DOI 10.1007/s10958-023-06285-5
Journal of Mathematical Sciences, Vol. 269, No. 3, January, 2023

CAYLEY–DICKSON SPLIT-ALGEBRAS: DOUBLY ALTERNATIVE ZERO DIVISORS AND RELATION GRAPHS

A. E. Guterman and S. A. Zhilina UDC 512.554

Abstract. Our paper is devoted to the investigations of doubly alternative zero divisors of the real Cayley–Dickson split-algebras. We describe their annihilators and orthogonalizers and also establish the relationship between centralizers and orthogonalizers for such elements. Then we obtain an analogue of the real Jordan normal form in the case of the split-octonions. Finally, we describe commutativity, orthogonality, and zero divisor graphs of the split-complex numbers, the split-quaternions, and the split-octonions in terms of their diameters and cliques.

To the memory of V. T. Markov

1. Introduction

The zero divisors of the Cayley–Dickson algebras are of special interest; however, the problem of their identification and description of their annihilators is rather difficult. Some attempts to classify the zero divisors of the algebras of the main sequence were made by Moreno in [20–22] and by Biss et al. in [11]. However, until now this problem is far from being solved. Particularly, [11] contains the description of top-dimensional zero divisors only. It should be noted that Moreno was the first to study doubly alternative zero divisors of the algebras of the main sequence, i.e., the elements whose both components are alternative. Then their annihilators were described in [11, Proposition 11.1]. Still, there is no simple criterion for doubly alternative elements to be zero divisors, except for Theorem 2.9 in [20].

An important approach to the visualization of numerous algebraic relations, such as zero division, commutativity, etc., is to define a relation graph via the algebraic properties under consideration. The research in the area of graphs determined by the relations in the algebraic systems originates from the group theory (see, e.g., [5]). Rings and algebras were first studied in this way by Beck in [9] (1988), where the zero divisor graph of a commutative ring was first introduced and investigated. This definition was later improved by Anderson and Livingston in [4]. As for zero divisor graphs of noncommutative rings, Redmond was the first to introduce them in [23]. Commutativity graphs of non-commutative rings were defined in [2] by Akbari et al. Orthogonality graphs represent a similar concept and were studied first in [8] by Bakhadly et al.

Relation graphs of matrix rings are of particular interest (see [12]) for zero divisor graphs, [1, 3, 14] for commutativity graphs, and [7, 8, 18] for orthogonality graphs.

In this paper, we consider doubly alternative zero divisors of arbitrary real Cayley–Dickson split-algebras. Our main goal is to provide a criterion for a doubly alternative element to be a zero divisor and to describe its left and right annihilators and its orthogonalizer. In the process, we classify alternative elements in the split-algebras. Also we generalize the relationship between the centralizer and the orthogonalizer of an arbitrary doubly alternative zero divisor obtained in [17, Proposition 8.15] from the split-sedenions to general split-algebras.

We apply the obtained results to relation graphs of the real low-dimensional Cayley–Dickson split-algebras and focus on their combinatorial characteristics such as the diameter and the description of cliques.

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 23, No. 3, pp. 95–129, 2020.

The structure of this paper is as follows. Section 2 contains essential definitions and notations. Section 3 is devoted to real Cayley–Dickson algebras. Particularly, we describe the Cayley–Dickson process in detail in Sec. 3.1 and mention some of their properties in Sec. 3.2. Split-complex numbers, the split-quaternions, and the split-octonions are introduced in Sec. 3.3. In Sec. 4, we study doubly alternative zero divisors of arbitrary real Cayley–Dickson split-algebras. We use these results to describe relation graphs of real low-dimensional Cayley–Dickson split-algebras in Sec. 5. Section 5.1 contains some elementary results about relation graphs of the split-complex numbers. In Sec. 5.2, we give our proof of some well-known results about relation graphs of the split-quaternions for the purpose of demonstrating an analogy between the split-quaternions and the split-octonions. We obtain an analogue of the real Jordan normal form for the split-octonions in Corollary 5.23 of Sec. 5.3, and then describe the orthogonality graph and the zero divisor graph of the split-octonions.

2. Definitions

Let $\mathbb{F}$ be an arbitrary field and $(\mathcal{A}, +, \cdot)$ be an algebra with an identity $1_{\mathcal{A}}$ over the field $\mathbb{F}$. $\mathcal{A}$ is not assumed to be commutative and associative. Let $a, b \in \mathcal{A}$. Then

- a and b *commute* if $ab = ba$,
- a and b *anticommute* if $ab + ba = 0$,
- a and b *are orthogonal* if $ab = ba = 0$,
- a is a *left zero divisor* if $a \neq 0$ and there exists nonzero $x \in \mathcal{A}$ such that $ax = 0$,
- a is a *right zero divisor* if $a \neq 0$ and there exists nonzero $x \in \mathcal{A}$ such that $xa = 0$,
- a is a *two-sided zero divisor* if it is both a left and a right zero divisor,
- a is a *zero divisor* if it is a left or a right zero divisor.

Definition 2.1.

- The *center* of an algebra $\mathcal{A}$ is the set $C_{\mathcal{A}} = \{a \in \mathcal{A} \mid ab = ba \text{ for all } b \in \mathcal{A}\}$.
- $Z_{\mathrm{L}}(\mathcal{A})$ denotes the set of left zero divisors of $\mathcal{A}$.
- $Z_{\mathrm{R}}(\mathcal{A})$ denotes the set of right zero divisors of $\mathcal{A}$.
- $Z(\mathcal{A}) = Z_{\mathrm{L}}(\mathcal{A}) \cup Z_{\mathrm{R}}(\mathcal{A})$ is the set of zero divisors of $\mathcal{A}$.
- $Z_{\mathrm{LR}}(\mathcal{A}) = Z_{\mathrm{L}}(\mathcal{A}) \cap Z_{\mathrm{R}}(\mathcal{A})$ is the set of two-sided zero divisors of $\mathcal{A}$.

Definition 2.2. Let a be an arbitrary element of $\mathcal{A}$.

- The *centralizer* of a is $C_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = ba\}$, namely, the set of all elements in $\mathcal{A}$ that commute with a.
- The *anticentralizer* of a is $\mathrm{Anc}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab + ba = 0\}$, namely, the set of all elements in $\mathcal{A}$ that anticommute with a.
- The *left annihilator* of a is the set $\mathrm{l.Ann}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ba = 0\}$.
- Similarly, the *right annihilator* of a is $\mathrm{r.Ann}_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = 0\}$.
- The *orthogonalizer* of a is $O_{\mathcal{A}}(a) = \{b \in \mathcal{A} \mid ab = ba = 0\}$, namely, the set of all elements in $\mathcal{A}$ that are orthogonal to a.

Remark 2.3. Let $a \in \mathcal{A}$. It can be easily seen that $C_{\mathcal{A}}$, $C_{\mathcal{A}}(a)$, $\mathrm{Anc}_{\mathcal{A}}(a)$, $\mathrm{l.Ann}_{\mathcal{A}}(a)$, $\mathrm{r.Ann}_{\mathcal{A}}(a)$, and $O_{\mathcal{A}}(a)$ are vector spaces over $\mathbb{F}$.

We can now introduce some relation graphs to be studied in this paper.

Definition 2.4. For an algebra $\mathcal{A}$, we define the following structures.

- The *commutativity graph* $\Gamma_C(\mathcal{A})$: its vertex set is $\mathcal{A} \backslash C_{\mathcal{A}}$, and distinct vertices a and b are adjacent if and only if $ab = ba$.
- The *orthogonality graph* $\Gamma_O(\mathcal{A})$: its vertex set is $Z_{LR}(\mathcal{A})$, and distinct vertices a and b are adjacent if and only if $ab = ba = 0$.
- The *directed zero divisor graph* $\Gamma_Z(\mathcal{A})$: its vertex set is $Z(\mathcal{A})$, and distinct vertices a and b are connected with an edge directed from a to b if and only if $ab = 0$.

We will need the following graph theory definitions.

Definition 2.5. Let Γ be a directed or an undirected graph.

- Γ is called *connected* if for each ordered pair of vertices (x, y) there exists a path leading from x to y.
- The *distance* $d(x, y) = d_\Gamma(x, y)$ between two vertices x and y in Γ is the number of edges in a shortest path from x to y. If there is no path from x to y, then $d(x, y) = \infty$.
- The *diameter* $d(\Gamma)$ of Γ is defined as $\sup\limits_{x,y\in\Gamma} d(x, y)$.

An undirected graph Γ has also the following invariants.

- A *connected component* of Γ is a maximal connected subgraph of Γ.
- A *clique* Q in Γ is a subset of vertices of Γ such that every two distinct vertices in Q are adjacent.
- A clique Q is called *maximal* if Q is maximal by inclusion.

3. An Overview of Real Cayley–Dickson Algebras

3.1. Constructing Cayley–Dickson Algebras.

Definition 3.1 ([19, p. 139, Definition 1.5.1])**.** Let $(\mathcal{A}, +, \cdot)$ be an algebra over a field $\mathbb{F}$. An *involution* $a \mapsto \bar{a}$ on $\mathcal{A}$ is an endomorphism of the vector space $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$ we have $\bar{\bar{a}} = a$ and $\overline{ab} = \bar{b}\bar{a}$.

Definition 3.2. Let $(\mathcal{A}, +, \cdot)$ be an algebra over a field $\mathbb{F}$ with an identity $1_\mathcal{A}$ and an involution $a \mapsto \bar{a}$. This involution is called *regular* if it satisfies $a + \bar{a} = t(a)1_\mathcal{A}$ and $a\bar{a} = \bar{a}a = n(a)1_\mathcal{A}$, where $t(a), n(a) \in \mathbb{F}$, for any $a \in \mathcal{A}$. Here $t(a)$ is called the *trace* of a and $n(a)$ is called the *norm* of a.

Henceforth we assume that $\mathcal{A}$ is an algebra over a field $\mathbb{F}$ with a regular involution $a \mapsto \bar{a}$. Below we provide several basic facts, which will be used later. For the completeness we give the proofs for some of them.

Proposition 3.3. *Let $a \in \mathcal{A}$, $\lambda \in \mathbb{F}$. Then $n(a - \lambda 1_\mathcal{A}) = \lambda^2 - t(a)\lambda + n(a)$.*

Proof. For any $b \in \mathcal{A}$, we have $\bar{b} = \overline{b1_\mathcal{A}} = \overline{1_\mathcal{A}} \cdot \bar{b}$, so $\overline{1_\mathcal{A}} = 1_\mathcal{A}$. Thus,

$$(a - \lambda 1_\mathcal{A})\overline{(a - \lambda 1_\mathcal{A})} = (a - \lambda 1_\mathcal{A})(\bar{a} - \lambda 1_\mathcal{A}) = a\bar{a} - \lambda(a + \bar{a}) + \lambda^2 1_\mathcal{A} = \big(\lambda^2 - t(a)\lambda + n(a)\big)1_\mathcal{A}. \qquad \square$$

Definition 3.4. The *characteristic polynomial* of $a \in \mathcal{A}$ is

$$p_a(\lambda) = n(a - \lambda 1_\mathcal{A}) = \lambda^2 - t(a)\lambda + n(a).$$

Its discriminant is $\operatorname{dis}(a) = \big(t(a)\big)^2 - 4n(a)$.

Proposition 3.5 ([24, p. 438])**.** *For any $a \in \mathcal{A}$, we have $p_a(a) = 0$.*

In this section, we use [19,24] to recall the classical nonassociative algebras, the so-called Cayley–Dickson algebras.

