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MAXIMALLY ACCRETIVE AND NONNEGATIVE EXTENSIONS OF A NONNEGATIVE LINEAR RELATION

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In terms of spaces of boundary values, we formulate and prove criteria of maximal θ-accretivity and maximal nonnegativity for the proper extension of a closed nonnegative linear relation in a Hilbert space. In the case of differential operators, this directly leads to boundary conditions.

Keywords: Hilbert space, linear relation, extension, accretive, nonnegative.

Introduction and Main Notation

For the last several decades, numerous mathematicians focus their attention on the theory of linear relations (multivalued maps) in Hilbert spaces. For any relation T of this kind, there exist adjoint T^* and inverse T^{-1} relations. This fact appears to be quite useful for the investigation, in particular, of different classes of extensions of nondensely defined operators.

Note that the theory of linear relations in Hilbert spaces was formulated by R. Arens in [10]. Various aspects of this theory (first of all, we mean the theory of extensions of the indicated relations) were later developed in the works of numerous researchers (see, e.g., [11, 13] and the references therein).

The present paper is a direct continuation of author's works [14, 15]. Our aim is to establish the conditions of maximal nonnegativity and maximal accretivity of the proper extension of a closed linear nonnegative relation in a Hilbert space in the terms of "abstract boundary conditions."

In what follows, we understood H as a fixed complex Hilbert space with a scalar product $(\cdot|\cdot)$ and the corresponding norm $\|\cdot\|$. Any (closed) linear manifold in $H^2 \stackrel{\text{def}}{=} H \oplus H$ is called a (closed) linear relation in H, and a linear operator is identified with its graph. For any linear relation (in particular, an operator) $T \subset H^2$, there exists an adjoint (closed linear) relation $T^* \subset H^2$ that can be defined as follows:

$$T^* = JT^{\perp} \; (= (JT)^{\perp}),$$

where, $\forall (y,y') \in H^2$, $J(y,y') = (-iy', iy)$ and "$\perp$" is the symbol of orthogonal complement in H^2.

We use the following notation:

– $D(T)$, $R(T)$, and $\ker T$ are the domain of definition, the range of values, and the manifold of zeros, respectively, of the relation (operator) T:

$$D(T) = \{ y \in H \,|\, (\exists y' \in H) : (y,y') \in T \},$$

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$$R(T) = \{y' \in H \,|\, (\exists y \in H): (y,y') \in T\},$$

$$\ker T = \{y \in H : (y,0) \in T\};$$

if $\lambda \in \mathbb{C}$, then

$$\lambda T = \{(y,\lambda y'): (y,y') \in T\},$$

$$T - \lambda = \{(y, y' - \lambda y): (y,y') \in T\}$$

(therefore, $\ker(T-\lambda) = \{y \in H : (y,0) \in T - \lambda\} \equiv \{y \in H : (y,\lambda y) \in T\}$),

$$\widehat{\ker}(T-\lambda) = \{(y,\lambda y): y \in \ker(T-\lambda)\};$$

$$T^{-1} = \{(y',y) \in H^2 : (y,y') \in T\};$$

if X and Y are Hilbert spaces, then $(\cdot|\cdot)_X$ is the symbol of scalar product on X and $\mathcal{B}(X,Y)$ is a collection of linear continuous operators $S: X \to Y$ such that $D(S) = X$;

$$\mathcal{B}(X) = \mathcal{B}(X,X);$$

- $\mathbb{I}_X$ is the identical transformation of the space X;
- $S \downarrow E$ is the restriction of the mapping S to the set E;
- SE is the image of set E under the mapping S;
- $\dot{+}$, $\oplus$, and Θ are the symbols of direct sum, orthogonal sum, and orthogonal complement, respectively;
- $\overline{E}$ is the closure of the set E;
- if $A_i : X \to Y_i$, $i = 1,\ldots,n$, are linear operators, then the notation $A = A_1 \oplus \ldots \oplus A_n$ means that $Ax = (A_1 x, \ldots, A_n x)$ for $\forall x \in X$.

Recall that the linear relation T in H is called nonnegative (and we write $T \geq 0$) if

$$\forall (y,y') \in T \qquad (y'|\,y) \geq 0,$$

positively definite $(T \gg 0)$ if, in addition,

$$\inf T \overset{\text{def}}{=} \inf\{(y'|\,y): (y,y') \in T, \|y\| = 1\} > 0,$$

and self-adjoint if $T = T^*$. This relation is called θ-accretive,

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

provided that

$$\forall \hat{y} = (y, y') \in T \qquad \arg(y' | y) \in \left[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right],$$

i.e.,

$$\forall \hat{y} \in T \qquad \arg(\pi_2 \hat{y} | \pi_1 \hat{y}) \in \left[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right],$$

where π_1 and π_2 are the orthoprojectors $H^2 \to H \oplus \{0\}$ and $H^2 \to \{0\} \oplus H$, respectively. Furthermore, if T has no θ-accretive extensions in H, then it is said that T is the maximally θ-accretive relation (compare with the corresponding definitions introduced in [7]). In the case where $\theta = 0$ $\left(\theta = \frac{\pi}{2}, \text{ or } \theta = -\frac{\pi}{2}\right)$, the θ-accretive relation is called accretive (dissipative, or accumulative).

In the present work, we choose a closed linear nonnegative relation $L_0 \subset H^2$ as the subject of inquiry. Our aim is to describe its maximally nonnegative and proper maximally accretive extensions (an extension L_1 of the relation L_0 is called proper if $L_0 \subset L_1 = \bar{L}_1 \subset L_0^*$). It is known [12] that there exist (nonnegative) self-adjoint extensions L_F and L_K of the relation L_0 with the following property:

a self-adjoint extension L_1 of the relation L_0 is nonnegative if and only if, for any $\varepsilon > 0$,

$$\forall y \in H \qquad ((L_F + \varepsilon)^{-1} y | y) \le ((L_1 + \varepsilon)^{-1} y | y) \le ((L_K + \varepsilon)^{-1} y | y).$$

For the case of densely defined operator L_0, this property was proved in [5].

The extensions L_F and L_K are called stiff and soft extensions of the relation L_0, respectively.

1. Preliminary Results and Formulation of the Problem

Everywhere in what follows, we set $L \stackrel{\text{def}}{=} L_0^*$.

