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OPTIMAL ACCUMULATION OF FACTORS WITH LINEAR-HOMOGENEOUS PRODUCTION FUNCTIONS ON INFINITE TIME HORIZON

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We study the extension of the problem of optimal investments in two factors of production (physical and human capital) for an arbitrary linearly homogeneous production function satisfying the limit conditions for partial derivatives, where the total discounted utility function of consumption for infinite time is maximized. We show that, with a sufficiently small discount rate of utility and a sufficiently large relative risk aversion of a consumer, it is optimal for the consumer to initially increase only physical capital until the rate of technological substitution of physical and human capital (the ratio of the corresponding partial derivatives) is equal to the efficiency of investments in human capital and then invest in both types of capital in a constant proportion, which leads to unlimited growth. Bibliography: 12 titles. Illustrations: 3 figures.

How much resources should be spent on production and how much on development of technologies is one of the main questions of the theory of economic growth. To solve this problem, as well as the Ramsey problem [1] of optimal investment in physical capital [2], methods of optimal control theory began to be applied [3] almost immediately after the publication of the maximum principle [4].

In works on optimal technological development, in addition to consumption, the shares of limited resources allocated to technological development are chosen as controls: labor [3], human capital [5], and physical capital. The target functional is usually the total discounted utility from the consumption of a representative household over infinite time.

Optimal control problems in infinite time present mathematical difficulties in choosing the correct condition on the conjugate variables of transversality type, as shown, for example, in [6]. This direction of the theory is still developing [7], even on the example of the classical Ramsey problem [8]. The optimal solution for an infinite planning horizon may not exist in optimal

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resource exploitation problems [9]. Therefore, for the mathematical justification of a solution, it is good if one manages to formulate a problem to which sufficient optimality conditions can be applied [10, 11].

Economic papers mainly study the balanced growth path of the model when all variables grow at a constant rate. The dependence on the parameters of the problem of a constant growth rate of average labor productivity on this trajectory helps to explain cross-country differences. However, the transitional trajectory to the balanced growth path is no less interesting to study since it proposes the current optimal policy of investment in capital and technological development, the ratio of which can radically differ on the transitional trajectory and on the balanced growth path.

In this paper, we consider a generalization of the model from [12], where, unlike a similar problem in [5], it was assumed that the rate of human capital growth is proportional to the amount of the final product spent on it, which, together with consumption, can temporarily exceed the current output by reducing the stock of physical capital. The assumption that control is unlimited from above admits the nonexistence of a locally bounded optimal control. However, in the present formulation, we can use the sufficient conditions of the Mangasaryan optimality [10], which guarantees the existence. The necessary optimality conditions are used to prove that there are no solutions with investment in human capital under certain parameters.

1 Statement of the Problem

We assume that the population and the labor supplied by the population in the economy are constant and normalize them to 1. We consider the problem of a central planner maximizing the utility of consumers in the economy discounted with the norm $\rho > 0$

$$\int_0^\infty e^{-\rho t} u(C(t))\, dt \to \max_{C(\cdot),G(\cdot)}, \quad C(t) \geqslant 0, \quad G(t) \geqslant 0, \quad K(t) > 0,$$

the growth speed of the physical capital K or the net investment flow

$$\dot{K}(t) = F(K(t), A(t)) - C(t) - G(t), \quad K(0) = K_0 > 0, \tag{1.1}$$

this is what remains of the net output F after the private consumption C and the government spending G. The growth speed of the human capital A

$$\dot{A}(t) = \gamma\, G(t), \quad A(0) = A_0 > 0, \tag{1.2}$$

is proportional to the government spending G, where $\gamma > 0$ is the efficiency coefficient of investment in human capital. [1)]

We assume that F is a linearly homogeneous and differentiable function in both variables $A, K > 0$ such that its partial derivative monotonically decreases with respect to the differentiation variable and monotonically increases with respect to the other variable.

[1)] The *net output* is the output minus the expenses due to depreciation of capital. Also, the capital growth is called the *net investment* which, in addition to the costs of capital depreciation, would amount to the gross investment.