Definition 3.6 ([24])**.** The *algebra* $\mathcal{A}\{\gamma\}$ produced by the Cayley–Dickson process, when applied to $\mathcal{A}$ with the parameter $\gamma \in \mathbb{F}$, $\gamma \neq 0$, is defined as the set of ordered pairs of elements of $\mathcal{A}$ with operations

$$\alpha(a, b) = (\alpha a, \alpha b),$$
$$(a, b) + (c, d) = (a + c, b + d),$$
$$(a, b)(c, d) = (ac + \gamma\bar{d}b, da + b\bar{c})$$

and the involution

$$\overline{(a, b)} = (\bar{a}, -b), \quad a, b, c, d \in \mathcal{A}, \quad \alpha \in \mathbb{F}.$$

Proposition 3.7 ([24]).

- *$\mathcal{A}\{\gamma\}$ is an algebra over $\mathbb{F}$ with the identity $1_{\mathcal{A}\{\gamma\}} = (1_{\mathcal{A}}, 0)$ and a regular involution.*
- *Let $\mathcal{A}$ be an n-dimensional algebra and $\{e_i\}_{i=1,\dots,n}$ be a basis in $\mathcal{A}$. Then $\mathcal{A}\{\gamma\}$ is $2n$-dimensional and $\{(e_i,0),(0,e_i)\}_{i=1,\dots,n}$ is a basis in $\mathcal{A}\{\gamma\}$.*

Thus if we begin with a one-dimensional algebra and successively apply the Cayley–Dickson process to it, we will get a 2^n-dimensional algebra in the nth step.

Proposition 3.8 ([19, p. 161, Exercise 2.5.1]). *Let $\gamma' = \alpha^2\gamma$ with $\alpha \neq 0$. Then $\mathcal{A}\{\gamma\}$ and $\mathcal{A}\{\gamma'\}$ are isomorphic.*

However, the converse implication in Proposition 3.8 fails, see Example 5.17.

Lemma 3.9 ([24, p. 435]). *Let $a, b \in \mathcal{A}$, $(a,b) \in \mathcal{A}\{\gamma\}$. Then*

$$t\big((a,b)\big) = t(a),$$
$$n\big((a,b)\big) = n(a) - \gamma n(b).$$

Henceforth we assume that $\mathbb{F} = \mathbb{R}$, and $\mathbb{R}1_{\mathcal{A}}$ is identified with $\mathbb{R}$. Consider the following definitions, which are analogous to those for complex numbers.

Definition 3.10. The *real part* of an element $a \in \mathcal{A}$ is

$$\mathfrak{Re}(a) = \frac{a+\bar{a}}{2},$$

the *imaginary part* of a is

$$\mathfrak{Im}(a) = \frac{a-\bar{a}}{2},$$

the *norm* of a is $n(a) = a\bar{a} = \bar{a}a$. Then a is said to be *pure* if $\mathfrak{Re}(a) = 0$.

Observe that $\mathfrak{Re}(a), n(a) \in \mathbb{R}1_{\mathcal{A}} = \mathbb{R}$, since the involution on $\mathcal{A}$ is regular. Clearly, the introduced notion of the norm is well-agreed with Definition 3.2.

Remark 3.11. The norm of a is often defined as $\sqrt{a\bar{a}}$, unlike $n(a) = a\bar{a}$, in this paper. However, most of the results can be easily extended to the norm modified in this way.

Lemma 3.12. *For any $a \in \mathcal{A}$ we have* $\operatorname{dis}(a) = -4n\big(\mathfrak{Im}(a)\big)$.

Proof. We have

$$\operatorname{dis}(a) = \big(t(a)\big)^2 - 4n(a) = (a+\bar{a})^2 - 2a\bar{a} - 2\bar{a}a = (a-\bar{a})^2 = -(a-\bar{a})\overline{(a-\bar{a})} = -4n\big(\mathfrak{Im}(a)\big). \qquad \square$$

Definition 3.13. For every integer $n \geq 0$ and nonzero real numbers $\gamma_0, \dots, \gamma_{n-1}$, the real Cayley–Dickson algebra $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is defined inductively:

(1) $\mathcal{A}_0 = \mathbb{R}$, and $e_0^{(0)} = 1$ is its only basis element;
(2) if $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is constructed, then $\mathcal{A}_{n+1}\{\gamma_0, \dots, \gamma_n\} = (\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\})\{\gamma_n\}$. Its basis elements are $e_0^{(n+1)}, \dots, e_{2^{n+1}-1}^{(n+1)}$ such that

$$e_m^{(n+1)} = \begin{cases} \big(e_m^{(n)}, 0\big), & 0 \leq m \leq 2^n - 1, \\ \big(0, e_{m-2^n}^{(n)}\big), & 2^n \leq m \leq 2^{n+1} - 1. \end{cases}$$

Lemma 3.14. *For every integer $n \geq 0$, the structure $\mathcal{A}_n$ in Definition 3.13 is a 2^n-dimensional algebra over $\mathbb{R}$ with the identity $e_0^{(n)}$ and a regular involution.*

Proof. It follows from Proposition 3.7 by induction on n. $\square$

We will use the notation $1 = 1^{(n)} = e_0^{(n)}$ and $r = r1^{(n)}$ for $r \in \mathbb{R}$.

3.2. Some Properties of Real Cayley–Dickson Algebras. Henceforth we assume that $\mathcal{A}$ is an arbitrary algebra over a field $\mathbb{F}$, and $\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is an arbitrary real Cayley–Dickson algebra. Proposition 3.8 implies that $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is isomorphic to $\mathcal{A}_n\{\mathrm{sgn}(\gamma_0), \dots, \mathrm{sgn}(\gamma_{n-1})\}$, so it is sufficient to consider only $\gamma_k \in \{\pm 1\}$, $k = 0, \dots, n-1$.

Notation 3.15. For every $m = 0, \dots, 2^n - 1$, we define

$$\delta_m^{(n)} = \prod_{l=0}^{n-1} (-\gamma_l)^{c_{m,l}},$$

where the degrees $c_{m,l} \in \{0, 1\}$ are the unique coefficients of the binary decomposition

$$m = \sum_{l=0}^{n-1} c_{m,l} 2^l$$

(cf. [17, Proposition 3.18]).

Lemma 3.16. *Let $a = a_0 + a_1 e_1^{(n)} + \dots + a_{2^n-1} e_{2^n-1}^{(n)} \in \mathcal{A}_n$. Then*

$$\begin{aligned}
&\bar{a} = a_0 - a_1 e_1^{(n)} - \dots - a_{2^n-1} e_{2^n-1}^{(n)}, \\
&\mathfrak{Re}(a) = a_0, \\
&\mathfrak{Im}(a) = a_1 e_1^{(n)} + \dots + a_{2^n-1} e_{2^n-1}^{(n)}, \\
&n(a) = \sum_{m=0}^{2^n-1} \delta_m^{(n)} a_m^2,
\end{aligned}$$

here we consider conjugation in the sense of Definition 3.6*, and norm, real and imaginary parts in the sense of Definition* 3.10*.*

Proof. Follows from Lemma 3.9 by direct calculations. □

Notation 3.17. Given

$$a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}, \ b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{A}_n,$$

let us define

$$\langle a, b \rangle = \sum_{m=0}^{2^n-1} \delta_m^{(n)} a_m b_m.$$

Proposition 3.18. *$\langle a, b \rangle$ is a real-valued symmetric bilinear form associated with the quadratic form $n(a)$, i.e.,*

$$\begin{aligned}
&\langle a, a \rangle = n(a), \\
&\langle a_1 + a_2, b \rangle = \langle a_1, b \rangle + \langle a_2, b \rangle, \\
&\langle \alpha a, b \rangle = \alpha \langle a, b \rangle, \\
&\langle a, b \rangle = \langle b, a \rangle, \\
&\langle a, b \rangle \in \mathbb{R},
\end{aligned}$$

for all $a, a_1, a_2, b \in \mathcal{A}$, $\alpha \in \mathbb{R}$.

Proof. The first equality follows from Lemma 3.16, while other properties can be verified directly. □

Proposition 3.19. *Let $a, b \in \mathcal{A}_n$. Then $2\langle a, b \rangle = a\bar{b} + b\bar{a} = \bar{a}b + \bar{b}a$.*

Proof. By Proposition 3.18 we have

$$2\langle a,b\rangle = \langle a+b,a+b\rangle - \langle a,a\rangle - \langle b,b\rangle = (a+b)\overline{(a+b)} - a\bar{a} - b\bar{b} = a\bar{b} + b\bar{a}.$$

Similarly,

$$2\langle a,b\rangle = \langle a+b,a+b\rangle - \langle a,a\rangle - \langle b,b\rangle = \overline{(a+b)}(a+b) - \bar{a}a - \bar{b}b = \bar{a}b + \bar{b}a. \qquad \square$$

Corollary 3.20. *Let $a,b \in \mathcal{A}_n$. Then $\langle a,b\rangle = \mathfrak{Re}(a\bar{b}) = \mathfrak{Re}(\bar{b}a)$.*

Proof. Indeed, by Proposition 3.19 we have $2\mathfrak{Re}(a\bar{b}) = a\bar{b} + \overline{a\bar{b}} = a\bar{b} + b\bar{a} = 2\langle a,b\rangle$. Similarly, $2\mathfrak{Re}(\bar{b}a) = \bar{b}a + \overline{\bar{b}a} = \bar{b}a + \bar{a}b = 2\langle a,b\rangle$. $\square$

The following lemma describes the anticentralizer of an arbitrary element of $\mathcal{A}_n$.

Lemma 3.21 ([17, Lemma 5.8])**.** *Let $a \in \mathcal{A}_n$, $a \neq 0$.*

(1) *If $\mathfrak{Re}(a) \neq 0$, $n(a) \neq 0$, then* $\operatorname{Anc}_{\mathcal{A}_n}(a) = \{0\}$.
(2) *If $\mathfrak{Re}(a) \neq 0$, $n(a) = 0$, then* $\operatorname{Anc}_{\mathcal{A}_n}(a) = \mathbb{R}\bar{a}$.
(3) *If $\mathfrak{Re}(a) = 0$, then* $\operatorname{Anc}_{\mathcal{A}_n}(a) = \{b \in \mathcal{A}_n \mid \mathfrak{Re}(b) = 0$ *and* $\langle a,b\rangle = 0\}$.

We now proceed to some concepts related to associativity.

Definition 3.22. The associator of $a,b,c \in \mathcal{A}$ is the element $[a,b,c] = (ab)c - a(bc)$.

Proposition 3.23 ([19, p. 141])**.** *The associator is a trilinear function of its arguments.*

Definition 3.24 ([19, Definition 2.1.1])**.**

- An algebra $\mathcal{A}$ is called *flexible* if for all $a,b \in \mathcal{A}$ the equality $(ab)a = a(ba)$ holds.
- An element $a \in \mathcal{A}$ is called *alternative* if for all $b \in \mathcal{A}$ the equalities $a(ab) = a^2b$ and $(ba)a = ba^2$ hold.
- An algebra $\mathcal{A}$ is called *alternative* if all its elements are alternative.

Proposition 3.25 ([19, Exercise 2.1.1])**.**

- *If $\mathcal{A}$ is flexible then for all $a,b,c \in \mathcal{A}$, we have $[a,b,c] = -[c,b,a]$.*
- *If $\mathcal{A}$ is alternative, then the associator in $\mathcal{A}$ is skew-symmetric, i.e., it changes sign if an argument transposition is performed.*

Lemma 3.26 ([24, p. 436, p. 438, Theorem 1])**.**

- *$\mathcal{A}_n$ is commutative if and only if $n \leq 1$.*
- *$\mathcal{A}_n$ is associative if and only if $n \leq 2$.*
- *$\mathcal{A}_n$ is alternative if and only if $n \leq 3$.*
- *$\mathcal{A}_n$ is flexible for all $n \in \mathbb{N} \cup \{0\}$ and $\gamma_0, \ldots, \gamma_{n-1} \in \mathbb{R} \setminus \{0\}$.*

We now recall some basic facts concerning algebra automorphisms.

