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DYNAMICS OF TWO BODIES WITH TRAJECTORIES ON A FIXED STRAIGHT LINE WITH REGARD FOR THE FINITE SPEED OF GRAVITY

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We study the dynamics of two bodies moving along an immobile straight line with regard for a finite speed of gravity. It is shown that the escape velocity is higher than the corresponding velocity in the classical celestial mechanics. We present estimates for this velocity.

1. Introduction

In the present paper, we establish the fact that the escape velocities in the Newton celestial mechanics and in the mechanics with a finite speed of gravity do not coincide. This property of cosmic velocities is obtained for the case of two celestial bodies by using a system of differential equations with delayed argument and functional equations, which, together with certain additional conditions imposed on the solutions of equations, form a mathematical model of motion of the analyzed bodies. These equations better describe the dynamics of bodies than the ordinary differential equations of Newton mechanics. The necessity of application of these equations is caused by the fact that the speed of gravity is finite. This property of gravity agrees both with the Einstein general theory of relativity in which, for the speed of gravity c_g, it is assumed that $c_g = c$, where c is the speed of light [1, 2] and with the experimental data of evaluation of the rate of influence of the gravitational fields on the results of measurement [3] and with the data of measurements of the speed of gravity performed by detecting gravitational waves from remote star sources simultaneously with their light signals [4].

In [5], the author studied the motion of two bodies with regard for a finite speed of gravity and revealed the non-Keplerian character and instability of motion of the bodies.

In the present paper, we investigate the motion of two bodies located on a fixed straight line in the case where $c_g = c$. It is shown that, in this case, the escape velocity is higher than in the classical celestial mechanics. We obtain estimates for their difference. The difference between the indicated escape velocities is caused by the delay of the gravitational field.

By using the law of gravitation with a finite speed of gravity and the second Newton's law, we derive the equations of motion of the bodies with regard for the delay of gravitational fields and analyze the dynamics of the bodies.

2. Law of Gravitation for a Finite Speed of Gravity

Consider two points M_1 and M_2 with masses m_1 and m_2, respectively. These points move in the space according to the law of gravitation and Newton's second law. We consider the process of motion of these points in an inertial Cartesian coordinate system x, y, z with origin at the point O. Assume that each point is subjected to the action of the gravitational field generated by the other point. The locations of the points M_1 and M_2 at time t are specified by their radius vectors $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

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To study the motion of the points M_1 and M_2, it is necessary to know the forces of their mutual attraction.

In the case where the speed of gravity is infinite, as in the classical celestial mechanics, according to the law of gravitation, the point M_2 attracts the point M_1 at time t by a force

$$\vec{F}_{2,1,\infty}(t) = \frac{Gm_1m_2}{\left|\vec{r}_2(t) - \vec{r}_1(t)\right|^3} \left(\vec{r}_2(t) - \vec{r}_1(t)\right),$$

where G is the gravitational constant and $\left|\vec{r}_2(t) - \vec{r}_1(t)\right|$ is the Euclidean length of the vector $\vec{r}_2(t) - \vec{r}_1(t)$. The direction of this force coincides with the direction of the vector $\vec{r}_2(t) - \vec{r}_1(t)$.

Similarly, at time t, the point M_1 attracts the point M_2 by the force

$$\vec{F}_{1,2,\infty}(t) = \frac{Gm_1m_2}{\left|\vec{r}_1(t) - \vec{r}_2(t)\right|^3} \left(\vec{r}_1(t) - \vec{r}_2(t)\right).$$

Since, according to Sec. 1, the speed of gravity is finite, the action of one point upon the other occurs with a certain delay of the gravitational field. Therefore, a somewhat different force

$$\vec{F}_{2,1,c}(t) = \frac{Gm_1m_2}{\left|\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right|^2} \left(\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right) \tag{1}$$

acts upon the point M_1. Here, the delay of gravity $\tau_{2,1}(t)$ in (1) satisfies the following relation:

$$c\tau_{2,1}(t) = \left|\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right|, \tag{2}$$

and c is the speed of gravity (see [5, 6]).

The *attracting point for the point M_1 at time t is not the point M_2 but the point that coincides with the end of the vector $\vec{r}_2(t-\tau_{2,1}(t))$.*

Similarly, the force

$$\vec{F}_{1,2,c}(t) = \frac{Gm_1m_2}{\left|\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right|^3} \left(\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right) \tag{3}$$

acts upon the point M_2. The *attractor for the point M_2 at time t is not the point M_1* but the point that coincides with the end of the vector $\vec{r}_1(t-\tau_{1,2}(t))$, and the delay of gravity $\tau_{1,2}(t)$ in relation (3) satisfies the relation

$$c\tau_{1,2}(t) = \left|\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right|. \tag{4}$$

The forces $\vec{F}_{2,1,c}(t)$ and $\vec{F}_{1,2,c}(t)$ may be different and not collinear. According to (2) and (4), for any time t,

$$\lim_{c\to+\infty} \vec{F}_{i,j,c}(t) = \vec{F}_{i,j,\infty}(t) \qquad \text{and} \qquad \lim_{c\to+\infty} \tau_{i,j}(t) = 0, \quad i \neq j. \tag{5}$$

3. Mathematical Model of Motion of the Points M_1 and M_2

In view of the second Newton's law, the law of gravitation with finite speed of gravity, and relations (1)–(4), the process of motion of the points M_1 and M_2 is described by the following system of differential equations with deviating argument and functional equations:

$$\begin{cases} m_1\ddot{\vec{r}}_1(t) = \dfrac{Gm_1m_2}{\left|\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right|^3}\left(\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right), \\ m_2\ddot{\vec{r}}_2(t) = \dfrac{Gm_1m_2}{\left|\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right|^3}\left(\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right), \\ c\tau_{2,1}(t) = \left|\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right|, \\ c\tau_{1,2}(t) = \left|\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right|. \end{cases} \tag{6}$$

The detailed description of system (6) and the results of its investigation can be found in [5–7].

It is clear that the following relation must be true in addition to system (6):

$$\left|\vec{r}_2(t-\tau_{2,1}(t)) - \vec{r}_1(t)\right| \cdot \left|\vec{r}_1(t-\tau_{1,2}(t)) - \vec{r}_2(t)\right| \neq 0. \tag{7}$$

According to (5), system (6) is a generalization of the classical model of motion of the points M_1 and M_2:

$$\begin{cases} m_1\ddot{\vec{r}}_1(t) = \dfrac{Gm_1m_2}{\left|\vec{r}_2(t) - \vec{r}_1(t)\right|^3}\left(\vec{r}_2(t) - \vec{r}_1(t)\right), \\ m_2\ddot{\vec{r}}_2(t) = \dfrac{Gm_1m_2}{\left|\vec{r}_1(t) - \vec{r}_2(t)\right|^3}\left(\vec{r}_1(t) - \vec{r}_2(t)\right), \end{cases} \tag{8}$$

which follows from (6) as $c \to +\infty$.

In view of (7), the inequality $\vec{r}_2(t) \neq \vec{r}_1(t)$ must hold for system (8).

For the determination of the trajectories of motion of the points M_1 and M_2, in addition to the system of equations (6), it is necessary to use initial or boundary conditions (see [5–7]). System (6), together with these conditions, forms a mathematical model of motion of the points M_1 and M_2 with regard for the finite speed of gravity.

4. Investigation of Rectilinear Motion of the Points M_1 and M_2

We study the motion of the points M_1 and M_2 located on a fixed straight line.

Without loss of generality, we can assume that the points M_1 and M_2 lie on the Ox-axis with directional unit vector $\vec{i}$, as shown in Fig. 1, and the velocity of the origin of coordinates (the point O) is equal to zero.

We assume that this coordinate system is inertial.

It is clear that the location of the points O, M_1, and M_2 on the Ox-axis depends on time t. An important requirement is that the location of the points M_1 and M_2 must be such that the vector $\overrightarrow{M_1M_2}$ be nonzero ($\overrightarrow{M_1M_2} \neq \vec{0}$; the absence of collisions of the points) and its direction coincide with the direction of the vector $\vec{i}$, as shown in Fig. 1.

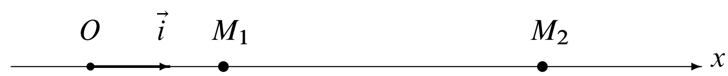

Fig. 1. Location of the points M_1 and M_2 at time t.

In this case, the motion of the points M_1 and M_2 can be described by the vector functions $\vec{r}_1(t) = x_1(t)\,\vec{i}$ and $\vec{r}_2(t) = x_2(t)\,\vec{i}$, where $x_1(t)$ and $x_2(t)$ are scalar functions with values in $\mathbb{R}$ whose properties should be investigated.

According to (6) (i.e., for $c_g = c$), the functions $x_1(t)$ and $x_2(t)$ are solutions of the system of equations

$$\begin{cases} \ddot{x}_1(t) = Gm_2 \dfrac{x_2(t-\tau_{2,1}(t)) - x_1(t)}{|x_2(t-\tau_{2,1}(t)) - x_1(t)|^3}, \\ \ddot{x}_2(t) = Gm_1 \dfrac{x_1(t-\tau_{1,2}(t)) - x_2(t)}{|x_1(t-\tau_{1,2}(t)) - x_2(t)|^3}, \\ c\tau_{2,1}(t) = |x_2(t-\tau_{2,1}(t)) - x_1(t)|\,, \\ c\tau_{1,2}(t) = |x_1(t-\tau_{1,2}(t)) - x_2(t)|\,. \end{cases} \tag{9}$$

At the same time, according to (8) ($c_g = \infty$), these functions are solutions of the following system of equations:

$$\begin{cases} \ddot{x}_1(t) = Gm_2 \dfrac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|^3}, \\ \ddot{x}_2(t) = Gm_1 \dfrac{x_1(t) - x_2(t)}{|x_1(t) - x_2(t)|^3}. \end{cases} \tag{10}$$

The properties of the solutions $x_1(t)$ and $x_2(t)$ of systems (9) and (10) depend on the initial time $t_0 \in \mathbb{R}$ and the initial conditions.

We investigate the motion of the point M_2 relative to the point M_1, i.e., the quantity $x_2(t) - x_1(t)$.

Consider properties of the quantity $x_2(t) - x_1(t)$ for each system (9) and (10) separately. First, we present some properties of solutions of the well investigated system (10) (see, e.g., [8–11]) required for analysis of the properties of solutions of the main system (9).

4.1. Case of System (10). We take into account the values $x_1(t_0)$ and $x_2(t_0)$ of the solutions $x_1(t)$ and $x_2(t)$ of system (10) and their derivatives $v_1(t_0) = \dot{x}_1(t_0)$ and $v_2(t_0) = \dot{x}_2(t_0)$ at the initial time t_0. Here, $v_1(t_0)$ and $v_2(t_0)$ are the velocities of the points M_1 and M_2 at the time t_0, respectively. Assume that

$$x_1(t_0) < x_2(t_0), \tag{11}$$

which agrees with the condition concerning the location of the points M_1 and M_2.

Consider the quantity

$$v_{2,\infty}^* = \sqrt{\frac{2G(m_1+m_2)}{x_2(t_0)-x_1(t_0)}}. \tag{12}$$

The following statement is true:

Theorem 1. *Suppose that relation (11) holds.*
Then the following assertions are true for system (10):

(i) *if*

$$v_2(t_0) - v_1(t_0) \in (-c, 0], \tag{13}$$

then there exits a number $T > t_0$ such that the function $x_2(t) - x_1(t)$ is strictly decreasing on the interval $[t_0, T)$ and $\lim\limits_{t\to T-0}(x_2(t) - x_1(t)) = 0$ (the points M_1 and M_2 collide at the time T);

(ii) *if*

$$v_2(t_0) - v_1(t_0) \in \big(0, v_{2,\infty}^*\big), \tag{14}$$

then, for some numbers T_1 and T_2 ($t_0 < T_1 < T_2$), the function $x_2(t) - x_1(t)$ is strictly increasing and strictly decreasing on the intervals $[t_0, T_1)$ and $[T_1, T_2)$, respectively, and $\lim\limits_{t\to T_2-0}(x_2(t) - x_1(t)) = 0$ (the points M_1 and M_2 collide at the time T_2);

(iii) *if*

$$v_2(t_0) - v_1(t_0) \geq v_{2,\infty}^*, \tag{15}$$

then the function $x_2(t) - x_1(t)$ strictly increases on $[t_0, +\infty)$ and $\lim\limits_{t\to+\infty}(x_2(t) - x_1(t)) = +\infty$.

