take a value $h \in (0, +\infty)$ at a point $x^* \in \mathscr{D}_+$. Then*

$$H(\overline{u}^*, X(0)) = h$$

for any $X(0) \in A(\varphi(d, x^*))$ and

$$\overline{u}^* = (u^*, u^*, \ldots), \quad u^* = \Big(1 - \frac{x_1^*}{\varphi_1(d, x^*)}, \ldots, 1 - \frac{x_n^*}{\varphi_n(d, x^*)}\Big). \tag{2.6}$$

Furthermore, if the maximum of $D(x)$ on $\mathbb{R}_+^n$ is attained at a point $x^* \in \mathscr{D}_+$, then for any $\overline{u} \in U$ and $X(0) \in \mathbb{R}_+^n$

$$H(\overline{u}, X(0)) \leqslant H(\overline{u}^*, X^*(0)) = D(x^*),$$

where $X^*(0) \in A(\varphi(d, x^*))$ and the control $\overline{u}^* \in U$ is given by (2.6).

Proof. If $\overline{u}^* = (u^*, u^*, \ldots)$, then the system (2.1) is of the form

$$X(k+1) = \varphi\Big(d, \frac{x^*}{\varphi(d, x^*)} X(k)\Big), \quad k = 0, 1, 2, \ldots \tag{2.7}$$

and has a fixed point $\xi(u^*) = \varphi(d, x^*)$. Therefore, (2.3) implies

$$H(\overline{u}^*, X(0)) = \sum_{i=1}^{n} C_i \varphi_i(d, x^*)\Big(1 - \frac{x_i^*}{\varphi_i(d, x^*)}\Big) = \sum_{i=1}^{n} C_i(\varphi_i(d, x^*) - x_i^*) = D(x^*) = h.$$

We show that if the maximum of the function $D(x)$ on $\mathbb{R}_+^n$ is attained at a point $x^* \in \mathscr{D}_+$, then $H(\overline{u}^*, x^*(0)) = D(x^*) = h$ is the maximal value of the time-average profit on the set of all controls U. Let $x_i(k) = (1 - u_i(k))X_i(k)$, $i = 1, \ldots, n$, be the resource size of the ith species after harvesting at time τ_k, and let $x(k) = (x_1(k), \ldots, x_n(k))$, $k = 0, 1, 2, \ldots$. For each $i = 1, \ldots, n$ we find $X_i(k)u_i(k) = X_i(k) - (1 - u_i(k))X_i(k) = \varphi_i(d, x(k-1)) - x_i(k)$, $k = 1, 2, \ldots$. Hence for any $\overline{u} \in U$ and $X(0) \in \mathbb{R}^n_+$

$$H_*(\overline{u}, X(0)) \doteq \varliminf_{k\to+\infty} \frac{1}{k}\sum_{j=0}^{k-1}\sum_{i=1}^{n} C_i X_i(j) u_i(j)$$

$$= \varliminf_{k\to+\infty} \frac{1}{k}\Big(\sum_{i=1}^{n} C_i X_i(0) u_i(0) + \sum_{j=1}^{k-1}\sum_{i=1}^{n} C_i(\varphi_i(d, x(j-1)) - x_i(j))\Big)$$

$$= \varliminf_{k\to+\infty} \frac{1}{k}\sum_{i=1}^{n} C_i(X_i(0)u_i(0) - x_i(k-1)) + \varliminf_{k\to+\infty} \frac{1}{k}\sum_{j=1}^{k-2}\sum_{i=1}^{n} C_i(\varphi_i(d, x(j)) - x_i(j)). \tag{2.8}$$

Since

$$\sum_{i=1}^{n} C_i(\varphi_i(d, x(j)) - x_i(j)) \leqslant D(x^*) \quad \forall\ x(j) \in \mathbb{R}^n_+, \quad j = 1, 2, \ldots,$$

we conclude that

$$\begin{aligned} H_*(\overline{u}, X(0)) &\leqslant \varliminf_{k\to+\infty} \frac{-1}{k}\sum_{i=1}^{n} C_i x_i(k-1) + \lim_{k\to+\infty} \frac{k-2}{k} D(x^*) \\ &\leqslant -\varlimsup_{k\to+\infty} \frac{1}{k}\sum_{i=1}^{n} C_i x_i(k-1) + D(x^*). \end{aligned} \tag{2.9}$$

Since

$$\varlimsup_{k\to+\infty} \frac{1}{k}\sum_{i=1}^{n} C_i x_i(k-1) \geqslant 0,$$

the right-hand side of (2.9) does not exceed $D(x^*)$, and, consequently, $H_*(\overline{u}, X(0)) \leqslant D(x^*)$ for any $\overline{u} \in U$ and $X(0) \in \mathbb{R}^n_+$. □

Remark 2.1. For a structured population the function $D(x)$ can attain the maximum at a point $x^* \in \mathbb{R}^n_+$ such that one of its coordinates satisfies the condition $x^*_i = \varphi_i(d, x^*) = 0$, i.e., in this case, x^* does not belong to $\mathscr{D}_+$ (see the corresponding example in [1]). In this case, it is reasonable to harvest only some of n species. The following assertion yields a control providing the maximal value of the time-average profit.