Definition 3.27. A mapping $\phi\colon \mathcal{A} \to \mathcal{A}$ is called an automorphism of $\mathcal{A}$ if ϕ is bijective and for any $a,b \in \mathcal{A}$ and $\gamma \in \mathbb{F}$ we have $\phi(a+b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$, and $\phi(\gamma a) = \gamma\phi(a)$. $\operatorname{Aut}_{\mathbb{F}}(\mathcal{A})$ denotes the set of all automorphisms of $\mathcal{A}$.

Lemma 3.28. *Let $\phi \in \operatorname{Aut}_{\mathbb{F}}(\mathcal{A})$. Then ϕ preserves pairs of commuting elements, pairs of orthogonal elements, and pairs of zero divisors.*

Proof. It follows directly from the definition of an algebra automorphism. $\square$

Notation 3.29. Let $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathcal{A}_n$. Then

$$\operatorname{Lin}(a_1, \ldots, a_m) = \mathbb{R}a_1 + \cdots + \mathbb{R}a_n,$$
$$\operatorname{Lin}^*(a_1, \ldots, a_m) = \operatorname{Lin}(a_1, \ldots, a_m) \setminus \{0\},$$
$$\operatorname{Lin}_I^*(a_1, \ldots, a_m) = \operatorname{Lin}(1, a_1, \ldots, a_m) \setminus \mathbb{R}.$$

3.3. Some Examples of Real Cayley–Dickson Algebras.

Definition 3.30.

- It is said that the algebra $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is an algebra of the *main sequence* if $\gamma_k = -1$ for each $k = 0, \dots, n-1$. We denote this algebra by $\mathcal{M}_n$.
- The algebra $\mathcal{A}_n\{\gamma_0, \dots, \gamma_{n-1}\}$ is called a *Cayley–Dickson split-algebra* if $\gamma_k = -1$ for each $k = 0, \dots, n-2$ and $\gamma_{n-1} = 1$. We denote it by $\mathcal{H}_n$, since the norm in $\mathcal{H}_n$ appears to be hyperbolic.

Proposition 3.31.

- *Let*
$$a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}, \; b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{M}_n.$$
Then
$$\langle a, b \rangle = \sum_{m=0}^{2^n-1} a_m b_m$$
is a Euclidean inner product. Particularly,
$$n(a) = \sum_{m=0}^{2^n-1} a_m^2,$$
so $n(a) = 0$ *if and only if* $a = 0$.
- *Let*
$$a = \sum_{m=0}^{2^n-1} a_m e_m^{(n)}, \; b = \sum_{m=0}^{2^n-1} b_m e_m^{(n)} \in \mathcal{H}_n.$$
Then
$$\langle a, b \rangle = \sum_{m=0}^{2^{n-1}-1} a_m b_m - \sum_{m=2^{n-1}}^{2^n-1} a_m b_m.$$

Proof. The expression for the inner product on the algebras of the main sequence can be obtained by substituting
$$\delta_m^{(n)} = \prod_{l=0}^{n-1} (-\gamma_l)^{c_{m,l}} = 1$$
for every $m = 0, \dots, 2^n - 1$ to the definition of $\langle a, b \rangle$.

In case of the split-algebras we have $\delta_m^{(n)} = 1$ for every $m = 0, \dots, 2^{n-1} - 1$ and $\delta_m^{(n)} = -1$ for every $m = 2^{n-1}, \dots, 2^n - 1$. □

Example 3.32.

- The complex numbers ($\mathbb{C}$), the quaternions ($\mathbb{H}$), the octonions ($\mathbb{O}$), and the sedenions ($\mathbb{S}$) are the algebras of the main sequence for $n = 1$, 2, 3, and 4, correspondingly. We refer the reader to [6] for the definition of $\mathbb{H}$ and $\mathbb{O}$, and to [13] for that of $\mathbb{S}$.
- The split-complex numbers ($\hat{\mathbb{C}}$), the split-quaternions (the coquaternions; $\hat{\mathbb{H}}$), and the split-octonions (the hyperbolic octonions; $\hat{\mathbb{O}}$) are the examples of real low-dimensional split-algebras, all of them being defined in [10].

Exact definitions and some basic properties of these algebras are given below.

Definition 3.33 ([10, pp. 3, 5, and 6])**.**

- The algebra of the split-complex numbers $\hat{\mathbb{C}}$ is the algebra of the elements of the form $a + b\ell$ with $a, b \in \mathbb{R}$, $\ell^2 = 1$ and an involution $\overline{a + b\ell} = a - b\ell$.

- The algebra of the split-quaternions $\hat{\mathbb{H}}$ is a four-dimensional algebra over $\mathbb{R}$, its basis elements being equal to 1, i, ℓ, ℓi. The involution in $\hat{\mathbb{H}}$ is given by the formula
$$\overline{a_0 + a_1 i + a_2\ell + a_3\ell i} = a_0 - a_1 i - a_2\ell - a_3\ell i,$$
and multiplication is given by Table 1.

Table 1. Multiplication table of the unit split-quaternions.

$\times$	1	i	ℓ	ℓi
1	1	i	ℓ	ℓi
i	i	-1	$-\ell i$	ℓ
ℓ	ℓ	ℓi	1	i
ℓi	ℓi	$-\ell$	$-i$	1

- The algebra of the split-octonions $\hat{\mathbb{O}}$ is an eight-dimensional algebra over $\mathbb{R}$, its basis elements being equal to 1, i, j, k, ℓ, ℓi, ℓj, ℓk. The involution in $\hat{\mathbb{O}}$ is given by the formula
$$\overline{a_0 + a_1 i + a_2 j + a_3 k + a_4\ell + a_5\ell i + a_6\ell j + a_7\ell k} = a_0 - a_1 i - a_2 j - a_3 k - a_4\ell - a_5\ell i - a_6\ell j - a_7\ell k,$$
and multiplication is given by Table 2.

Table 2. Multiplication table of the unit split-octonions.

$\times$	1	i	j	k	ℓ	ℓi	ℓj	ℓk
1	1	i	j	k	ℓ	ℓi	ℓj	ℓk
i	i	-1	k	$-j$	$-\ell i$	ℓ	$-\ell k$	ℓj
j	j	$-k$	-1	i	$-\ell j$	ℓk	ℓ	$-\ell i$
k	k	j	$-i$	-1	$-\ell k$	$-\ell j$	ℓi	ℓ
ℓ	ℓ	ℓi	ℓj	ℓk	1	i	j	k
ℓi	ℓi	$-\ell$	$-\ell k$	ℓj	$-i$	1	k	$-j$
ℓj	ℓj	ℓk	$-\ell$	$-\ell i$	$-j$	$-k$	1	i
ℓk	ℓk	$-\ell j$	ℓi	$-\ell$	$-k$	j	$-i$	1

Proposition 3.34 ([10, p. 2]). *The following isomorphisms hold*: $\hat{\mathbb{C}} \cong \mathcal{H}_1$, $\hat{\mathbb{H}} \cong \mathcal{H}_2$, *and* $\hat{\mathbb{O}} \cong \mathcal{H}_3$.

Proof. By [20, p. 1], we have the isomorphisms $\mathbb{R} \cong \mathcal{M}_0$, $\mathbb{C} \cong \mathcal{M}_1$, and $\mathbb{H} \cong \mathcal{M}_2$. Now the desired isomorphisms can be defined on the generating bases as follows:
$$\hat{\mathbb{C}} \cong \mathbb{R}\{1\}\colon 1 \mapsto (1,0),\ \ell \mapsto (0,1);$$
$$\hat{\mathbb{H}} \cong \mathbb{C}\{1\}\colon 1 \mapsto (1,0),\ i \mapsto (i,0),\ \ell \mapsto (0,1),\ \ell i \mapsto -(0,i);$$
$$\hat{\mathbb{O}} \cong \mathbb{H}\{1\}\colon 1 \mapsto (1,0),\ i \mapsto (i,0),\ j \mapsto (j,0),\ k \mapsto (k,0),$$
$$\ell \mapsto (0,1),\ \ell i \mapsto -(0,i),\ \ell j \mapsto -(0,j),\ \ell k \mapsto -(0,k).$$
□

Corollary 3.35.
- $\hat{\mathbb{C}}$ *is both commutative and associative*;
- $\hat{\mathbb{H}}$ *is noncommutative associative*;
- $\hat{\mathbb{O}}$ *is noncommutative non-associative but alternative.*

Proof. Follows immediately from Lemma 3.26 and Proposition 3.34. □

Proposition 3.36. *Let* $a = a_0 + a_1 i + a_2 \ell + a_3 \ell i \in \hat{\mathbb{H}}$. *Then*

$$\begin{aligned} \mathfrak{Re}(a) &= a_0, \\ \mathfrak{Im}(a) &= a_1 i + a_2 \ell + a_3 \ell i, \\ n(a) &= a_0^2 + a_1^2 - a_2^2 - a_3^2. \end{aligned}$$

Proof. It is a special case of Lemma 3.16 for $n = 2$, $\gamma_0 = -1$, $\gamma_1 = 1$, since by Proposition 3.34 we have $\hat{\mathbb{H}} \cong \mathcal{H}_2$. □

Lemma 3.37 ([19, p. 66])**.** $\hat{\mathbb{H}}$ *is isomorphic to* $\mathrm{M}_2(\mathbb{R})$*, i.e., the algebra of real* 2×2 *matrices. The required isomorphism* $\sigma_{\hat{\mathbb{H}}} \colon \hat{\mathbb{H}} \to \mathrm{M}_2(\mathbb{R})$ *is defined as follows:*

$$1 \mapsto I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \ell \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \ell i \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we translate the concepts of conjugate, real part, and norm to the elements of $\mathrm{M}_2(\mathbb{R})$.

Proposition 3.38 ([19, p. 157])**.** *For any*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}),$$

we have

$$\bar{A} = \mathrm{tr}(A)I - A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

$$2\mathfrak{Re}(A) = \mathrm{tr}(A) = a + d, \quad n(A) = \det(A) = ad - bc.$$

Below we identify the elements of $\hat{\mathbb{H}}$ with their $\sigma_{\hat{\mathbb{H}}}$-images.

Proposition 3.39. *Let*

$$a = a_0 + a_1 i + a_2 j + a_3 k + a_4 \ell + a_5 \ell i + a_6 \ell j + a_7 \ell k \in \hat{\mathbb{O}}.$$

Then

$$\begin{aligned} \mathfrak{Re}(a) &= a_0, \\ \mathfrak{Im}(a) &= a_1 i + a_2 j + a_3 k + a_4 \ell + a_5 \ell i + a_6 \ell j + a_7 \ell k, \\ n(a) &= a_0^2 + a_1^2 + a_2^2 + a_3^2 - a_4^2 - a_5^2 - a_6^2 - a_7^2. \end{aligned}$$

Proof. It is a special case of Lemma 3.16 for $n = 3$, $\gamma_0 = -1$, $\gamma_1 = -1$, and $\gamma_2 = 1$, since by Proposition 3.34 we have $\hat{\mathbb{O}} \cong \mathcal{H}_3$. □

Notation 3.40. Let $a \in \mathbb{H}$, $\mathfrak{Re}(a) = 0$, $a = a_1 i + a_2 j + a_3 k$. Then a can be identified with a vector $\mathbf{a} = (a_1, a_2, a_3)^t \in \mathbb{R}^3$.