Proof. We first present some relations necessary in what follows.
An important equality

$$\ddot{x}_2(t) - \ddot{x}_1(t) = -G(m_1+m_2)\frac{x_2(t)-x_1(t)}{|x_2(t)-x_1(t)|^3} \tag{16}$$

follows from (10). By using this equality, we get

$$\frac{d(\dot{x}_2(t)-\dot{x}_1(t))^2}{dt} = -2G(m_1+m_2)\,\frac{x_2(t)-x_1(t)}{|x_2(t)-x_1(t)|^3}\,(\dot{x}_2(t)-\dot{x}_1(t)).$$

In view of (11) and the equality $(\dot{x}_2(t) - \dot{x}_1(t))\,dt = d(x_2(t) - x_1(t))$, we conclude that, for all $t > t_0$,

$$(\dot{x}_2(t)-\dot{x}_1(t))^2 = (v_2(t_0)-v_1(t_0))^2 - 2G(m_1+m_2)\int\limits_{t_0}^{t}\frac{x_2(s)-x_1(s)}{|x_2(s)-x_1(s)|^3}\,d(x_2(s)-x_1(s))$$

$$= (v_2(t_0) - v_1(t_0))^2 - 2G(m_1 + m_2) \int_{t_0}^{t} \frac{d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s))^2}$$

$$= (v_2(t_0) - v_1(t_0))^2 - \frac{2G(m_1 + m_2)}{x_2(t_0) - x_1(t_0)} + \frac{2G(m_1 + m_2)}{x_2(t) - x_1(t)} \tag{17}$$

if $x_2(s) > x_1(s)$ for every $s \in [t_0, t)$. The set of these t is nonempty by virtue of (11) and the continuity of the functions $x_1(t)$ and $x_2(t)$ at the point t_0.

Relation (16) also implies that, for all $t > t_0$,

$$\dot{x}_2(t) - \dot{x}_1(t) = v_2(t_0) - v_1(t_0) - \int_{t_0}^{t} \frac{G(m_1 + m_2)\, ds}{|x_2(s) - x_1(s)|^2} \tag{18}$$

and

$$x_2(t) - x_1(t) = x_2(t_0) - x_1(t_0) + (v_2(t_0) - v_1(t_0))(t - t_0) - \int_{t_0}^{t} \int_{t_0}^{\tau} \frac{G(m_1 + m_2)\, ds}{|x_2(s) - x_1(s)|^2}\, d\tau \tag{19}$$

provided that $x_2(s) > x_1(s)$ for any $s \in [t_0, t)$.

We use these relations to prove the assertions of the theorem.

Assume that inclusion (13) is true.

In view of inequality (11) and Eqs. (16) and (18), the point M_2 moves relative to the point M_1 with a deceleration equal to

$$-\frac{G(m_1 + m_2)}{(x_2(t) - x_1(t))^2} < 0.$$

Hence, in view of (18) and (19), for some $T > t_0$, the functions $\dot{x}_2(t) - \dot{x}_1(t)$ and $x_2(t) - x_1(t)$ are strictly decreasing on the interval $[t_0, T)$ and $\lim\limits_{t \to T-0} (x_2(t) - x_1(t)) = 0$.

From the mechanical point of view, this corresponds to the collision of the points M_1 and M_2 at time T.

Thus, the first part of the assertion of the theorem is true.

Assume that inclusion (14) holds.

In this case, the function $x_2(t) - x_1(t)$ cannot be both positive and unbounded on $[t_0, +\infty)$ because, in view of (17), the function $(\dot{x}_2(t) - \dot{x}_1(t))^2$ takes negative values for some sufficiently large t [here, we take into account (12)], which is impossible.

The function $x_2(t) - x_1(t)$ cannot also be positive and bounded on the interval $[t_0, +\infty)$ because, in this case, in view of (18), for some $t_1 > t_0$, the function $\dot{x}_2(t) - \dot{x}_1(t)$ takes negatives values for $t > t_1$ and $\dot{x}_2(t) - \dot{x}_1(t)$ monotonically decreases. Hence, $\lim\limits_{t \to t_2-0} (x_2(t) - x_1(t)) = 0$ for some $t_2 > t_1$ and $x_2(t_2) - x_1(t_2) > 0$, which is impossible.

Thus, the continuous function $x_2(t) - x_1(t)$ is positive and bounded on some interval $[t_0, T_2)$ and $\lim\limits_{t \to T_2-0} (x_2(t) - x_1(t)) = 0$.

Indeed, in view of (14) and (18), the function $x_2(t) - x_1(t)$ is strictly increasing on some interval $[t_0, T_1)$, where T_1 is the time for which $\dot{x}_2(T_1) - \dot{x}_1(T_1) = 0$, and strictly decreasing on the interval $[T_1, T_2)$, where T_2 is the time for which $\lim\limits_{t \to T_2-0} (x_2(t) - x_1(t)) = 0$.

The equality $\lim_{t\to T_2-0}(x_2(t)-x_1(t))=0$ corresponds to the collision of points M_1 and M_2 at the time T_2. Thus, the second assertion of the theorem is also true.

Assume that inclusion (15) is true. Thus, by virtue of (12) and (17), we get

$$(\dot{x}_2(t)-\dot{x}_1(t))^2 \geq \frac{2G(m_1+m_2)}{x_2(t)-x_1(t)} > 0 \quad \text{for all} \quad t\geq t_0,$$

whence, by using relation (18), we conclude that

$$\dot{x}_2(t)-\dot{x}_1(t) > 0 \quad \text{for all} \quad t\geq t_0. \tag{20}$$

Hence, the function $x_2(t)-x_1(t)$ is strictly increasing on the interval $[t_0,+\infty)$.

The function $x_2(t)-x_1(t)$ cannot be bounded on the interval $[t_0,+\infty)$ because, in this case, in view of (18), the function $\dot{x}_2(t)-\dot{x}_1(t)$ takes negative values for sufficiently large t, which contradicts (20) and, hence,

$$\lim_{t\to+\infty}(x_2(t)-x_1(t))=+\infty. \tag{21}$$

Thus, the third part of assertion of the theorem is also true.

Theorem 1 is proved.

Remark 1. In Theorem 1, the quantity $v^*_{2,\infty}$ given by equality (12) is the *escape velocity* in the case of classical celestial mechanics. This is the minimal relative velocity $v_2(t_0)-v_1(t_0)$ of motion of the point M_2 relative to the point M_1 (with coordinates $x_2(t_0)$ and $x_1(t_0)$, respectively) at the time t_0 for which relation (21) can be true.

4.2. The Case of System (9). We now show that *the assertion similar to Theorem 1 is true for system (9) and the corresponding escape velocity is higher than* $v^*_{2,\infty}$.

The trajectory of motion of the point M_2 relative to the point M_2 depends on the values of functions $x_1(t)$ and $x_2(t)$ and their derivatives on the segments $[t_0-\tau_{2,1}(t_0),t_0]$ and $[t_0-\tau_{1,2}(t_0),t_0]$, respectively (we do not indicate the values of the derivatives $v_1(t)=dx_1(t)/dt$ and $v_2(t)=dx_2(t)/dt$ on the corresponding segments because they are determined by the functions $x_1(t)$ and $x_2(t)$).

In view of (9), we find

$$\ddot{x}_2(t)-\ddot{x}_1(t) = -Gm_1\frac{x_2(t)-x_1(t-\tau_{1,2}(t))}{|x_1(t-\tau_{1,2}(t))-x_2(t)|^3} - Gm_2\frac{x_2(t-\tau_{2,1}(t))-x_1(t)}{|x_2(t-\tau_{2,1}(t))-x_1(t)|^3}. \tag{22}$$

Note that the numerators of the terms on the right-hand side of (22) are positive.

Indeed, assume that

$$x_2(t)-x_1(t-\tau_{1,2}(t)) < 0. \tag{23}$$

Then, in view of the fourth equation in system (9), we get

$$c\tau_{1,2}(t) = |x_2(t)-x_1(t-\tau_{1,2}(t))| = -(x_2(t)-x_1(t-\tau_{1,2}(t)))$$

$$= -(x_2(t) - x_1(t)) - (x_1(t) - x_1(t - \tau_{1,2}(t))) = -(x_2(t) - x_1(t)) - v_1\tau_{1,2}(t),$$

where

$$v_1 = \frac{x_1(t) - x_1(t - \tau_{1,2}(t))}{\tau_{1,2}(t)},$$

and, hence, $(-c - v_1)\tau_{1,2}(t) = x_2(t) - x_1(t)$. By using this result and the positivity of $\tau_{1,2}(t)$ and $x_2(t) - x_1(t)$, we obtain $-c - v_1 > 0$. This inequality is true only for $-v_1 > c$, which is impossible according to [1] (the velocity of the point M_1 cannot be greater than c).

Thus, the assumption that relation (23) holds is not true. Therefore, by using (7), we get

$$x_2(t) - x_1(t - \tau_{1,2}(t)) > 0. \tag{24}$$

Similarly, we find

$$x_2(t - \tau_{2,1}(t)) - x_1(t) > 0. \tag{25}$$

In view of (24) and (25), relation (22) can be rewritten in the form

$$\ddot{x}_2(t) - \ddot{x}_1(t) = -\frac{Gm_1}{(x_2(t) - x_1(t - \tau_{1,2}(t)))^2} - \frac{Gm_2}{(x_2(t - \tau_{2,1}(t)) - x_1(t))^2}. \tag{26}$$

In what follows, we need the closed and open initial intervals

$$I_{t_0} = [t_0 - \max\{\tau_{1,2}(t_0), \tau_{2,1}(t_0)\}, t_0] \qquad \text{and} \qquad \operatorname{int} I_{t_0} = (t_0 - \max\{\tau_{1,2}(t_0), \tau_{2,1}(t_0)\}, t_0).$$

To study the motion of the point M_2 relative to the point M_1, we specify the initial values $\psi_1(s)$ and $\psi_2(s)$ for $x_1(t)$ and $x_2(t)$ on the interval I_{t_0} by assuming that:

(1) $\psi_1(s)$ and $\psi_2(s)$ are continuously differentiable on int I_{t_0} and continuous on I_{t_0};

(2) $\psi_2(s) - \psi_1(s) > 0$ for all $s \in I_{t_0}$ (the point M_2 cannot coincide with the point M_1);

(3) the derivatives $\dot{\psi}_1(s)$ and $\dot{\psi}_2(s)$ are bounded and integrable on int I_{t_0} and the limits $\lim\limits_{s \to t_0 - 0} \dot{\psi}_i(s)$, $i = \overline{1, 2}$, exist.

According to the third and fourth equations in system (9), the deviations $\tau_{1,2}(t_0)$ and $\tau_{2,1}(t_0)$ satisfy the relations

$$c\tau_{2,1}(t_0) = \psi_2(t_0 - \tau_{2,1}(t_0)) - \psi_1(t_0),$$

$$c\tau_{1,2}(t_0) = \psi_2(t_0) - \psi_1(t_0 - \tau_{1,2}(t_0)),$$

$$x_1(t - \tau_{1,2}(t)) = \psi_1(t - \tau_{1,2}(t))$$

for all $t \geq t_0$ such that $t - \tau_{1,2}(t) \in [t_0 - \tau_{1,2}(t_0), t_0]$ and

$$x_2(t - \tau_{2,1}(t)) = \psi_2(t - \tau_{2,1}(t))$$

for all $t \geq t_0$ such that $t - \tau_{2,1}(t) \in [t_0 - \tau_{2,1}(t_0), t_0]$.