Definition 1. *Suppose that G is a Hilbert space and $\Gamma_1, \Gamma_2 \in \mathcal{B}(L, G)$. A triple (G, Γ_1, Γ_2) is called the space of boundary values (SBV) of the relation L_0 if*

$$R(\Gamma_1 \oplus \Gamma_2) = G \oplus G, \qquad \ker(\Gamma_1 \oplus \Gamma_2) = L_0,$$

and, for any $\hat{y} = (y, y')$, $\hat{z} = (z, z') \in L$,

$$(y' | z) - (y | z') = (\Gamma_1 \hat{y} | \Gamma_2 \hat{z})_G - (\Gamma_2 \hat{y} | \Gamma_1 \hat{z})_G.$$

If, in addition, $\ker\Gamma_1 = L_K$ and $\ker\Gamma_2 = L_F$, then we say that (G,Γ_1,Γ_2) is a *stiff* SBV of relation L_0. Everywhere in what follows, we assume that (G,Γ_1,Γ_2) is a fixed SBV for L_0 such that

$$L_2 \overset{\text{def}}{=} \ker\Gamma_2 \geq 0.$$

Hence, for any $\lambda < \inf L_2$, the following operators are correctly defined:

$$L_\lambda = (L_2-\lambda)^{-1} \in \mathcal{B}(H), \qquad \hat{L}_\lambda = \begin{pmatrix} L_\lambda \\ \mathbb{I}_H + \lambda L_\lambda \end{pmatrix} \in \mathcal{B}(H,H^2),$$

$$Z_\lambda = (\Gamma_1\hat{L}_\lambda)^* \in \mathcal{B}(G,H), \qquad \hat{Z}_\lambda = \begin{pmatrix} Z_\lambda \\ \lambda Z_\lambda \end{pmatrix} \in \mathcal{B}(G,H^2).$$

Definition 2. *A $\mathcal{B}(G)$-valued function*

$$M(\lambda) = \Gamma_1\hat{Z}_\lambda, \qquad \lambda < \inf L_2,$$

is called the Weyl function of relation L_0 corresponding to its SBV (G,Γ_1,Γ_2).

Remark 1. The notion of SBV was introduced in [4] under the assumption that L_0 is a densely defined symmetric operator with identical defective numbers. In [6], it was extended to the case of nondensely defined operators. The notion of Weyl function corresponding to a given SBV was proposed in [3] and found its subsequent development in the works of numerous mathematicians (see [2] and the references therein). It is easy to see that Definition 2 is equivalent to the corresponding definitions from cited works.

We set

$$U(\lambda) = (M(\lambda)-i)(M(\lambda)+i)^{-1}$$

and present some results from [15] necessary in what follows and formulated, for the sake of convenience, in the form of a theorem.

Theorem 1. *There exist unitary operators $U_{-\infty}, U_0 \in \mathcal{B}(G)$, defined as follows:*

$$U_{-\infty} = s-\lim_{\lambda\to-\infty} U(\lambda), \qquad U_0 = s-\lim_{\lambda\to-0} U(\lambda); \tag{1}$$

moreover,

$$L_F = \{\hat{y} \in L : (U_{-\infty} - \mathbb{I}_G)\Gamma_1\hat{y} + i(U_{-\infty} + \mathbb{I}_G)\Gamma_2\hat{y} = 0\}, \tag{2}$$

$$L_K = \{\hat{y} \in L : (U_0 - \mathbb{I}_G)\Gamma_1\hat{y} + i(U_0 + \mathbb{I}_G)\Gamma_2\hat{y} = 0\}, \tag{3}$$

where $U_{-\infty}$ and U_0 are the same as in (1).

Corollary 1. *If* $L_2 \gg 0$, *then there exists*

$$B \overset{\text{def}}{=} s - \lim_{\lambda \to -\infty} (M(\lambda) - M(0))^{-1}, \tag{4}$$

where $B \in \mathcal{B}(G)$ *and* $B \le 0$. *In this situation, the space* (G, γ_1, γ_2), *where*

$$\gamma_1 \hat{y} = \Gamma_1 \hat{y} - M(0) \Gamma_2 \hat{y}, \qquad \hat{y} \in L, \tag{5}$$

$$\gamma_2 \hat{y} = \Gamma_2 \hat{y} - B \gamma_1 \hat{y} \equiv -B \Gamma_1 \hat{y} + (\mathbb{I}_G + BM(0)) \Gamma_2 \hat{y}, \qquad \hat{y} \in L, \tag{6}$$

and B *is defined according to (4), is a stiff SBV of the relation* L_0 *and, in particular,*

$$L_F = \ker \gamma_2 \equiv \{\hat{y} \in L : \gamma_2 \hat{y} = 0\}, \tag{7}$$

$$L_K = \ker \gamma_1 \equiv \{\hat{y} \in L : \gamma_1 \hat{y} = 0\}. \tag{8}$$

In the present paper, we consider the problem of establishing the criteria of maximal accretivity and maximal nonnegativity of the relation

$$L_1 = \{\hat{y} \in L : A_1 \Gamma_1 \hat{y} + A_2 \Gamma_2 \hat{y} = 0\}, \tag{9}$$

where $A_1, A_2 \in \mathcal{B}(G)$.

2. Main Results

Theorem 2. *Suppose that* $L_2 \equiv \ker \Gamma_2 \gg 0$, *the operator* B *is defined according to (4), and the relation* L_1 *is given by formula (9). Also let*

$$a_1 = A_1(\mathbb{I}_G + M(0)B) + A_2 B, \qquad a_2 = A_1 M(0) + A_2. \tag{10}$$

(i) The following assertions are equivalent:

(1°) L_1 *is a maximally* θ-*accretive relation;*

(2°) $\operatorname{Re}(e^{i\theta} a_1 a_2^*) \le 0$, $\ker(a_1 - e^{-i\theta} a_2) = \{0\}$;

(3°) *there exists a contraction* $K \in \mathcal{B}(G)$ *such that*

$$L_1 = \{\hat{y} \in L : (K - \mathbb{I}_G) \gamma_1 \hat{y} + e^{i\theta} (K + \mathbb{I}_G) \gamma_2 \hat{y} = 0\},$$

where γ_1 *and* γ_2 *are defined according to (5), (6).*

(ii) The following propositions are equivalent:

(1°) L_1 *is a maximally nonnegative relation;*

(2°) $a_1 a_2^* \le 0$, $\ker(a_1 - a_2) = \{0\}$;

