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WEIGHTED MEANS AND CHARACTERIZATION OF BALLS

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Weighted mean value identities over balls are considered for harmonic functions and their derivatives. Logarithmic and other weights are involved in these identities, some applications of which are presented. Also, new analytic characterizations of balls are proved; each of them requires the volume mean of a single weight function over the domain under consideration to be equal to a prescribed number depending on the weight. Bibliography: 9 titles.

1 Weighted Means of Harmonic Functions

1.1. Background. A function $u \in C^2(D)$ is *harmonic* (the origin of this term is described in [1, p. 25]) if it satisfies the equation $\nabla^2 u(x) = 0$ in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$, where $\nabla = (\partial_1, \dots, \partial_m)$ denotes the gradient operator, $\partial_i = \partial / \partial x_i$ and $x = (x_1, \dots, x_m)$ is a point of $\mathbb{R}^m$. Studies of mean value properties of harmonic functions date back to the Gauss theorem of the arithmetic mean over a sphere ([2, Article 20]). Nowadays, its standard formulation is as follows.

Theorem 1.1. *Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. Then for every $x \in D$ the equality $M^\circ(x, r, u) = u(x)$ holds for each admissible sphere $S_r(x)$.*

Here and below, the following notation and terminology are used. The open ball of radius r centered at x is denoted by $B_r(x) = \{y : |y - x| < r\}$ and is called *admissible* with respect to a domain D provided that $\overline{B_r(x)} \subset D$. We denote by $S_r(x) = \partial B_r(x)$ the corresponding admissible sphere. If $u \in C^0(D)$, then its spherical mean value over $S_r(x) \subset D$ is

$$M^\circ(x, r, u) = \frac{1}{|S_r|} \int\limits_{S_r(x)} u(y)\, \mathrm{d}S_y = \frac{1}{\omega_m} \int\limits_{S_1(0)} u(x + ry)\, \mathrm{d}S_y \,, \tag{1.1}$$

where $|S_r| = \omega_m r^{m-1}$ and $\omega_m = 2\pi^{m/2}/\Gamma(m/2)$ is the total area of the unit sphere (as usual, Γ stands for the Gamma function) and $\mathrm{d}S$ is the surface area measure.

Integrating $M^\circ(x, r, u)$ with respect to r, one obtains the following mean value property over balls.

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Theorem 1.2. *Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. Then for every $x \in D$ the equality $M^\bullet(x, r, u) = u(x)$ holds for each admissible ball $B_r(x)$.*

The volume mean of a locally Lebesgue integrable function u is defined as in (1.1):

$$M^\bullet(x, r, u) = \frac{1}{|B_r|} \int\limits_{B_r(x)} u(y)\, \mathrm{d}y = \frac{m}{\omega_m r^m} \int\limits_{|y|<r} u(x+y)\, \mathrm{d}y.$$

In their extensive article [3], Netuka and Veselý reviewed many other assertions involving various mean value properties of harmonic functions. The survey [4] published several years ago substantially complemented [3] with old and new results not covered in [3]. However, to the best author's knowledge, no results concerning weighted means of harmonic functions have appeared so far. The goal of the present note is to fill in this gap, at least, partially.

1.2. Weighted means.

Theorem 1.3. *Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. Then*

$$u(x) = \frac{m}{|B_r|} \int\limits_{B_r(x)} u(y) \log \frac{r}{|x-y|}\, \mathrm{d}y \tag{1.2}$$

for every $x \in D$ and each admissible ball $B_r(x)$.

Proof. Theorem 1.2 implies that the right-hand side of (1.2) is equal to

$$u(x)\, m \log r - \frac{m}{|B_r|} \int\limits_{B_r(x)} u(y) \log |x-y|\, \mathrm{d}y\,. \tag{1.3}$$

In the polar coordinates (ρ, θ) centered at x, we have

$$\int\limits_{B_r(x)} u(y) \log |x-y|\, \mathrm{d}y = \int\limits_0^r \int\limits_{S_1(0)} u(\rho, \theta)\, \rho^{m-1} \log \rho\, \mathrm{d}S\, \mathrm{d}\rho = |S_1|\, u(x) \int\limits_0^r \rho^{m-1} \log \rho\, \mathrm{d}\rho\,,$$

where the last equality is a consequence of Theorem 1.1. Since

$$\int\limits_0^r \rho^{m-1} \log \rho\, \mathrm{d}\rho = r^m (m \log r - 1)/m^2\,,$$

the expression (1.3) is equal to $u(x)$, which completes the proof. □

Remark 1.1. The two-dimensional version of Theorem 1.3 was heuristically derived by the author (arXiv: 2209.10281v1) in 2022 and proved later by F. Oschmann (arXiv:2209.110741v1).

Corollary 1.1. *Let D be a domain in $\mathbb{R}^m$. If u is harmonic in D, then*

$$\partial_i u(x) = \frac{m}{|B_r|} \int\limits_{B_r(x)} u(y)\, \frac{y_i - x_i}{|x-y|^2}\, \mathrm{d}y\,, \quad i = 1, \ldots, m,$$

for every $x \in D$ and each admissible ball $B_r(x)$.