Here, $t - \tau_{2,1}(t) > t_0 - \tau_{2,1}(t_0)$ for all $t > t_0$. Indeed, in view of the fact that $c\tau_{2,1}(t) = x_2(t - \tau_{2,1}(t)) - x_1(t)$, for all $t > t_0$ and a sufficiently small number $\Delta > 0$, we get

$$\begin{aligned} c\,\frac{\tau_{2,1}(t+\Delta) - \tau_{2,1}(t)}{\Delta} &= \frac{x_2(t+\Delta-\tau_{2,1}(t+\Delta)) - x_2(t-\tau_{2,1}(t))}{\Delta - (\tau_{2,1}(t+\Delta) - \tau_{2,1}(t))} \\ &\quad\times \frac{\Delta - (\tau_{2,1}(t+\Delta) - \tau_{2,1}(t))}{\Delta} - \frac{x_1(t+\Delta) - x_1(t)}{\Delta} \\ &= v_2^*\left(1 - \frac{\tau_{2,1}(t+\Delta) - \tau_{2,1}(t)}{\Delta}\right) - v_1^*, \end{aligned}$$

where

$$v_1^* = \frac{x_1(t+\Delta) - x_1(t)}{\Delta} \qquad \text{and} \qquad v_2^* = \frac{x_2(t+\Delta-\tau_{2,1}(t+\Delta)) - x_2(t-\tau_{2,1}(t))}{\Delta - (\tau_{2,1}(t+\Delta) - \tau_{2,1}(t))}.$$

According to [1], the velocities of the points M_1 and M_2 are lower than c, which means that, for every $t > t_0$ and a sufficiently small number $\Delta > 0$, we have

$$1 - \frac{\tau_{2,1}(t+\Delta) - \tau_{2,1}(t)}{\Delta} = \frac{c + v_1^*}{c + v_2^*} > 0.$$

Therefore, the function $t - \tau_{2,1}(t)$ is strictly increasing and the corresponding relation is true.

Similarly, $t - \tau_{1,2}(t) > t_0 - \tau_{1,2}(t_0)$ for all $t > t_0$.

At the time t_0, we also specify the velocities $v_1(t_0)$ and $v_2(t_0)$ of the points M_1 and M_2, respectively. We do not impose the following requirements:

$$\lim_{s \to t_0 - 0} \dot{\psi}_1(s) = v_1(t_0) \qquad \text{and} \qquad \lim_{s \to t_0 - 0} \dot{\psi}_2(s) = v_2(t_0). \tag{27}$$

The initial velocities $v_1(s)$ and $v_2(s)$ on int I_{t_0} are determined by the functions $\psi_1(s)$ and $\psi_1(s)$.

By analogy with [7], we can show that system (9) with the initial conditions introduced above is uniquely solvable. The solution of the system is twice continuously differentiable at the points $t > t_0$, where relation (7) is true, and continuous at the point t_0. The derivatives $dx_1(t)/dt$ and $dx_2(t)/dt$ have jumps at this point if equalities (27) are not true.

According to the initial conditions given above and relation (26), $v_2(t) - v_1(t)$ and $x_2(t) - x_1(t)$ satisfy the integral relations

$$\begin{aligned} v_2(t) - v_1(t) &= v_2(t_0) - v_1(t_0) \\ &\quad - \int_{t_0}^{t} \left(\frac{Gm_1}{(x_2(s) - x_1(s - \tau_{1,2}(s)))^2} + \frac{Gm_2}{(x_2(s - \tau_{2,1}(s)) - x_1(s))^2} \right) ds \end{aligned} \tag{28}$$

Fig. 2. Location of the points M_1 and M_2 at time t.

and

$$x_2(t) - x_1(t) = x_2(t_0) - x_1(t_0) + (t - t_0)(v_2(t_0) - v_1(t_0))$$

$$- \int_{t_0}^{t} \left(\int_{t_0}^{\tau} \left(\frac{Gm_1}{(x_2(s) - x_1(s - \tau_{1,2}(s)))^2} + \frac{Gm_2}{(x_2(s - \tau_{2,1}(s)) - x_1(s))^2} \right) ds \right) d\tau, \quad (29)$$

where t is an arbitrary time from an interval $\left[t_0, t^*\right) \subset [t_0, +\infty)$ on which $x_2(t) - x_1(t) > 0$.

To study the dynamics of the point M_2 relative to the point M_1, we use relations (28) and (29).

Further, we fix the functions $\psi_1(s)$ and $\psi_2(s)$ and the velocity $v_1(t_0)$. The velocity $v_2(t_0)$ may take arbitrary values. For the velocities $v_1(t_0)$ and $v_2(t_0)$, by $\mathcal{V}_1$ and $\mathcal{V}_2$ we denote the sets of all differences $v_2(t_0) - v_1(t_0)$ for each of which the function $x_2(t) - x_1(t)$ that describes the motion of the point M_2 relative to the point M_1 is bounded or unbounded on $[t_0, +\infty)$, respectively.

Further, we use one more important property of system (9). Up to this point, we performed the required preparatory work by using the inertial coordinate system corresponding to Fig. 1. In what follows, we consider an inertial system $\tilde{O}\tilde{x}$ in which the point $\tilde{O}$ moves relative to the point O with a constant velocity $\vec{v} = \tilde{v}\vec{i}$ ($\tilde{v}$ can be an arbitrary element of the set $\mathbb{R}$). The directional vector of the $\tilde{O}\tilde{x}$-axis coincides with the vector $\vec{i}$ and the points M_1 and M_2 are located on the $\tilde{O}\tilde{x}$-axis (Fig. 2).

We describe the motion of the points M_1 and M_2 by new vector functions $\vec{\tilde{r}}_1(t) = \tilde{x}_1(t)\vec{i}$ and $\vec{\tilde{r}}_2(t) = \tilde{x}_2(t)\vec{i}$.

An obvious statement presented below is important for our subsequent presentation.

Lemma 1. *For any $\tilde{v} \in \mathbb{R}$, the following identities are true for the points M_1 and M_2 :*

$$\begin{gathered} \tilde{x}_2(t) - \tilde{x}_1(t) \equiv x_2(t) - x_1(t), \\ \dot{\tilde{x}}_2(t) - \dot{\tilde{x}}_1(t) \equiv \dot{x}_2(t) - \dot{x}_1(t), \\ \ddot{\tilde{x}}_2(t) - \ddot{\tilde{x}}_1(t) \equiv \ddot{x}_2(t) - \ddot{x}_1(t), \\ \ddot{\tilde{x}}_1(t) \equiv \ddot{x}_1(t) \quad \text{and} \quad \ddot{\tilde{x}}_2(t) \equiv \ddot{x}_2(t). \end{gathered} \quad (30)$$

Rewriting relation (26) in the new inertial system (Fig. 2), we get

$$\ddot{\tilde{x}}_2(t) - \ddot{\tilde{x}}_1(t) = -\frac{Gm_1}{(\tilde{x}_2(t) - \tilde{x}_1(t - \tau_{1,2}(t)))^2} - \frac{Gm_2}{(\tilde{x}_2(t - \tau_{2,1}(t)) - \tilde{x}_1(t))^2}. \quad (31)$$

Note that the deviations $\tau_{1,2}(t)$ and $\tau_{2,1}(t)$ and the time variable t are independent of $\tilde{v}$.

By virtue of Lemma 1 and relations (26) and (31), the following statement is true:

Corollary 1. *For the points M_1 and M_2, the relation*

$$\frac{Gm_1}{(x_2(t)-x_1(t-\tau_{1,2}(t)))^2}+\frac{Gm_2}{(x_2(t-\tau_{2,1}(t))-x_1(t))^2}$$
$$=\frac{Gm_1}{(\tilde{x}_2(t)-\tilde{x}_1(t-\tau_{1,2}(t)))^2}+\frac{Gm_2}{(\tilde{x}_2(t-\tau_{2,1}(t))-\tilde{x}_1(t))^2}$$

is true for all $\tilde{v}\in\mathbb{R}$.

Further, we consider system (9) in the inertial coordinate system corresponding to Fig. 1.

The following auxiliary statements explaining the properties of motion of the point M_2 relative to the point M_1 are true:

Lemma 2. $\mathcal{V}_1\neq\varnothing$ *and* $v^*_{2,\infty}\in\mathcal{V}_1$.

Proof. Let $v_2(t_0)-v_1(t_0)\in(-c,0]$. According to (26), the velocity of the point M_2 relative to the point M_1 is negative and the value of this velocity strictly decreases because

$$\frac{Gm_1}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}+\frac{Gm_2}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2}>0.$$

Hence, the function $x_2(t)-x_1(t)$ that describes the motion of the point M_2 relative to the point M_1 is strictly decreasing and $\lim\limits_{t\to T-0}(x_2(t)-x_1(t))=0$ for some $T>t_0$.

From the mechanical point of view, this means that the points M_1 and M_2 collide at the time T. Thus, $\mathcal{V}_1\neq\varnothing$.

In what follows, we need the relations

$$\begin{aligned}&\frac{d\big(v_2(t)-v_1(t)\big)^2}{dt}\\&\quad=2\big(v_2(t)-v_1(t)\big)(\dot{v}_2(t)-\dot{v}_1(t))\\&\quad=2\big(v_2(t)-v_1(t)\big)\left(-\frac{Gm_1}{(x_2(t)-x_1(t-\tau_{1,2}(t)))^2}-\frac{Gm_2}{(x_2(t-\tau_{2,1}(t))-x_1(t))^2}\right)\end{aligned}\tag{32}$$

and

$$\begin{aligned}\big(v_2(t)-v_1(t)\big)^2&=(v_2(t_0)-v_1(t_0))^2\\&\quad-\int_{t_0}^{t}\frac{2Gm_1\,d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}-\int_{t_0}^{t}\frac{2Gm_2\,d(x_2(s)-x_1(s))}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2},\end{aligned}\tag{33}$$

which can be deduced by using (26) and the equality

$$\big(v_2(t)-v_1(t)\big)\,dt=d(x_2(t)-x_1(t)).$$

We now show that the inclusion $v^*_{2,\infty} \in \mathcal{V}_1$ is true. Assume that this inclusion does not hold, i.e., in the case where $v_2(t_0) - v_1(t_0) = v_{2,\infty}$, the following relations are true:

$$v_2(t) - v_1(t) > 0 \quad \text{for all} \quad t > t_0 \tag{34}$$

and

$$\lim_{t\to+\infty} (x_2(t) - x_1(t)) = +\infty. \tag{35}$$

Note that, for $v_{2,\infty} \notin \mathcal{V}_1$, according to (28) and (29), equality (35) and the inequality

$$v_2(t) - v_1(t) \le 0, \quad t \ge t_1,$$

are not true for some $t_1 > t_0$.

In view of the fact that the functions $t - \tau_{1,2}(t)$ and $t - \tau_{2,1}(t)$ are strictly increasing on the interval $(t_0, +\infty)$, relations (34) and (35), and Lemma 1, for any $t > t_2$, where t_2 is a number such that $\min\{t_2 - \tau_{1,2}(t_2), t_2 - \tau_{2,1}(t_2)\} > t_0$, there exists $\tilde{v} \in \mathbb{R}$ for which

$$v_2(t) - v_1(t) = \tilde{v}_2(t) - \tilde{v}_1(t) > 0,$$

where $\tilde{v}_1(t) = \dot{\tilde{x}}_1(t)$ and $\tilde{v}_2(t) = \dot{\tilde{x}}_2(t)$,

$$\lim_{t\to+\infty} (\tilde{x}_2(t) - \tilde{x}_1(t)) = +\infty, \tag{36}$$

$$\tilde{v}_1(t) < 0, \quad \tilde{v}_2(t) > 0, \tag{37}$$

$$\tilde{x}_1(t) < \tilde{x}_1(t - \tau_{1,2}(t)), \tag{38}$$

and

$$\tilde{x}_2(t) > \tilde{x}_2(t - \tau_{2,1}(t)). \tag{39}$$

Inequalities (37) follow from (34) for the proper choice of $\tilde{v} \in \mathbb{R}$.

By using the Taylor formula [12], inequalities (24), (25), and (37), and identities (30), we derive inequalities (38) and (39) from the relations

$$\tilde{x}_1(t - \tau_{1,2}(t)) - \tilde{x}_1(t) = -\tau_{1,2}(t)\tilde{v}_1(t) + \frac{1}{2}\,\tau^2_{1,2}(t)\,\frac{Gm_2}{(\tilde{x}_2(\xi_1 - \tau_{2,1}(\xi_1)) - \tilde{x}_1(\xi_1))^2} > 0$$

and

$$\tilde{x}_2(t - \tau_{2,1}(t)) - \tilde{x}_2(t) = -\tau_{2,1}(t)\tilde{v}_2(t) + \frac{1}{2}\tau^2_{2,1}(t)\frac{Gm_1}{(\tilde{x}_1(\xi_2 - \tau_{1,2}(\xi_2)) - \tilde{x}_2(\xi_2))^2} < 0,$$

where ξ_1 and ξ_2 are certain numbers from the intervals $(t - \tau_{1,2}(t), t)$ and $(t - \tau_{2,1}(t), t)$, respectively, where the functions $x_1(t)$ and $x_2(t)$ are twice continuously differentiable.