Assumption 1.1. For all $A > 0$

$$\lim_{K\to\infty} F'_K(K,A) \leqslant \min\{\rho, \gamma \cdot \lim_{K\to\infty} F'_A(K,A)\},$$

$$\lim_{K\to 0} F'_K(K,A) \geqslant \max\{\rho, \gamma \cdot \lim_{K\to 0} F'_A(K,A)\}.$$

We assume that the consumers have a constant relative risk aversion $\theta > 0$, i.e.,

$$u'(C) = C^{-\theta}. \tag{1.3}$$

This means that we consider instantaneous utility functions, for example,

$$u(C) = \frac{C^{1-\theta}}{1-\theta},$$

where $\theta \neq 1$ and $u(C) = \ln C$ for which $\theta = 1$.

2 Optimality Conditions

The Hamilton–Pontryagin function

$$\mathscr{H}(K, A, C, G, \lambda, \psi) = u(C) + \lambda(F(K,A) - C - G) + \psi\,\gamma\, G$$

is concave over the set of its variables (K, A, C, G). Therefore, we can use the Mangasarian sufficient optimality conditions [10] consisting of the following conditions.

1. The maximum condition

$$(\widehat{C}(t), \widehat{G}(t)) \in \arg\max_{C,G\geqslant 0} \mathscr{H}(\widehat{K}(t), \widehat{A}(t), C, G, \lambda(t), \psi(t)) \tag{2.1}$$

which can be written as

$$u'(\widehat{C}(t)) - \lambda(t) = 0, \tag{2.2}$$

$$\widehat{G}(t) \in \arg\max_{G\geqslant 0} [\gamma\,\psi(t) - \lambda(t)]G \tag{2.3}$$

in view of the strict concavity of $u(C)$.

2. The adjoint equations

$$\rho\lambda(t) - \dot{\lambda}(t) = \frac{\partial \mathscr{H}}{\partial K},$$

$$\rho\psi(t) - \dot{\psi}(t) = \frac{\partial \mathscr{H}}{\partial A}$$

in the form

$$\rho\,\lambda(t) - \dot{\lambda}(t) = \lambda(t)\, F'_K(\widehat{K}(t), \widehat{A}(t)), \tag{2.4}$$

$$\rho\,\psi(t) - \dot{\psi}(t) = \lambda(t)\, F'_A(\widehat{K}(t), \widehat{A}(t)). \tag{2.5}$$

3. The limit conditions of transversality type in the form

$$\liminf_{t\to\infty} e^{-\rho t}\lambda(t)(K(t)-\widehat{K}(t)) \geqslant 0,$$

$$\liminf_{t\to\infty} e^{-\rho t}\psi(t)(A(t)-\widehat{A}(t)) \geqslant 0,$$

where $K(\cdot)>0$ and $A(\cdot)>0$ are admissible trajectories satisfying Equations (1.1) and (1.2) for nonnegative controls $C(\cdot)$ and $G(\cdot)$.

Since $K(t)$ and $A(t)$ are positive, the limit conditions are satisfied if

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)\,\widehat{K}(t)=0, \qquad \lim_{t\to\infty} e^{-\rho t}\psi(t)\,\widehat{A}(t)=0, \tag{2.6}$$

where only optimal variables are involved. We will omit the symbol $\widehat{\ }$ over the optimal variables.

It is convenient to pass to the new specific variables per unit of human capital

$$f(k):=\frac{F(K,A)}{A}=F(k,1), \quad k:=\frac{K}{A}, \quad g:=\frac{G}{A}, \quad c:=\frac{C}{A}.$$

From (1.1) and (1.2), we can express the growth rates of the physical and human capital in the new variables as

$$\frac{\dot K(t)}{K(t)}=\frac{f(k(t))-g(t)-c(t)}{k(t)}, \quad \frac{\dot A(t)}{A(t)}=\gamma\,g(t).$$

Then for the dynamics of physical capital per unit of human capital

$$\dot k(t)=\left(\frac{\dot K(t)}{K(t)}-\frac{\dot A(t)}{A(t)}\right)k(t)$$

we obtain the equation

$$\dot k(t)=f(k(t))-c(t)-(1+\gamma\,k(t))g(t), \quad k(0)=k_0:=\frac{K_0}{A_0}>0. \tag{2.7}$$