Theorem 2.2. *Assume that $n \geqslant 2$, the function $D(x)$ takes the value $h \in (0, +\infty)$ at a point $x^* \in \mathbb{R}^n_+$, and there exists a nonempty set $I \subset \{1, \ldots, n\}$ such that $x^*_i = \varphi_i(d, x^*) = 0$ for $i \in I$ and $x^*_i \leqslant \varphi_i(d, x^*) \neq 0$ for $i \in \{1, \ldots, n\} \setminus I$. Then $H(\overline{u}^*, X(0)) = h$ for any $X(0) \in A(\varphi(d, x^*))$ with the stationary control $\overline{u}^* = (u^*, u^*, \ldots)$, where*

$$\begin{aligned} u^*_i &= 0, \quad i \in I, \\ u^*_i &= 1 - \frac{x^*_i}{\varphi_i(d, x^*)}, \quad i \in \{1, \ldots, n\} \setminus I. \end{aligned} \tag{2.10}$$

Moreover, if h is the maximum of $D(x)$ on $\mathbb{R}^n_+$, then for any $\overline{u} \in U$, $X(0) \in \mathbb{R}^n_+$

$$H(\overline{u}, X(0)) \leqslant H(\overline{u}^*, X^*(0)) = D(x^*) = h,$$

where $X^*(0) \in A(\varphi(d, x^*))$ and the control $\overline{u}^* \in U$ is given by (2.10).

Proof. Without loss of generality we can assume that $I = \{1, \dots, m\}$, $1 \leqslant m \leqslant n-1$, $n \geqslant 2$. If $\overline{u}^*$ is given by (2.10), then the system (2.1) can be written as

$$X(k+1) = \varphi\Big(d, X_1(k), \dots, X_m(k), \frac{x^*_{m+1}}{\varphi_{m+1}(d, x^*)} X_{m+1}(k), \dots, \frac{x^*_n}{\varphi_n(d, x^*)} X_n(k)\Big) \tag{2.11}$$

for $k = 0, 1, 2, \dots$. The system (2.11) has the fixed point $\xi(u^*) = \varphi(d, x^*)$, where $\varphi_i(d, x^*) = 0$ for $i = 1, \dots, m$ and $x^*_i \leqslant \varphi_i(d, x^*) \neq 0$ for $i = m+1, \dots, n$. Consequently, from (2.3) we find

$$H(\overline{u}^*, X(0)) = \sum_{i=m+1}^{n} C_i(\varphi_i(d, x^*) - x^*_i) = D(x^*) = h.$$

Then we argue in the same way as in the proof of Theorem 2.1. □

Remark 2.2. If $D(x)$ is not bounded, then we can construct (nonstationary) controls such that the time-average profit takes the value $+\infty$ (see [3]).

3 Construction of Controls to Achieve Given Values of Harvesting Efficiency

We recall that $\varphi(t, x)$ denotes a solution to the system (1.1) satisfying the initial condition $\varphi(0, x) = x$, where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n_+$. We consider the set

$$\mathscr{E}_+ \doteq \Big\{x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \frac{x_i}{\varphi_i(d, x)} \neq n; x_i \leqslant \varphi_i(d, x) \neq 0, i = 1, \dots, n\Big\}$$

and for any $x \in \mathscr{E}_+$ introduce the function

$$E(x) \doteq \sum_{i=1}^{n} C_i(\varphi_i(d, x) - x_i)\Big(n - \sum_{i=1}^{n} \frac{x_i}{\varphi_i(d, x)}\Big)^{-1}. \tag{3.1}$$

Theorem 3.1. *Let $E(x^*) = \ell \in (0, +\infty)$ for some $x^* \in \mathscr{E}_+$. Then $E(\overline{u}^*, X(0)) = \ell$ for any $X(0) \in A(\varphi(d, x^*))$ and $\overline{u}^*$ given by (2.6). Furthermore, if the maximum of $E(x)$ on $\mathbb{R}^n_+$ is attained at a point $x^* \in \mathscr{E}_+$, then $E(\overline{u}, X(0)) \leqslant E(x^*)$ for any stationary control $\overline{u} \in U$ and any $X(0) \in \mathbb{R}^n_+$.*

Proof. Let $\overline{u}^*$ be defined by (2.6). Then Equation (2.1) has the fixed point $\xi(u^*) = \varphi(d, x^*)$. Taking into account (2.4), for any $X(0) \in A(\varphi(d, x^*))$ we find

$$\begin{aligned}
E(\overline{u}^*, X(0)) &= \sum_{i=1}^{n} C_i \xi_i(u^*) u^*_i \Bigg(\sum_{i=1}^{n} u^*_i\Bigg)^{-1} \\
&= \sum_{i=1}^{n} C_i \varphi_i(d, x^*)\Big(1 - \frac{x^*_i}{\varphi_i(d, x^*)}\Big)\Bigg(n - \sum_{i=1}^{n} \frac{x^*_i}{\varphi_i(d, x^*)}\Bigg)^{-1} \\
&= \sum_{i=1}^{n} C_i(\varphi_i(d, x^*) - x^*_i)\Bigg(n - \sum_{i=1}^{n} \frac{x^*_i}{\varphi_i(d, x^*)}\Bigg)^{-1} = E(x^*) = \ell.
\end{aligned}$$