Further, for the solutions $x_1(t)$ and $x_2(t)$ of system (9), we use the equalities

$$\int_{t_2}^{t} \frac{2G(m_1+m_2)}{(\tilde{x}_2(s)-\tilde{x}_1(s))^2}\, d(\tilde{x}_2(s)-\tilde{x}_1(s))$$

$$= \left(v_{2,\infty}^*\right)^2 - \int_{t_0}^{t_2} \frac{2G(m_1+m_2)}{(\tilde{x}_2(s)-\tilde{x}_1(s))^2}\, d(\tilde{x}_2(s)-\tilde{x}_1(s))$$

$$- \frac{2G(m_1+m_2)}{\tilde{x}_2(t)-\tilde{x}_1(t)} = \frac{2G(m_1+m_2)}{\tilde{x}_2(t_2)-\tilde{x}_1(t_2)} - \frac{2G(m_1+m_2)}{\tilde{x}_2(t)-\tilde{x}_1(t)}, \quad t \ge t_2,$$

which are true for every $\tilde{\nu} \in \mathbb{R}$ in view of (12), (34), (35), and Lemma 1.

By using these equalities in the case where $v_2(t_0) - v_1(t_0) = v_{2,\infty}^*$, we represent (33) in the form

$$\left(v_2(t)-v_1(t)\right)^2 = (v_2(t_0)-v_1(t_0))^2 - \int_{t_0}^{t_2} \frac{2Gm_1\, d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}$$

$$- \int_{t_0}^{t_2} \frac{2Gm_2\, d(x_2(s)-x_1(s))}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2} - \int_{t_2}^{t} \frac{2Gm_1\, d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}$$

$$- \int_{t_2}^{t} \frac{2Gm_2\, d(x_2(s)-x_1(s))}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2} = \frac{2G(m_1+m_2)}{x_2(t_2)-x_1(t_2)}$$

$$- \int_{t_2}^{t} \frac{2Gm_1\, d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2} - \int_{t_2}^{t} \frac{2Gm_2\, d(x_2(s)-x_1(s))}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2}$$

$$= \frac{2G(m_1+m_2)}{\tilde{x}_2(t)-\tilde{x}_1(t)} - \int_{t_2}^{t} \frac{2Gm_1\, d(\tilde{x}_2(s)-\tilde{x}_1(s))}{(\tilde{x}_2(s)-\tilde{x}_1(s-\tau_{1,2}(s)))^2}$$

$$- \int_{t_2}^{t} \frac{2Gm_2\, d(\tilde{x}_2(s)-\tilde{x}_1(s))}{(\tilde{x}_2(s-\tau_{2,1}(s))-\tilde{x}_1(s))^2} + \int_{t_2}^{t} \frac{2G(m_1+m_2)}{(\tilde{x}_2(s)-\tilde{x}_1(s))^2}\, d(\tilde{x}_2(s)-\tilde{x}_1(s)).$$

Here, we have also used Lemma 1 and Corollary 1.

Thus,

$$\left(v_2(t)-v_1(t)\right)^2 = \frac{2G(m_1+m_2)}{\tilde{x}_2(t)-\tilde{x}_1(t)} - \int_{t_2}^{t} \Bigg(\frac{2Gm_1}{(\tilde{x}_2(s)-\tilde{x}_1(s-\tau_{1,2}(s)))^2}$$

$$+\frac{2Gm_2}{(\tilde{x}_2(s-\tau_{2,1}(s))-\tilde{x}_1(s))^2}$$

$$-\frac{2G(m_1+m_2)}{(\tilde{x}_2(s)-\tilde{x}_1(s))^2}\Bigg)d\,(\tilde{x}_2(s)-\tilde{x}_1(s)), \quad t\geq t_2. \tag{40}$$

By Lemma 1 and Corollary 1, the integral on the right-hand side of (40) is independent of $\tilde{\nu}\in\mathbb{R}$. Hence, by virtue of (38) and (39), for any $s\in[t_0,t]$, we can choose $\tilde{\nu}$ such that

$$|\tilde{x}_2(s)-\tilde{x}_1(s-\tau_{1,2}(s)))|<|\tilde{x}_2(s)-\tilde{x}_1(s)|$$

and

$$|\tilde{x}_2(s-\tau_{2,1}(s)))-\tilde{x}_1(s)|<|\tilde{x}_2(s)-\tilde{x}_1(s)|\,.$$

Therefore, the function

$$I(t)=\int_{t_0}^{t}\Bigg(\frac{2Gm_1}{(\tilde{x}_2(s)-\tilde{x}_1(s-\tau_{1,2}(s)))^2}+\frac{2Gm_2}{(\tilde{x}_2(s-\tau_{2,1}(s))-\tilde{x}_1(s))^2}$$

$$-\frac{2G(m_1+m_2)}{(\tilde{x}_2(s)-\tilde{x}_1(s))^2}\Bigg)d\,(\tilde{x}_2(s)-\tilde{x}_1(s))$$

takes positive values for any $t>t_2$ and is strictly increasing on $[t_0,+\infty)$. Thus, in view of (36) and (40), we have $v_2(t_1)-v_1(t_1)=0$ for some $t_1>t_2$. However, according to the reasoning used at the beginning of the proof of the lemma, this contradicts (35).

Thus, the assumption $v^*_{2,\infty}\notin\mathcal{V}_1$ is not true.

Lemma 2 is proved.

Lemma 3. $\mathcal{V}_2\neq\varnothing$.

Proof. We use relation (33) in the case where

$$v_2(t_0)-v_1(t_0)=2v^*_{2,\infty}. \tag{41}$$

This relation has the form

$$\big(v_2(t)-v_1(t)\big)^2=\frac{8G(m_1+m_2)}{x_2(t_0)-x_1(t_0)}-\int_{t_0}^{t}\Bigg(\frac{2Gm_1}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}$$

$$+\frac{2Gm_2}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2}\Bigg)d(x_2(s)-x_1(s)). \tag{42}$$

The following inclusion is true:

$$2v^*_{2,\infty}\in\mathcal{V}_2.$$

Indeed, the function $v_2(t) - v_1(t)$ satisfying equality (41) is continuous at the point t_0. According to (42), the set of intervals $[t_0, \theta)$, $\theta > t_0$, on each of which $v_2(t) - v_1(t) > 0$ is nonempty. Assume that, for some $T > t_0$, we have

$$v_2(t) - v_1(t) > 0 \quad \text{for all} \quad t \in [t_0, T) \tag{43}$$

and

$$v_2(T) - v_1(T) = 0. \tag{44}$$

In view of (43), we obtain

$$x_2(T) - x_1(T) > 0. \tag{45}$$

We use the relation

$$x_1(s - \tau_{1,2}(s)) < \frac{x_1(s) + x_2(s)}{2} < x_2(s - \tau_{2,1}(s)), \tag{46}$$

which can be easily deduced by using the following formulas for the location of the points M_1 and M_2 on the Ox-axis and the velocities of their motion:

$$\max\{x_1(s), x_1(s - \tau_{1,2}(s))\} < x_2(s), \qquad x_1(s) < \min\{x_2(s), x_2(s - \tau_{2,1}(s))\}$$

[see (24) and (25)] and

$$\max\left\{\left|\dot{x}_1(s)\right|, \left|\dot{x}_2(s)\right|\right\} < c$$

[the restriction imposed on the velocities of points M_1 and M_2 is required for the substantiation of relations (24) and (25)]. In the presented auxiliary relations, s is an arbitrary element from $[t_0 - \max\{\tau_{1,2}(t_0), \tau_{2,1}(t_0)\}, +\infty)$ such that $x_2(s) > x_1(s)$.

It follows from (46) that

$$\max\left\{\frac{x_2(s) - x_1(s)}{x_2(s) - x_1(s - \tau_{1,2}(s))}, \frac{x_2(s) - x_1(s)}{x_2(s - \tau_{2,1}(s)) - x_1(s)}\right\} < 2 \tag{47}$$

for all $s \in [t_0 - \max\{\tau_{1,2}(t_0), \tau_{2,1}(t_0)\}, +\infty)$ such that $x_2(s) > x_1(s)$.

According to (47), for all $t \in (t_0, T]$, we get

$$\int\limits_{t_0}^{t} \left(\frac{2Gm_1}{(x_2(s) - x_1(s - \tau_{1,2}(s)))^2} + \frac{2Gm_2}{(x_2(s - \tau_{2,1}(s)) - x_1(s))^2}\right) d(x_2(s) - x_1(s))$$

$$< 8G(m_1 + m_2) \int\limits_{t_0}^{t} \frac{d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s))^2} = \frac{8G(m_1 + m_2)}{x_2(t_0) - x_1(t_0)} - \frac{8G(m_1 + m_2)}{x_2(t) - x_1(t)}.$$

By using (42) and (45), we arrive at the relation

$$\big(v_2(t) - v_1(t)\big)^2 > \frac{8G(m_1 + m_2)}{x_2(T) - x_1(T)} > 0,$$

which contradicts (44).

Thus, relation (43) is also true for $T = +\infty$. By using (28), we get

$$\lim_{t\to+\infty} (x_2(t) - x_1(t)) = +\infty.$$

Hence, $2v^*_{2,\infty} \in \mathcal{V}_2$ and $\mathcal{V}_2 \neq \varnothing$.

Lemma 3 is proved.

Lemma 4. *The set $\mathcal{V}_2$ is closed and connected.*

Proof. We use the standard initial conditions imposed in the previous steps. Let $x_1(t)$ and $x_2(t)$ be the corresponding solutions of system (9) and let

$$v_2(t_0) - v_1(t_0) \in \mathcal{V}_2. \tag{48}$$

In view of this inclusion, we obtain

$$v_2(t) - v_1(t) > 0, \quad t \geq t_0, \tag{49}$$

and

$$\lim_{t\to+\infty} (x_2(t) - x_1(t)) = +\infty. \tag{50}$$

Consider an arbitrary number $\delta \geq 0$ and the solutions $x_{1,\delta}(t)$ and $x_{2,\delta}(t)$ of system (9) such that

$$x_{1,\delta}(t) = x_1(t) \qquad \text{and} \qquad x_{2,\delta}(t) = x_2(t) \quad \text{for all} \quad t \in I_{t_0}, \tag{51}$$

$$x_{1,0}(t) = x_1(t) \qquad \text{and} \qquad x_{2,0}(t) = x_2(t) \quad \text{for all} \quad t \in I_{t_0} \cup [t_0, +\infty), \tag{52}$$

$$\lim_{t\to t_0+0} \dot{x}_{2,\delta}(t) = v_2(t_0) + \delta \tag{53}$$

and, in view of (27), the requirement $\lim\limits_{t\to t_0-0} \dot{x}_{1,\delta}(t) = v_1(t_0)$ can be violated.

Hence, the functions $x_{2,\delta}(t) - x_{1,\delta}(t)$ and $v_{2,\delta}(t) - v_{1,\delta}(t) = \dot{x}_{2,\delta}(t) - \dot{x}_{1,\delta}(t)$ satisfy the integral relations

$$v_{2,\delta}(t) - v_{1,\delta}(t) = v_2(t_0) + \delta - v_1(t_0)$$

$$- \int_{t_0}^{t} \left(\frac{Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))^2} + \frac{Gm_2}{(x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^2} \right) ds \tag{54}$$

and

$$x_{2,\delta}(t) - x_{1,\delta}(t) = x_2(t_0) - x_1(t_0) + (t - t_0)(v_2(t_0) + \delta - v_1(t_0))$$
$$-\int_{t_0}^{t}\left(\int_{t_0}^{\tau}\left(\frac{Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))^2}\right.\right.$$
$$\left.\left.+\frac{Gm_2}{(x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^2}\right)ds\right)d\tau, \tag{55}$$

similar to relations (28) and (29), where t is an arbitrary time from an interval $[t_0, t^*) \subset [t_0, +\infty)$ on which $x_{2,\delta}(t) - x_{1,\delta}(t) > 0$, the quantities $\tau_{1,2,\delta}(s)$ and $\tau_{2,1,\delta}(s)$ are delays satisfying, according to the third and fourth equations in system (9) and the location of the points M_1 and M_2 on the Ox-axis, the following relations:

$$c\tau_{2,1,\delta}(s) = x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s), \tag{56}$$

$$c\tau_{1,2,\delta}(s) = x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)). \tag{57}$$

Note that, for $\delta = 0$, relations (54) and (55) coincide with relations (28) and (29), respectively, and relations (56) and (57) coincide with the third and fourth equations in system (9), $\tau_{1,2,0}(s) = \tau_{1,2}(s)$, and $\tau_{2,1,0}(s) = \tau_{2,1}(s)$.

