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AN INVERSE PROBLEM FOR A QUASILINEAR ELLIPTIC EQUATION

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We consider the inverse problem for a quasilinear second order elliptic equation with the unknown coefficient at the lower-order term and the first kind boundary condition and the integral overedetermination condition on the boundary. We prove a theorem on the local existence and uniqueness of a strong solution to the problem. The result is illustrated by an example of a nonlinear equation satisfying all the assumptions of the theorem. Bibliography: 6 *titles.*

1 Introduction

We study the following inverse problem. For given functions $f(x)$, $\beta(x)$, $h(x)$ and a constant μ it is required to find a function u and a constant k such that

$$-\operatorname{div}(\mathscr{M}(x)\nabla u) + m(x)u + kr(u) = f, \tag{1.1}$$

$$u|_{\partial\Omega} = \beta(x), \tag{1.2}$$

$$\int\limits_{\partial\Omega} \frac{\partial u}{\partial \overline{N}} h(x)ds = \mu, \tag{1.3}$$

where $\Omega \cap \mathbf{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^2$, $\mathscr{M}(x) = m_{ij}(x)$ is a matrix-valued function, m_{ij}, $i, j = 1, 2, \dots, n$, and $m(x)$ is a scalar function. We set

$$\frac{\partial}{\partial \overline{N}} = (\mathscr{M}(x)\nabla, \mathbf{n}),$$

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where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

The main goal of the paper is to establish the existence, uniqueness, and stability (in the sense of the continuous dependence on the data) of a strong solution to the problem (1.1)–(1.3). We use the method developed in [1]–[3] and based on reducing the inverse problem to an operator equation of the second kind for the unknown coefficient [4]. We note that the inverse problem for the linear equation $Mu + ku = f$ with unknown constant coefficient k was studied in [5] and [6] in the case of boundary conditions of the first and third kind respectively.

2 Preliminaries

For the norm and inner product we use the notation $\|\cdot\|_R$ and $(\cdot,\cdot)_R$ in $\mathbb{R}^n$ and $\|\cdot\|$ and $(\cdot,\cdot)$ in $L^2(\Omega)$. We write $\|\cdot\|_j$ and $\langle\cdot,\cdot\rangle_1$ for the $W_2^j(\Omega)$-norm, $j = 1, 2$, and the duality relation between $\overset{\circ}{W}{}_2^1(\Omega)$ and $W_2^{-1}(\Omega)$ respectively. We introduce the linear operator $M : W_2^1(\Omega) \to W_2^1(\Omega)$ by

$$M = -\operatorname{div}(\mathscr{M}(x)\nabla) + m(x)I,$$

where I is the identity operator. We set

$$\langle Mv_1, v_2\rangle_M = \int\limits_\Omega ((\mathscr{M}(x)\nabla v_1, \nabla v_2)_R + m(x)v_1v_2)dx \quad \forall\ v_1, v_2 \in W_2^1(\Omega).$$

In what follows, we assume that the following assumptions hold.

(I) $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i, j, l = 1, 2, \ldots, n$, $m(x)$ are bounded in Ω. The operator M is elliptic, i.e., there exist positive constants m_0 and m_1 such that for any $v \in W_2^1(\Omega)$

$$m_0\|v\|_1^2 \leqslant \langle Mv, v\rangle_M \leqslant m_1\|v\|_1^2. \tag{2.1}$$

(II) M is a selfadjoint operator, i.e., $m_{ij}(x) = m_{ji}(x)$, $i, j = 1, \ldots, n$.

(III) $r(\rho)$ is continuous and strictly monotone on $(-\infty, +\infty)$, i.e., $(r(\rho_1) - r(\rho_2))(\rho_1 - \rho_2) > 0$ for any $\rho_1, \rho_2 \in (-\infty, +\infty)$, $\rho_1 \neq \rho_2$, and $r(0) = 0$.

The existence and uniqueness of a solution to the direct problem (1.1), (1.2) is guaranteed by the following assertion.

Lemma 2.1 ([3, Lemma 1])**.** *Let Assumptions* (I)–(III) *hold, and let $\partial\Omega \in C^2$. Assume that $k \geqslant 0$ is a given number, $f \in L^2(\Omega)$, $\beta \in W_2^{3/2}(\partial\Omega)$,*

$$|r(\rho)| \leqslant C_r|\rho|^{p-1} \tag{2.2}$$

for any $\rho \in (-\infty, +\infty)$, where $C_r > 0$, p are constants, $p > 0$ for $n \leqslant 2$ and $0 < p \leqslant n/(n-2)$ for $n > 2$. Then there exists a unique solution u to the problem (1.1), (1.2) *in $W_2^2(\Omega)$.*

The proof of the existence and uniqueness of a solution to the inverse problem (1.1)–(1.3) is based on the following two lemmas for the direct problem (1.1), (1.2).