Assume that the maximum of $E(x)$ on $\mathbb{R}_+^n$ is attained at a point $x^* \in \mathscr{E}_+$ and $E(x^*) = \ell$. By Proposition 2.1, to find the maximum of $E(\overline{u}, X(0))$ in the case of the stationary exploitation regime $u(k) \equiv u$, $k = 0, 1, \ldots$, we need to find the largest value of the function

$$W(u) \doteq \sum_{i=1}^{n} C_i \xi_i(u) u_i \left(\sum_{i=1}^{n} u_i \right)^{-1}, \tag{3.2}$$

where $\xi(u) = \varphi(d, (1-u)\xi(u))$ is a fixed point of the system (2.1). We set $v(u) = (1-u)\xi(u)$. Then $\xi(u) = \varphi(d, v(u))$ and the equality $\xi(u) = \varphi(d, (1-u)\xi(u))$ can be written as $v(u) = (1-u)\varphi(d, v(u))$. Further, from (3.2) we find

$$W(u) = \sum_{i=1}^{n} C_i \varphi_i(d, v(u)) u_i \left(\sum_{i=1}^{n} u_i \right)^{-1}$$

$$= \frac{\sum_{i=1}^{n} C_i \varphi_i(d, v(u)) - \sum_{i=1}^{n} C_i (1-u_i) \varphi_i(d, v(u))}{n - \sum_{i=1}^{n} (1-u_i)} = \frac{\sum_{i=1}^{n} C_i (\varphi_i(d, v(u)) - v_i(u))}{n - \sum_{i=1}^{n} \dfrac{v_i(u)}{\varphi_i(d, v(u))}} = E(v(u)).$$

Consequently, the maxima of $W(u)$ and $E(v(u))$ coincide and are attained at $v(u) = x^*$. From $v(u) = (1-u)\varphi(d, v(u))$ and $v(u) = x^*$ we get $x_i^* = (1-u_i^*)\varphi_i(d, x^*)$. Therefore,

$$u_i^* = 1 - \frac{x_i^*}{\varphi_i(d, x^*)}, \quad i = 1, \ldots, n,$$

i.e., the stationary control $\overline{u}^*$ is given by (2.6). □

Remark 3.1. Considering arbitrary (not necessarily stationary) controls, we can obtain the value of the harvesting efficiency larger than $E(x^*)$. For example, if

$$\sum_{i=1}^{n} C_i X_i(0) > n E(x^*),$$

then for the controls $u(0) = (1, 1, \ldots, 1)$ and $u(k) = (0, 0, \ldots, 0)$, $k = 1, 2, \ldots$, we have

$$E(\overline{u}, X(0)) = \frac{1}{n} \sum_{i=1}^{n} C_i X_i(0) > E(x^*).$$

We consider the problem of constructing a method of exploitation of a population $\overline{u} \in U$ guaranteeing that $E(\overline{u}, X(0)) = +\infty$. One of such controls is given in the proof of the following theorem. We set $\widetilde{\mathbb{R}}_+^n \doteq \{x \in \mathbb{R}_n : x_1 > 0, \ldots, x_n > 0\}$.

Theorem 3.2. *Assume that there exists $\widehat{x} \in \widetilde{\mathbb{R}}_+^n$ satisfying the following conditions*:

(1) *the sequences $\{\varphi_i(kd, \widehat{x})\}_{k=0}^{+\infty}$, $i = 1, \ldots, n$, are nondecreasing,*

(2) *the following relation holds*:

$$\lim_{k \to +\infty} \frac{1}{k} \sum_{i=1}^{n} C_i \varphi_i(kd, \widehat{x}) = +\infty.$$

Then for any $X(0)$ such that $X_i(0) \geqslant \widehat{x}_i$, $i = 1, \dots, n$, there exists an exploitation regime $\overline{u} \in U$ such that $E(\overline{u}, X(0)) = +\infty$.

Proof. We propose a method for exploiting a population $\overline{u} \in U$, owing to which an infinite time-average profit can be achieved. We begin by choosing the control

$$u(0) = (u_1(0), \dots, u_n(0)) = \Big(1 - \frac{\widehat{x}_1}{X_1(0)}, \dots, 1 - \frac{\widehat{x}_n}{X_n(0)}\Big) \tag{3.3}$$

and note that $u(0) \in [0, 1)^n$ since $X_i(0) \geqslant \widehat{x}_i > 0$, $i = 1, \dots, n$. The controls at the further times are constructed as follows:

$$\begin{aligned} &u(2k-1) = (0, \dots, 0), \quad k = 1, 2, \dots, \\ &u(2k) = \Big(1 - \frac{\varphi_1(kd, \widehat{x})}{\varphi_1((k+1)d, \widehat{x})}, \dots, 1 - \frac{\varphi_n(kd, \widehat{x})}{\varphi_n((k+1)d, \widehat{x})}\Big), \quad k = 1, 2, \dots. \end{aligned} \tag{3.4}$$

By the quasiperiodicity condition and the first assumption of theorem, we have $u(2k) \in [0, 1)^n$.