Lemma 3.41 ([19, p. 158])**.** *The algebra of the split-octonions* $\hat{\mathbb{O}}$ *is isomorphic to the Zorn vector-matrix algebra, which consists of all* 2×2 *matrices of the following form:*

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix}, \quad \text{where} \quad a, b \in \mathbb{R}, \quad \mathbf{v}, \mathbf{w} \in \mathbb{R}^3,$$

while addition and multiplication are given by the formulas

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} + \begin{pmatrix} a' & \mathbf{v}' \\ \mathbf{w}' & b' \end{pmatrix} = \begin{pmatrix} a + a' & \mathbf{v} + \mathbf{v}' \\ \mathbf{w} + \mathbf{w}' & b + b' \end{pmatrix},$$

$$\begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \begin{pmatrix} a' & \mathbf{v}' \\ \mathbf{w}' & b' \end{pmatrix} = \begin{pmatrix} aa' + \mathbf{v} \cdot \mathbf{w}' & a\,\mathbf{v}' + b'\mathbf{v} + \mathbf{w} \times \mathbf{w}' \\ a'\mathbf{w} + b\mathbf{w}' - \mathbf{v} \times \mathbf{v}' & bb' + \mathbf{v}' \cdot \mathbf{w} \end{pmatrix},$$

where $\cdot$ and $\times$ denote the dot product and the cross product of the elements of $\mathbb{R}^3$, respectively. The identity of this algebra is

$$I = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}.$$

The required isomorphism $\sigma_{\hat{\mathbb{O}}}$ is defined as follows:

$$(a+c) + \ell(b+d) \mapsto \begin{pmatrix} a+b & \mathbf{c}+\mathbf{d} \\ -\mathbf{c}+\mathbf{d} & a-b \end{pmatrix},$$

where $a, b \in \mathbb{R}$, $c, d \in \mathbb{H}$, and $\mathfrak{Re}(c) = \mathfrak{Re}(d) = 0$.

Proposition 3.42 ([19, p. 158])**.** *Now we carry over the concepts of conjugate, real part, and norm to the elements of the Zorn vector-matrix algebra, so that for any*

$$A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix},$$

we have

$$\bar{A} = \operatorname{tr}(A)I - A = \begin{pmatrix} b & -\mathbf{v} \\ -\mathbf{w} & a \end{pmatrix},$$

$$2\mathfrak{Re}(A) = \operatorname{tr}(A) = a + b, \quad n(A) = \det(A) = ab - \mathbf{v}\cdot\mathbf{w}.$$

We will identify the elements of $\hat{\mathbb{O}}$ with their $\sigma_{\hat{\mathbb{O}}}$-images.

4. Doubly Alternative Zero Divisors in Arbitrary Split-Algebras

Lemma 4.1 ([24, Lemma 2])**.** *For any $x, y, z \in \mathcal{A}_n$, we have $\mathfrak{Re}([x, y, z]) = 0$.*

The following lemma is proved in [20] for the case of the algebras of the main sequence only; however, the proof is valid for an arbitrary real Cayley–Dickson algebra.

Lemma 4.2 ([20, Lemma 1.3])**.** *Let $x, y, z \in \mathcal{A}_n$. Then $\langle x, yz\rangle = \langle x\bar{z}, y\rangle = \langle \bar{y}x, z\rangle$.*

Proof. By using Corollary 3.20 and Lemma 4.1, we obtain

$$\langle x, yz\rangle = \mathfrak{Re}\big(x(\overline{yz})\big) = \mathfrak{Re}\big(x(\bar{z}\bar{y})\big) = \mathfrak{Re}\big((x\bar{z})\bar{y}\big) = \langle x\bar{z}, y\rangle.$$

Similarly,

$$\langle \bar{y}x, z\rangle = \mathfrak{Re}\big((\bar{y}x)\bar{z}\big) = \mathfrak{Re}\big(\bar{y}(x\bar{z})\big) = \langle x\bar{z}, y\rangle.$$

□

Notation 4.3. Let $a \in \mathcal{A}_n$. Then the mappings $L_a, R_a \colon \mathcal{A}_n \to \mathcal{A}_n$ are given by

$$L_a(x) = ax,$$
$$R_a(x) = xa,$$

for all $x \in \mathcal{A}_n$.

Proposition 4.4 ([19, p. 55])**.** *For any $a \in \mathcal{A}_n$ the mappings L_a and R_a are $\mathbb{R}$-linear operators on the 2^n-dimensional vector space $\mathcal{A}_n$.*

The following lemma demonstrates that in case of real Cayley–Dickson algebras all zero divisors appear to be two-sided zero divisors.

Lemma 4.5. *Let $a \in \mathcal{A}_n$. Then* $\dim(\operatorname{Ker} L_a) = \dim(\operatorname{Ker} R_a)$.

Proof. Let $b \in \operatorname{Ker} L_a$, i.e., $ab = 0$. Then, by Lemma 4.2, we have

$$\langle \bar{b}, xa \rangle = \langle \bar{b}\bar{a}, x \rangle = \langle \overline{ab}, x \rangle = \langle 0, x \rangle = 0,$$

for all $x \in \mathcal{A}_n$, whence $\bar{b} \perp \operatorname{Im} R_a$. Hence $\overline{\operatorname{Ker} L_a} \perp \operatorname{Im} R_a$ and $\dim(\overline{\operatorname{Ker} L_a}) + \dim(\operatorname{Im} R_a) \leq \dim(\mathcal{A}_n)$. Then

$$\dim(\operatorname{Ker} R_a) = \dim(\mathcal{A}_n) - \dim(\operatorname{Im} R_a) \geq \dim(\overline{\operatorname{Ker} L_a}) = \dim(\operatorname{Ker} L_a).$$

Similarly, $\dim(\operatorname{Ker} L_a) \geq \dim(\operatorname{Ker} R_a)$, so we have $\dim(\operatorname{Ker} L_a) = \dim(\operatorname{Ker} R_a)$. □

Corollary 4.6. $Z(\mathcal{A}_n) = Z_{\mathrm{LR}}(\mathcal{A}_n)$.

Proof. Let $a \in \mathcal{A}_n$, $a \neq 0$. Then, by Lemma 4.5, $\operatorname{Ker} L_a \neq \{0\}$ if and only if $\operatorname{Ker} R_a \neq \{0\}$. Equivalently, $a \in Z_{\mathrm{L}}(\mathcal{A}_n)$ if and only if $a \in Z_{\mathrm{R}}(\mathcal{A}_n)$. Hence $Z_{\mathrm{L}}(\mathcal{A}_n) = Z_{\mathrm{R}}(\mathcal{A}_n)$, and thus $Z(\mathcal{A}_n) = Z_{\mathrm{LR}}(\mathcal{A}_n)$. □

Lemma 4.7. *Let $a, b, c \in \mathcal{M}_n$ satisfy $\mathfrak{Re}(a) = \mathfrak{Re}(b) = 0$, $[a, b, b] = 0$, and $b = [a, c, b]$. Then $b = 0$.*

Proof. We use here considerations from [20, p. 21].

Consider a mapping $S\colon \mathcal{M}_n \to \mathcal{M}_n$ such that $S(x) = [a, x, b]$ for all $x \in \mathcal{M}_n$. Then $S = R_b L_a - L_a R_b$. By [20, Proposition 1.7], L_a and R_b are skew-symmetric with respect to the Euclidean inner product $\langle \cdot, \cdot \rangle$, and thus S is also skew-symmetric. Then $S\big(S(x)\big) = 0$ if and only if $S(x) = 0$. We have $b = S(c)$ and $0 = [a, b, b] = S(b) = S\big(S(c)\big)$, whence $b = S(c) = 0$. □

The arguments of the following lemma are similar to those in [21, p. 15].

Lemma 4.8. *Let $a, b \in \mathcal{A}_n$, $[a, a, b] = 0$. Then $n(ab) = n(ba) = n(a)n(b)$.*

Proof. Note that

$$[\bar{a}, a, b] = [2\mathfrak{Re}(a) - a, a, b] = -[a, a, b] = 0,$$

so $\bar{a}(ab) = (\bar{a}a)b$. By Proposition 3.18, $n(ab) = \langle ab, ab \rangle$. It follows from Lemma 4.2 for $x = ab$, $y = a$, and $z = b$ that

$$\langle ab, ab \rangle = \langle \bar{a}(ab), b \rangle = \langle (\bar{a}a)b, b \rangle = \langle n(a)b, b \rangle = n(a)\langle b, b \rangle = n(a)n(b).$$

Now we can apply Lemma 4.2 for $x = ba$, $y = b$, and $z = a$ to get $n(ba) = \langle ba, ba \rangle = \langle (ba)\bar{a}, b \rangle$. Since, by flexibility, $[b, a, \bar{a}] = -[b, a, a] = [a, a, b] = 0$, we have

$$\langle (ba)\bar{a}, b \rangle = \langle b(a\bar{a}), b \rangle = \langle n(a)b, b \rangle = n(a)\langle b, b \rangle = n(a)n(b).$$

Therefore, $n(ba) = n(a)n(b)$. □

Now consider zero divisors $(a, b) \in \mathcal{A}_n$ such that both elements a and b are alternative elements in $\mathcal{A}_{n-1}$.

Definition 4.9. The set of *doubly alternative elements* of $\mathcal{A}_n$ is

$$\mathrm{DA}(\mathcal{A}_n) = \{(a, b) \in \mathcal{A}_n \mid \text{both } a \text{ and } b \text{ are alternative elements in } \mathcal{A}_{n-1}\}.$$

Clearly, this definition makes sense for $n \geq 1$ only. The next proposition determines the condition under which all elements of $\mathcal{A}_n$ are doubly alternative. Particularly, it implies that all elements of the split-complex numbers, the split-quaternions, and the split-octonions are doubly alternative.

Proposition 4.10. *Let $n \geq 1$. Then $\mathrm{DA}(\mathcal{A}_n) = \mathcal{A}_n$ if and only if $n \leq 4$.*

Proof. By Lemma 3.26, the algebra $\mathcal{A}_n$ is alternative if and only if $n \leq 3$, whence the proposition follows immediately. □

Note that doubly alternative elements need not be alternative (see Lemma 4.16).

Lemma 4.11. *Let $(a, b) \in \mathrm{DA}(\mathcal{H}_n) \setminus \{0\}$. Then $(a, b) \in Z(\mathcal{H}_n)$ if and only if $n\big((a, b)\big) = n(a) - n(b) = 0$.*

Proof. Let $n\big((a,b)\big)=0$. Then

$$(a,b)\overline{(a,b)}=\overline{(a,b)}(a,b)=n\big((a,b)\big)=0,$$

hence $(a,b)\in Z(\mathcal{H}_n)$.