(3°) *there exists a self-adjoint contraction* $K \in \mathcal{B}(G)$ *such that*

$$L_1 = \{\hat{y} \in L : (K - \mathbb{I}_G)\gamma_1 \hat{y} + (K + \mathbb{I}_G)\gamma_2 \hat{y} = 0\}.$$

Proof. Since [see (5), (6), and (10)]

$$\begin{aligned} a_1\gamma_1 + a_2\gamma_2 &= \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} \mathbb{I}_G + M(0)B & M(0) \\ B & \mathbb{I}_G \end{pmatrix} \begin{pmatrix} \mathbb{I}_G & -M(0) \\ -B & \mathbb{I}_G + BM(0) \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \\ &= \begin{pmatrix} A_1 & A_2 \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = A_1\Gamma_1 + A_2\Gamma_2, \end{aligned}$$

the validity of this theorem follows from the results presented in [14] and Corollary 1, in particular, with (7), (8).

Remark 2. As follows from (10), $\operatorname{Im}(a_1 a_2^*) = \operatorname{Im}(A_1 A_2^*)$. Therefore, the conditions of maximal dissipativity and maximal accumulativity of the relation L_1 can be formulated in a much simpler form than the conditions of maximal accretivity of this relation.

Theorem 3. *Let* L_1 *be an arbitrary proper maximally accretive extension of the relation* L_0. *Then*

$$\forall \varepsilon > 0 \qquad (L_F + \varepsilon)^{-1} \le \operatorname{Re}(L_1 + \varepsilon)^{-1} \le (L_K + \varepsilon)^{-1}. \tag{11}$$

Proof. First, we suppose that $L_0 \gg 0$ and (G, Γ_1, Γ_2) is a stiff SBV of the relation L_0.
It is known (see either [14] or Theorem 2 with $B = M(0) = 0$) that, for a certain contraction $K \in \mathcal{B}(G)$,

$$L_1 = \{\hat{y} \in L : (K - \mathbb{I}_G)\Gamma_1 \hat{y} + (K + \mathbb{I}_G)\Gamma_2 \hat{y} = 0\}. \tag{12}$$

Applying Theorem 1 from [8] with $M_0 = L_0$, $\lambda = -\varepsilon < 0$, $A_1 = K - \mathbb{I}_G$, and $A_2 = K + \mathbb{I}_G$, we obtain

$$(L_1 + \varepsilon)^{-1} = (L_F + \varepsilon)^{-1} - Z_{-\varepsilon}[(K - \mathbb{I}_G)M(-\varepsilon) + (K + \mathbb{I}_G)]^{-1}(K - \mathbb{I}_G)Z_{-\varepsilon}^*. \tag{13}$$

In view of the inequalities $KK^* \le \mathbb{I}_G$ and $M(-\varepsilon) \le M(0) = 0$, we get

$$\operatorname{Re}\left\{(K-\mathbb{I}_G)\left[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)\right]^*\right\}$$

$$=\operatorname{Re}\left\{(K-\mathbb{I}_G)\left[M(-\varepsilon)(K^*-\mathbb{I}_G)+(K^*+\mathbb{I}_G)\right]\right\}$$

$$=(K-\mathbb{I}_G)M(-\varepsilon)(K^*-\mathbb{I}_G)+(KK^*-\mathbb{I}_G)\le 0\,.$$

Since the nonpositivity of the operator $\operatorname{Re}\{(K-\mathbb{I}_G)[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)]^*\}$ is equivalent to the nonpositivity of the operator $\operatorname{Re}\{[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)]^{-1}(K-\mathbb{I}_G)\}$, relation (13) implies that the first inequality in (11) is true.

Further, L_K with $K=-\mathbb{I}_G$ is defined by a condition of the form

$$(L_K+\varepsilon)^{-1} = (L_F+\varepsilon)^{-1}-Z_{-\varepsilon}M(-\varepsilon)^{-1}Z_{-\varepsilon}^*. \tag{14}$$

It is clear from (13) and (14) that

$$\operatorname{Re}(L_1+\varepsilon)^{-1} \le (L_K+\varepsilon)^{-1}$$

$$\Leftrightarrow \operatorname{Re}\left\{\left[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)\right]^{-1}(K-\mathbb{I}_G)\right\}\ge M(-\varepsilon)^{-1}$$

$$\Leftrightarrow \operatorname{Re}\left\{(K-\mathbb{I}_G)\left[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)\right]^*\right\}$$

$$\ge\left[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)\right]M(-\varepsilon)^{-1}$$

$$\times\left[(K-\mathbb{I}_G)M(-\varepsilon)+(K+\mathbb{I}_G)\right]^*$$

$$\Leftrightarrow (KK^*-\mathbb{I}_G)+(K+\mathbb{I}_G)M(-\varepsilon)^{-1}(K+\mathbb{I}_G)\le 0\,;$$

hence, the second inequality in (11) is true.

Now let $\inf L_0=0$, $\xi>0$. By $(L+\xi)_F$ and $(L+\xi)_K$ we denote the stiff and soft extensions of the relation $L_0+\xi$, respectively. It follows from the propositions proved above that

$$\forall\varepsilon>0,\qquad \forall\lambda\in(-\varepsilon,0)\qquad (L_F+\varepsilon)^{-1} = \left(\left(L_F+\frac{\varepsilon}{2}\right)+\frac{\varepsilon}{2}\right)^{-1}$$

$$\le \operatorname{Re}\left(\left(L_1+\frac{\varepsilon}{2}\right)+\frac{\varepsilon}{2}\right)^{-1} = \operatorname{Re}(L_1+\varepsilon)^{-1}.$$

We introduce the notation

$$L^{(\lambda)} \overset{\text{def}}{=} L_0 \dotplus \widehat{\ker}(L-\lambda)M(\lambda), \quad \lambda < 0.$$

It is known (see, e.g. [1, 2, 12]) that $L^{(\lambda)} = (L-\lambda)_K + \lambda$. Therefore, in view of the propositions proved above, for any $\varepsilon > 0$ and $\lambda \in (-\varepsilon, 0)$, we find

$$(L^{(\lambda)} + \varepsilon)^{-1} = ((L-\lambda)_K + (\varepsilon+\lambda))^{-1}$$

$$\geq \operatorname{Re}((L_1 - \lambda) + (\varepsilon + \lambda))^{-1} = \operatorname{Re}(L_1 + \varepsilon)^{-1}.$$