By (3.3), the resource amount of the ith species harvested at time $t = 0$ is equal to $X_i(0)u_i(0) = X_i(0) - \widehat{x}_i$, whereas the amount of the remaining resource is equal to $x_i(0) = (1 - u_i(0))X_i(0) = \widehat{x}_i$, $i = 1, \dots, n$. Thus, $x(0) = \widehat{x}$. We find

$$X(1) = \varphi(d, x(0)) = \varphi(d, \widehat{x}), \quad x(1) = X(1) = \varphi(d, \widehat{x}),$$

$$X(2) = \varphi(d, x(1)) = \varphi(2d, \widehat{x}), \quad x(2) = (1 - u(2))X(2) = \frac{\varphi(d, \widehat{x})}{\varphi(2d, \widehat{x})} X(2) = \varphi(d, \widehat{x}).$$

Similarly, we find

$$x(2k-1) = x(2k) = \varphi(kd, \widehat{x}), \quad X(2k-1) = \varphi(kd, \widehat{x}), \quad X(2k) = \varphi((k+1)d, \widehat{x})$$

for $k = 1, 2, \dots$. We emphasize that no resource is extracted at times $t = d, 3d, 5d, \dots$ according to the chosen exploitation regime. The resource amount extracted at time $t = 0$ is equal to

$$X(0)u(0) = X(0) - \widehat{x},$$

and its cost is equal to

$$\sum_{i=1}^{n} C_i X_i(0) u_i(0) = \sum_{i=1}^{n} C_i (X_i(0) - \widehat{x}_i).$$

The resource amount extracted at time $t = 2jd$, $j = 1, 2, \dots$, is equal to

$$X(2j)u(2j) = \varphi((j+1)d, \widehat{x}) - \varphi(jd, \widehat{x}),$$

and its cost is equal to

$$\sum_{i=1}^{n} C_i X_i(2j) u_i(2j) = \sum_{i=1}^{n} C_i (\varphi_i((j+1)d, \widehat{x}) - \varphi_i(jd, \widehat{x})).$$

We find the total cost of the resource extracted at times $t = 0, 2d, \dots, 2kd$:

$$\begin{aligned} \sum_{j=0}^{k} \sum_{i=1}^{n} C_i X_i(2j) u_i(2j) &= \sum_{i=1}^{n} C_i (X_i(0) - \widehat{x}_i) + \sum_{j=1}^{k} \sum_{i=1}^{n} C_i (\varphi_i((j+1)d, \widehat{x}) - \varphi_i(jd, \widehat{x})) \\ &= \sum_{i=1}^{n} C_i (X_i(0) - \widehat{x}_i) + \sum_{i=1}^{n} C_i (\varphi_i((k+1)d, \widehat{x}) - \varphi_i(d, \widehat{x})). \end{aligned}$$

Thus, taking into account the obvious inequality

$$\sum_{j=0}^{2k}\sum_{i=1}^{n} u_i(j) \leqslant (2k+1)n,$$

we see that for $\overline{u}$ given by (3.3), (3.4) we have

$$E_*(\overline{u}, X(0)) \doteq \varliminf_{k\to+\infty} \frac{\sum\limits_{j=0}^{2k}\sum\limits_{i=1}^{n} C_i X_i(j) u_i(j)}{\sum\limits_{j=0}^{2k}\sum\limits_{i=1}^{n} u_i(j)} \geqslant \varliminf_{k\to+\infty} \frac{1}{(2k+1)n}\sum_{j=0}^{k}\sum_{i=1}^{n} C_i X_i(2j) u_i(2j)$$

$$= \varliminf_{k\to+\infty} \frac{1}{2kn}\sum_{i=1}^{n} C_i(X_i(0) - \widehat{x}_i) + \varliminf_{k\to+\infty} \frac{1}{2kn}\sum_{i=1}^{n} C_i(\varphi_i((k+1)d, \widehat{x}) - \varphi_i(d, \widehat{x}))$$

$$= \varliminf_{k\to+\infty} \frac{1}{2kn}\sum_{i=1}^{n} C_i\varphi_i((k+1)d, \widehat{x}).$$

By the second assumption of the theorem, $E_*(\overline{u}, X(0)) = +\infty$. Hence $E(\overline{u}, X(0)) = +\infty$. □

4 Resource Harvesting for Homogeneous Population

We consider a homogeneous population such that, in the absence of exploitation, its dynamics is described by the differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}_+ = [0, +\infty).$$

It is assumed that some share $u(k) \in [0, 1]$ of biological resources is extracted from the population at time $\tau_k = kd$, $d > 0$. Then we obtain an exploited homogeneous population with dynamics described by the following equation with impulse action:

$$\begin{aligned} &\dot{x} = f(x), \quad t \neq kd, \\ &x(kd) = (1 - u(k)) \cdot x(kd - 0), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{4.1}$$

where $x(kd - 0)$ and $x(kd)$ denote the population size before and after harvesting at time kd, $k = 0, 1, 2, \ldots$, respectively. We assume that the solutions to Equation (4.1) are continuous from the right and the function $f(x)$ is continuously differentiable for all $x \geqslant 0$. In addition, let $f(0) \geqslant 0$ that the quasipositivity condition for homogenious population is satisfied. For such an exploitation regime the population size $X(k) = x(kd - 0)$ before harvesting at time kd, $k = 0, 1, 2, \ldots$, is determined by the difference equation

$$X(k+1) = \varphi(d, (1 - u)X(k)), \quad k = 0, 1, 2, \ldots, \tag{4.2}$$

where $\varphi(t, x)$, $t \geqslant 0$, $x \geqslant 0$, is a solution to the equation $\dot{x} = f(x)$ satisfying the initial condition $\varphi(0, x) = x$.