To prove the converse, we use the argument similar to that in [20, p. 25]. By Corollary 4.6, we may assume that $(a,b)\in Z_{\mathrm{L}}(\mathcal{H}_n)$. Then there exist $c,d\in\mathcal{M}_{n-1}$ such that $(c,d)\in\mathcal{H}_n\setminus\{0\}$ and $(a,b)(c,d)=0$. Since $(a,b)(c,d)=(ac+\bar{d}b, da+b\bar{c})$, we have $ac+\bar{d}b=0$ and $da+b\bar{c}=0$. If $c=0$, then $\bar{d}b=da=0$ implies $d=0$, since $n(b)\bar{d}=\bar{d}(b\bar{b})=(\bar{d}b)\bar{b}=0$, $n(a)d=d(a\bar{a})=(da)\bar{a}=0$, and $(a,b)\neq 0$. Therefore, $c\neq 0$. The elements a and b are alternative, whence $[a,a,c]=0$, $[a,a,d]=0$, $[b,b,\bar{c}]=0$, and $[b,b,\bar{d}]=0$. Then we can use Lemma 4.8 to obtain that

$$\begin{aligned}
&n(a)n(c)=n(ac)=n(-\bar{d}b)=n(\bar{d}b)=n(\bar{d})n(b)=n(b)n(d),\\
&n(a)n(d)=n(da)=n(-b\bar{c})=n(b\bar{c})=n(b)n(\bar{c})=n(b)n(c),\\
&\big(n(a)\big)^2n(c)=n(a)\big(n(a)n(c)\big)=n(a)\big(n(b)n(d)\big)=n(b)\big(n(a)n(d)\big)=n(b)\big(n(b)n(c)\big)=\big(n(b)\big)^2n(c).
\end{aligned}$$

Since $c\neq 0$, we have $n(c)\neq 0$, and thus the equality $\big(n(a)\big)^2n(c)=\big(n(b)\big)^2n(c)$ implies that $\big(n(a)\big)^2=\big(n(b)\big)^2$. Moreover, $n(a)\geq 0$ and $n(b)\geq 0$; so $n(a)=n(b)$, whence $n\big((a,b)\big)=n(a)-n(b)=0$. □

Corollary 4.12. *If* $1\leq n\leq 4$, *then*

$$Z(\mathcal{H}_n)=\{x\in\mathcal{H}_n\setminus\{0\}\mid n(x)=0\}.$$

Proof. By Proposition 4.10, the equality $\mathrm{DA}(\mathcal{H}_n)=\mathcal{H}_n$ holds, so we may use Lemma 4.11. □

Now we show that the set of elements of zero norm is in general strictly smaller than the set of zero divisors.

Proposition 4.13. *For* $n\geq 5$ *the set* $Z(\mathcal{H}_n)$ *contains elements of nonzero norm.*

Proof. If $n\geq 5$, then there exist $c,d\in Z(\mathcal{M}_{n-1})$ such that $cd=dc=0$ (see [20, Corollary 1.6]). Then $(c,0)(d,0)=(d,0)(c,0)=0$; however, $n\big((c,0)\big)=n(c)\neq 0$ and $n\big((d,0)\big)=n(d)\neq 0$. □

Lemma 4.14 ([24, Lemma 4])**.** *For any* $m=1,\dots,2^n-1$, *the element* $e_m^{(n)}\in\mathcal{A}_n$ *is alternative.*

Lemma 4.15. *Let* $a,b\in\mathcal{A}_n$, $\mathfrak{Re}(a)=0$. *Then* $a(ab)=(ba)a$.

Proof. Since $\mathfrak{Re}(a)=0$ implies $a^2\in\mathbb{R}$, by flexibility we have

$$0=[a,a,b]+[b,a,a]=a^2b-a(ab)+(ba)a-ba^2=(ba)a-a(ab).$$ □

The following auxiliary Lemma 4.16 is similar to Theorem 3.3 in [21] for $\mathcal{M}_n$, except for item (3).

Lemma 4.16. *Let* $n\geq 4$, $a,b\in\mathcal{M}_{n-1}$, $\mathfrak{Re}(a)=\mathfrak{Re}(b)=0$. *Consider the following statements*:

(1) $(a,b)\in\mathcal{H}_n$ *is alternative*;
(2) *a and b are alternative and linearly dependent*;
(3) $a=\pm b$.

Then the condition (1) *is equivalent to either* (2) *or* (3).

Proof. Let us consider the alternativity condition for the element (a,b). For an arbitrary $(c,d)\in\mathcal{H}_n$, we have

$$\begin{aligned}
(a,b)\big((a,b)(c,d)\big)&=(a,b)(ac+\bar{d}b, da+b\bar{c})=\big(a(ac+\bar{d}b)+\overline{(da+b\bar{c})}b, (da+b\bar{c})a+b\overline{(ac+\bar{d}b)}\big)\\
&=(a(ac)-(cb)b-[a,\bar{d},b], (da)a-b(bd)+[b,\bar{c},a])=(A,B).
\end{aligned}$$

It holds that $[a,\bar{d},b]=-[a,d,b]$ and, by flexibility, $[b,\bar{c},a]=-[b,c,a]=[a,c,b]$. Moreover, it follows from Lemma 4.15 that $(cb)b=b(bc)$ and $(da)a=a(ad)$. Hence $A=a(ac)-b(bc)+[a,d,b]$, and $B=a(ad)-b(bd)+[a,c,b]$.

Now consider

$$(a,b)^2(c,d) = -n\big((a,b)\big)(c,d) = -\big(n(a)-n(b)\big)(c,d) = (a^2-b^2)(c,d).$$

Note that B can be obtained by interchanging c and d in A. Thus, (a,b) is alternative if and only if for any $c,d \in \mathcal{M}_{n-1}$ we have

$$a(ac) - b(bc) + [a,d,b] = (a^2-b^2)c. \tag{1}$$

By substituting $c=0$ and $d=0$ independently, we obtain that this condition is equivalent to the system

$$\begin{cases} a(ac) - b(bc) = (a^2-b^2)c, \\ [a,d,b] = 0, \end{cases}$$

for any $c,d \in \mathcal{M}_{n-1}$.

By [21, Theorem 2.3] and flexibility, $[a,d,b]=0$ for any $d \in \mathcal{M}_{n-1}$ if and only if a and b are linearly dependent.

Now let a and b be linearly dependent. We may suppose, without loss of generality, that $b = \beta a$ for some $\beta \in \mathbb{R}$, whence (a,b) is alternative if and only if $(1-\beta^2)a(ac) = (1-\beta^2)a^2c$ for any $c \in \mathcal{M}_{n-1}$. This condition holds if and only if either $\beta = \pm 1$ or a is alternative. □

Example 4.17 below demonstrates that we can not replace $\mathcal{H}_n$ in Lemma 4.11 with an arbitrary algebra $\mathcal{A}_n$ such that $\gamma_{n-1} = 1$.

Example 4.17. Let $n \geq 4$, $\mathcal{A}_n = \mathcal{H}_{n-1}\{1\}$. Consider

$$a = \big(e_1^{(n-2)}, 0\big),\ b = \big(e_1^{(n-2)}, e_1^{(n-2)}\big),\ c = \big(e_2^{(n-2)}, -e_2^{(n-2)}\big) \in \mathcal{H}_{n-1}.$$

Then $(a+b,a)$ and $(c,-c)$ are doubly alternative and orthogonal in $\mathcal{A}_n$, whence $(a+b,a) \in Z(\mathcal{A}_n)$; however, $n\big((a+b,a)\big) \neq 0$.

Indeed, by applying Lemma 4.16 to the result of Lemma 4.14 we obtain that $(a+b,a)$ and $(c,-c)$ are doubly alternative. Moreover, by item (3) of Lemma 3.21, a and c anticommute. One can also verify that b and c are orthogonal. Then

$$\begin{aligned}(a+b,a)(c,-c) &= \big((a+b)c - \bar{c}a, -c(a+b) + a\bar{c}\big) = \big(ac+ca, -(ac+ca)\big) = 0,\\ (c,-c)(a+b,a) &= \big(c(a+b) - \bar{a}c, ac - c\overline{(a+b)}\big) = (ac+ca, ac+ca) = 0,\end{aligned}$$

so $(a+b,a)$ and $(c,-c)$ are orthogonal. Finally, $n\big((a+b,a)\big) = n(a+b) - n(a) = 2$, since

$$n(a+b) = n\Big(\big(2e_1^{(n-2)}, e_1^{(n-2)}\big)\Big) = n\big(2e_1^{(n-2)}\big) - n\big(e_1^{(n-2)}\big) = 4 - 1 = 3$$

and $n(a) = n(e_1^{(n-2)}) - n(0) = 1$.

Lemmas 4.18 and 4.21 originate from [11, Proposition 11.1].

Lemma 4.18. *Let* $(a,b) \in \mathrm{DA}(\mathcal{H}_n) \cap Z(\mathcal{H}_n)$. *Then*

$$\begin{aligned}\mathrm{l.Ann}_{\mathcal{H}_n}\big((a,b)\big) &= \left\{ \left(c, -\frac{(bc)a}{n(a)}\right) \;\middle|\; [a,c,b] = 0 \right\},\\ \mathrm{r.Ann}_{\mathcal{H}_n}\big((a,b)\big) &= \left\{ \left(c, -\frac{(b\bar{c})\bar{a}}{n(a)}\right) \;\middle|\; [a,c,b] = 0 \right\}.\end{aligned}$$

Proof. We consider $\mathrm{l.Ann}_{\mathcal{H}_n}\big((a,b)\big)$ first. Let $(c,d) \in \mathcal{H}_n$ such that $(c,d)(a,b) = (ca + \bar{b}d, bc + d\bar{a}) = 0$. Then $bc + d\bar{a} = 0$, so $n(a)d = d(\bar{a}a) = (d\bar{a})a = -(bc)a$. Moreover, $ca + \bar{b}d = 0$ and, by Lemma 4.11, $n(a) = n(b)$, hence

$$b(ca) = -b(\bar{b}d) = -(b\bar{b})d = -n(b)d = -n(a)d = (bc)a.$$

Thus, $[b,c,a]=0$ or, by flexibility, $[a,c,b]=0$. Reasoning in the opposite way, we may conclude that for any $c \in \mathcal{M}_{n-1}$ such that $[a,c,b]=0$ we have that

$$\left(c, -\frac{(bc)a}{n(a)}\right) \in \mathrm{l.Ann}_{\mathcal{H}_n}\big((a,b)\big).$$

So, the converse is also true.

We now proceed to $\mathrm{r.Ann}_{\mathcal{H}_n}\big((a,b)\big)$. Let $(c,d) \in \mathcal{H}_n$ such that $(a,b)(c,d) = (ac + \bar{d}b, da + b\bar{c}) = 0$. Then $da + b\bar{c} = 0$, whence

$$n(a)d = d(a\bar{a}) = (da)\bar{a} = -(b\bar{c})\bar{a}.$$

Since $ac + \bar{d}b = 0$ and $n(a) = n(b)$, we have

$$(ac)\bar{b} = -(\bar{d}b)\bar{b} = -\bar{d}(b\bar{b}) = -n(b)\bar{d} = \overline{-n(a)d} = \overline{(b\bar{c})\bar{a}} = a(c\bar{b}).$$

Thus, $[a,c,\bar{b}]=0$ or, equivalently, $[a,c,b]=0$. Clearly, the converse is also true in this case, i.e., for any $c \in \mathcal{M}_{n-1}$ such that $[a,c,b]=0$ we have

$$\left(c, -\frac{(b\bar{c})\bar{a}}{n(a)}\right) \in \mathrm{r.Ann}_{\mathcal{H}_n}\big((a,b)\big).$$

□

Proposition 4.19. *Let $x \in Z(\mathcal{A}_n)$, $\mathfrak{Re}(x) \neq 0$, $O_{\mathcal{A}_n}(x) \neq \{0\}$. Then $n(x)=0$, $O_{\mathcal{A}_n}(x) = \mathrm{Lin}(\bar{x})$ and the connected component of $\Gamma_O(\mathcal{A}_n)$ that contains x is a complete bipartite graph, its parts being $\mathrm{Lin}^*(x)$ and $\mathrm{Lin}^*(\bar{x})$.*

Proof. $O_{\mathcal{A}_n}(x) \subset \mathrm{Anc}_{\mathcal{A}_n}(x)$, whence $\mathrm{Anc}_{\mathcal{A}_n}(x) \neq \{0\}$. Then by Lemma 3.21 we have $n(x)=0$ and $\mathrm{Anc}_{\mathcal{A}_n}(x) = \mathrm{Lin}(\bar{x})$, so we also have $O_{\mathcal{A}_n}(x) = \mathrm{Lin}(\bar{x})$. Similarly, $O_{\mathcal{A}_n}(\bar{x}) = \mathrm{Lin}(x)$. The proposition follows immediately. □

Due to Proposition 4.19, it is natural to give the following definition.