Lemma 2.2. *Let the assumptions of Lemma* 2.1 *hold, and let $u \in W_2^2(\Omega)$ be a solution to the problem* (1.1), (1.2). *If $f \geqslant 0$, $\beta \geqslant 0$, $k > 0$, then $u \geqslant 0$ almost everywhere in Ω.*

Proof. We take the inner product of (1.1) and $\overline{u} = \min\{u, 0\}$ in $L_2(\Omega)$ and integrate by parts the first term:

$$\langle M\overline{u}, \overline{u}\rangle_M + k\int\limits_\Omega r(\overline{u})\overline{u}dx - \int\limits_\Omega f\overline{u}dx = 0.$$

By (2.1), we have $m_1\|\overline{u}\|_1^2 \leqslant 0$. Thus, $\overline{u} = 0$ almost everywhere in Ω. □

Lemma 2.3. *Let the assumptions of Lemma* 2.1 *hold, and let* $u_1, u_2 \in W_2^2(\Omega)$ *be solutions to the problems*

$$Mu_i + k_i r(u_i) = f_i,$$

$$u_i|_{\partial\Omega} = \beta_i,$$

where $i = 1, 2$*. If* $0 \leqslant k_1 \leqslant k_2$*,* $0 \leqslant \beta_2 \leqslant \beta_1$*,* $0 \leqslant f_2 \leqslant f_1$*, then* $0 \leqslant u_2 \leqslant u_1$ *for almost all* $x \in \overline{\Omega}$*.*

Proof. By Lemma 2.2, $u_i \geqslant 0$, $i = 1, 2$, for almost all $x \in \overline{\Omega}$, which, in view of Assumption (III), implies the nonnegativity of $r(u_i)$, $i = 1, 2$. The difference $u_1 - u_2$ satisfies the equation

$$M(u_1 - u_2) + k_1(r(u_1) - r(u_2)) = r(u_2)(k_2 - k_1) + f_1 - f_2$$

and the boundary condition $u_2 - u_1|_{\partial\Omega} = \beta_1 - \beta_2$. We take the inner product of the equation for $u_1 - u_2$ and $\widetilde{u} = \min\{u_1 - u_2, 0\}$ in $L_2(\Omega)$ and integrate by parts the first term:

$$\langle M\widetilde{u}, \widetilde{u}\rangle_M + k_1\int\limits_\Omega (r(u_1) - r(u_2))\widetilde{u}dx + (k_1 - k_2)\int\limits_\Omega r(u_2)\widetilde{u}dx - \int\limits_\Omega f\widetilde{u}dx = 0,$$

which, in view of the definition of $\widetilde{u}$, implies that $u_1 - u_2 \geqslant 0$ for almost all $x \in \overline{\Omega}$. □

We note that $0 \leqslant r(u_2) \leqslant r(u_1)$ under the assumptions of Lemma 2.3.

3 Existence and Uniqueness Theorem

By a solution to the problem (1.1)–(1.3) we mean a pair $\{u, k\}$ of a function $u \in W_2^2(\Omega)$ and a constant $k \in \mathbf{R}_+$ satisfying Equation (1.1) and the conditions (1.2), (1.3), where $\mathbf{R}_+$ denotes the set of nonnegative real numbers.

We introduce the following additional condition on the function r.

(IV) For any number $R > 0$ and functions $v_1, v_2 \in W_2^1(\Omega)$ such that $\|v_i\|_{L^{2(p-1)}(\Omega)} \leqslant R$, $i = 1, 2$,

$$\|r(v_1) - r(v_2)\| \leqslant c(R)\|u - v\|_1,$$

where the constant $c(R) > 0$ depends on R.

We introduce auxiliary functions a, a^σ, b as solutions to the problem

$$Ma = f(x), \quad a|_{\partial\Omega} = \beta(x), \tag{3.1}$$

$$Ma^\sigma + \sigma r(a^\sigma) = f(x), \quad a^\sigma|_{\partial\Omega} = \beta(x), \tag{3.2}$$

$$Mb = 0, \quad b|_{\partial\Omega} = h(x), \tag{3.3}$$

where $\sigma > 0$ is a real number.