To complete the proof, it suffices to apply the relation

$$s - \lim_{\lambda \to -0} (L^{(\lambda)} + \varepsilon)^{-1} = (L_K + \varepsilon)^{-1}$$

established in the cited works [1, 2, 12]. **Q.E.D.**

In what follows, we always assume that L_0 is a nonnegative (generally speaking, nonpositively definite) relation and (G, Γ_1, Γ_2) is the SBV of this relation such that

$$L_2 \equiv \ker \Gamma_2 = L_F. \tag{15}$$

It follows from Theorem 1 [in particular, from (2)] that, in this case,

$$U_{-\infty} \equiv s - \lim_{\lambda \to -\infty} U(\lambda) = \mathbb{I}_G.$$

Remark 3. Assumption (15) does not lead to the loss of generality. Indeed, let (G, Γ_1, Γ_2) be the SBV of the relation L_0 (satisfying the condition $\ker \Gamma_2 \geq 0$). We set

$$\hat{\Gamma}_1 = \frac{i}{2}(U_{-\infty} + \mathbb{I}_G)\Gamma_1 - \frac{1}{2}(U_{-\infty} - \mathbb{I}_G)\Gamma_2,$$

$$\hat{\Gamma}_2 = \frac{1}{2}(U_{-\infty} - \mathbb{I}_G)\Gamma_1 + \frac{i}{2}(U_{-\infty} + \mathbb{I}_G)\Gamma_2.$$

By direct calculations, we can show that $(G, \hat{\Gamma}_1, \hat{\Gamma}_2)$ is the SBV of relation L_0. Thus, in view of Theorem 1, we get $\ker \hat{\Gamma}_2 = L_F$.

Theorem 4. *The relation* L_1 *is maximally accretive (maximally nonnegative) if and only if*

(i) there exists $s - \lim_{\lambda \to -0} A_1 M(\lambda) A_1^* \overset{\text{def}}{=} A_0 \in \mathcal{B}(G)$,

(ii) $A_0 + \operatorname{Re}(A_1 A_2^*) \leq 0, \quad A_0 + A_1 A_2^* \leq 0,$

(iii) for some (and, hence, for any) $\lambda < 0$,

$$\ker(A_1 - A_2 - A_1 M(\lambda)) = \{0\}. \tag{16}$$

Proof. Since the maximal accretivity (maximal nonnegativity) of a closed linear relation $T \subset H^2$ is equivalent to the simultaneous accretivity (nonnegativity) of the relations T and T^*, the relation L_1 is maximally accretive (maximally nonnegative) if and only if the extension $L_1 + \varepsilon$ has the same property for any $\varepsilon > 0$. Further, as follows from the definition of the function $M(\lambda)$, $(G, \Gamma_1 - M(-\varepsilon)\Gamma_2, \Gamma_2)$ is a stiff SBV of the positive definite relation $L_0 + \varepsilon$. In addition, relation (9) is equivalent to the following relation:

$$L_1 = \{\hat{y} \in L : A_1(\Gamma_1 - M(-\varepsilon)\Gamma_2)\hat{y} + (A_2 + A_1 M(-\varepsilon))\Gamma_2 \hat{y} = 0\}.$$

Therefore, by applying either the results obtained in [14] or Theorem 2 with $B = M(0) = 0$, we conclude that $L_1 + \varepsilon$ is maximally accretive (maximally nonnegative) if and only if

$$A_1 M(-\varepsilon) A_1^* + \mathrm{Re}(A_1 A_2^*) \le 0, \qquad A_1 M(-\varepsilon) A_1^* + (A_1 A_2^*) \le 0.$$

By using the monotonicity of the function $M(\lambda)$ and the theorem on the limit of a nondecreasing sequence of self-adjoint operators bounded above (see [9, Problem 94]), we conclude that L_1 belongs to one of the analyzed classes if and only if conditions (i) and (ii) are true and equality (16) is satisfied for all $\lambda < 0$.

Suppose now that conditions (i) and (ii) are satisfied and (16) is true for some $\lambda = -\varepsilon < 0$. Then $L_1^* + \varepsilon$ is maximally accretive (maximally nonnegative) and, hence, $R(L_1^* + 2\varepsilon) = H$. This fact and the accretivity (nonnegativity) of L_1^* yield the maximal accretivity (maximal nonnegativity) of the relation L_1 (see [1, 12]) and, hence, the validity of relation (12) for all $\lambda < 0$.

3. Relationship between $L_{\min}$ and $L_{\max}$

We set

$$L_{\min} = L_F \cap L_K, \qquad L_{\max} = L_F \hat{+} L_K. \tag{17}$$

It is easy to see that $L_{\min}^* = \overline{L_{\max}}$ and $L_{\max}^* = L_{\min}$, where $\overline{L_{\max}}$ is the closure of relation $L_{\max}$. It is clear that, in this case, L_F is a stiff extension and L_K is a soft extension (not only of the relation L_0 but also of $L_{\min}$). In the case where $L_F \gg 0$, we get $L_{\min} = L_0$ and $L_{\max} = L$ (for details, see, e.g., [1, 2, 12]). In the general case, in view of (3) and (15), we obtain

$$\hat{y} \in L_{\min} \Leftrightarrow \Gamma_2 \hat{y} = 0, \qquad (U_0 - \mathbb{I}_G)\Gamma_1 \hat{y} = 0.$$

Remark 4. If X and Y are linear relations, then

$$X \hat{+} Y \overset{\text{def}}{=} \{(x+u, y+v) : x, u \in X, y, v \in Y\}$$

(component-wise sum). In the case where $X \bigcap Y = \{0\}$, we write $X \dot{+} Y$ instead of $X \hat{+} Y$ [compare (17) with the definition of relation $L^{(\lambda)}$ in Theorem 3].