According to (9) and (53), the relations

$$x_{2,\delta}(t) = x_2(t_0) + (t - t_0)(v_2(t_0) + \delta) - \int_{t_0}^{t}\left(\int_{t_0}^{\tau}\frac{Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))^2}\,ds\right)d\tau \tag{58}$$

and

$$x_{1,\delta}(t) = x_1(t_0) + (t - t_0)v_1(t_0) + \int_{t_0}^{t}\left(\int_{t_0}^{\tau}\frac{Gm_2}{(x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^2}\,ds\right)d\tau \tag{59}$$

are also true.

We study the influence of the quantity δ on the functions $x_{2,\delta}(t) - x_{1,\delta}(t)$ and $v_{2,\delta}(t) - v_{1,\delta}(t)$ for $t > t_0$.

In view of (54) and (55), these functions depend on $x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s))$ and $x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s)$. In addition, at points s of the set

$$R_\delta = \{s\colon s - \tau_{1,2,\delta}(s) = t_0\} \cup \{s\colon s - \tau_{2,1,\delta}(s) = t_0\},$$

the right-hand sides of (56) and (57) are continuous and can be not differentiable. By virtue of the fact that the functions $s - \tau_{1,2,\delta}(s)$ and $s - \tau_{2,1,\delta}(s)$ are strictly increasing, the set R_δ contains one or two points for every $\delta \geq 0$.

We differentiate both sides of (56) and (57) with respect to δ. For $s \in [t_0, t^*) \setminus R_\delta$, we obtain

$$c\,\frac{d\tau_{2,1,\delta}(s)}{d\delta} = \frac{d(x_{2,\delta}(s-\tau_{2,1,\delta}(s)) - x_{1,\delta}(s))}{d\delta} = -v_{2,\delta}(s-\tau_{2,1,\delta}(s))\,\frac{d\tau_{2,1,\delta}(s)}{d\delta} - \frac{dx_{1,\delta}(s)}{d\delta},$$

$$c\,\frac{d\tau_{1,2,\delta}(s)}{d\delta} = \frac{d(x_{2,\delta}(s) - x_{1,\delta}(s-\tau_{1,2,\delta}(s)))}{d\delta} = \frac{dx_{2,\delta}(s)}{d\delta} + v_{1,\delta}(s-\tau_{1,2,\delta}(s))\,\frac{d\tau_{1,2,\delta}(s)}{d\delta}.$$

This implies that

$$\frac{d\tau_{2,1,\delta}(s)}{d\delta} = -\frac{1}{c + v_{2,\delta}(s-\tau_{2,1,\delta}(s))}\,\frac{dx_{1,\delta}(s)}{d\delta}, \tag{60}$$

$$\frac{d\tau_{1,2,\delta}(s)}{d\delta} = \frac{1}{c - v_{1,\delta}(s-\tau_{1,2,\delta}(s))}\,\frac{dx_{2,\delta}(s)}{d\delta} \tag{61}$$

for $s \in [t_0, t^*) \setminus R_\delta$.

Further, we differentiate both sides of (58) and (59) with respect to the variable δ. By using (60) and (61), we obtain

$$\begin{aligned}\frac{dx_{2,\delta}(t)}{d\delta} &= t - t_0 \\ &\quad + \int_{t_0}^{t}\left(\int_{t_0}^{\tau} \frac{2Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s-\tau_{1,2,\delta}(s)))^3}\,\frac{d((x_{2,\delta}(s) - x_{1,\delta}(s-\tau_{1,2,\delta}(s)))}{d\delta}\,ds\right)d\tau \\ &= t - t_0 + \int_{t_0}^{t}\left(\int_{t_0}^{\tau} \frac{2Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s-\tau_{1,2,\delta}(s)))^3}\,\frac{c\,d\tau_{1,2,\delta}(s)}{d\delta}\,ds\right)d\tau \\ &= t - t_0 + \int_{t_0}^{t}\left(\int_{t_0}^{\tau} \frac{2Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s-\tau_{1,2,\delta}(s)))^3}\,\frac{c}{c - v_{1,\delta}(s-\tau_{1,2,\delta}(s))}\,\frac{dx_{2,\delta(s)}}{d\delta}\,ds\right)d\tau\end{aligned} \tag{62}$$

and

$$\begin{aligned}\frac{dx_{1,\delta}(t)}{d\delta} &= -\int_{t_0}^{t}\left(\int_{t_0}^{\tau} \frac{2Gm_2}{(x_{2,\delta}(s-\tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^3}\,\frac{d(x_{2,\delta}(s-\tau_{2,1,\delta}(s)) - x_{1,\delta}(s))}{d\delta}\,ds\right)d\tau \\ &= -\int_{t_0}^{t}\left(\int_{t_0}^{\tau} \frac{2Gm_2}{(x_{2,\delta}(s-\tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^3}\,\frac{c\tau_{2,1,\delta}(s)}{d\delta}\,ds\right)d\tau \\ &= \int_{t_0}^{t}\left(\int_{t_0}^{\tau} \frac{2Gm_2}{(x_{2,\delta}(s-\tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^3}\,\frac{c}{c + v_{2,\delta}(s-\tau_{2,1,\delta}(s))}\,\frac{dx_{1,\delta(s)}}{d\delta}\,ds\right)d\tau.\end{aligned} \tag{63}$$

The operation of differentiation under the integral signs in (62) and (63) is possible by virtue of the Lebesgue theorem on limit transition under the integral sign and the Riemann integrability of a bounded almost everywhere continuous function [13, pp. 120, 125] {the integrands in (62) and (63) are continuous and bounded on $[t_0, t) \setminus R_\delta$, at points of the set R_δ, these functions have finite upper and lower (right and left) derivatives [14] that do not affect the values of the corresponding integrals}.

By virtue of (56), (57) and the positivity of the delays $\tau_{2,1,\delta}(s)$ and $\tau_{1,2,\delta}(s)$, the functions

$$\frac{2Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))^3} \frac{c}{c - v_{1,\delta}(s - \tau_{1,2,\delta}(s))} \tag{64}$$

and

$$\frac{2Gm_2}{(x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^3} \frac{c}{c + v_{2,\delta}(s - \tau_{2,1,\delta}(s))} \tag{65}$$

in (62) and (63) are continuous and positive on each interval $[t_0, t)$ for all points of which we have

$$(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))(x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s)) \neq 0. \tag{66}$$

For this reason, by virtue of (62), the function $dx_{2,\delta}(t)/d\delta$ takes positive values on each interval (t_0, t) at every point of which relation (66) is true. On this interval, the function $dx_{1,\delta}(t)/d\delta$ becomes equal to zero because relation (63) with respect to $dx_{1,\delta}(t)/d\delta$ is a linear homogeneous equation with quasinilpotent operator.

Thus,

$$\frac{d(x_{2,\delta}(t) - x_{1,\delta}(t))}{d\delta} = \frac{dx_{2,\delta}(t)}{d\delta} - \frac{dx_{1,\delta}(t)}{d\delta} = \frac{dx_{2,\delta}(t)}{d\delta} > 0 \tag{67}$$

for every $t > t_0$ such that, relation (66) is true for all points $s \in (t_0, t)$.

Further, differentiating both sides of (54) with respect to the variable δ and using relations (56), (57), (60), and (61) and the properties of functions (64) and (65), $dx_{2,\delta}(t)/d\delta$, and $dx_{1,\delta}(t)/d\delta$, we obtain

$$\begin{aligned}
&\frac{d(v_{2,\delta}(t) - v_{1,\delta}(t))}{d\delta} \\
&= 1 + \int_{t_0}^{t} \frac{2Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))^3} \frac{c}{c - v_{1,\delta}(s - \tau_{1,2,\delta}(s))} \frac{dx_{2,\delta(s)}}{d\delta}\, ds \\
&\quad + \int_{t_0}^{t} \frac{2Gm_2}{(x_{2,\delta}(s - \tau_{2,1,\delta}(s)) - x_{1,\delta}(s))^3} \frac{c}{c + v_{2,\delta}(s - \tau_{2,1,\delta}(s))} \frac{dx_{1,\delta(s)}}{d\delta}\, ds \\
&= 1 + \int_{t_0}^{t} \frac{2Gm_1}{(x_{2,\delta}(s) - x_{1,\delta}(s - \tau_{1,2,\delta}(s)))^3} \frac{c}{c - v_{1,\delta}(s - \tau_{1,2,\delta}(s))} \frac{dx_{2,\delta(s)}}{d\delta}\, ds \geq 1
\end{aligned} \tag{68}$$

for any $t > t_0$ such that relation (66) holds for all points $s \in (t_0, t)$.

Inequality (68) is true for all $\delta \geq 0$.

By using the same reasoning as above and relations (49), (50), (67), and (68), we get

$$\lim_{t\to+\infty}(x_{2,\delta}(t) - x_{1,\delta}(t)) = +\infty$$

for all $\delta \geq 0$.

Hence, in view of (48), we get $[v_2(t_0) - v_1(t_0), +\infty) \subset \mathcal{V}_2$.

By using the same reasoning as above, we also conclude that $\mathcal{V}_2 = \bigcup_{v\in\mathcal{V}_2}[v, +\infty)$. Therefore, $\mathcal{V}_2$ is a connected set.

Consider the velocity $v^*_{2,c} = v_2(t_0) - v_1(t_0)$ of the point M_2 relative to the point M_1 given by the equality $v^*_{2,c} = \inf_{v\in\mathcal{V}_2} v$.

We want to show that

$$v^*_{2,c} \in \mathcal{V}_2. \tag{69}$$

This would imply that the set $\mathcal{V}_2$ is closed.

Assume that relation (69) is not true, i.e.,

$$v^*_{2,c} \in \mathcal{V}_1. \tag{70}$$

According to Lemma 2, we get $v^*_{2,\infty} \leq v^*_{2,c}$. Thus, $v^*_{2,c} > 0$. By virtue of (32), (33), and (70), on some interval $[t_0, t_1)$, $t_1 > t_0$, the velocity $v_2(t) - v_1(t)$ of the point M_2 relative to the point M_1 is positive, strictly decreases, and $v_2(t_1) - v_1(t_1) = 0$. Moreover, the distance $x_2(t) - x_1(t)$ between the points M_1 and M_2 on the interval $[t_0, t_1)$ is strictly increasing. By using the arguments from the proof of Lemma 2, on a certain interval $[t_1, t_2)$, $t_2 > t_1$, the difference $x_2(t) - x_1(t)$ is strictly decreasing and, moreover, $\lim_{t\to t_2-0}(x_2(t) - x_1(t)) = 0$ (the points M_1 and M_2 collide at the time t_2).

We fix an arbitrary $t_2^+ \in (t_1, t_2)$.

Consider an arbitrary number $\delta > 0$ and the solutions $x_{1,\delta}(t)$ and $x_{2,\delta}(t)$ of system (9) satisfying conditions (51)–(53) under which

$$v_{2,\delta}(t_0) - v_{1,\delta}(t_0) = v_2(t_0) - v_1(t_0) + \delta = v^*_{2,c} + \delta \in \mathcal{V}_2.$$

It follows from the continuous dependence of the solutions $x_{1,\delta}(t)$ and $x_{2,\delta}(t)$ on δ that

$$\lim_{\delta\to+0}\max_{t\in[t_0,t_2^+]}\left|(x_{2,\delta}(t) - x_{1,\delta}(t)) - (x_2(t) - x_1(t))\right| = 0.$$

Hence, for sufficiently small values of $\delta > 0$, the distance $x_{2,\delta}(t) - x_{1,\delta}(t)$ between the points M_1 and M_2 is not strictly increasing on $[t_0, +\infty)$, which is impossible because $\left(v^*_{2,c}, +\infty\right) \subset \mathcal{V}_2$.