Theorem 3.1. *Let the assumptions of Lemma* 2.1 *and Assumption* (IV) *hold. Assume that*

(i) $f(x) \in L^2(\Omega)$, $\beta(x), h(x) \in W_2^{3/2}(\partial\Omega)$,

(ii) $f(x) \geqslant 0$ *almost everywhere in* Ω, $\beta(x) \geqslant 0$, $h(x) \geqslant 0$ *for almost all* $x \in \partial\Omega$ *and there exists a part* Γ *of the boundary* $\partial\Omega$ *and a constant* $\delta > 0$ *such that* $\beta \geqslant \delta$ *and* $h(x) \geqslant \delta$ *almost everywhere on* Γ.

We also assume that

$$0 \leqslant \Phi \equiv (f, b) - \langle Ma, b\rangle_1 + \mu \leqslant \frac{m_1\,(r(a), b)^2}{4c_0^p\, C_r^{p/(p-1)}\,\Psi},$$

where $\Psi = c(\|a\|_{L^{2p-2}(\Omega)})\|a\|_1^{p-1}\|b\|_1$, c_0 *is the constant of the embedding of* $W_2^1(\Omega)$ *to* $L^p(\Omega)$. *Then the problem* (1.1)–(1.3) *has a solution* $\{u, k\}$. *Furthermore, if*

$$0 \leqslant \Phi \equiv (f, b) - \langle Ma, b\rangle_1 + \mu < \frac{m_1\,(r(a), b)^2}{4c_0^p\, C_r^{p/(p-1)}\,\Psi}, \tag{3.4}$$

then the solution is unique and for some $\sigma > 0$ *the following estimates hold*:

$$0 \leqslant k \leqslant \sigma, \quad a^\sigma \leqslant u \leqslant a, \quad \|u\|_2 \leqslant C_M(\sigma\, C_r\, \|a\|_1^{p-1} + \|a\|) + \|a\|_2, \tag{3.5}$$

where the constant C_M *depends on* m_0, σ, *and* mes Ω.

Proof. Following [4] and [1], we reduce the inverse problem to the operator equation for the coefficient k. For this purpose we introduce the function $w = a - u$. By (1.1)–(1.3), the function w and constant k satisfy the relations

$$Mw + k(r(a) - r(a - w)) = kr(a), \tag{3.6}$$

$$w|_{\partial\Omega} = 0, \tag{3.7}$$

$$\int\limits_{\partial\Omega} \frac{\partial w}{\partial \overline{N}} h(x)ds = \int\limits_{\partial\Omega} \frac{\partial a}{\partial \overline{N}} h(x)ds - \mu = \langle Ma, b\rangle - (f, b) - \mu. \tag{3.8}$$

We take the inner product of (3.6) and b in $L^2(\Omega)$ and twice integrate by parts. Then, taking into account (3.7) and (3.8), we get

$$k(r(u), b) = \Phi.$$

We introduce the operator $A : R_+ \to R$ sending each number $y \in R$ to a number Ay by the rule

$$Ay = \frac{\Phi}{(r(u_y), b)}, \tag{3.9}$$

where u_y is a solution to the problem (1.1), (1.2) for $k = y$. We obtain the assertion of the theorem if we establish the existence of $\sigma > 0$ such that the operator A defined for all $k \in [0, \sigma]$ is continuous on $[0, \sigma]$ and maps $[0, \sigma]$ into itself. From Lemma 2.3 it follows that for $0 \leqslant y \leqslant \sigma$

$$a^\sigma \leqslant u_y \leqslant a. \tag{3.10}$$

By the assumptions of the theorem, $0 < (a^\sigma, b) \leqslant (a,b)$ and, consequently, $0 < (r(a^\sigma), b) \leqslant (r(a), b)$, which implies

$$0 \leqslant Ay \leqslant \frac{\Phi}{(r(a^\sigma), b)}.$$

On the other hand, the difference $a - a^\sigma$ satisfies the equation

$$M(a - a^\sigma) + \sigma(r(a) - r(a^\sigma)) = \sigma r(a)$$

and vanishes on the boundary. Taking the inner product of this equality and $a - a^\sigma$ in $L^2(\Omega)$ and integrating by parts the first term, we find

$$\langle M(a - a^\sigma), a - a^\sigma\rangle_1 + \sigma(r(a) - r(a^\sigma), a - a^\sigma) = \sigma(r(a), a - a^\sigma).$$