The dynamics of human capital (1.2) takes the form

$$\dot A(t)=\gamma\,g(t)\,A(t), \quad A(0)=A_0>0. \tag{2.8}$$

By the Euler theorem, for a homogeneous function of the first degree we have [2)]

$$F'_K(K,A)=f'(k), \qquad F'_A(K,A)=f(k)-f'(k)\,k. \tag{2.9}$$

Then the adjoint equations (2.4) and (2.5) can be written as

$$\rho\,\lambda(t)-\dot\lambda(t)=f'(k(t))\,\lambda(t), \tag{2.10}$$

$$\rho\,\psi(t)-\dot\psi(t)=\lambda(t)[f(k(t))-f'(k(t))\,k(t)]. \tag{2.11}$$

[2)] By the Euler theorem , $F(K,A)=F'_K(K,A)\,K+F'_A(K,A)\,A$ and $F'_K(K,A)=F'_k(k,1)$. Dividing the first identity by A and taking into account the second one $F'_K(K,A)=f'_k(k)$, we get $f(k)=F'_K(K,A)\,k+F'_A(K,A)=f'(k)\,k+F'_A(K,A)$.

From the maximum condition it follows that $\lambda(t) = u'(C(t)) = [C(t)]^{-\theta} > 0$, $C(t) > 0$, since $\theta > 0$. Substituting $\lambda(t) = [A(t)\, c(t)]^{-\theta}$ into (2.10) and taking into account (2.8), we write the equation of the dynamics of specific consumption

$$\theta\, \dot{c}(t) = c(t)[f'(k(t)) - \rho - \theta\, \gamma\, g(t)]. \tag{2.12}$$

Introducing the variable $z(t) = \psi(t)/\lambda(t)$, from (2.10) and (2.11) we obtain the equation

$$\dot{z}(t) = f'(k(t))\, z(t) + f'(k(t))\, k(t) - f(k(t)). \tag{2.13}$$

We write the maximum condition (2.3) in the new variables as

$$g(t) \in \arg\max_{g \geqslant 0} [\gamma\, z(t) - 1] g. \tag{2.14}$$

It is clear that $0 < g(t) < \infty$ for all $t \geqslant T$ only if $z(t) = 1/\gamma$ for all $t \geqslant T$. In (2.13) this condition holds for the stationary value $k = \overline{k}$ which exists and is unique in view of Assumption 1.1 and formula (2.9),

$$\underbrace{f'(\overline{k})}_{F'_K(K,A)} = \underbrace{(f(\overline{k}) - f'(k)\, \overline{k})}_{F'_A(K,A)}\, \gamma. \tag{2.15}$$

Using the differential equation in (2.7) with $k = \overline{k}$ and taking into account that

$$f(\overline{k}) = \Big(\frac{1}{\gamma} + \overline{k}\Big) f'(\overline{k}),$$

it is possible to express the specific consumption from (2.15) as

$$c(t) = f(\overline{k}) - (1 + \gamma\, \overline{k}) g(t) = \Big(\frac{1}{\gamma} + \overline{k}\Big)(f'(\overline{k}) - \gamma g(t)). \tag{2.16}$$

Substituting (2.16) into (2.12), we get the differential equation

$$\dot{g}(t) = -\Big[\frac{f'(\overline{k})}{\gamma} - g(t)\Big]\Big[\frac{f'(\overline{k}) - \rho}{\theta} - \gamma\, g(t)\Big]. \tag{2.17}$$

Equation (2.17) has two stationary solutions. The first solution

$$\widetilde{g} = \frac{f'(\overline{k})}{\gamma}$$

corresponds to the zero stationary consumption $\widetilde{c} = 0$ according to (2.16). The second one

$$\overline{g} = \frac{f'(\overline{k}) - \rho}{\theta\, \gamma}, \tag{2.18}$$

which is positive if

$$f'(\overline{k}) - \rho > 0, \tag{2.19}$$

corresponds to the stationary consumption

$$\overline{c} = \Big(\frac{1}{\gamma} + \overline{k}\Big)\frac{(\theta - 1) f'(\overline{k}) + \rho}{\theta}, \tag{2.20}$$

which is positive if $\overline{g} < \widetilde{g}$, i.e.,

$$(\theta - 1)f'(\overline{k}) + \rho > 0. \tag{2.21}$$

We note that for $\theta \geqslant 1$ the inequality (2.21) holds if (2.19) is fulfilled.