Thus, the assumption that relation (70) holds is not true.

Lemma 4 is proved.

Lemma 5. *The set $\mathcal{V}_1$ is open and connected.*

This statement is a corollary of Lemma 4.

By virtue of the presented lemmas and their proofs, the following statement is true:

Theorem 2. *Suppose that the functions $\psi_1(s)$ and $\psi_2(s)$ are initial values of the solution $x_1(t)$ and $x_2(t)$ of system (9), these functions are continuously differentiable on* $\operatorname{int} I_{t_0}$ *and continuous on I_{t_0}, $\psi_2(s) - \psi_1(s) > 0$ for all $s \in I_{t_0}$, the derivatives $\dot\psi_1(s)$ and $\dot\psi_2(s)$ are bounded and integrable on* $\operatorname{int} I_{t_0}$*, the limits $\lim\limits_{s\to t_0-0} \dot\psi_i(s)$, $i = \overline{1,2}$, exist, $v_1(t_0)$ and $v_2(t_0)$ are arbitrary numbers from the interval $(-c, c)$ for which $v_2(t_0) - v_1(t_0) \in (-c, c)$, and the limits $\lim\limits_{s\to t_0-0} \dot\psi_1(s))$ and $\lim\limits_{s\to t_0-0} \dot\psi_2(s)$ cannot coincide with $v_1(t_0)$ and $v_2(t_0)$, respectively.*

Suppose that:

(i) $x_1(t-\tau_{1,2}(t)) = \psi_1(t-\tau_{1,2}(t))$ for all $t \ge t_0$ such that $t-\tau_{1,2}(t) \in [t_0-\tau_{1,2}(t_0), t_0]$ and $x_2(t-\tau_{2,1}(t)) = \psi_2(t-\tau_{2,1}(t))$ for all $t \ge t_0$ for which $t-\tau_{2,1}(t) \in [t_0-\tau_{2,1}(t_0), t_0]$;

(ii) $c\tau_{2,1}(t_0) = \psi_2(t_0-\tau_{2,1}(t_0)) - \psi_1(t_0)$ and $c\tau_{1,2}(t_0) = \psi_2(t_0) - \psi_1(t_0-\tau_{1,2}(t_0))$.

Then:

(i) if $v_2(t_0) - v_1(t_0) \in (-c, 0]$, then there exists a number $T > t_0$ such that, for the solution $x_1(t)$, $x_2(t)$ of system (9), the difference $x_2(t) - x_1(t)$ is strictly decreasing on the interval $[t_0, T)$ and $\lim\limits_{t\to T-0}(x_2(t) - x_1(t)) = 0$ (the points M_1 and M_2 collide at time T);

(ii) there exists a number $v^*_{2,c} > v^*_{2,\infty}$ such that

(a) if $v_2(t_0) - v_1(t_0) \in (0, v^*_{2,c})$, then, for some numbers T_1 and T_2 ($t_0 < T_1 < T_2$), the difference $x_2(t) - x_1(t)$ is strictly increasing and strictly decreasing, respectively, for the solution $x_1(t)$, $x_2(t)$ of system (9) on the intervals $[t_0, T_1)$ and $[T_1, T_2)$ and, in addition, $\lim\limits_{t\to T_2-0}(x_2(t) - x_1(t)) = 0$ (collision of the points M_1 and M_2 at the time T_2);

(b) if $v_2(t_0) - v_1(t_0) \ge v^*_{2,c}$, then, for the solution $x_1(t)$, $x_2(t)$ of system (9), the difference $x_2(t) - x_1(t)$ is strictly increasing on $[t_0, +\infty)$ and $\lim\limits_{t\to+\infty}(x_2(t) - x_1(t)) = +\infty$.

Remark 2. In Theorem 2, $x_2(t) - x_1(t)$ is the distance between the points M_1 and M_2.

Remark 3. The velocity $v^*_{2,c}$ in Theorem 2 is the escape velocity with regard for the speed of gravity. This is the minimal velocity $v_2(t_0)-v_1(t_0)$ of motion of the point M_2 relative to the point M_1 at the time t_0 for coordinates $x_2(t)$ and $x_1(t)$ at the time t such that the relation $\lim\limits_{t\to+\infty}(x_2(t) - x_1(t)) = +\infty$ is true. By virtue of Lemma 2 and the proof of Lemma 4, we get $v^*_{2,c} > v^*_{2,\infty}$, i.e., the *actual escape velocity $v^*_{2,c}$ (due to finiteness of the speed of gravity) is higher than the escape velocity $v^*_{2,\infty}$ in the classical celestial mechanics.*

4.3. Estimates of the Difference $v^*_{2,c} - v^*_{2,\infty}$.

To find the difference $v^*_{2,c} - v^*_{2,\infty}$, we need some auxiliary statements.

Lemma 6. *Suppose that the solution $x_1(t)$, $x_2(t)$ of system (9) satisfies the conditions of Theorem 2. If*

$$(x_2(s) - x_1(s-\tau_{1,2}(s)))(x_2(s-\tau_{2,1}(s)) - x_1(s)) \ne 0$$

for every $s \in [t_0, T)$, then the following equalities are true:

$$\frac{1}{(x_2(s) - x_1(s-\tau_{1,2}(s)))^2} = \left(1 + \frac{v_1^*(s)}{c}\right)^2 \frac{1}{(x_2(s) - x_1(s))^2} \tag{71}$$

and

$$\frac{1}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2}=\left(1+\frac{v_2^*(s)}{c}\right)^2\frac{1}{(x_2(s)-x_1(s))^2} \tag{72}$$

for all $s\in[t_0,T)$, *where*

$$v_1^*(s)=\frac{x_1(s-\tau_{1,2}(s))-x_1(s)}{\tau_{1,2}(s)} \qquad \textit{and} \qquad v_2^*(s)=\frac{x_2(s)-x_2(s-\tau_{2,1}(s))}{\tau_{2,1}(s)}.$$

Note that $v_1^*(s)$ and $v_2^*(s)$ are the mean velocities of the points M_1 and M_2 on the segments $[s-\tau_{1,2}(s),s]$ and $[s-\tau_{2,1}(s),s]$, respectively.

Proof. We fix an arbitrary $s\in[t_0,T)$. It is easy to see that the following equalities are true:

$$\frac{1}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}=\left(1+\frac{-x_1(s)+x_1(s-\tau_{1,2}(s))}{x_2(s)-x_1(s-\tau_{1,2}(s))}\right)^2\frac{1}{(x_2(s)-x_1(s))^2} \tag{73}$$

and

$$\frac{1}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2}=\left(1+\frac{x_2(s)-x_2(s-\tau_{2,1}(s))}{x_2(s-\tau_{2,1}(s))-x_1(s)}\right)^2\frac{1}{(x_2(s)-x_1(s))^2}. \tag{74}$$

According to (9), (24), and (25), we find

$$x_2(s)-x_1(s-\tau_{1,2}(s))=c\tau_{1,2}(s) \qquad \text{and} \qquad x_2(s-\tau_{2,1}(s))-x_1(s)=c\tau_{2,1}(s).$$

Therefore,

$$\frac{-x_1(s)+x_1(s-\tau_{1,2}(s))}{x_2(s)-x_1(s-\tau_{1,2}(s))}=\frac{v_1^*(s)}{c} \qquad \text{and} \qquad \frac{x_2(s)-x_2(s-\tau_{2,1}(s))}{x_2(s-\tau_{2,1}(s))-x_1(s)}=\frac{v_2^*(s)}{c}.$$

Hence, in view of (73) and (74), we obtain (71) and (72).

Lemma 6 is proved.

Lemma 7. *If the velocities of the points* M_1 *and* M_2 *are bounded on the interval* $[t_0,+\infty)$ *and the numbers* ε_1 *and* ε_2 *are such that*

$$\frac{m_1\varepsilon_1}{m_1+m_2}\ge\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_1^*(s)\right|}{c} \qquad \textit{and} \qquad \frac{m_2\varepsilon_2}{m_1+m_2}\ge\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_2^*(s)\right|}{c}, \tag{75}$$

then

$$\sqrt{\left(1+\frac{m_1\varepsilon_1}{m_1+m_2}\right)^2\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}+\left(1+\frac{m_2\varepsilon_2}{m_1+m_2}\right)^2\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}}\in\mathcal{V}_2. \tag{76}$$

Proof. We use relations (33) in the case where

$$v_2(t_0) - v_1(t_0)$$
$$= \sqrt{\left(1 + \frac{m_1\varepsilon_1}{m_1+m_2}\right)^2 \frac{2Gm_1}{x_2(t_0) - x_1(t_0)} + \left(1 + \frac{m_2\varepsilon_2}{m_1+m_2}\right)^2 \frac{2Gm_2}{x_2(t_0) - x_1(t_0)}}. \tag{77}$$

By virtue of (12), this relation takes the form

$$\begin{aligned}\big(v_2(t) - v_1(t)\big)^2 &= \left(1 + \frac{m_1\varepsilon_1}{m_1+m_2}\right)^2 \frac{2Gm_1}{x_2(t_0) - x_1(t_0)} \\ &\quad + \left(1 + \frac{m_2\varepsilon_2}{m_1+m_2}\right)^2 \frac{2Gm_2}{x_2(t_0) - x_1(t_0)} \\ &\quad - \int_{t_0}^{t} \frac{2Gm_1\, d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s - \tau_{1,2}(s)))^2} - \int_{t_0}^{t} \frac{2Gm_2\, d(x_2(s) - x_1(s))}{(x_2(s - \tau_{2,1}(s)) - x_1(s))^2}.\end{aligned} \tag{78}$$

We now show that inclusion (76) is true.

The function $v_2(t) - v_1(t)$ satisfying equality (77) is continuous at the point t_0. Therefore, according to (78), the set of intervals $[t_0, \theta)$, $\theta > t_0$, in each of which $v_2(t) - v_1(t) > 0$, is nonempty.

Assume that, for some $T > t_0$,

$$v_2(t) - v_1(t) > 0 \quad \text{for all} \quad t \in [t_0, T) \tag{79}$$

and

$$v_2(T) - v_1(T) = 0. \tag{80}$$

Note that, in view of (79), we have $x_2(T) - x_1(T) > 0$.

By virtue of (71), (72), and (75), for all $t \in (t_0, T]$, we obtain

$$\begin{aligned}&\int_{t_0}^{t} \frac{2Gm_1\, d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s - \tau_{1,2}(s)))^2} + \int_{t_0}^{t} \frac{2Gm_2\, d(x_2(s) - x_1(s))}{(x_2(s - \tau_{2,1}(s)) - x_1(s))^2} \\ &\quad = \int_{t_0}^{t} \left(2Gm_1\left(1 + \frac{v_1^*(s)}{c}\right)^2 + 2Gm_2\left(1 + \frac{v_2^*(s)}{c}\right)^2\right) \frac{d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s))^2} \\ &\quad \le \left(1 + \frac{\sup_{s > t_0 - \tau_{1,2}(t_0)} \left|v_1^*(s)\right|}{c}\right)^2 2Gm_1 \int_{t_0}^{t} \frac{d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s))^2}\end{aligned}$$

$$+\left(1+\frac{\sup_{s>t_0-\tau_{2,1}(t_0)}\left|v_2^*(s)\right|}{c}\right)^2 2Gm_2\int\limits_{t_0}^{t}\frac{d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s))^2}$$

$$=\left(1+\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_1^*(s)\right|}{c}\right)^2\left(\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}-\frac{2Gm_1}{x_2(T)-x_1(T)}\right)$$

$$+\left(1+\frac{\sup_{s>t_0-\tau_{2,1}(t_0)}\left|v_2^*(s)\right|}{c}\right)^2\left(\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}-\frac{2Gm_2}{x_2(T)-x_1(T)}\right).$$

By virtue of (75) and (78), we get

$$\big(v_2(t)-v_1(t)\big)^2\geq\left(1+\frac{m_1\varepsilon_1}{m_1+m_2}\right)^2\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}$$

$$+\left(1+\frac{m_2\varepsilon_2}{m_1+m_2}\right)^2\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}$$

$$-\left(1+\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_1^*(s)\right|}{c}\right)^2\left(\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}-\frac{2Gm_1}{x_2(T)-x_1(T)}\right)$$

$$-\left(1+\frac{\sup_{s>t_0-\tau_{2,1}(t_0)}\left|v_2^*(s)\right|}{c}\right)^2\left(\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}-\frac{2Gm_2}{x_2(T)-x_1(T)}\right)$$

$$=\left(\left(1+\frac{m_1\varepsilon_1}{m_1+m_2}\right)^2-\left(1+\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_1^*(s)\right|}{c}\right)^2\right)\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}$$

$$+\left(\left(1+\frac{m_2\varepsilon_2}{m_1+m_2}\right)^2-\left(1+\frac{\sup_{s>t_0-\tau_{2,1}(t_0)}\left|v_2^*(s)\right|}{c}\right)^2\right)\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}$$

$$+\left(1+\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_1^*(s)\right|}{c}\right)^2\frac{2Gm_1}{x_2(T)-x_1(T)}$$

$$+\left(1+\frac{\sup_{s>t_0-\tau_{2,1}(t_0)}\left|v_2^*(s)\right|}{c}\right)^2\frac{2Gm_2}{x_2(T)-x_1(T)}>0,$$

which contradicts (80).