By the ellipticity of M, the inequality (2.2), and Assumptions (III), (IV), we have

$$\langle M(a - a^\sigma), a - a^\sigma\rangle_1 \geqslant m_1\|a - a^\sigma\|^2$$

$$\sigma(r(a^\sigma), a - a^\sigma) \leqslant \frac{\sigma^2}{2m_1} C_r^{2p/p-1}\|a\|_1^{2(p-1)} \cdot c_0^2 + \frac{m_1}{2}\|a - a^\sigma\|_1^2.$$

Thus,

$$\|a - a^\sigma\|_1 \leqslant \frac{\sigma}{m_1} C_r^{p/(p-1)} c_0^p \|a\|_1^{p-1}.$$

Using the obtained estimate and the inequality (3.10), we estimate the denominator in (3.9):

$$(r(u_y), b) \geqslant (r(a), b) + (r(a^\sigma) - r(a), b) \geqslant (r(a), b) - \sigma \frac{C_r^{p/(p-1)} c_0^p}{m_1} c(\|a\|_{L^{2p-2}(\Omega)})\|a\|_1^{p-1}\|b\|_1. \quad (3.11)$$

Hence it is clear that the right-hand side of (3.11) is strictly positive if

$$0 < \sigma \leqslant \frac{m_1\ (r(a), b)}{C_r^{p/(p-1)}\ c_0^p\ \Psi}.$$

The operator A transforms the segment $[0, \sigma]$ into itself provided that

$$0 \leqslant Ay \leqslant \frac{\Phi}{(r(a), b) - \frac{C_r^{p/(p-1)} c_0^p}{m_1} \cdot \Psi\sigma},$$

i.e., if σ satisfies the inequality

$$\frac{C_r^{p/(p-1)} c_0^p}{m_1} \Psi \cdot \sigma^2 - (r(a), b) \cdot \sigma + \Phi \leqslant 0. \quad (3.12)$$

Since

$$D = (r(a), b)^2 - 4\Phi \frac{C_r^{p/(p-1)} c_0^p}{m_1} \Psi \geqslant 0,$$

the inequality (3.12) is valid, i.e., the operator A maps $[0, \sigma]$ into itself for

$$\frac{((r(a), b) - \sqrt{D})m_1}{2\Psi C_r^{p/(p-1)} c_0^p} \leqslant \sigma \leqslant \frac{((r(a), b) + \sqrt{D})m_1}{2\Psi C_r^{p/(p-1)} c_0^p}.$$

Now, we estimate u_y in $W_2^1(\Omega)$ for $0 \leqslant y \leqslant \sigma$. Let the function $w_y = a - u_y$ satisfy (3.6)–(3.8) with $k = y$. Taking the inner product of (3.6) and w_y in $L^2(\Omega)$ and integrating by parts the first term, we get

$$\langle Mw, w\rangle_1 + k \int\limits_\Omega (r(a) - r(a-w))w\, dx = k \int\limits_\Omega r(a)w\, dx.$$

By Assumptions (I)–(IV),

$$\langle Mw, w\rangle_1 + k \int\limits_\Omega (r(a) - r(a-w))w\, dx \geqslant m_0 \|w\|_1^2,$$

$$k \int\limits_\Omega r(a)w\, dx \leqslant \sigma \|w\|_{L^p(\Omega)} \|r(a)\|_{L^{p/p-1}(\Omega)} \leqslant \sigma\, C_r^{p/p-1} c_0\, \|a\|_{L^p(\Omega)}^{p-1} \|w\|_1.$$

Thus,

$$\|w\|_1 \leqslant \frac{\sigma}{m_0}\, C_r^{p/(p-1)}\, c_0\, \|a\|_{L^p(\Omega)}^{p-1}.$$

Consequently,

$$\|u\|_1 \leqslant \frac{\sigma}{m_0}\, C_r^{p/(p-1)}\, c_0\, \|a\|_{L^p(\Omega)}^{p-1} + \|a\|_1.$$