From (2.17) we can see that for $g(t) < \widetilde{g} \leqslant \overline{g}$ the derivative is negative, $\dot{g}(t) < 0$, i.e., if (2.21) holds, then the solution of Equation (2.17) cannot converge to a trajectory with the zero consumption $\widetilde{c} = 0$, having a positive consumption $c(t) > 0$ In view of (2.16). The other stationary consumption is negative $\overline{c} < 0$. It is clear that $\dot{g}(t) < 0$ for $g(t) < \overline{g} \leqslant \widetilde{g}$ or $\overline{g} \leqslant \widetilde{g} < g(t)$ and $\dot{g}(t) > 0$ for $\overline{g} < g(t) < \widetilde{g}$, i.e., if (2.21) holds and the specific investment in the human capital is $g(t) \neq \overline{g}$, then it becomes nonpositive in a finite time or converges to $\widetilde{g}$ as the consumption converges to zero. Therefore, if both conditions (2.19) and (2.21) hold, the only way to have positive consumption is to have stationary $g(t) = \overline{g} > 0$ and $c(t) = \overline{c} > 0$ for all $t \geqslant T$.

If the condition (2.19) fails, then $g \equiv 0$ satisfies the maximum condition (2.14) at $z(t) \leqslant 1/\gamma$, and from the stationarity conditions $f'(k^*) = \rho$ and $c^* = f(k^*)$ of the system (2.7), (2.12) we find its saddle point (k^*, c^*) which exists and is unique by Assumption 1.1 and to which the solution of the system should converge along the saddle path (2.7), (2.12) with $g \equiv 0$, which coincides with the solution of the Ramsey model without exogenous technological growth.

Now, we are ready to formulate the main theorem.

Theorem 2.1. *Let Assumption* 1.1 *hold. Then there is a ratio of the physical K and human A capitals $\dfrac{K}{A} = \overline{k} > 0$ determined by* (2.15) *such that*

$$\frac{F'_K(K, A)}{F'_A(K, A)} = \gamma \quad \forall\, A > 0, \quad K > 0, \quad \frac{K}{A} = \overline{k}.$$

If (2.19) *and* (2.21) *hold, $0 < 1 - \rho/f'(\overline{k}) < \theta$, and $k_0 \leqslant \overline{k}$, then the optimal solution is the dynamics of* (2.7), (2.12) *with $g(t) = 0$, $t \in [0, T]$, and $k(0) = k_0$:*

$$\dot{k}(t) = f(k(t)) - c(t), \tag{2.22}$$

$$\theta\, \frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \rho, \tag{2.23}$$

until the moment $T = T(k_0)$ when the specific capital and consumption reach the stationary values $k(T) = \overline{k}$ and $c(T) = \overline{c}$ calculated from (2.15) *and* (2.20) *when the investment $g(\cdot)$ jumps from zero to $\overline{g}$ in* (2.18) *and, like the other specific variables, $c(t) = \overline{c}$ and $k(t) = \overline{k}$ remain at its stationary level $g(t) = \overline{g}$ for all $t > T$.*

If (2.19) *is not satisfied and $f'(\overline{k}) \leqslant \rho$, i.e., $1 - \rho/f'(\overline{k}) \leqslant 0 < \theta$, then the optimal dynamics does not include the human capital investment, $g(t) \equiv 0$, and is described by Equations* (2.22), (2.23) *for $T = \infty$, where $c(t)$ and $k(t)$ converge to the stationary values c^* and k^* determined from the conditions $c^* = f(k^*)$ and $f'(k^*) = \rho$.*

Proof. Let (2.19) and (2.21) be satisfied. We show that, up to the moment $t = T$, the maximum condition leads to $g(t) = 0$. Equation (2.13) with the condition $z(T) = 1/\gamma$ for $k < \overline{k}$ for $t < T$ gives $\dot{z}(t) > 0$ due to the fact that the marginal productivity of physical capital $f'(k)$ decreases by k and the marginal productivity of human capital $f(k) - f'(k)\,k$ increases by k. Hence $z(t) < 1/\gamma$ for $t < T$ and from the maximum condition (2.14) we have