Proof. Since $\partial_i u$ is also harmonic in D, Theorem 1.3 implies

$$\partial_i u(x) = \frac{m}{|B_r|} \int\limits_{B_r(x)} \frac{\partial u}{\partial y_i} \log \frac{r}{|x-y|} \, \mathrm{d}y \,, \quad i = 1, \dots, m,$$

for every $x \in D$, where $B_r(x)$ is an arbitrary admissible ball. Integrating by parts on the right-hand side, one arrives at the required assertion because the integral over $S_r(x)$ vanishes. Indeed, $\log r/|x-y| = 0$ when $y \in S_r(x)$. □

Corollary 1.1 makes obvious the following.

Proposition 1.1. *Let u be harmonic in D. If D' is a compact subset of D, then*

$$\max_{x \in D'} |\partial_i u| \leqslant \frac{m}{d} \sup_{x \in D} |u|, \quad i = 1, \dots, m,$$

where d is the distance between D' and ∂D. Moreover, if $u \geqslant 0$ in D, then

$$|\partial_i u(x_0)| \leqslant \frac{m}{d_0} u(x_0), \quad i = 1, \dots, m,$$

where d_0 is the distance from x_0 to ∂D.

Along with $\log \frac{r}{|x-y|}$, there are other weights with mean value properties analogous to (1.2); a couple of them is adduced in the following.

Theorem 1.4. *Let $u \in C^2(D)$ be harmonic in a domain $D \subset \mathbb{R}^m$, $m \geqslant 2$. Then*

$$(m\, \alpha^{-1} - 1)\, r^{\alpha - m}\, u(x) = \frac{1}{|B_r|} \int\limits_{B_r(x)} u(y) [|x-y|^{\alpha - m} - r^{\alpha - m}] \mathrm{d}y \,, \quad \alpha \in (0, m) \,, \tag{1.4}$$

and

$$[1 - \frac{m}{m+\beta}] r^\beta\, u(x) = \frac{1}{|B_r|} \int\limits_{B_r(x)} u(y) \left[r^\beta - |x-y|^\beta \right] \mathrm{d}y \,, \quad \beta > 0 \,, \tag{1.5}$$

for every $x \in D$ and each admissible ball $B_r(x)$. Here, both weights are integrable functions of y over any bounded domain for any $x \in \mathbf{R}^m$.

The proof literally repeats that of Theorem 1.3. We note that the coefficients at u are positive on the left-hand side of (1.4) and (1.5) in view of the assumptions on α and β respectively.

2 Characterization of Balls via Averaging Weights

In 1962, Epstein published the one-page long note [5], in which he proved the following theorem. *Let D be a simply connected plane domain of finite area and t a point of D such that, for every function u harmonic in D and integrable over D, the mean value of u over the area of D equals $u(t)$. Then D is a disc and t is its center.*

Further studies of inverse mean value properties (this widely accepted term, coined by Hansen and Netuka [6], concerns analytic characterizations of various domains via their volume or/and boundary area means) for almost six decades were restricted to those of harmonic functions (see the extensive survey [3, Sections 7 and 8]). Recently, inverse mean value properties were obtained for real-valued solutions of the modified Helmholtz and Helmholtz equations (see [7] and [8] respectively).

However, it occurs that one can characterize balls without involving any solutions of partial differential equations. Establishing some characterizations of this kind, which are based on weighted means of the domains volume, is the second goal of this paper.

The motivation behind this is the observation that for nonzero constants the weighted mean value formula (1.2) takes the form

$$m^{-1} = \frac{1}{|B_r|} \int\limits_{B_r(x)} \log \frac{r}{|x-y|} \, \mathrm{d}y \, . \tag{2.1}$$

Another essential point is that the spherically symmetric weight $\log(r/|x-y|)$ has the following properties: it is a continuous function of y going along any ray emanating from x, it monotonically decreases from $+\infty$ to $-\infty$, and it is positive when y belongs to $B_r(x)$, being negative outside this ball. This particular behavior of the weight is used in the following.

Theorem 2.1. *Let $D \subset \mathbb{R}^m$ be a bounded domain. If the identity*

$$\frac{1}{m} = \frac{1}{|D|} \int\limits_{D} \log \frac{r}{|x-y|} \, \mathrm{d}y \tag{2.2}$$

holds with $x \in D$ and $r > 0$ such that $|D| \geqslant |B_r|$, then $D = B_r(x)$.

Proof. Without loss of generality we suppose that the domain D is located in such a way that x coincides with the origin. Let us show that the assumption $D \neq B_r(0)$ leads to a contradiction. For this purpose we consider two bounded open sets $G_i = D \setminus \overline{B_r(0)}$ (nonempty by the assumption on D and r) and $G_e = B_r(0) \setminus \overline{D}$ (possibly empty).