By definition, a homogeneous population consists of one species, i.e. $n = 1$. Without loss of generality we can assume that $C_1 = 1$, where C_1 is the cost of the species. The formulas for the time-average profit from the resource extraction and the harvesting efficiency take the form

$$H_*(\overline{u}, X(0)) \doteq \varliminf_{k\to+\infty} \frac{1}{k}\sum_{j=0}^{k-1} X(j)u(j),$$

$$E_*(\overline{u}, X(0)) \doteq \varliminf_{k\to+\infty} \frac{\sum_{j=1}^{k} X(j)u(j)}{\sum_{j=0}^{k-1} u(j)}$$

respectively. If the limits in these expressions exist, we use the notation $H(\overline{u}, X(0))$ and $E(\overline{u}, X(0))$ for the time-average profit and the harvesting efficiency respectively.

For homogeneous populations the following consequence of Proposition 2.1 holds.

Corollary 4.1. *Assume that the stationary exploitation regime $u(k) \equiv u$, $k = 0, 1, \ldots$, holds and Equation (4.2) has a fixed point $\xi(u)$. Then for any initial point $X(0) \in A(\xi(u))$*

$$H(\overline{u}, X(0)) = \xi(u)u, \quad E(\overline{u}, X(0)) = \xi(u).$$

The following assertion provides conditions for the existence of a positive fixed point of Equation (4.2) in the case of homogeneous populations.

Proposition 4.1. *Assume that there exists $K > 0$ such that $\varphi(t) \equiv K$ is a solution to the equation $\dot{x} = f(x)$. Let one of the following conditions hold:*

(1) $\varphi(d, 0) > 0$ *and* $u \neq 1$,

(2) $\varphi(d, 0) = 0$ *and* $(1 - u)\varphi'_x(d, 0) > 1$.

Then Equation (4.2) has a fixed point $\xi(u)$ such that $0 < \xi(u) \leqslant K$.

Proof. We first show that for fixed $d > 0$ the function $x \mapsto \varphi(d, x)$ is increasing. Indeed, if there exist $x_1 < x_2$ such that $\varphi(d, x_1) \geqslant \varphi(d, x_2)$, then there exists a point $t_* \in (0, d]$ such that $\varphi(t_*, x_1) = \varphi(t_*, x_2)$, which contradicts the uniqueness condition for solutions to the differential equation.

Let condition (1) be satisfied. Consider the increasing function $h(x) \doteq \varphi(d, (1 - u)x)$. We have $h(0) = \varphi(d, 0) > 0$ and $h(K) = \varphi(d, (1 - u)K) \leqslant \varphi(d, K) = K$. Hence there exists a point where the graphs of the functions $y = h(x)$ and $y = x$ intersect, i.e., there exists a fixed point $\xi(u)$ of Equation (4.2) such that $0 < \xi(u) \leqslant K$.

Let condition (2) be satisfied. We note that the inequality $(1 - u)\varphi'_x(d, 0) > 1$ fails for $u = 1$. Therefore, we consider $u \in [0, 1)$. We have $h(0) = \varphi(d, 0) = 0$, $h(K) \leqslant K$, and $h'(0) = \varphi'_x(d, 0)(1 - u)$. Then $(1 - u)\varphi'_x(d, 0) > 1$ implies $h'(0) > 1$. By the continuity of $h'(x)$, we have $h'(x) > 1$ for all $x \in [0, \delta)$ and some $\delta > 0$. Therefore, the graph of the function $y = h(x)$ lies above the graph of the function $y = x$ for $x \in [0, \delta)$. Hence there exists $x_* > 0$ such that $h(x_*) > x_*$. Thus, there exists a point $\xi(u)$ where the graphs of the functions $y = h(x)$ and $y = x$ intersect. This point is a fixed point of Equation (2.1) such that $0 < \xi(u) \leqslant K$. □

Remark 4.1. By (2.5) and (3.1), for a homogeneous population we have $D(x) = \varphi(d,x) - x$ and $E(x) = \varphi(d,x)$, $x \geqslant 0$. Therefore, from the proof of Proposition 4.1 it follows that for fixed $d > 0$ the function $E(x) = \varphi(d,x)$ is increasing and can have a finite or infinite limit as $x \to +\infty$. The behavior of the function $D(x)$ is more complicated. In particular, $D(x)$ can attain its maximum at some point $x^* \geqslant 0$.

Example 4.1. We find the time-average profit and the harvesting efficiency for a population the dynamics of which is described, in the absence of exploitation, by the logistic equation

$$\dot{x} = (a - bx)x, \quad x \geqslant 0, \tag{4.3}$$

where $x = x(t)$ is the population size at time $t \geqslant 0$, whereas $a > 0$ and $b > 0$ characterize the population growth and the intraspecific competition respectively. Properties of solutions to Equation (4.3) are described in [9].

The function

$$\varphi(t,x) = \frac{ae^{at}x}{a + bx\big(e^{at} - 1\big)}, \quad t \geqslant 0,$$

is a solution to Equation (4.3) and satisfies the initial condition $\varphi(0,x) = x$. A simple computation shows that for $u \in [0, 1 - e^{-ad})$ Equation (4.2) has two fixed points

$$\xi_1(u) = 0, \quad \xi_2(u) = \frac{ae^{ad}(1-u) - a}{b(e^{ad} - 1)(1-u)} > 0$$

such that the first one is unstable, whereas the second one is stable and has the attraction domain $(0, +\infty)$. For $u \in [1 - e^{-ad}, 1]$ Equation (4.2) has only one fixed point $\xi(u) = 0$ with the attraction domain $[0, +\infty)$.