Definition 4.20. $Z_{\mathfrak{Im}}(\mathcal{A}_n) = \{x \in Z(\mathcal{A}_n) \mid \mathfrak{Re}(x) = 0\}$ is the set of all zero divisors with zero real part. $\Gamma_O^{\mathfrak{Im}}(\mathcal{A}_n)$ is the subgraph of $\Gamma_O(\mathcal{A}_n)$ on the vertex set $Z_{\mathfrak{Im}}(\mathcal{A}_n)$.

Lemma 4.21. *Let $(a,b) \in \mathrm{DA}(\mathcal{H}_n) \cap Z_{\mathfrak{Im}}(\mathcal{H}_n)$. Then*

$$O_{\mathcal{H}_n}\big((a,b)\big) = \left\{\left(c, -\frac{(bc)a}{n(a)}\right) \,\middle|\, \mathfrak{Re}(c) = 0,\ [a,c,b]=0\right\}.$$

Proof. $O_{\mathcal{H}_n}\big((a,b)\big) \subset \mathrm{Anc}_{\mathcal{H}_n}\big((a,b)\big)$, so it follows from Lemma 3.21 that for any $(c,d) \in O_{\mathcal{H}_n}\big((a,b)\big) \subset \mathrm{Anc}_{\mathcal{H}_n}\big((a,b)\big)$, we have $\mathfrak{Re}(c) = 0$. Now we use the representation of $\mathrm{l.Ann}_{\mathcal{H}_n}\big((a,b)\big)$ and $\mathrm{r.Ann}_{\mathcal{H}_n}\big((a,b)\big)$ from Lemma 4.18. For any $(c,d) \in O_{\mathcal{H}_n}\big((a,b)\big) \subset \mathrm{l.Ann}_{\mathcal{H}_n}\big((a,b)\big)$, we have $[a,c,b]=0$ and

$$d = -\frac{(bc)a}{n(a)}.$$

Conversely, for any $c \in \mathcal{M}_{n-1}$ such that $\mathfrak{Re}(c) = 0$ and $[a,c,b]=0$ we have

$$\left(c, -\frac{(bc)a}{n(a)}\right) \in \mathrm{l.Ann}_{\mathcal{H}_n}\big((a,b)\big) \cap \mathrm{r.Ann}_{\mathcal{H}_n}\big((a,b)\big) = O_{\mathcal{H}_n}\big((a,b)\big).$$

Thus, the equality

$$O_{\mathcal{H}_n}\big((a,b)\big) = \left\{\left(c, -\frac{(bc)a}{n(a)}\right) \,\middle|\, \mathfrak{Re}(c) = 0,\ [a,c,b]=0\right\}$$

holds. □

Lemma 4.22 ([17, Lemma 8.11])**.** *Let $x \in \mathcal{A}_n \setminus \{0\}$, $\mathfrak{Re}(x) = 0$. Then*

(1) *if $n(x) = 0$ and $n \leq 3$, then $C_{\mathcal{A}_n}(x) = \mathbb{R} \oplus O_{\mathcal{A}_n}(x)$;*
(2) *if $n(x) \neq 0$, then $C_{\mathcal{A}_n}(x) = \mathbb{R} \oplus \mathbb{R}x \oplus O_{\mathcal{A}_n}(x)$.*

The following theorem is a generalization of Proposition 8.15 in [17]. Proposition 8.19 in [17] demonstrates that its conditions are essential.

Theorem 4.23. *Let* $(a,b) \in \mathrm{DA}(\mathcal{H}_n) \cap Z_{\mathfrak{Im}}(\mathcal{H}_n)$. *Then* $C_{\mathcal{H}_n}\big((a,b)\big) = \mathbb{R} \oplus O_{\mathcal{H}_n}\big((a,b)\big)$.

Proof. Assume that there exists $(c,d) \in C_{\mathcal{H}_n}\big((a,b)\big) \setminus \Big(\mathbb{R} \oplus O_{\mathcal{H}_n}\big((a,b)\big)\Big)$. Without loss of generality, we may suppose that $n(a) = n(b) = 1$ and $\mathfrak{Re}(c) = 0$. Then

$$\overline{(a,b)(c,d)} = \overline{(c,d)} \cdot \overline{(a,b)} = (c,d)(a,b) = (a,b)(c,d),$$

i.e., $(a,b)(c,d) = r \in \mathbb{R}$. Since $(c,d) \notin O_{\mathcal{H}_n}\big((a,b)\big)$, we have $r \neq 0$. Assume, without loss of generality, that $r = 1$. We have

$$(1,0) = (a,b)(c,d) = (ac + \bar{d}b, da + b\bar{c}) = (ac + \bar{d}b, da - bc).$$

Then $da = bc$ implies $d = d(a\bar{a}) = (da)\bar{a} = (bc)\bar{a}$, and thus $\bar{d} = a(\bar{c}\bar{b}) = -a(c\bar{b})$. We next multiply the equality $1 = ac + \bar{d}b$ by $\bar{b}$ on the right and substitute the expression for $\bar{d}$, whence

$$\bar{b} = (ac)\bar{b} + (\bar{d}b)\bar{b} = (ac)\bar{b} + \bar{d}(b\bar{b}) = (ac)\bar{b} + \bar{d} = (ac)\bar{b} - a(c\bar{b}) = [a,c,\bar{b}].$$

It follows from Lemma 4.1 that $\mathfrak{Re}(\bar{b}) = \mathfrak{Re}([a,c,\bar{b}]) = 0$, hence $\bar{b} = -b$ and $b = [a,c,b]$. By Lemma 4.7, we have $b = 0$, a contradiction. $\square$

Corollary 4.24. *Let* $(a,b) \in \mathrm{DA}(\mathcal{H}_n) \cap Z_{\mathfrak{Im}}(\mathcal{H}_n)$. *Then* $x \in O_{\mathcal{H}_n}\big((a,b)\big)$ *if and only if* $\mathfrak{Re}(x) = 0$ *and* $x \in C_{\mathcal{H}_n}\big((a,b)\big)$.

Proof. Note that $O_{\mathcal{H}_n}\big((a,b)\big) \subset \mathrm{Anc}_{\mathcal{H}_n}\big((a,b)\big)$, so by Lemma 3.21 we have $\mathfrak{Re}(x) = 0$, for any $x \in O_{\mathcal{H}_n}\big((a,b)\big)$. By Theorem 4.23, we have $C_{\mathcal{H}_n}\big((a,b)\big) = \mathbb{R} \oplus O_{\mathcal{H}_n}\big((a,b)\big)$, thus $\mathfrak{Im}\Big(C_{\mathcal{H}_n}\big((a,b)\big)\Big) = O_{\mathcal{H}_n}\big((a,b)\big)$. $\square$

Lemma 4.25 ([24, p. 438])**.**

1. *If* $n \leq 1$, *then* $C_{\mathcal{A}_n} = \mathcal{A}_n$, *so the vertex set of* $\Gamma_C(\mathcal{A}_n)$ *is the empty set.*
2. *If* $n \geq 2$, *then* $C_{\mathcal{A}_n} = \mathbb{R}$, *so the vertex set of* $\Gamma_C(\mathcal{A}_n)$ *is* $\mathcal{A}_n \setminus \mathbb{R}$.

Proposition 4.26. *Any path between any two vertices in* $\Gamma_C(\mathcal{A}_n)$ *can be modified so that all intermediate elements are pure.*

Proof. It follows immediately from the fact that $\mathbb{R} \subseteq C_{\mathcal{A}_n}$. $\square$

Proposition 4.27. *Let* $2 \leq n \leq 4$, $x \in \mathcal{H}_n \setminus \mathbb{R}$, $n\big(\mathfrak{Im}(x)\big) \neq 0$. *Then* $C_{\mathcal{H}_n}(x) = \mathrm{Lin}(1,x)$ *and the connected component of* $\Gamma_C(\mathcal{H}_n)$ *which contains* x *is the complete graph on the vertex set* $\mathrm{Lin}_I^*(x)$.

Proof. By Corollary 4.12, $\mathfrak{Im}(x) \notin Z(\mathcal{H}_n)$, so $O_{\mathcal{H}_n}\big(\mathfrak{Im}(x)\big) = \{0\}$. Thus, Lemma 4.22 implies that

$$C_{\mathcal{H}_n}(x) = C_{\mathcal{H}_n}\big(\mathfrak{Im}(x)\big) = \mathbb{R} \oplus \mathbb{R}\mathfrak{Im}(x) = \mathbb{R} \oplus \mathbb{R}x.$$

Then the proposition follows immediately. $\square$

Notation 4.28. Let Γ be an undirected graph. Then $\mathcal{C}(\Gamma)$ is the set of all connected components of Γ, and $\mathcal{Q}(\Gamma)$ is the set of all maximal cliques in Γ.

Notation 4.29. Let S be a subset of $\mathfrak{Im}(\mathcal{H}_n)$. Then $\mathbb{R} + S = \{x \in \mathcal{H}_n \mid \mathfrak{Im}(x) \in S\}$.

Theorem 4.30. *Let* $2 \leq n \leq 4$. *Let us denote the subgraph of* $\Gamma_C(\mathcal{H}_n)$ *on the vertex set* $\mathbb{R} + Z_{\mathfrak{Im}}(\mathcal{H}_n) = \{x \in \mathcal{H}_n \setminus \mathbb{R} \mid n\big(\mathfrak{Im}(x)\big) = 0\}$ *by* Γ_C. *We also denote here* $\Gamma_O = \Gamma_O^{\mathfrak{Im}}(\mathcal{H}_n)$ *for short. Then* Γ_C *and* Γ_O *are related as follows.*

1. *Consider a mapping* $\phi_{\mathcal{C}} \colon \mathcal{C}(\Gamma_O) \to \mathcal{C}(\Gamma_C)$ *such that* $\phi_{\mathcal{C}}(C) = \mathbb{R} + C$ *for any* $C \in \mathcal{C}(\Gamma_O)$. *Then* $\phi_{\mathcal{C}}$ *is a bijection and preserves diameters.*

(2) *Similarly, we define a mapping* $\phi_{\mathcal{Q}}\colon \mathcal{Q}(\Gamma_O) \to \mathcal{Q}(\Gamma_C)$ *such that* $\phi_{\mathcal{Q}}(Q) = \mathbb{R} + Q$ *for any* $Q \in \mathcal{Q}(\Gamma_O)$. *Then* $\phi_{\mathcal{Q}}$ *is a bijection.*

Proof. We now show that $\phi_{\mathcal{C}}$ is a bijection and preserves diameters. It can be easily seen that the diameter of any connected component of Γ_C can be achieved on the pair of elements with distinct imaginary parts. Now let $x, y \in \big(\mathbb{R} + Z_{\mathfrak{Im}}(\mathcal{H}_n)\big)$, $\mathfrak{Im}(x) \neq \mathfrak{Im}(y)$. Then $d_{\Gamma_C}(x, y) = d_{\Gamma_C}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big)$. Orthogonality implies commutativity, whence any path between $\mathfrak{Im}(x)$ and $\mathfrak{Im}(y)$ in Γ_O is also a path in Γ_C, hence $d_{\Gamma_C}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big) \leq d_{\Gamma_O}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big)$. Moreover, it follows from Proposition 4.26 that any path between $\mathfrak{Im}(x)$ and $\mathfrak{Im}(y)$ in Γ_C can be modified so that all intermediate elements are pure, and by Corollary 4.24 this new path is also a path in Γ_O. Thus we have $d_{\Gamma_O}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big) \leq d_{\Gamma_C}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big)$. Hence $d_{\Gamma_C}(x, y) = d_{\Gamma_C}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big) = d_{\Gamma_O}\big(\mathfrak{Im}(x), \mathfrak{Im}(y)\big)$.