Let us write (2.2) as follows:

$$\frac{|D|}{m} = \int\limits_{D} \log \frac{r}{|y|} \, \mathrm{d}y \, . \tag{2.3}$$

Since the identity (2.1) holds for $x = 0$ and $B_r(0)$, we write it in the same way:

$$\frac{|B_r|}{m} = \int\limits_{B_r(0)} \log \frac{r}{|y|} \, \mathrm{d}y \, . \tag{2.4}$$

Subtracting (2.4) from (2.3), we find

$$\frac{|D| - |B_r|}{m} = \int\limits_{G_i} \log \frac{r}{|y|} \, \mathrm{d}y - \int\limits_{G_e} \log \frac{r}{|y|} \, \mathrm{d}y \, .$$

Here, the difference on the right-hand side is negative. Indeed, $\log(r/|y|) < 0$ on $G_i \neq \varnothing$ because $|y| > r$ there. Hence, the first term is negative. If $G_e \neq \varnothing$, then the second integral is positive because $\log(r/|y|) > 0$ on G_e, where $|y| < r$. On the other hand, the expression on the left-hand side is nonnegative. The obtained contradiction proves the theorem. □

An analogue of Theorem 2.1 ensues by averaging the weight

$$|x-y|^{\alpha-m} - r^{\alpha-m}, \quad r>0, \quad \alpha \in (0,m), \tag{2.5}$$

which is an integrable function of y over any bounded domain for any $x \in \mathbb{R}^m$. A direct calculation shows (cf. (1.4) with $u \equiv 1$)

$$\frac{1}{|B_r|} \int\limits_{B_r(x)} [|x-y|^{\alpha-m} - r^{\alpha-m}] \mathrm{d}y = (m\,\alpha^{-1} - 1)\, r^{\alpha-m} \quad \forall\, x \in \mathbb{R}^m. \tag{2.6}$$

This value is positive in view of the assumption on α. This identity is similar to (2.1) and allows us to prove the following.

Theorem 2.2. *Let $D \subset \mathbb{R}^m$ be a bounded domain. If the identity*

$$(m\,\alpha^{-1} - 1)\, r^{\alpha-m} = \frac{1}{|D|} \int\limits_{D} [|x-y|^{\alpha-m} - r^{\alpha-m}] \mathrm{d}y \tag{2.7}$$

holds with $x \in D$ and $r > 0$ such that $|D| \geqslant |B_r|$, then $D = B_r(x)$.

The proof literally follows that of Theorem 2.1, but the identities (2.6) and (2.7) must be used instead of (2.1) and (2.2) respectively. Therefore, assuming that $D \neq B_r(x)$ for $x \in D$, one arrives at the equality

$$(|D| - |B_r|)\,(m\,\alpha^{-1} - 1)\, r^{\alpha-m} = \left[\, \int\limits_{G_i} - \int\limits_{G_e} \right] [|x-y|^{\alpha-m} - r^{\alpha-m}] \mathrm{d}y\,,$$

which is impossible. Indeed, the expression on the left-hand side is nonnegative, whereas the integral over $G_i \neq \varnothing$ is negative because $|x-y|^{\alpha-m} < r^{\alpha-m}$ on G_i and the integral over G_e (possibly empty) is positive since $|x-y|^{\alpha-m} > r^{\alpha-m}$ on G_e when it is nonempty.

It is worth mentioning the result obtained by O'Hara [9], which seems to be related to Theorem 2.2. It is known that the first term in (2.5) integrated over $D \times D$ defines the so-called Riesz $(\alpha - m)$-energy of a domain D. Its generalization proposed in [9] serves for characterization of balls (see Theorem 3.4 in [9]). However, the proof of this theorem is rather technical.

Finally, let us provide an example of a nonsingular weight function, namely, $r^\alpha - |x-y|^\alpha$, $\alpha > 0$. It is straightforward to calculate (cf. (1.5) with $u \equiv 1$) that for any $x \in \mathbb{R}^m$

$$\frac{1}{|B_r|} \int\limits_{B_r(x)} [r^\alpha - |x-y|^\alpha] \mathrm{d}y = \left[1 - \frac{m}{m+\alpha}\right] r^\alpha > 0, \quad r > 0\,,$$

which allows us to prove an assertion analogous to Theorems 2.1 and 2.2, but using this weight. Thus, one arrives at another characterization of balls via averaging the last weight over D with $|D| \geqslant |B_r|$.

It is easy to continue the list of weights (singular and nonsingular) that characterize balls via averaging. Indeed, one just has to take into account that such a weight is any continuous (of course, this requirement can be relaxed) function $(0,\infty) \times (0,\infty) \ni (t,r) \mapsto w \in \mathbb{R}$ with the following properties:

(i) $w(t, r) > 0$ if $t < r$, $w(t, r) < 0$ if $t > r$, and $w(r, r) = 0$,

(ii) for every $x \in \mathbb{R}^m$ and each $r > 0$ the superposition $w(|x - y|, r)$ is locally integrable in y.

It is interesting to find out whether every weight characterizing balls via averaging yields a mean value identity for harmonic functions. Also, the converse of this assertion is worth considering.

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