$g(t) = 0$ if $k_0 < \overline{k}$. We note that for $k_0 > \overline{k}$ the maximum condition would yield $g(0) = \infty$. We show that the solution satisfies (2.6). Starting from the time T, the variables K and A grow at the same rate $\gamma\,\overline{g}$. The adjoint variables λ and ψ also grow at the same rate because $\psi(t)/\lambda(t) = z = 1/\gamma$. Both conditions in (2.6) are fulfilled if $\dot{\lambda}(t)/\lambda(t) + \dot{K}(t)/K(t) - \rho < 0$ which, along with $\dot{K}(t)/K(t) = \gamma\,\overline{g}$, (2.10), $\rho - \dot{\lambda}(t)/\lambda(t) = f'(\overline{k})$, (2.18), $\rho + \theta\,\gamma\,g = f'(\overline{k})$, amounts to (2.21).

If (2.19) is not satisfied and $f'(\overline{k}) \leqslant \rho$, then for $g \equiv 0$ we consider the solution which monotonically tends to (k^*, c^*) along the saddle trajectory of the system (2.22), (2.23), i.e., $A \equiv A_0$, $K \to A_0\,k^*$, $C \to A_0\,f(k^*)$, $\lambda = u'(C) \to u'(A_0\,f(k^*))$ as $t \to \infty$. The variable $z(t)$ should tend to the stationary value

$$z(t) \to z^* = \frac{f(k^*) - f'(k^*)\,k^*}{f'(k^*)} < \frac{f(\overline{k}) - f'(\overline{k})\,\overline{k}}{f'(\overline{k})} = \frac{1}{\gamma}$$

less than $1/\gamma$ due to the relation $f'(\overline{k}) \leqslant \rho = f'(k^*)$ and the fact that the marginal productivity of physical capital (in the denominator) is a decreasing function and the marginal productivity of human capital (in the numerator) is an increasing function of k. In view of the inequality obtained, the maximum condition for $g \equiv 0$ holds. For the adjoint variable we have $\psi(t) = \lambda(t)\,z(t) \to$ const. Thus, both conditions in (2.6) are satisfied. □

Remark 2.1. In the case of (2.19), (2.21) with a large initial specific capital $k_0 > \overline{k}$, it is not possible to find locally bounded controls $g(\cdot)$ leading to a stationary solution. One can consider a problem with restriction from above on the values of the function $g(\cdot)$, which would be physically justified, or with the possibility to exchange at the initial moment the excess capital $k_0 - \overline{k}$ for technologies at the price γ. However, the latter case does not seem to be economically natural.

Remark 2.2. If (2.19) holds, but (2.21) fails, i.e., $0 < 1 - \rho/f'(\overline{k}) \geqslant \theta$, when the consumer risk aversion θ is small, there is no optimal solution with positive investment in human capital. Otherwise, there should be the zero consumption. One can say that the consumer would indefinitely postpone the consumption. Moreover, there is no optimal solution at all if, in addition, we have

$$\max_k \Big(f(k) - (1 + \gamma\,k)\frac{\rho}{1-\theta} \Big) > 0.$$

Indeed, having a positive capital, it is not optimal to stop consumption forever because the capital could be "consumed." The conditions (2.2), (2.3) and (2.4), (2.5) are also included in the necessary optimality conditions of the Pontryagin maximum principle in the normal form (see, for example,

[7]). In this case, Equation (2.17) has no solution corresponding to a positive consumption, and the stationary solution $\widetilde{g}$ with the zero consumption contradicts the maximum condition (2.2) because of the infinite derivative of the utility function at zero.

In the abnormal form of the maximum principle, when the Pontryagin function has the form

$$\mathscr{H}(K, A, C, G, \lambda, \psi) = \lambda(F(K, A) - C - G) + \psi\,\gamma\,G,$$

from the maximum condition (2.1) it follows that for positive values of C and G the identities $\lambda(t) \equiv 0$ and $\gamma\,\psi(t) - \lambda(t) \equiv 0$ are satisfied, which implies $\psi(t) = \lambda(t) \equiv 0$. But this contradicts the assumptions of the Pontryagin maximum principle.