Lemma 1. *Let* $\hat{y} \in L$. *Then*

$$\text{(i)} \quad \hat{y} \in \overline{L_{\max}} \Leftrightarrow \Gamma_2 \hat{y} \in \overline{R(U_0 - \mathbb{I}_G)}; \tag{18}$$

$$\text{(ii)} \quad \hat{y} \in L_{\max} \Leftrightarrow \Gamma_2 \hat{y} \in R(U_0 - \mathbb{I}_G). \tag{19}$$

Proof (i). We denote by P_1 the orthoprojector $G \to \overline{R(U_0 - \mathbb{I}_G)}$. It is clear that

$$L_{\min} = \ker \Gamma_2 \bigcap \ker P_1 \Gamma_1.$$

Since (G, Γ_1, Γ_2) is the SBV of the relation L_0, we have $\overline{L_{\max}} = L_{\min}^* = \ker(\mathbb{I}_G - P_1)\Gamma_2$, i.e., condition (18) is satisfied.

(ii). Let $\hat{y} = \hat{y}_1 + \hat{y}_2$, where $\hat{y}_1 \in L_F$ and $\hat{y}_2 \in L_K$. In view of (3), it is easy to see that there exists $h \in G$ satisfying the equalities

$$\Gamma_1 \hat{y} = -i(U_0 + \mathbb{I}_G)h, \qquad \Gamma_2 \hat{y} = (U_0 - \mathbb{I}_G)h. \tag{20}$$

Conditions (20) are satisfied by the solution of the system

$$(U_0 - \mathbb{I}_G)\Gamma_1 \hat{y}_2 + i(U_0 + \mathbb{I}_G)\Gamma_2 \hat{y}_2 = 0,$$

$$(U_0^* + \mathbb{I}_G)\Gamma_1 \hat{y}_2 - i(U_0^* - \mathbb{I}_G)\Gamma_2 \hat{y}_2 = -4ih.$$

Furthermore, $\Gamma_2 \hat{y}_1 = 0$, and, hence,

$$\Gamma_2 \hat{y} = (U_0 - \mathbb{I}_G)h \in R(U_0 - \mathbb{I}_G).$$

On the contrary, assume that $h \in G$ and $\Gamma_2 \hat{y} = (U_0 - \mathbb{I}_G)h$. There exists $\hat{y}_2 \in L$ satisfying conditions (20). It is clear that $\hat{y}_2 \in L_K$ and $\hat{y} - \hat{y}_2 \in L_F$.

Corollary 2.

(i) $L_{\min} = L_0 \Leftrightarrow \ker(U_0 - \mathbb{I}_G) = \{0\}$;

(ii) $L_{\max} = \overline{L_{\max}} \Leftrightarrow R(U_0 - \mathbb{I}_G) = \overline{R(U_0 - \mathbb{I}_G)}$;

(iii) $L_{\max} = L \Leftrightarrow R(U_0 - \mathbb{I}_G) = G$.

Corollary 3.

(i) $L_1 \subset \overline{L_{\max}} \Leftrightarrow R(A_1^*) \subset \overline{R(U_0 - \mathbb{I}_G)}$;

(ii) $L_1 \subset L_{\max} \Rightarrow R(A_1^*) \subset R(U_0 - \mathbb{I}_G)$

in the case where $A \in \mathcal{B}(G^2, G)$ *defined by the condition*

$$A(h_1, h_2) = A_1 h_1 + A_2 h_2, \qquad h_1, h_2 \in G,$$

is normally solvable, and the converse assertion is also true.

Proof. First, let the operator A be normally solvable. Then (see [8, Lemma 9])

$$L_1^* = \{\hat{z} \in L \mid \exists h \in G : \Gamma_1 \hat{z} = A_2^* h, \Gamma_2 \hat{z} = -A_1^* h\},$$

and, therefore, $\{\Gamma_2 \hat{z} : \hat{z} \in L_1^*\} = R(A_1^*)$. We see that, in this case, the validity of Corollary 3 follows from Lemma 1. In the general situation, one should use the fact that there exist $C \in \mathcal{B}(G)$ and $\hat{A} \in \mathcal{B}(G \oplus G, G)$ such that $\ker C = \{0\}$, A is a normally solvable operator, and $A = C\hat{A}$.

Lemma 2. *Suppose that* $e \in G$ *and* $\|e\| = 1$. *If*

$$\sup_{\varepsilon > 0} (M(-\varepsilon) e \mid e) < +\infty,$$

then there exists a nonnegative self-adjoint relation $L_e \subset L$ *such that, for some* $\hat{y} \in L_e$, *the equality* $\Gamma_2 \hat{y} = e$ *is true.*

Proof. Let P be the orthoprojector $G \to \mathrm{sp}\{e\}$, i.e., $Ph = (h \mid e)e$, $\alpha > 0$, $L_e = \ker(\alpha P \Gamma_1 - \Gamma_2)$. It is clear that $L_e^* = L_e$. Further, by using the theorem on the limit of a monotone bounded operator function cited above (see [9, p. 246]), Theorem 4 with $A_1 = \alpha P$ and $A_2 = -\mathbb{I}_G$, and the equalities

$$\big((A_1 M(-\varepsilon) A_1^* + A_1 A_2^*) h \mid h\big)_G$$

$$= \alpha^2 \big((M(-\varepsilon) h \mid e)_G \, e \mid (h \mid e)_G \, e\big)_G - \alpha (h \mid e)_G (e \mid h)_G$$

$$= \alpha |(h \mid e)|^2 \big(\alpha (M(-\varepsilon) e \mid e)_G - 1\big),$$

we conclude that, for

$$0 < \alpha < (\max\{1, \sup (M(-\varepsilon) e \mid e)_G\})^{-1},$$

the relation L_e is nonnegative. Finally, there exists $\hat{y} \in L$ such that $\Gamma_2 \hat{y} = e$ and $\Gamma_1 \hat{y} = \frac{1}{\alpha} e$, and hence, $\hat{y} \in L_e$.