Thus, relation (79) remains true for $T=+\infty$. Hence, by analogy with the proof of Lemma 3, we get

$$\lim_{t\to+\infty}(x_2(t)-x_1(t))=+\infty.$$

Thus, inclusion (76) is true.
Lemma 7 is proved.

Remark 4. The velocities of the points M_1 and M_2 are bounded in the interval $[t_0, +\infty)$ if these points do not collide, i.e., $\inf\limits_{t \geq t_0} (x_2(t) - x_2(t)) > 0$.

Remark 5. By Lemma 7 and the definition of $v^*_{2,c}$, the following inequality is true:

$$v^*_{2,c} \leq v^*_{2,c,\varepsilon_1,\varepsilon_2},$$

where

$$v^*_{2,c,\varepsilon_1,\varepsilon_2} = \sqrt{\left(1 + \frac{m_1\varepsilon_1}{m_1 + m_2}\right)^2 \frac{2Gm_1}{x_2(t_0) - x_1(t_0)} + \left(1 + \frac{m_2\varepsilon_2}{m_1 + m_2}\right)^2 \frac{2Gm_2}{x_2(t_0) - x_1(t_0)}}. \quad (81)$$

Lemma 8. *If the positive numbers μ_1 and μ_2 are such that*

$$\frac{m_1\mu_1}{m_1 + m_2} < \frac{\sup_{s > t_0 - \tau_{1,2}(t_0)} \left|v_1^*(s)\right|}{c} \qquad \textit{and} \qquad \frac{m_2\mu_2}{m_1 + m_2} < \frac{\sup_{s > t_0 - \tau_{1,2}(t_0)} \left|v_2^*(s)\right|}{c}, \quad (82)$$

then

$$\sqrt{\left(1 + \frac{m_1\mu_1}{m_1 + m_2}\right)^2 \frac{2Gm_1}{x_2(t_0) - x_1(t_0)} + \left(1 + \frac{m_2\mu_2}{m_1 + m_2}\right)^2 \frac{2Gm_2}{x_2(t_0) - x_1(t_0)}} \in \mathcal{V}_1. \quad (83)$$

Proof. We use relation (33) in the case where

$$v_2(t_0) - v_1(t_0) = \sqrt{\left(1 + \frac{m_1\mu_1}{m_1 + m_2}\right)^2 \frac{2Gm_1}{x_2(t_0) - x_1(t_0)} + \left(1 + \frac{m_2\mu_2}{m_1 + m_2}\right)^2 \frac{2Gm_2}{x_2(t_0) - x_1(t_0)}}.$$

This relation takes the form

$$\begin{aligned}\big(v_2(t) - v_1(t)\big)^2 = {} & \left(1 + \frac{m_1\mu_1}{m_1 + m_2}\right)^2 \frac{2Gm_1}{x_2(t_0) - x_1(t_0)} \\ & + \left(1 + \frac{m_2\mu_2}{m_1 + m_2}\right)^2 \frac{2Gm_2}{x_2(t_0) - x_1(t_0)} \\ & - \int_{t_0}^{t} \frac{2Gm_1\, d(x_2(s) - x_1(s))}{(x_2(s) - x_1(s - \tau_{1,2}(s)))^2} - \int_{t_0}^{t} \frac{2Gm_2\, d(x_2(s) - x_1(s))}{(x_2(s - \tau_{2,1}(s)) - x_1(s))^2}. \end{aligned} \quad (84)$$

As in the proof of Lemma 2, we assume that inclusion (83) is not true, i.e., the relations

$$v_2(t) - v_1(t) > 0 \quad \text{for all} \quad t > t_0 \quad (85)$$

and

$$\lim_{t\to+\infty}(x_2(t)-x_1(t))=+\infty \tag{86}$$

hold.

By using the equality

$$\int\limits_{t_0}^{t}\frac{2G(m_1+m_2)}{(x_2(s)-x_1(s))^2}\,d(x_2(s)-x_1(s))=\frac{2G(m_1+m_2)}{x_2(t_0)-x_1(t_0)}-\frac{2G(m_1+m_2)}{x_2(t)-x_1(t)},\quad t\ge t_0,$$

which follows from (85) and (86), we represent (84) in the form

$$\begin{aligned}\big(v_2(t)-v_1(t)\big)^2=&\left(\left(1+\frac{m_1\mu_1}{m_1+m_2}\right)^2-1\right)\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}\\&+\left(\left(1+\frac{m_2\mu_2}{m_1+m_2}\right)^2-1\right)\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}\\&+\frac{2G(m_1+m_2)}{x_2(t)-x_1(t)}-\int\limits_{t_0}^{t}\frac{2Gm_1\,d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s-\tau_{1,2}(s)))^2}\\&-\int\limits_{t_0}^{t}\frac{2Gm_2\,d(x_2(s)-x_1(s))}{(x_2(s-\tau_{2,1}(s))-x_1(s))^2}+\int\limits_{t_0}^{t}\frac{2G(m_1+m_2)}{(x_2(s)-x_1(s))^2}\,d(x_2(s)-x_1(s)).\end{aligned}\tag{87}$$

Applying Lemma 6 to (87), for all $t\ge t_0$, we conclude that

$$\begin{aligned}\big(v_2(t)-v_1(t)\big)^2=&\left(\left(1+\frac{m_1\mu_1}{m_1+m_2}\right)^2-1\right)\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}\\&+\left(\left(1+\frac{m_2\mu_2}{m_1+m_2}\right)^2-1\right)\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}+\frac{2G(m_1+m_2)}{x_2(t)-x_1(t)}\\&-2Gm_1\int\limits_{t_0}^{t}\left(\left(1+\frac{v_1^*(s)}{c}\right)^2-1\right)\frac{d(x_2(s)-x_1\qquad)}{(x_2(s)-x_1(s))^2}\\&-2Gm_2\int_{t_0}^{t}\left(\left(1+\frac{v_2^*(s)}{c}\right)^2-1\right)\frac{d(x_2(s)-x_1(s))}{(x_2(s)-x_1(s))^2}\\\ge&\left(\left(1+\frac{m_1\mu_1}{m_1+m_2}\right)^2-1\right)\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}\end{aligned}$$

$$+\left(\left(1+\frac{m_2\mu_2}{m_1+m_2}\right)^2-1\right)\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}+\frac{2G(m_1+m_2)}{x_2(t)-x_1(t)}$$

$$-\left(\left(1+\frac{\sup_{s>t_0-\tau_{1,2}(t_0)}\left|v_1^*(s)\right|}{c}\right)^2-1\right)\left(\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}-\frac{2Gm_1}{x_2(t)-x_1(t)}\right)$$

$$-\left(\left(1+\frac{\sup_{s>t_0-\tau_{2,1}(t_0)}\left|v_2^*(s)\right|}{c}\right)^2-1\right)\left(\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}-\frac{2Gm_2}{x_2(t)-x_1(t)}\right).$$

Therefore, by using the equality $\lim_{t\to+\infty} 1/(x_2(t)-x_1(t))=0$ and requirements (82) imposed on μ_1 and μ_2, we conclude that there exists a number $t_1>t_0$ such that $v_2(t_1)-v_1(t_1)=0$, which contradicts (85).

Thus, inclusion (83) is valid in the case where relations (82) are true.

Lemma 8 is proved.

Remark 6. The following inequality is true:

$$v_{2,\infty}^* < v_{2,\infty,\mu_1,\mu_2}^*,$$

where

$$v_{2,\infty,\mu_1,\mu_2}^*=\sqrt{\left(1+\frac{m_1\mu_1}{m_1+m_2}\right)^2\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}+\left(1+\frac{m_2\mu_2}{m_1+m_2}\right)^2\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}}. \tag{88}$$

This inequality follows from the relations

$$\left(1+\frac{m_1\mu_1}{m_1+m_2}\right)^2\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}+\left(1+\frac{m_2\mu_2}{m_1+m_2}\right)^2\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}$$
$$>\frac{2Gm_1}{x_2(t_0)-x_1(t_0)}+\frac{2Gm_2}{x_2(t_0)-x_1(t_0)}=v_{2,\infty}^*.$$

According to the results of investigations presented above, we find

$$0<v_{2,\infty}^*<v_{2,\infty,\mu_1,\mu_2}^*<v_{2,c}^*\le v_{2,c,\varepsilon_1,\varepsilon_2}^*<+\infty$$

and

$$0<v_{2,\infty,\mu_1,\mu_2}^*-v_{2,\infty}^*<v_{2,c}^*-v_{2,\infty}^*\le v_{2,c,\varepsilon_1,\varepsilon_2}^*-v_{2,\infty}^*, \tag{89}$$

whence, in view of (89) and equalities (12), (81), and (88), we arrive at the following assertion for the estimate of the difference $v_{2,c}^*-v_{2,\infty}^*$:

Theorem 3. *The escape velocities $v^*_{2,c}$ and $v^*_{2,\infty}$ satisfy the relation*

$$0 < \sqrt{\left(1+\frac{m_1\mu_1}{m_1+m_2}\right)^2 \frac{2Gm_1}{x_2(t_0)-x_1(t_0)} + \left(1+\frac{m_2\mu_2}{m_1+m_2}\right)^2 \frac{2Gm_2}{x_2(t_0)-x_1(t_0)}} - \sqrt{\frac{2G(m_1+m_2)}{x_2(t_0)-x_1(t_0)}} < v^*_{2,c} - v^*_{2,\infty} \le -\sqrt{\frac{2G(m_1+m_2)}{x_2(t_0)-x_1(t_0)}} + \sqrt{\left(1+\frac{m_1\varepsilon_1}{m_1+m_2}\right)^2 \frac{2Gm_1}{x_2(t_0)-x_1(t_0)} + \left(1+\frac{m_2\varepsilon_2}{m_1+m_2}\right)^2 \frac{2Gm_2}{x_2(t_0)-x_1(t_0)}} \tag{90}$$

for all ε_1, ε_2, μ_1, and μ_2 satisfying (75) and (82).

Remark 7. Since the differences $\varepsilon_1 - \mu_1$ and $\varepsilon_2 - \mu_2$ can be arbitrarily small, it follows from (90) and (12) that

$$v^*_{2,c} = \sqrt{\left(1+\frac{m_1\varepsilon_1}{m_1+m_2}\right)^2 \frac{2Gm_1}{x_2(t_0)-x_1(t_0)} + \left(1+\frac{m_2\varepsilon_2}{m_1+m_2}\right)^2 \frac{2Gm_2}{x_2(t_0)-x_1(t_0)}}, \tag{91}$$

where

$$\frac{m_1\varepsilon_1}{m_1+m_2} = \frac{\sup_{s>t_0-\tau_{1,2}(t_0)} \left|v^*_1(s)\right|}{c} \quad \text{and} \quad \frac{m_2\varepsilon_2}{m_1+m_2} = \frac{\sup_{s>t_0-\tau_{1,2}(t_0)} \left|v^*_2(s)\right|}{c}. \tag{92}$$

Remark 8. The escape velocity $v^*_{2,\infty}$ in the Newton celestial mechanics is obtained from the escape velocity $v^*_{2,c}$ in the celestial mechanics constructed with regard for the finite speed of gravity by setting $c = +\infty$ [relation (91) is a generalization of relation (12)]. Indeed, in view of (12), (91), and (92), we obtain

$$\lim_{c\to+\infty} v^*_{2,c} = v^*_{2,\infty}.$$

5. Escape Velocity on Earth's Surface

We determine the escape velocity $v^*_{2,c}$ on Earth's surface. Recall that, in the Newton mechanics, we have $v^*_{2,\infty} = 11.2\ \text{km}\cdot\text{sec}^{-1}$ (see [15, p. 28]).