By Corollary 4.12, any element of $Z_{\mathfrak{Im}}(\mathcal{H}_n)$ has zero norm and thus is orthogonal to itself. Let now $x, y \in \big(\mathbb{R} + Z_{\mathfrak{Im}}(\mathcal{H}_n)\big)$. By Corollary 4.24, x and y commute if and only if $\mathfrak{Im}(x)$ and $\mathfrak{Im}(y)$ are orthogonal. That is, x and y are connected in Γ_C if and only if either $\mathfrak{Im}(x) = \mathfrak{Im}(y)$ or $\mathfrak{Im}(x)$ and $\mathfrak{Im}(y)$ are connected in Γ_O. Thus we may infer that $\phi_{\mathcal{Q}}$ is a bijection. ☐

5. Applications to Graphs

5.1. Split-Complex Numbers.

5.1.1. Orthogonality.

Proposition 5.1. $\Gamma_O(\hat{\mathbb{C}})$ *is a complete bipartite graph, its parts being* $\mathrm{Lin}^*(1+\ell)$ *and* $\mathrm{Lin}^*(1-\ell)$.

Proof. By Corollary 4.12,

$$Z(\hat{\mathbb{C}}) = \{a \in \hat{\mathbb{C}} \setminus \{0\} \mid n(a) = 0\} = \mathrm{Lin}^*(1+\ell) \cup \mathrm{Lin}^*(1-\ell).$$

Then we use Proposition 4.19. ☐

5.1.2. Zero divisors.

Proposition 5.2. $\Gamma_Z(\hat{\mathbb{C}})$ *is obtained from* $\Gamma_O(\hat{\mathbb{C}})$ *by replacing every undirected edge with a pair of directed edges.*

Proof. $\hat{\mathbb{C}}$ is commutative, whence the conditions $ab = 0$ and $ba = 0$ are equivalent. ☐

5.2. Split-Quaternions.

5.2.1. Real Jordan normal form.

Remark 5.3. Let $A \in \hat{\mathbb{H}}$. Then the characteristic polynomial $p_A(\lambda)$ introduced in Definition 3.4 coincides with the standard characteristic polynomial of A as a matrix, and its discriminant is $\mathrm{dis}(A) = \big(\mathrm{tr}(A)\big)^2 - 4\det(A)$.

Lemma 5.4. *Let* $A \in \hat{\mathbb{H}} \setminus \mathbb{R}I$. *Then there exists* $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{H}})$ *such that*

(1) *if* $\mathrm{dis}(A) > 0$, *i.e.,* $\mathrm{dis}(A) = d^2$ *for some* $d \neq 0$, *then*

$$\phi(A) = \frac{\mathrm{tr}(A) + d\ell}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{tr}(A) + d & 0 \\ 0 & \mathrm{tr}(A) - d \end{pmatrix};$$

(2) *if* $\mathrm{dis}(A) < 0$, *i.e.,* $\mathrm{dis}(A) = -d^2$ *for some* $d \neq 0$, *then*

$$\phi(A) = \frac{\mathrm{tr}(A) + di}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{tr}(A) & d \\ -d & \mathrm{tr}(A) \end{pmatrix};$$

(3) *if* $\mathrm{dis}(A) = 0$, *then*

$$\phi(A) = \frac{\mathrm{tr}(A) + i + \ell i}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{tr}(A) & 2 \\ 0 & \mathrm{tr}(A) \end{pmatrix}.$$

Proof. Let J_A be the real Jordan normal form of A.

(1) If $\operatorname{dis}(A) > 0$, i.e., $\operatorname{dis}(A) = d^2$ for some $d \neq 0$, then the roots of $p_A(\lambda)$ are $(\operatorname{tr}(A) \pm d)/2$, whence

$$J_A = \frac{1}{2}\begin{pmatrix} \operatorname{tr}(A) + d & 0 \\ 0 & \operatorname{tr}(A) - d \end{pmatrix}.$$

(2) If $\operatorname{dis}(A) < 0$, i.e., $\operatorname{dis}(A) = -d^2$ for some $d \neq 0$, then the roots of $p_A(\lambda)$ are $(\operatorname{tr}(A) \pm di)/2$, whence

$$J_A = \frac{1}{2}\begin{pmatrix} \operatorname{tr}(A) & d \\ -d & \operatorname{tr}(A) \end{pmatrix}.$$

(3) If $\operatorname{dis}(A) = 0$, then $p_A(\lambda)$ has a root $\operatorname{tr}(A)/2$ of multiplicity 2, whence

$$J_A = \frac{1}{2}\begin{pmatrix} \operatorname{tr}(A) & 2 \\ 0 & \operatorname{tr}(A) \end{pmatrix},$$

since $A \notin \mathbb{R}I$.

There exists $C \in \hat{\mathbb{H}}$ such that C is invertible and $A = C^{-1}J_AC$. Consider $\phi\colon \hat{\mathbb{H}} \to \hat{\mathbb{H}}$ such that $\phi(B) = CBC^{-1}$ for any $B \in \hat{\mathbb{H}}$. Then $\phi \in \operatorname{Aut}_{\mathbb{R}}(\hat{\mathbb{H}})$ and $\phi(A) = J_A$. ☐

5.2.2. Orthogonality. It should be noted that Theorem 5.6 is a particular case of Lemma 4.1 in [8]; however, we suppose that the formulation of Theorem 5.6 is more convenient.

Lemma 5.5. *Let $A \in Z(\hat{\mathbb{H}})$. Then $O_{\hat{\mathbb{H}}}(A) = \operatorname{Lin}(\bar{A})$.*

Proof. Consider two cases.

If $\mathfrak{Re}(A) \neq 0$, then we use Proposition 4.19.

If $\mathfrak{Re}(A) = 0$, then it follows from Lemma 4.21 that $\dim\big(O_{\hat{\mathbb{H}}}(A)\big) = 1$. Since $\operatorname{Lin}(\bar{A}) \subset O_{\hat{\mathbb{H}}}(A)$, we have $O_{\hat{\mathbb{H}}}(A) = \operatorname{Lin}(\bar{A}) = \operatorname{Lin}(A)$. ☐

Theorem 5.6. *The connected components of $\Gamma_O(\hat{\mathbb{H}})$ are as follows:*

(1) *a complete bipartite graph, its parts being* $\operatorname{Lin}^*(A)$ *and* $\operatorname{Lin}^*(\bar{A})$, *where* $\det(A) = 0$, $\operatorname{tr}(A) \neq 0$;
(2) *a complete graph on the vertex set* $\operatorname{Lin}^*(A)$, *where* $\det(A) = 0$, $\operatorname{tr}(A) = 0$.

Proof. The theorem follows immediately from Lemma 5.5. ☐

5.2.3. Zero divisors. Note that Theorem 5.9 is a particular case of Lemma 4.2 in [12]. But still we give our proof of this statement to draw an analogy between the cases of $\hat{\mathbb{H}}$ and $\hat{\mathbb{O}}$ (see Theorems 5.9 and 5.32, respectively). Moreover, it is shown in [12, Proposition 3.2] that for any commutative ring R and any $n \geq 2$, there exists a directed 3-cycle in $\Gamma\big(\mathrm{M}_n(R)\big)$. However, Lemma 5.11 provides a more concrete result for $R = \mathbb{R}$ and $n = 2$.

Notation 5.7. Let us denote

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 5.8. *Let $A \in Z(\hat{\mathbb{H}})$. Then there exists $\phi \in \operatorname{Aut}_{\mathbb{R}}(\hat{\mathbb{H}})$ such that*

(1) *if* $\operatorname{tr}(A) \neq 0$, *then* $\phi(A) = \operatorname{tr}(A)E_{11}$;
(2) *if* $\operatorname{tr}(A) = 0$, *then* $\phi(A) = E_{12}$.

Proof. By Corollary 4.12, $A \in Z(\hat{\mathbb{H}})$ implies $\det(A) = 0$, so $\operatorname{dis}(A) = \big(\operatorname{tr}(A)\big)^2 - 4\det(A) = \big(\operatorname{tr}(A)\big)^2$. Thus both cases follow immediately from Lemma 5.4. ☐

Theorem 5.9. *The diameter of $\Gamma_Z(\hat{\mathbb{H}})$ equals* 2.

Proof. First, we show that $d\big(\Gamma_Z(\hat{\mathbb{H}})\big) \le 2$. It follows from Lemmas 3.28 and 5.4 that it is sufficient to prove the following. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det(A) = ad - bc = 0.$$

Then

(1) $d(E_{11}, A) \le 2$:
- if $a \neq 0$ or $c \neq 0$, then
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ c & -a \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$
- if $a = 0$ and $c = 0$, then either $b \neq 0$ or $d \neq 0$, whence
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ d & -b \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

(2) $d(E_{12}, A) \le 2$:
- if $a \neq 0$ or $c \neq 0$, then
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} c & -a \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$
- if $a = 0$ and $c = 0$, then either $b \neq 0$ or $d \neq 0$, whence
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} d & -b \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Moreover, $d\big(\Gamma_Z(\hat{\mathbb{H}})\big) \ge 2$, since there exists a pair of vertices that are not adjacent (for example, $E_{11} \not\to E_{12}$). Hence $d\big(\Gamma_Z(\hat{\mathbb{H}})\big) = 2$. □

Proposition 5.10.

(1) $\mathrm{l.Ann}_{\hat{\mathbb{H}}}(E_{11}) = \mathrm{Lin}(E_{12}, E_{22})$, $\mathrm{r.Ann}_{\hat{\mathbb{H}}}(E_{11}) = \mathrm{Lin}(E_{21}, E_{22})$;
(2) $\mathrm{l.Ann}_{\hat{\mathbb{H}}}(E_{12}) = \mathrm{Lin}(E_{12}, E_{22})$, $\mathrm{r.Ann}_{\hat{\mathbb{H}}}(E_{12}) = \mathrm{Lin}(E_{11}, E_{12})$.