It could be neither optimal to stop investment in human capital forever because the objective functional would be finite, while we can find admissible controls such that the functional is infinite

$$\int_0^{\infty} e^{-\rho t}\, u(C(t))\, dt = \int_0^{t_0} e^{-\rho t}\, u(C(t))\, dt + \frac{A_0^{1-\theta} c^{1-\theta}}{1-\theta} \int_{t_0}^{\infty} e^{((1-\theta)g-\rho)\, t}\, dt = \infty,$$

where we take $c(t) = g(t) = 0$ for $t \in [0, t_0)$ and $g(t) = g > \rho/(1-\theta)$, $c(t) = c > 0$ for all $t \geqslant t_0$. It is possible if the maximal specific consumption for stationary k in (2.7) is positive. Then there is no optimal solution.

3 Example

We consider the Cobb-Douglas production function with a constant rate $\delta > 0$ of capital depreciation (see [12]) $F(K, A) = K^{\alpha} A^{1-\alpha} - \delta\, K$, $\alpha \in (0, 1)$, and the logarithmic instantaneous utility function, i.e., $\theta = 1$. In the specific variables, the net output is $f(k) = k^{\alpha} - \delta\, k$.

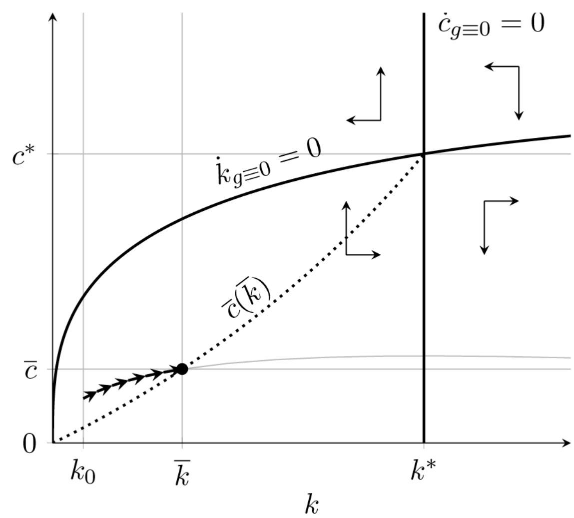

Figure 1. The phase diagram of the system (3.1), (3.2). The stationary curves are drawn by the bold solid lines. The optimal trajectory, indicated by arrows converging to the point $(\overline{k}, \overline{c})$, which becomes stationary at time $t = T$ when the human capital investment $g(t)$, increases from 0 to $\overline{g}$ in (3.4). The dots denote the stationary positions $(\overline{k}, \overline{c})$ at different values of the human capital investment efficiency γ.

The dynamics of specific capital and consumptions (2.22) and (2.23)

$$\dot k(t) = (k(t))^{\alpha} - c(t) - \delta\, k(t), \quad k(0) = k_0,\, k(T) = \overline{k}, \tag{3.1}$$

$$\frac{\dot c(t)}{c(t)} = \alpha\, (k(t))^{\alpha-1} - \rho - \delta, \quad c(T) = \overline{c}, \tag{3.2}$$

is represented in the phase diagram of Figure 1, where $\overline{k}$ is the solution to Equation (2.15)

$$\alpha\, k^{\alpha-1} - \delta = (1-\alpha)\, k^{\alpha} \gamma, \tag{3.3}$$

which exists and is unique because the left-hand side of (3.3) strictly monotonically decreases from $+\infty$ to $-\delta$, and the right-hand side strictly monotonically increases from 0 to $+\infty$ at $k \in (0, +\infty)$, i.e., Assumption 1.1 is satisfied.

The steady-state level of investment in human capital is found by formula (2.18)

$$\overline{g} = \frac{\alpha\, \overline{k}^{\alpha-1} - \delta - \rho}{\gamma} \tag{3.4}$$

and the stationary consumption is determined by (2.20)

$$\overline{c} = \left(\frac{1}{\gamma} + \overline{k}\right)\rho. \tag{3.5}$$

The optimal trajectory of the absolute values of K and C is depicted in the phase diagram of Figure 2.