Now, we estimate the second order derivatives. We first write (3.6) in the form $Mw = kr(a-w)$. Then

$$\|Mw\| \leqslant \sigma \|r(a-w)\|.$$

Using the definition of w, (2.2), (3.10), and the known inequality

$$\|v\|_2 \leqslant C_M(\|Mv\| + \|v\|),$$

we obtain the required estimate

$$\|w\|_2 \leqslant C_M(\sigma\|r(a-w)\| + \|w\|) \leqslant C_M(\sigma\, C_r \|a\|_1^{p-1} + \|a\|),$$

where the constant C_M depends on m_0, m_1, and mes Ω. Thus,

$$C_r \|a\|_1^{p-1} + \|a\|) + \|a\|_2.$$

We show that the operator A is continuous on $[0, \sigma]$. Let $y_1, y_2 \in [0, \sigma]$. Assume that u_{y_1} and u_{y_2} are solutions to the problem (3.6), (3.7) with $k = y_1$ and $k = y_2$ respectively. By the definition of A,

$$Ay_1 - Ay_2 = \frac{\Phi}{(r(u_{y_1}), b)} - \frac{\Phi}{(r(u_{y_2}), b)} = \frac{\Phi(r(u_{y_2}) - r(u_{y_1}), b)}{(r(u_{y_1}), b)(r(u_{y_2}), b)}.$$

Let us estimate in modulus the right-hand side of the last relation. Taking into account the inequality (3.11) and the definition of Ψ, we find

$$|Ay_1 - Ay_2| \leqslant \Big| \frac{\Phi(r(u_{y_2}) - r(u_{y_1}), b)}{(r(a^\sigma), b)^2} \Big| \leqslant \frac{|\Phi(r(u_{y_2}) - r(u_{y_1}), b)|}{((r(a), b) - \Psi\sigma \frac{C_r^{p/(p-1)} c_0^p}{m_1})^2}. \qquad (3.13)$$

On the other hand, taking the inner product of Equation (3.6) with $k = y_1$, $k = y_2$ and $u_{y_1} - u_{y_2}$ in $L^2(\Omega)$ and integrating by parts the first term, we get

$$\langle M(u_{y_1} - u_{y_2}), u_{y_1} - u_{y_2}\rangle_1 + y_1(r(u_{y_1}) - r(u_{y_2}), u_{y_1} - u_{y_2}) = (y_2 - y_1)(r(u_{y_1}), u_{y_1} - u_{y_2}).$$

By Assumptions (I)–(III) and the nonnegativity of y_1,

$$\langle M(u_{y_1} - u_{y_2}), u_{y_1} - u_{y_2}\rangle_1 + y_1(r(u_{y_1}) - r(u_{y_2}), u_{y_1} - u_{y_2}) \geqslant m_1\|u_{y_1} - u_{y_2}\|_1^2.$$

Using (2.2) for u_{y_1}, the Cauchy inequality, and the Cauchy–Schwarz inequality, we get

$$(y_2 - y_1)(r(u_{y_1}), u_{y_1} - u_{y_2}) \leqslant \frac{c_0^2}{2m_1}|y_2 - y_1|^2 C_r^{2p/(p-1)}\|a\|_{L^p(\Omega)}^{2p-2} + \frac{m_1}{2}\|u_{y_1} - u_{y_2}\|_1^2$$

which implies

$$\|u_{y_1} - u_{y_2}\|_1 \leqslant C_r^{p/(p-1)} c_0 \frac{\|a\|_{L^p(\Omega)}^{p-1}}{m_1}|y_2 - y_1|.$$

Consequently,

$$\|r(u_{y_1}) - r(u_{y_2})\| \leqslant C_r^{p/(p-1)} c_0 \frac{\|a\|_{L^p(\Omega)}^{p-1}}{m_1} c(\|a\|_{L^{2p-2}(\Omega)})|y_2 - y_1|. \tag{3.14}$$

Combining (3.13) with (3.14) and setting

$$\sigma = \frac{((r(a), b) - \sqrt{D})m_1}{2\Psi C_r^{p/(p-1)} c_0^p} = \sigma_0,$$

we obtain the inequality

$$|Ay_1 - Ay_2| \leqslant 4\sqrt{C_r^{2p/(p-1)} c_0^2}\, \frac{\Phi\|b\|\|a\|_{L^p(\Omega)}^{p-1}}{m_1((r(a), b) + \sqrt{D})^2}\, |y_2 - y_1|, \tag{3.15}$$

which implies that the operator A is Lipschitz. By the Brower fixed-point theorem, the operator A has a fixed point $k^* \in [0, \sigma]$ and the pair $\{u^*, k^*\}$ is a solution to the problem (3.6), (3.7) with $k = k^*$.