Lemma 3. *For any nonnegative self-adjoint extension* L_1 *of the relation* L_0, *the following relation is true:*

$$L_{\min} \subset L_1 \subset \overline{L_{\max}}. \tag{21}$$

Proof. As follows from the definition of stiff and soft extensions of a nonnegative relation, for all $f \in H$, we get

$$0 \le \left([(L_1+1)^{-1} - (L_F+1)^{-1}] f | f \right) \le \left([(L_K+1)^{-1} - (L_F+1)^{-1}] f | f \right). \tag{22}$$

Suppose that

$$\hat{y}_0 = (y_0, y_0') \in L_{\min} \equiv L_F \cap L_K \qquad \text{and} \qquad f_0 \overset{\text{def}}{=} y_0' + y_0.$$

Then $(y_0, f_0) \in L_F + 1$ and $(y_0, f_0) \in L_K + 1$, i.e., $(L_K+1)^{-1} f_0 = (L_F+1)^{-1} f_0 = y_0$. Substituting $f = f_0$ in (22), we conclude that

$$\left([(L_1+1)^{-1} - (L_F+1)^{-1}] f_0 | f_0 \right) = 0.$$

However, $(L_1+1)^{-1} \ge (L_F+1)^{-1}$ and, therefore, $(L_1+1)^{-1} f_0 = y_0$, which follows from the following series of implications:

$$\left(W = W^* \in \boldsymbol{\mathcal{B}}(G), W \ge 0, (Wf_0 | f_0) = 0 \right) \Rightarrow W^{1/2} f_0 = 0 \Rightarrow W f_0 = 0.$$

Hence, $(y_0, y_0') \in L_1$.

Thus, $L_{\min} \subset L_1$ but L_1^* is also a nonnegative self-adjoint extension of the relation L_0. Therefore, $L_{\min} \subset L_1^*$ and, hence, $L_1 \subset L_{\min}^* = \overline{L_{\max}}$.

Corollary 4. *If* $\sup_{\varepsilon > 0} (M(-\varepsilon) h | h)_G < +\infty$, *then* $h \in \overline{R(U_0 - 1_G)}$.

Proof. We set

$$e = \frac{h}{||h||}.$$

According to Lemma 2, there exists a nonnegative self-adjoint relation $L_e \subset L$ such that, for some $\hat{y} \in L$, the equality $\Gamma_2 \hat{y} = e$ is true. Since, by virtue of Lemma 3, $\hat{y} \in \overline{L_{\max}}$, we have (see (18)) $e \in \overline{R(U_0 - \mathbb{I}_G)}$ and, therefore, $h \in \overline{R(U_0 - \mathbb{I}_G)}$.

Theorem 5. *For any proper maximally accretive extension* L_1 *of the relation* L_0, *conditions (21) are satisfied.*

Proof. Assume that the relation $L_1 = \ker(A_1\Gamma_1 + A_2\Gamma_2)$ is maximally accretive. Theorem 4 implies that, for any $h \in R(A_1^*)$, $\sup_{\varepsilon>0}(M(-\varepsilon)h|\, h)_G < +\infty$ and, therefore (in view of Corollary 4), $h \in \overline{R(U_0 - \mathbb{I}_G)}$. Thus,

$$R(A_1^*) \subset \overline{R(U_0 - \mathbb{I}_G)},$$

i.e., (see Corollary 3), $L_1^* \subset \overline{L_{\max}}$ and, hence, $L_{\min} = L_{\max}^* \subset L_1$. Since L_1^* is also a proper maximally accretive extension of the relation L_0, we conclude that $L_1 = L_1^{**} \subset \overline{L_{\max}}$.

4. Some Corollaries

In what follows, we use the following notation:

$$G_1 = \overline{R(U_0 - \mathbb{I}_G)}, \qquad G_2 = G \ominus G_1,$$

$$\gamma_{11} = P_1\Gamma_1, \qquad \gamma_{12} = P_2\Gamma_1, \qquad \gamma_{21} = P_1\Gamma_2, \qquad \gamma_{22} = P_2\Gamma_2,$$

where P_i is the orthoprojector $G \to G_i$, $i = 1,2$,

$$M(\lambda) = \begin{pmatrix} m(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_0 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}$$

are the matrix representations of the corresponding operators as mappings $G_1 \oplus G_2 \to G_1 \oplus G_2$ (see [9]), and $\mathbb{I}_i = \mathbb{I}_{G_i}$, $i = 1,2$.

It is easy to see that

$$L_F = \{\hat{y} \in L : \gamma_{21}\hat{y} = 0, \gamma_{22}\hat{y} = 0\},$$

$$L_K = \{\hat{y} \in L : (u_0 - \mathbb{I}_1)\gamma_{11}\hat{y} + i(u_0 + \mathbb{I}_1)\gamma_{21}\hat{y} = 0, \gamma_{22}\hat{y} = 0\},$$

$$L_{\min} = \{\hat{y} \in L : \gamma_{21}\hat{y} = \gamma_{22}\hat{y} = \gamma_{11}\hat{y} = 0\},$$

$$\overline{L_{\max}} = \{\hat{y} \in L : \gamma_{22}\hat{y} = 0\},$$

and $(G_1, \gamma_{11}, \gamma_{21})$ is the SBV of the relation $L_{\min}$. Therefore, in view of Theorem 5, any proper maximally accretive extension L_1 of the relation L_0 can be represented in the following form:

$$L_1 = \{\hat{y} \in \overline{L_{\max}} : \alpha_1 \gamma_{11} \hat{y} + \alpha_2 \gamma_{21} \hat{y} = 0\}, \tag{23}$$

where $\alpha_1, \alpha_2 \in \mathcal{B}(G_1)$. Hence, the problem formulated at the beginning of the present work is now, in fact, reduced to the case where $G = G_1$, i.e., where $L_{\min} = L_0$.

Lemma 4. *If* $h \in R(u_0 - \mathbb{I}_1)$, *then there exists* $\lim_{\varepsilon \to +0} (m(-\varepsilon)h| h)_{G_1}$ *and*

$$\lim_{\varepsilon \to +0} (m(-\varepsilon)h| h)_{G_1} \le (-i(u_0 + \mathbb{I}_1)(u_0 - \mathbb{I}_1)^{-1} h| h)_{G_1}.$$

Proof. Since L_K is a proper maximally nonnegative extension of the relation L_0, it follows from (3) and Theorem 4 (more exactly, from its proof) that

$$\forall \varepsilon > 0, \quad \forall g \in G \quad \left((U_0 - \mathbb{I}_G) M(-\varepsilon)(U_0 - \mathbb{I}_G)^* g| g\right)_G$$

$$\le \left(i(U_0 - \mathbb{I}_G)(U_0 + \mathbb{I}_G)^* g| g\right)_G$$

$$= \left(-i(U_0 + \mathbb{I}_G)(U_0 - \mathbb{I}_G)^* g| g\right)_G \tag{24}$$

(the last equality follows from the self-adjointness of the operator $i(U_0 - \mathbb{I}_G)(U_0 + \mathbb{I}_G)^*$). Assume that h belongs to

$$R(u_0 - \mathbb{I}_1) \left(= R(U_0 - \mathbb{I}_G) = R((U_0 - \mathbb{I}_G)^*) = R((u_0 - \mathbb{I}_1)^*)\right).$$