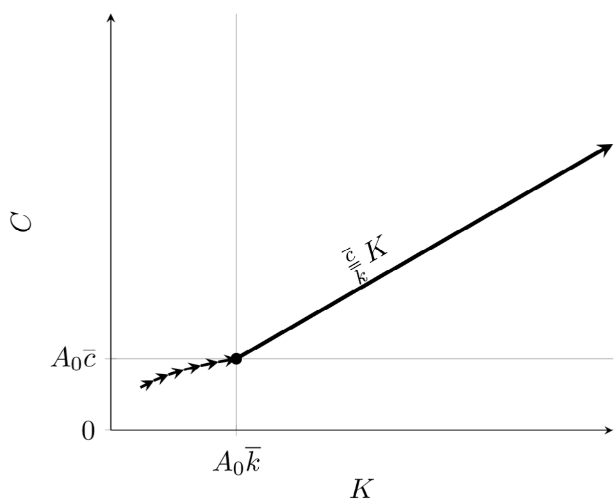

Figure 2. The phase diagram for absolute variables K and C. The optimal trajectory moves along the arrow-marked transition path to the balanced growth trajectory point $(A_0\overline{k}, A_0\overline{c})$, which is reached at time $t = T$, then the human capital investment $g(t)$ increases from 0 to $\overline{g}$ from (3.4) and the optimal trajectory follows the balanced growth trajectory indicated by the bold black line.

Figure 3 depicts the capital and consumption trajectories over time. It can be seen that, after time $t = T$ when the investment in human capital begins, the growth rate of physical capital falls and the previously slowing growth of consumption begins to accelerate.

The condition (2.21) is satisfied since $\theta = 1$. From the condition (2.19) of positive human capital investment (3.4) $\alpha\overline{k}^{\alpha-1} - \delta - \rho > 0$ and the stationarity condition of this regime (3.3) we can deduce that for the unlimited economic growth in this example it is sufficient for the efficiency of human capital investment to be above a certain value:

$$\gamma > \frac{\rho}{1-\alpha}\left(\frac{\delta+\rho}{\alpha}\right)^{\frac{\alpha}{1-\alpha}}.$$

Otherwise, the country will never invest in human capital.

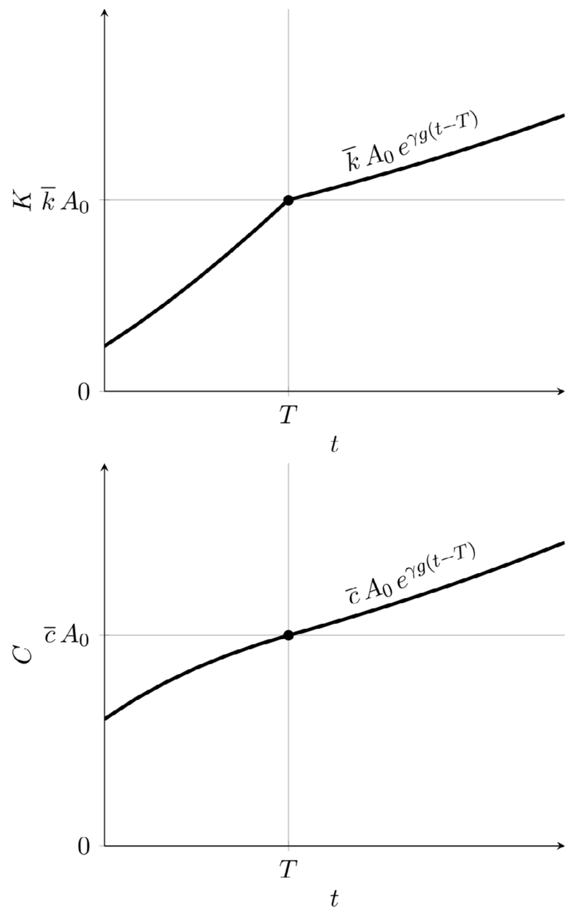

Figure 3. Trajectories of the capital $K(t)$ and consumption $C(t)$ over time.

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