Let us prove that the fixed point is unique on $[0, \sigma]$. Indeed if (3.5) holds, then

$$q = 4C_r^{p/(p-1)} c_0^p \cdot \frac{\Phi\|b\|\|a\|_{L^p(\Omega)}^{p-1}}{m_1((r(a), b) + \sqrt{D})^2} < \frac{(r(a), b)^2}{((r(a), b) + \sqrt{D})^2} < 1,$$

i.e., the operator A is a contraction on $[0, \sigma]$ with the coefficient of contraction q. Let (u', k') and (u'', k'') be two solutions to the problem (3.6)–(3.8). Then k' and k'' are fixed points of the operator A. By (3.15),

$$|k' - k''| = |Ak' - Ak''| = q|k' - k''|$$

which implies $k' = k''$. Repeating the proof of (3.14), we can obtain a similar relation for $u' - u''$ which means that $u' = u''$ almost everywhere in Ω. Under the assumption (3.4), the solution $\{u, k\}$ continuously depends on the data of the problem. □

Remark 3.1. The set of data satisfying the assumptions of Theorem 3.1 is nonempty. Indeed, let us consider the inverse problem (1.1)–(1.3) for the nonlinear stationary equation governing the anisotropic dissipation in a semiconductor

$$-\Big(\alpha_1\Delta_2 u+\alpha_2\frac{\partial^2 u}{\partial x_3^2}\Big)+k|u|^{p-2}u=0, \tag{3.16}$$

where Δ_2 is the Laplace operator with respect to the variables x_1 and x_2, the parameter k depends on the electric susceptibility, and the constants $\alpha_1>0$ and $\alpha_2>0$ are determined by the tensor of electric polarizability of the semiconductor. Without loss of generality we can assume that the inverse problem is considered in the domain $\Omega\subset\mathbf{R}^3_+=\{x|x=(x_1,x_2,x_3)\in\mathbf{R}^3, x_i\geqslant 0, i=1,2,3\}$. In this case, the operator $M=-(\alpha_1\Delta_2+\alpha_2\partial^2/\partial x_3^2)$ satisfies Assumptions (I) and (II). The function $r(u)=|u|^{p-2}u$ with $p=3$ satisfies Assumptions (III) and (IV) with $c(R)=2c_0R$ and the inequality (2.2) with $C_r=1$. In (1.2) and (1.3), we set $\beta(x)=d_1x_1+d_2x_2+d_3x_3+d_4$ and $h(x)=h_1x_1+h_2x_2+h_3x_3+h_4$, where $d_4>0$, $h_4>0$, $d_i\geqslant 0$, $h_i\geqslant 0$, $i=1,2,3$. Then the solutions to the problems (3.1) and (3.3) take the form $a(x)=d_1x_1+d_2x_2+d_3x_3+d_4$ and $b(x)=h_1x_1+h_2x_2+h_3x_3+h_4$ in the whole domain Ω and satisfy (i) and the nonnegativity condition in (ii). If $\mu>0$ and the domain Ω is sufficiently small, then (3.4) holds. Indeed,

$$\langle Ma,b\rangle_1=[\alpha_1(d_1h_1+d_2h_2)+\alpha_2d_3h_3]\int\limits_\Omega dx\equiv\alpha \text{ mes }\Omega.$$

The constant c_0 of the embedding of $W_2^1(\Omega)$ to $L^3(\Omega)$ can be estimated as follows:

$$c_0\leqslant\frac{\sqrt{5}}{2}(1+\text{mes }\Omega)^{1/2}+(\text{mes }\Omega)^{5/6}.$$

We obtain (3.4) from the relations

$$0\leqslant\mu-\alpha\text{ mes }\Omega\leqslant\frac{m_1d_4^4h_4^2(\text{mes }\Omega)^{-1/4}}{4K[\sqrt{5}(1+\text{mes }\Omega)^{1/2}+2(\text{mes }\Omega)^{5/6}]^4}$$

if $\mu>0$ and mes Ω is sufficietnly small. Here,

$$K=[(\max_{x\in\overline{\Omega}}a)^2+d_1^2+d_2^2+d_3^2]^3[(\max_{x\in\overline{\Omega}}b)^2+h_1^2+h_2^2+h_3^2]\max_{x\in\overline{\Omega}}a.$$

According to Theorem 3.1, the inverse problem for Equation (3.16) with conditions (1.2), (1.3) has a unique solution.

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