As in Sec. 4, we use the inertial coordinate system (see Fig. 1).

Consider a body with mass m located, up to time t_0 (including this time), on Earth's surface with mass $M_\oplus$ and radius R. Assume that the body begins to move at a time t_0 with a velocity $v_2(t_0) > 0$ in the vertical direction (along the coordinate axis Ox). Then the Earth begins to move in the opposite direction with a velocity $v_1(t_0) < 0$. Assume that

$$v_2(t_0) - v_1(t_0) = v^*_{2,c} \tag{93}$$

and the resistance forces are absent.

Under the assumption that Earth's center coincides with the point O, the motion of the Earth and the body is described by the system of equations (9) with $m_1 = M_\oplus$ and $m_2 = m$ for which $x_1(s) = 0$ and $x_2(s) = R$ for $s \le t_0$, $\dot{x}_1(t_0 - 0) = \dot{x}_2(t_0 - 0) = 0$, $\dot{x}_1(t_0 + 0) = v_1(t_0)$, and $\dot{x}_2(t_0 + 0) = v_2(t_0)$. In the analyzed case, $x_2(t_0) - x_1(t_0) = R$ and the considered initial conditions satisfy the general requirements to system (9) and the initial values of its solutions (see Sec. 4.2).

By using (93) and the proof of Lemma 4, we get

$$\lim_{t\to+\infty} (x_2(t) - x_1(t)) = +\infty \qquad \text{and} \qquad \lim_{t\to+\infty} \big(v_2(t) - v_1(t)\big) = 0. \tag{94}$$

In view of the equations of system (9), the velocities of the body and the Earth monotonically decrease and, according to the initial values of these velocities, $v_1(t) < 0$ and $v_2(t) > 0$ for $t > t_0$. By using (94), we get $\lim\limits_{t\to+\infty} v_i(t) = 0$, $i = \overline{1,2}$. Hence,

$$\sup_{s>t_0-\tau_{1,2}(t_0)} \left|v_1^*(s)\right| = -v_1(t_0) \qquad \text{and} \qquad \sup_{s>t_0-\tau_{2,1}(t_0)} \left|v_2^*(s)\right| = v_2(t_0). \tag{95}$$

We now determine the relationship between $v_{2,c}^*$ and $v_{2,\infty}^*$. To this end, in addition to (93), we use one more relation that connects $v_2(t_0)$ and $v_1(t_0)$. According to (9) and the requirements imposed on the motion of the body and the Earth, for any sufficiently small number $\delta > 0$, we get the relation

$$\begin{aligned} &m v_2(t_0 + \delta) + M_\oplus v_1(t_0 + \delta) - (m v_2(t_0 - 0) + M_\oplus v_1(t_0 - 0)) \\ &\qquad = G m M_\oplus \int\limits_{t_0-0}^{t_0+\delta} \left(\frac{x_1(t - \tau_{1,2}(t)) - x_2(t)}{|x_1(t - \tau_{1,2}(t)) - x_2(t)|^3} + \frac{x_2(t - \tau_{2,1}(t)) - x_1(t)}{|x_2(t - \tau_{2,1}(t)) - x_1(t)|^3} \right) dt. \end{aligned} \tag{96}$$

Since the integrand in (96) is bounded on the segment $[t_0, t_0 + \delta]$, according to $v_2(t_0 - 0) = v_1(t_0 - 0) = 0$, we find

$$m v_2(t_0) + M_\oplus v_1(t_0) = 0. \tag{97}$$

By using (93) and (97), we get

$$v_1(t_0) = \frac{-m v_{2,c}^*}{M_\oplus + m}, \qquad v_2(t_0) = \frac{M_\oplus v_{2,c}^*}{M_\oplus + m}. \tag{98}$$

Hence, in view of (92), (95), and (98), we obtain

$$\frac{M_\oplus \varepsilon_1}{M_\oplus + m} = \frac{m v_{2,c}^*}{c(M_\oplus + m)} \qquad \text{and} \qquad \frac{m \varepsilon_2}{M_\oplus + m} = \frac{M_\oplus v_{2,c}^*}{c(M_\oplus + m)},$$

whence, by virtue of (91) and (12), we conclude that

$$v_{2,c}^* = \sqrt{\left(1 + \frac{m v_{2,c}^*}{c(M_\oplus + m)}\right)^2 \frac{2GM_\oplus}{R} + \left(1 + \frac{M_\oplus v_{2,c}^*}{c(M_\oplus + m)}\right)^2 \frac{2Gm}{R}}. \tag{99}$$

By using (99), we determine $v^*_{2,c}$. To do this, we square both sides of (99), perform necessary transformations, and represent the relation obtained as a result in the form of a quadratic equation for $v^*_{2,c}$:

$$\left(1 - \frac{2GmM_\oplus}{c^2(M_\oplus + m)}\right)\left(v^*_{2,c}\right)^2 + \frac{8GmM_\oplus}{c(M_\oplus + m)R}\, v^*_{2,c} - \left(v^*_{2,\infty}\right)^2 = 0.$$

We obtain the following relationship between $v^*_{2,c}$ and $v^*_{2,\infty}$:

$$v^*_{2,c} = \frac{-\dfrac{4GmM_\oplus}{c(M_\oplus + m)R} + \sqrt{\dfrac{16G^2m^2M_\oplus^2}{c^2(M_\oplus + m)^2R^2} + \left(1 - \dfrac{2GmM_\oplus}{c^2(M_\oplus + m)}\right)\left(v^*_{2,\infty}\right)^2}}{1 - \dfrac{2GmM_\oplus}{c^2(M_\oplus + m)}}.$$

After necessary transformations, we can represent this formula in the form

$$v^*_{2,c} = \left(\sqrt{1 - \frac{2GmM_\oplus}{c^2(M_\oplus + m)} + \frac{8Gm^2M_\oplus^2}{c^2(M_\oplus + m)^3R}} + \sqrt{\frac{8Gm^2M_\oplus^2}{c^2(M_\oplus + m)^3R}}\right)^{-1} v^*_{2,\infty}. \tag{100}$$

In view of the fact that $G = 6.67408 \times 10^{-11}\ \mathrm{m}^3\cdot\mathrm{kg}^{-1}\cdot\mathrm{sec}^{-2}$, $M_\oplus = 5.9722 \times 10^{24}\cdot\mathrm{kg}$, $c = 2.99792458 \times 10^8\ \mathrm{m}\cdot\mathrm{sec}^{-1}$, and $R = 6.371 \times 10^6$ m, for $m \ll M_\oplus$, we obtain

$$v^*_{2,c} \approx \left(1 + 7.4259154861063335 \times 10^{-28}\{m\}\right) v^*_{2,\infty}, \tag{101}$$

where $\{m\}$ is the numerical value of mass m of the body. The error in (101) is smaller than $10^{-28}\{m\}v^*_{2,\infty}$.

By using (100), (101), and the fact that

$$v^*_{2,\infty} = \sqrt{2G(M_\oplus + m)/R},$$

we conclude that the mass m of the body strongly affects the value of the quantity $v^*_{2,c}$ and, moreover,

$$\lim_{m\to 0} \frac{v^*_{2,c}}{v^*_{2,\infty}} = 1 \qquad \text{and} \qquad \lim_{m\to +\infty} \frac{v^*_{2,c}}{v^*_{2,\infty}} = \left(1 - \frac{2GM_\oplus}{c^2}\right)^{-1/2} \approx 1.0044646.$$

6. Instability of Motion of the Points M_1 and M_2

The process of motion of the points M_1 and M_2 described by the system of equations (9) is *Lyapunov unstable* if the corresponding solution $x_1(t)$, $x_2(t)$ of system (9) is Lyapunov unstable [16].

Theorem 4. *The rectilinear motion of the points M_1 and M_2 described by the system of equations (9) for which $\lim_{t\to+\infty}(x_2(t) - x_1(t)) = +\infty$ is Lyapunov unstable.*

Proof. The following cases are possible:

1) $v_2(t_0) - v_1(t_0) = v^*_{2,c}$;

2) $v_2(t_0) - v_1(t_0) > v^*_{2,c}$.

In the first case, arbitrarily small perturbations of the initial values of solutions of system (9) and arbitrarily small perturbations δ_1 and δ_2 of the velocities $v_1(t_0)$ and $v_2(t_0)$ may lead to the situation when the difference $(v_2(t_0)+\delta_2)-(v_1(t_0)+\delta_1)$ becomes smaller than $v^*_{2,c}$. In this case, by Theorem 2, the points M_1 and M_2 collide. At the same time, in the absence of perturbations, these points move so that $\lim_{t\to+\infty}(x_2(t)-x_1(t)) = +\infty$. Hence, the process of motion of the points M_1 and M_2 is Lyapunov unstable.

In the second case, we fix an arbitrarily small $\delta > 0$ and use the solutions $x_1(t)$, $x_2(t)$ and $x_{1,\delta}(t)$, $x_{2,\delta}(t)$ of system (9) considered in the proof of Lemma 4.

According to relations (68), we get

$$v_{2,\delta}(t) - v_{1,\delta}(t) \geq v_2(t) - v_1(t) + \delta$$

for all $t \geq t_0$.

Since we also have $v_2(t) - v_1(t) > 0$ for all $t \geq t_0$ and $\lim_{t\to+\infty}(x_2(t) - x_1(t)) = +\infty$, it is possible to conclude that

$$\lim_{t\to+\infty}\big((x_{2,\delta}(t) - x_{1,\delta}(t)) - (x_2(t) - x_1(t))\big) = +\infty.$$

Therefore, in view of the arbitrary choice of the number $\delta > 0$, we see that the solution $x_1(t)$, $x_2(t)$ of system (9) is Lyapunov unstable.

Thus, in the second case, the process of motion of the points M_1 and M_2 is also Lyapunov unstable.

Theorem 4 is proved.

7. Additional Remarks and References

1. In the classical celestial mechanics, the two-body problem was studied by numerous mathematicians and mechanicians (see, e.g., [17–19]).

Fundamental results in this field were obtained by Kepler [17] and Newton [9].

Thus, Kepler constructed the kinematic picture of motion of two bodies and presented it in the form of three laws [17, 20]. On the basis of these laws, Newton deduced the law of gravitation. By using this law, together with his three laws of motion, he constructed the dynamical picture of motion of the bodies. According to Newton's results, the trajectories of two bodies are curves called conic sections or straight lines [9, 10].

2. The results of investigation of the rectilinear motion of two bodies with regard for the finite speed of gravity and conclusions concerning the escape velocity (for two bodies) are presented in the present paper for the first time. The indicated speed obtained in the celestial mechanics with finite speed of gravity is higher than the corresponding speed in the classical celestial mechanics (Sec. 5).

3. For the first time, the difference between the escape velocities in the Newton celestial mechanics and the celestial mechanics with finite speed of gravity was shown by the author in [21] in analyzing the dynamics of three bodies located on a straight line in the case where the masses of outer bodies and their distances from the central body are identical. In [21], it is also shown that the motion of these bodies is unstable.

4. In the general case, the Lyapunov instability of star systems with unbounded trajectories studied with regard for finiteness of the speed of gravity was shown in [22–24]. For the first time, this result in the case of two bodies was obtained in [5]. The non-Keplerian behavior of motion of two bodies was demonstrated in [5].

5. The law of gravitation for the finite speed of gravity was obtained with the use of Newton's law of gravitation and, for the first time, used in [6]. This law is a generalization of Newton's law and coincides with it in the limit case ($c = +\infty$).

6. For problems of celestial mechanics with the use of the theory of relativity, see, e.g., [11, 25].

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