By Corollary 4.1, in the case of the stationary exploitation regime $u(k) \equiv u$, $k = 0, 1, \ldots$, $u \in [0, 1 - e^{-ad})$, for any initial point $X(0) \in (0, +\infty)$

$$\begin{aligned} H(\overline{u}, X(0)) &= \frac{ae^{ad}(u - u^2) - au}{b(e^{ad} - 1)(1-u)}, \\ E(\overline{u}, X(0)) &= \frac{ae^{ad}(1-u) - a}{b(e^{ad} - 1)(1-u)}. \end{aligned} \tag{4.4}$$

If $X(0) = 0$, then

$$H(\overline{u}, 0) = E(\overline{u}, 0) = 0.$$

If $u \in [1 - e^{-ad}, 1]$, then

$$H(\overline{u}, X(0)) = E(\overline{u}, X(0)) = 0 \quad \forall\, X(0) \in [0, +\infty).$$

Thus, we can conclude that for the large stationary control $u(k) \equiv u$, where $u \in [1 - e^{-ad}, 1]$, the time-average profit $H(\overline{u}, X(0))$ and the harvesting efficiency $E(\overline{u}, X(0))$ vanish.

Computing the maximum of $D(x) = \varphi(d,x) - x$ on $\mathbb{R}_+$ and using Theorem 2.1, we find that for any $\overline{u} \in U$ and $X(0) \in \mathbb{R}_+$

$$H(\overline{u}, X(0)) \leqslant H(\overline{u}^*, X^*(0)) = D(x^*) = \frac{a(e^{ad/2} - 1)}{b(e^{ad/2} + 1)}, \quad x^* = \frac{a}{b(e^{ad/2} + 1)},$$

where $X^*(0) \in A(\varphi(d, x^*)) = (0, +\infty)$ and the stationary control $\overline{u}^*$ is given by $u^*(k) \equiv u^* = 1 - e^{-ad/2}$, $k = 0, 1, \ldots$. The harvesting efficiency $E(\overline{u}, X(0))$ can take any value $\ell \in [0, a/b]$.

5 Other Problems of Optimal Resource Harvesting

In this section, we consider two problems. In the first problem, the time-average profit takes the same value h for different stationary controls $u(k) \equiv u = (u_1, \ldots, u_n)$, $k = 0, 1, 2, \ldots$, and we study how to choose a control to guarantee the best harvesting efficiency. In the second problem, the harvesting efficiency is fixed and we look for a control for which the time-average profit attains its maximum. We discuss these problems by considering examples of two-species homogeneous populations.

Example 5.1. We consider a homogeneous population whose dynamics is described by Equation (4.3). By (4.4), $H(\overline{u}, X(0)) = h \in (0, D(x^*)$ for the controls $u(k) \equiv u$, $k = 0, 1, \ldots$, where

$$u = u^{1,2} = \frac{(a+bh)(e^{ad}-1) \pm \sqrt{(a+bh)^2(e^{ad}-1)^2 - 4abhe^{ad}(e^{ad}-1)}}{2ae^{ad}}. \tag{5.1}$$

Then the harvesting efficiency takes the values

$$E(\overline{u^p}, X(0)) = \frac{ae^{ad}}{b(e^{ad}-1)} - \frac{a}{b(e^{ad}-1)(1-u^p)}, \quad \overline{u^p} = (u^p, u^p, \ldots), \quad p = 1, 2.$$

Thus, if $H(\overline{u}, X(0)) = h$, then the maximum of the harvesting efficiency on the set of stationary controls is attained at the control $\overline{u} = \overline{u^1} = (u^1, u^1, \ldots)$, where $u^1 < u^2$ are given by (5.1).

Now, let $E(\overline{u}, X(0)) = \ell \in (0, a/b]$ for $u(k) \equiv u$, where $u \in [0, 1 - e^{-ad})$, $k = 0, 1, \ldots$. By (4.4), we get

$$u = \frac{(a-b\ell)(e^{ad}-1)}{(a-b\ell)e^{ad}+b\ell}, \quad H(\overline{u}, X(0)) = \frac{\ell(a-b\ell)(e^{ad}-1)}{(a-b\ell)e^{ad}+b\ell}.$$

Thus, if the harvesting efficiency $E(\overline{u}, X(0))$ takes a fixed value $ell \in (0, a/b]$, then the time-average profit $H(\overline{u}, X(0))$ can be uniquely found.

Figure 1 presents the graphs of the dependence of $H(\overline{u}, X(0))$ and $E(\overline{u}, X(0))$ on the controls $\overline{u} = (u, u, \ldots)$ in the stationary exploitation regime. Assume that the population dynamics is described by the logistic equation (4.3) with parameters $a = 1$, $b = 1$, $d = \ln 2$. Then the maximum of the function $D(x)$ is equal to $3 - 2\sqrt{2} \approx 0.172$, and attained at the point $x^* = \sqrt{2} - 1 \approx 0.414$. Moreover,

$$u^* = 1 - \frac{x^*}{\varphi(t, x^*)} = 1 - \frac{1}{\sqrt{2}} \approx 0.293$$

and for this control the harvesting efficiency $E(\overline{u}, X(0))$ takes the value $2 - \sqrt{2} \approx 0.586$.