There exists $g \in G_1$ such that

$$h = (U_0 - \mathbb{I}_G)^* = (u_0 - \mathbb{I}_1)^* g. \tag{25}$$

In view of (24) and (25), we obtain

$$(m(-\varepsilon)h| h)_{G_1} = (M(-\varepsilon)h| h)_G = \left(M(-\varepsilon)(U_0 - \mathbb{I}_G)^* g | (U_0 - \mathbb{I}_G)^* g\right)_G$$

$$= \left((U_0 - \mathbb{I}_G) M(-\varepsilon)(U_0 - \mathbb{I}_G)^* g| g\right)_G$$

$$\le \left(-i(U_0 + \mathbb{I}_G)(U_0 - \mathbb{I}_G)^* g| g\right)_G$$

$$= \left(-i(u_0 + \mathbb{I}_1)(u_0 - \mathbb{I}_1)^* g| g\right)_{G_1}$$

$$= \left(-i(u_0 + \mathbb{I}_1)h \middle| ((u_0 - \mathbb{I}_1)^*)^{-1} h\right)_{G_1}$$

$$= \left(-i(u_0 - \mathbb{I}_1)^{-1}(u_0 + \mathbb{I}_1)h \middle| h\right)_{G_1}$$

$$= \left(-i(u_0 + \mathbb{I}_1)(u_0 - \mathbb{I}_1)^{-1} h \middle| h\right)_{G_1}.$$

To complete the proof, it is sufficient to apply the theorem on the limit of a bounded monotone operator function.

Remark 5. It is easy to see that $m(\lambda)$ is the Weyl function of the relation $L_{\min}$ corresponding to its SBV $(G_1, \gamma_{11,}\gamma_{21})$. Further, since L_F and L_K are the stiff and soft extensions of this relation, we find

$$u_0 = s - \lim_{\lambda \to -0} (m(\lambda) - i\mathbb{I}_1)(m(\lambda) + i\mathbb{I}_1)^{-1}$$

(this follows from the definition of the function $U(\lambda)$ and Theorem 1) and, hence,

$$u_0 - \mathbb{I}_1 = s - \lim_{\lambda \to -0} (-2i)(m(\lambda) + i\mathbb{I}_1)^{-1}. \tag{26}$$

Lemma 5. *If there exists* $\lim_{\varepsilon \to +0} m(-\varepsilon)h$, *then* $h \in R(u_0 - \mathbb{I}_1)$ *and*

$$\lim_{\varepsilon \to +0} m(-\varepsilon)h = -i(u_0 + \mathbb{I}_1)(u_0 - \mathbb{I}_1)^{-1} h. \tag{27}$$

Proof. Let $\lim_{\varepsilon \to +0} m(-\varepsilon)h = g$. Then

$$\lim_{\varepsilon \to +0} (m(-\varepsilon) + i\mathbb{I}_1)h = g + ih.$$

Further, the identities

$$-2i(m(-\varepsilon) + i\mathbb{I}_1)^{-1}(m(-\varepsilon) + i\mathbb{I}_1)h = -2ih,$$

$$\lim_{\varepsilon \to +0} (-2i)(m(-\varepsilon) + i\mathbb{I}_1)^{-1}(g + ih) = (u_0 - \mathbb{I}_1)(g + ih)$$

are satisfied [see (26)]. Since, for any $\varepsilon > 0$, we have

$$\left\| (m(-\varepsilon) + i\mathbb{I}_1)^{-1}(m(-\varepsilon) + i\mathbb{I}_1)h - (m(-\varepsilon) + i\mathbb{I}_1)^{-1}(g + ih) \right\|$$

$$\leq \left\| (m(-\varepsilon) + i\mathbb{I}_1)^{-1} \right\| \| m(-\varepsilon)h + ih - g - ih \|$$

$$= \left\| (m(-\varepsilon) + i\mathbb{I}_1)^{-1} \right\| \| m(-\varepsilon)h - g \| \le \| m(-\varepsilon)h - g \| \underset{\varepsilon \to +0}{\to} 0$$

(here, we have used the theorem on the spectrum of self-adjoint operator), i.e.,

$$-2i(m(-\varepsilon) + i\mathbb{I}_1)^{-1}(g + ih) \underset{\varepsilon \to +0}{\to} -2ih$$

and, as follows from (26),

$$-2i(m(-\varepsilon) + i\mathbb{I}_1)^{-1}(g + ih) \underset{\varepsilon \to +0}{\to} (u_0 - \mathbb{I}_1)(g + ih),$$

we obtain

$$-2ih = (u_0 - \mathbb{I}_1)(g + ih)$$

and, in particular, $h \in R(u_0 - \mathbb{I}_1)$. Furthermore, the last equality implies that

$$g + ih = -2i(u_0 - \mathbb{I}_1)^{-1} h,$$

$$g = -2i(u_0 - \mathbb{I}_1)^{-1} h - ih = -i(2(u_0 - \mathbb{I}_1)^{-1} h + (u_0 - \mathbb{I}_1)^{-1}(u_0 - \mathbb{I}_1)h)$$

$$= -i(u_0 - \mathbb{I}_1)^{-1}(2h + u_0 h - h) = -i(u_0 - \mathbb{I}_1)^{-1}(u_0 + \mathbb{I}_1)h,$$

i.e., (27) is satisfied.

Corollary 5. $R(u_0 - \mathbb{I}_1) = G_1$ *if and only if, for every* $h \in G_1$ *there exists* $\lim_{\varepsilon \to +0} m(-\varepsilon)h$. *In this case,*

$$s - \lim_{\varepsilon \to +0} m(-\varepsilon) = -i(u_0 + \mathbb{I}_1)(u_0 - \mathbb{I}_1)^{-1} \in \mathcal{B}(G_1). \tag{28}$$

Proof. The necessity of the condition $R(u_0 - \mathbb{I}_1) = G_1$ and the validity of relation (28) follow from Lemma 5. On the contrary, if $R(u_0 - \mathbb{I}_1) = G_1$, then, in view of the Banach theorem on inverse operator,

$$-i(u_0 + \mathbb{I}_1)(u_0 - \mathbb{I}_1)^{-1} \in \mathcal{B}(G_1).$$

Hence, as follows from Lemma 4 and the theorem on the limit of monotone operator function, there exists an operator $m_0 \in \mathcal{B}(G_1)$ such that $s - \lim_{\varepsilon \to +0} m(-\varepsilon) = m_0$. To complete the proof, it remains to apply Lemma 5 once again.

Corollary 6. *Suppose that* $L_{\max} = \overline{L_{\max}}$ *and the relation* L_1 *is defined according to (9):*

$$L_1 = \{\hat{y} \in L : A_1\Gamma_1\hat{y} + A_2\Gamma_2\hat{y} = 0\},$$

where $A_1, A_2 \in \mathcal{B}(G)$ *and* $\ker\Gamma_2 = L_F$.

The relation L_1 *is maximally accretive (maximally nonnegative) if and only if*

(i) $A_1 M_0 A_1^* + \mathrm{Re}(A_1 A_2^*) \le 0$ (*resp.*, $A_1 M_0 A_1^* + A_1 A_2^* \le 0$), *where*

$$M_0 = \begin{pmatrix} -i(u_0 - \mathbb{I}_1)^{-1}(u_0 + \mathbb{I}_1) & 0 \\ 0 & 0 \end{pmatrix};$$

(ii) *for some (and, hence, for any)* $\varepsilon > 0$,

$$\ker(A_1 - A_2 - A_1 M(-\varepsilon)) = \{0\}.$$

Proof. As follows from Corollaries 2 and 5, the operator $M_0 \in \mathcal{B}(G)$ is correctly defined and, in addition, $s-\lim_{\varepsilon\to+0} M(-\varepsilon) = M_0$. Therefore, the validity of the required assertion directly follows from Theorem 4.

Remark 6. We recall that [see (23)], in the case where $L_{\max} = \overline{L_{\max}}$, any proper maximally accretive extension L_1 of the relation L_0 can be represented in the form

$$L_1 = \{\hat{y} \in L_{\max} : \alpha_1\Gamma_1\hat{y} + \alpha_2\Gamma_2\hat{y} = 0\},$$

where $\alpha_1, \alpha_2 \in \mathcal{B}(G_1)$. Corollary 6 implies that this extension L_1 of the relation L_0 is maximally accretive (maximally nonnegative) if and only if

(i) $\alpha_1 m_0 \alpha_1^* + \mathrm{Re}(\alpha_1\alpha_2^*) \le 0$,

(ii) for some (and, hence, for any) $\varepsilon > 0$,

$$\ker(\alpha_1 - \alpha_2 - \alpha_1 m(-\varepsilon)) = \{0\}.$$

REFERENCES

1. Yu. M. Arlins'kyi, *Maximal Accretive Extensions of Sectorial Operators* [in Ukrainian], Author's Abstract of the Doctoral-Degree Thesis (Phys., Math.), Kyiv (2000).
2. V. A. Derkach and M. M. Malamud, *Theory of Extensions of Symmetric Operators and Boundary Problems* [in Russian], Proceedings of the Institute of Mathematics, National Academy of Science of Ukraine, Mathematics and Its Applications, Vol. 104, Kyiv (2017)
3. V. A. Derkach and M. M. Malamud, *Weyl Function of a Hermitian Operator and Its Relationship with the Characteristic Function* [in Russian], Preprint No. 85–9, Donetsk Fiz.-Tekh. Inst., Akad. Nauk Ukr. SSR, Donetsk (1985).
4. A. N. Kochubei, "Extensions of symmetric operators and symmetric binary relations," *Mat. Zametki*, **17**, No. 1, 41–48 (1975), ***English translation:*** *Math. Notes Acad. Sci. USSR*, **17**, No. 1, 25–28 (1975); https://doi.org/10.1007/BF01093837.
5. M. G. Krein, "Theory of the self-adjoint extensions of semibounded Hermitian operators and its applications, I, II," *Mat. Sb.*, **20(62)**, No.3, 431–495; **21(63)**, No. 3, 365–404 (1947).

6. M. M. Malamud, "One approach to the theory of extensions of a nondensely defined Hermitian operator," *Dokl. Akad. Nauk Ukr. SSR,* No. 3, 20–25 (1990).
7. V. A. Mikhailets, "Spectra of operators and boundary problems," in: *Spectral Analysis of Differential Operators* [in Russian], Institute of Mathematics, Akad. Nauk Ukr. SSR, Kiev (1980), pp. 106–131.
8. O. Pihura and O. Storozh, "Resolvent and the conditions of solvability of proper extensions of linear relations in Hilbert spaces," *Visn. Lviv Univ., Ser. Mekh.-Mat.,* Issue 82, 174–185 (2016).
9. P. R. Halmos, *A Hilbert Space Problem Book,* Van Nostrand Company, Princeton (1967).
10. R. Arens, "Operational calculus of linear relations," *Pacific J. Math.,* **11**, No. 1, 9–23 (1961); DOI: 10.2140/pjm.1961.11.9.
11. E. A. Coddington, "Self-adjoint subspace extensions of nondensely defined symmetric operators," *Bull. Amer. Math. Soc.,* **79**, No. 4, 712–715 (1973); https://doi.org/10.1090/S0002-9904-1973-13275-6.
12. E. A. Coddington and H. S. V. de Snoo, "Positive self-adjoint extensions of positive symmetric subspaces," *Math. Zeit.,* **159**, No. 3, 203–214 (1978); https://doi.org/10.1007/BF01214571.
13. A. Dijksma and H. S. V. de Snoo, "Self-adjoint extensions of symmetric subspaces," *Pacific J. Math.,* **54**, No. 1, 71–100 (1974); DOI: 10.2140/pjm.1974.54.71.
14. O. G. Storozh, "Maximal accretive extensions of positively definite linear relation in a Hilbert space," in: *Book of Abstracts of the Internat. Conf. Devoted to the 70th Birthday of Prof. Oleh Lopushansky "Infinite-Dimensional Analysis and Topology"* [in Ukrainian], Ivano-Frankivsk (2019), pp. 49–50.
15. O. G. Storozh, "On an approach to the construction of the Friedrichs and Neumann–Krein extensions of nonlinear relations," *Karpath. Mat. Publ.,* **10,** No. 2, 387–394 (2018); https://doi.org/10.15330/cmp.10.2.387-394.