We find the maximum of the harvesting efficiency $E(\overline{u}, X(0))$ in the stationary exploitation regime under the assumption that the time-average profit $H(\overline{u}, X(0)) = h < D(x^*)$ is fixed. By (5.1), the time-average profit takes the value $H(\overline{u}, X(0)) = 0.15$ at the controls $u^1 = 0.2$, $u^2 = 0.375$. Calculating the harvesting efficiency at these controls, we find $E(\overline{u^1}, X(0)) = 0.75$ and $E(\overline{u^2}, X(0)) = 0.4$. Further, the time-average profit takes the value $H(\overline{u}, X(0)) = 0.1$ at the controls $u^1 \approx 0.115$ and $u^2 \approx 0.435$, and the harvesting efficiency calculated at these controls has the values $E(\overline{u^1}, X(0)) \approx 0.870$ and $E(\overline{u^2}, X(0)) \approx 0.23$. Thus, the maximal harvesting efficiency is equal to 0.75 if $H(\overline{u}, X(0)) = 0.15$ and is approximately equal to 0.870. if $H(\overline{u}, X(0)) = 0.1$.

Now, we consider the case where the harvesting efficiency $E(\overline{u}, X(0)) = \ell \leqslant 1$ is fixed. We find that, if $E(\overline{u}, X(0)) = 0.9$, then $H(\overline{u}, X(0)) \approx 0.082$ at $u \approx 0.091$; if $E(\overline{u}, X(0)) = 0.6$, then

$H(\overline{u}, X(0)) \approx 0.171$ at $u \approx 0.286$; if $E(\overline{u}, X(0)) = 0.3$, then $H(\overline{u}, X(0)) \approx 0.124$ at $u \approx 0.412$. We note that the harvesting efficiency attains the maximal value $E(\overline{u}, X(0)) = 1$ at the control $u = 0$ and, at this control, the time-average profit vanishes: $H(\overline{u}, X(0)) = 0$ (see Figure 1).

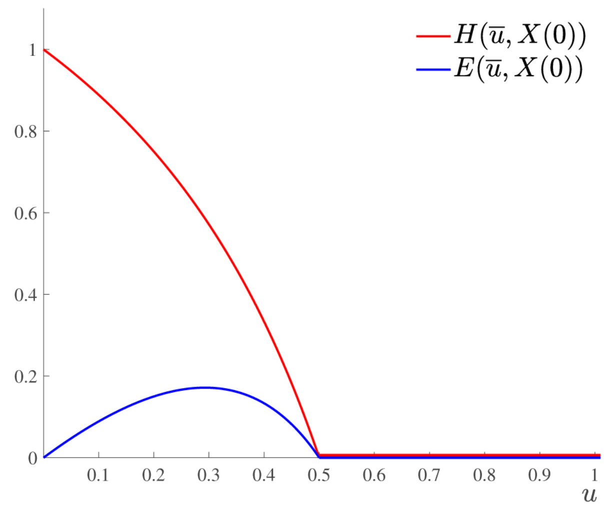


Figure 1. The dependence of $H(\overline{u}, X(0))$ and $E(\overline{u}, X(0))$ on control $\overline{u} = (u, u, \dots)$ for the parameters $a = 1$, $b = 1$, $d = \ln 2$ in Equation (4.3) in the stationary exploitation regime.

Example 5.2. Let the dynamics of a two-species population be described by the system

$$\begin{aligned} \dot{x_1} &= (a_1 - b_1 x_1)x_1, \\ \dot{x_2} &= (a_2 - b_2 x_2)x_2, \end{aligned} \tag{5.2}$$

where $x_i = x_i(t) \geqslant 0$ is the population size at time $t \geqslant 0$ and the coefficients $a_i > 0$ and $b_i > 0$ are interpreted as the population growth and the intraspecific competition respectively for the ith species. The system (5.2) is related to the situation where two species or two age classes interact, but do not influence each other in any way [9].

For graphical illustration we take the following values of the coefficients in (5.2): $a_1 = 2$, $a_2 = 2$, $b_1 = 5$, $b_2 = 2.5$. Let $C_1 = 1$ and $C_2 = 2$ be the resource costs of the first and second species. The solution $\varphi(t, x)$ to the system (5.2) satisfying the initial condition $\varphi(0, x) = x = (x_1, x_2)$ at time $t = d = 0.5 \ln 2$ has the form

$$\varphi_1(d, x) = \frac{4x_1}{2 + 5x_1}, \quad \varphi_2(d, x) = \frac{4x_2}{2 + 2.5x_2}.$$

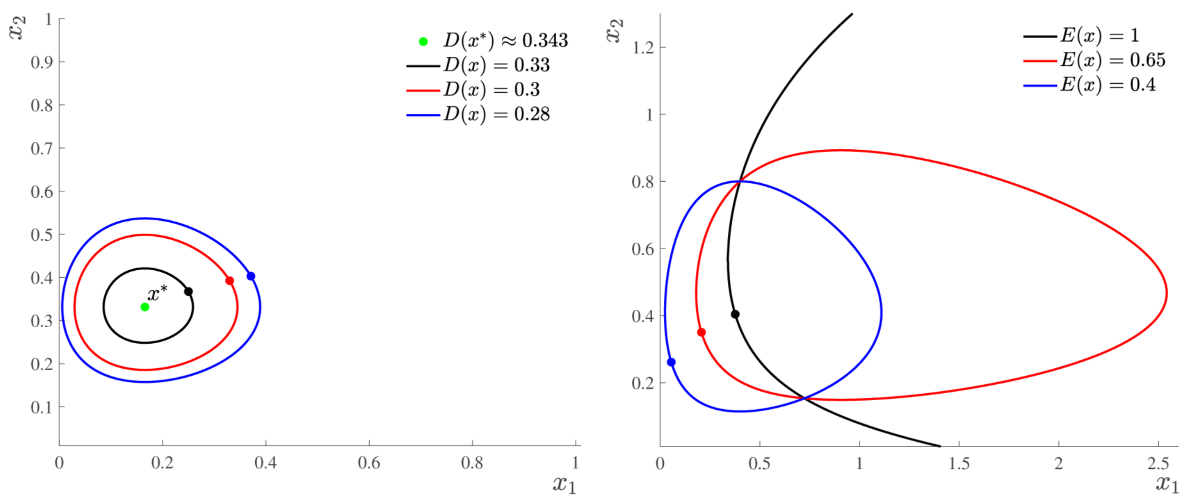
By (2.5) and (3.1), the functions $D(x)$ and $E(x)$ for the system (5.2) can be written as

$$D(x) = \frac{2x_1 - 5x_1^2}{2 + 5x_1} + \frac{4x_2 - 5x_2^2}{2 + 2.5x_2}, \quad E(x) = \frac{4D(x)}{4 - 5x_1 - 2.5x_2}.$$

The maximum of $D(x)$ is approximately equal to 0.343 and is attained at $x^* \approx (0.166, 0.331)$; moreover, $E(x^*) \approx 0.586$ at this point x^*.

Let $D(x)$ be fixed. We find the maximum of the harvesting efficiency $E(x)$. If $D(x) = 0.33$, then $\max E(x) \approx 0.721$ is attained at $(x_1, x_2) \approx (0.249, 0.367)$ with controls $u_1 \approx 0.188$ and $u_2 \approx 0.270$. If $D(x) = 0.3$, then $\max E(x) \approx 0.876$ is attained at $(x_1, x_2) \approx (0.329, 0.392)$ with controls $u_1 \approx 0.088$ and $u_2 \approx 0.255$. Finally, if $D(x) = 0.28$, then $\max E(x) \approx 0.984$ is attained at $x_1 \approx 0.371$ and $x_2 \approx 0.403$ with controls $u_1 \approx 0.036$ and $u_2 \approx 0.248$.

Now, let the harvesting efficiency $E(x)$ be fixed. We find the maximum of the function $D(x)$. If $E(x) = 1$, then $\max D(x) \approx 0.277$ is attained at $(x_1, x_2) \approx (0.376, 0.404)$ with controls $u_1 \approx 0.029$ and $u_2 \approx 0.247$. If $E(x) = 0.65$, then $\max D(x) \approx 0.339$ is attained at $(x_1, x_2) \approx (0.207, 0.350)$ with controls $u_1 \approx 0.242$ and $u_2 \approx 0.281$. Finally, if $E(x) = 0.4$, then $\max D(x) \approx 0.307$ at $(x_1, x_2) \approx (0.055, 0.261)$ with controls taking the values 0.431 and 0.337 respectively. The level curves of $D(x)$ and $E(x)$ are shown in Figure 2.



(a) (b)

Figure 2. (a) The level curves of the function $D(x)$ for the system (5.2). The marked points on the curve, mean the points where $E(x)$ attains the maximum for a given value $D(x)$; x^* is the maximum point of $D(x)$. (b) The level curve of the function $E(x)$ for the system (5.2). The marked points on the curves mean the points where the function $D(x)$ takes the largest value.

References

1. M. S. Woldeab and L. I. Rodina, "About the methods of extracting a biological resource that provide the maximum average time benefit," *Russ. Math.* **66**, No. 1, 8–18 (2022).

2. A. O. Belyakov, A. A. Davydov, and V. M. Veliov, "Optimal cyclic harvesting of renewable resource," *Dokl. Math.* **96**, No. 2, 472–474 (2017).

3. M. S. Woldeab and L. I. Rodina, "Methods of renewable resource extraction from a structured population" [in Russian], *Vestn. Ross. Univ., Mat.* **27**, No. 137, 16–26 (2022).

4. O. A. Kuzenkov and E. A. Ryabova, *Mathematical Modelling of Selection Processes* [in Russian], Nizhegorod. Univ. Press, Nizhnii Novgorod (2007).

5. L. I. Rodina, "Optimization of average time profit for a probability model of the population subject to a craft" [in Russian], *Vestn. Udmurt. Univ., Math. Mekh. Komp'yut. Nauki* **28**, No. 1, 48–58 (2018).

6. A. O. Belyakov and A. A. Davydov, "Efficiency optimization for the cyclic use of a renewable resource," *Proc. Steklov Inst. Math.* **299**, Suppl. 1, 14–21 (2017).

7. Yu. M. Svirezhev and D. O. Logofet, *Stability of Biological Communities* [in Russian], Nauka, Moscow (1978).

8. A. V. Egorova and L. I. Rodina, "On optimal harvesting of renewable resource from structured population" [in Russian], *Vestn. Udmurt. Univ., Math. Mekh. Komp'yut. Nauki* **29**, No. 4, 501–517 (2019).

9. G. Yu. Riznichenko and A. B. Rubin, *Mathematical Models of Biological Production Processes* [in Russian], Moscow State Univ. Press, Moscow (1993).

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