Proof. All equalities can be verified directly. □

Lemma 5.11. *Let $A, B \in Z(\hat{\mathbb{H}})$ be linearly independent, $AB = 0$. Then there exists $C \in Z(\hat{\mathbb{H}})$ such that A, B, and C are linearly independent and form a directed 3-cycle in $\Gamma_Z(\hat{\mathbb{H}})$, i.e., $BC = CA = 0$.*

Proof. It follows from Lemma 5.8 that we can take $A \in \{E_{11}, E_{12}\}$. Consider the following cases:

(1) $A = E_{11}$. Then $B = \alpha E_{21} + \beta E_{22}$, $(\alpha, \beta) \in \mathbb{R}^2 \setminus (0, 0)$, whence $C = \beta E_{12} - \alpha E_{22}$;
(2) $A = E_{12}$. Then $B = \alpha E_{11} + \beta E_{12}$, $(\alpha, \beta) \in \mathbb{R}^2$. Since A and B are linearly independent, we have $\alpha \neq 0$, whence $C = \beta E_{12} - \alpha E_{22}$. □

5.3. Split-Octonions.

5.3.1. Real Jordan normal form.

Theorem 5.12 (Artin's theorem [25, Theorem 3.1])**.** *Let $\mathcal{A}$ be an alternative algebra. Then the subalgebra generated by any two elements of $\mathcal{A}$ is associative.*

Lemma 5.13. *Let $n \le 3$, $a, b \in \mathcal{A}_n$. Then the set $\mathrm{Lin}(1, a, b, ab)$ is closed under multiplication and conjugation.*

Proof. By Proposition 3.5, $a^2 = 2\mathfrak{Re}(a)a - n(a) \in \mathrm{Lin}(1, a, b, ab)$ and $b^2 = 2\mathfrak{Re}(b)b - n(b) \in \mathrm{Lin}(1, a, b, ab)$. Moreover,

$$\begin{aligned} ba &= (2\mathfrak{Re}(b) - \bar{b})a = 2\mathfrak{Re}(b)a - \bar{b}a = 2\mathfrak{Re}(b)a - 2\langle a, b\rangle + \bar{a}b \\ &= 2\mathfrak{Re}(b)a - 2\langle a, b\rangle + (2\mathfrak{Re}(a) - a)b \\ &= 2\mathfrak{Re}(b)a + 2\mathfrak{Re}(a)b - 2\langle a, b\rangle - ab \in \mathrm{Lin}(1, a, b, ab). \end{aligned}$$

By Lemma 3.26, $\mathcal{A}_n$ is alternative, so Theorem 5.12 implies that the subalgebra generated by a and b is associative, and thus any word consisting of a and b belongs to $\mathrm{Lin}(1, a, b, ab)$.

For any $x \in \mathcal{A}_n$ we have $\bar{x} = 2\mathfrak{Re}(x) - x$, so $\mathrm{Lin}(1, a, b, ab)$ is also closed under conjugation. □

Definition 5.14. Let $\mathcal{A}$ have a regular involution. The algebra $\mathcal{A}$ is called a *composition algebra* if for all $a, b \in \mathcal{A}$, the equality $n(ab) = n(a)n(b)$ holds.

Lemma 5.15 ([6, p. 9])**.** *Let $\mathcal{A}$ be an alternative algebra with a regular involution. Then $\mathcal{A}$ is a composition algebra.*

Theorem 5.16 (Jacobson necessity theorem [19, Theorem 2.6.1])**.** *Let $\mathcal{A}$ be a proper finite-dimensional subalgebra of a composition algebra $\mathcal{B}$ over a field $\mathbb{F}$ such that norm form $n(\cdot)$ is nondegenerate on $\mathcal{A}$. Let also $x \in \mathcal{A}^{\perp}$ with $n(x) = -\gamma \neq 0$ (such x always exists). Consider a mapping $\sigma\colon \mathcal{A} + \mathcal{A}x \to \mathcal{A}\{\gamma\}$ such that for any $a, b \in \mathcal{A}$ we have $\sigma(a + bx) = (a, b)$. Then σ is isomorphic and isometric.*

Example 5.17. The converse implication in Proposition 3.8 fails, since $\hat{\mathbb{C}}\{1\} \cong \hat{\mathbb{C}}\{-1\}$.

We will identify the subalgebra $\mathrm{Lin}(1, \ell) \subset \hat{\mathbb{H}}$ with $\hat{\mathbb{C}}$. Then we use Theorem 5.16 for $\mathcal{A} = \hat{\mathbb{C}}$ and $x = i$ first, whence $\hat{\mathbb{H}} \supseteq \hat{\mathbb{C}} + \hat{\mathbb{C}} \cdot i \cong \hat{\mathbb{C}}\{-1\}$. Similarly, for $x = \ell i$ we have $\hat{\mathbb{H}} \supseteq \hat{\mathbb{C}} + \hat{\mathbb{C}} \cdot (\ell i) \cong \hat{\mathbb{C}}\{1\}$. Then, by reasons of dimension, we obtain that $\hat{\mathbb{C}}\{-1\} \cong \hat{\mathbb{H}} \cong \hat{\mathbb{C}}\{1\}$.

Lemma 5.18. *Let $\tilde{\imath}, \tilde{\jmath}, \tilde{\ell} \in \hat{\mathbb{O}}$ such that*

(1) $\mathfrak{Re}(\tilde{\imath}) = \mathfrak{Re}(\tilde{\jmath}) = \mathfrak{Re}(\tilde{\ell}) = 0$;
(2) $n(\tilde{\imath}) = n(\tilde{\jmath}) = 1$, $n(\tilde{\ell}) = -1$;
(3) $\langle \tilde{\imath}, \tilde{\jmath} \rangle = \langle \tilde{\ell}, \tilde{\imath} \rangle = \langle \tilde{\ell}, \tilde{\jmath} \rangle = \langle \tilde{\ell}, \tilde{\imath}\tilde{\jmath} \rangle = 0$.

Then there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that $\phi(\tilde{\imath}) = i$, $\phi(\tilde{\jmath}) = j$, and $\phi(\tilde{\ell}) = \ell$.

Proof. Let $\mathcal{A}$ be a subalgebra of $\hat{\mathbb{O}}$ generated by $\tilde{\imath}$ and $\tilde{\jmath}$. By Lemma 5.13, $\mathcal{A} = \mathrm{Lin}(1, \tilde{\imath}, \tilde{\jmath}, \tilde{\imath}\tilde{\jmath})$. It follows from Theorem 5.12 that $\mathcal{A}$ is associative. By Lemma 3.21, $\tilde{\imath}$ and $\tilde{\jmath}$ anticommute, whence one can verify that there exists an isomorphism $\psi\colon \mathcal{A} \to \mathbb{H}$ such that $\psi(1) = 1$, $\psi(\tilde{\imath}) = i$, $\psi(\tilde{\jmath}) = j$, and $\psi(\tilde{\imath}\tilde{\jmath}) = k$.

Then we use Theorem 5.16 for $\mathcal{A}$ and $x = \tilde{\ell}$, whence $\hat{\mathbb{O}} \supseteq \mathcal{A} + \mathcal{A}\tilde{\ell} \cong \mathcal{A}\{1\} \cong \mathbb{H}\{1\} = \hat{\mathbb{O}}$, and thus $\mathcal{A} + \mathcal{A}\tilde{\ell} = \hat{\mathbb{O}}$. Consider a mapping $\phi\colon \hat{\mathbb{O}} \to \hat{\mathbb{O}}$ defined by $\phi(a + b\tilde{\ell}) = \psi(a) + \psi(b)\ell$ for all $a, b \in \mathcal{A}$. Then ϕ is the desired automorphism. □

Lemma 5.19. *Let $\tilde{\ell}, \widetilde{\ell i}, \widetilde{\ell j} \in \hat{\mathbb{O}}$ such that*

(1) $\mathfrak{Re}(\tilde{\ell}) = \mathfrak{Re}(\widetilde{\ell i}) = \mathfrak{Re}(\widetilde{\ell j}) = 0$;
(2) $n(\tilde{\ell}) = n(\widetilde{\ell i}) = n(\widetilde{\ell j}) = -1$;
(3) $\langle \tilde{\ell}, \widetilde{\ell i} \rangle = \langle \widetilde{\ell j}, \tilde{\ell} \rangle = \langle \widetilde{\ell j}, \widetilde{\ell i} \rangle = \langle \widetilde{\ell j}, \tilde{\ell} \cdot \widetilde{\ell i} \rangle = 0$.

Then there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that $\phi(\tilde{\ell}) = \ell$, $\phi(\widetilde{\ell i}) = \ell i$, and $\phi(\widetilde{\ell j}) = \ell j$.

Proof. Let $\mathcal{A}$ be a subalgebra of $\hat{\mathbb{O}}$ generated by $\tilde{\ell}$ and $\widetilde{\ell i}$. By Lemma 5.13, $\mathcal{A} = \mathrm{Lin}(1, \tilde{\ell}, \widetilde{\ell i}, \tilde{\ell} \cdot \widetilde{\ell i})$. It follows from Theorem 5.12 that $\mathcal{A}$ is associative. By Lemma 3.21, $\tilde{\ell}$ and $\widetilde{\ell i}$ anticommute, whence one can verify that there exists an isomorphism $\psi\colon \mathcal{A} \to \hat{\mathbb{H}}$ such that $\psi(1) = 1$, $\psi(\tilde{\ell} \cdot \widetilde{\ell i}) = i$, $\psi(\tilde{\ell}) = \ell$, and $\psi(\widetilde{\ell i}) = \ell i$. We will identify the subalgebra $\mathrm{Lin}(1, i, \ell, \ell i) \subset \hat{\mathbb{O}}$ with $\hat{\mathbb{H}}$.

We use Theorem 5.16 for $\hat{\mathbb{H}}$ and $x = \ell j$ first, whence $\hat{\mathbb{O}} \supseteq \hat{\mathbb{H}} + \hat{\mathbb{H}} \cdot (\ell j) \cong \hat{\mathbb{H}}\{1\}$, and thus for reasons of dimension we have $\hat{\mathbb{O}} = \hat{\mathbb{H}} + \hat{\mathbb{H}} \cdot (\ell j)$. Then we use Theorem 5.16 for $\mathcal{A}$ and $x = \widetilde{\ell j}$, whence

$$\hat{\mathbb{O}} \supseteq \mathcal{A} + \mathcal{A} \cdot (\widetilde{\ell j}) \cong \mathcal{A}\{1\} \cong \hat{\mathbb{H}}\{1\} \cong \hat{\mathbb{H}} + \hat{\mathbb{H}} \cdot (\ell j) = \hat{\mathbb{O}},$$

and thus $\mathcal{A} + \mathcal{A} \cdot (\widetilde{\ell j}) = \hat{\mathbb{O}}$. Consider a mapping $\phi\colon \hat{\mathbb{O}} \to \hat{\mathbb{O}}$ defined by $\phi\big(a + b \cdot (\widetilde{\ell j})\big) = \psi(a) + \psi(b) \cdot (\ell j)$ for all $a, b \in \mathcal{A}$. Then ϕ is the desired automorphism. □

Lemma 5.20. *Let $a \in \hat{\mathbb{O}} \setminus \{0\}$, $\mathfrak{Re}(a) = 0$. Then there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that*

(1) *if $n(a) > 0$, then $\phi(a) = \sqrt{n(a)}\, i$;*

(2) *if* $n(a) < 0$, *then* $\phi(a) = \sqrt{-n(a)}\,\ell$;
(3) *if* $n(a) = 0$, *then* $\phi(a) = (i + \ell i)/2$.

Proof. Let us denote $A_1 = \mathrm{Lin}(i, j, k)$ and $A_2 = \mathrm{Lin}(\ell, \ell i, \ell j, \ell k)$. Then for any $a_1 \in A_1 \setminus \{0\}$ and $a_2 \in A_2 \setminus \{0\}$, we have $n(a_1) > 0$ and $n(a_2) < 0$.

(1) If $n(a) > 0$, then we choose $b \in A_1$ such that $b \in \mathrm{Lin}(a)^\perp$ and $n(b) = 1$. Then we choose $c \in A_2$ such that $c \in \mathrm{Lin}(a, b, ab)^\perp$ and $n(c) = -1$. By Lemma 5.18, there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that

$$\phi\left(\frac{a}{\sqrt{n(a)}}\right) = i, \quad \phi(b) = j, \quad \phi(c) = \ell.$$

Then

$$\phi(a) = \sqrt{n(a)}\,i.$$

(2) If $n(a) < 0$, then we choose $b \in A_2$ such that $b \in \mathrm{Lin}(a)^\perp$ and $n(b) = -1$. Then we choose $c \in A_2$ such that $c \in \mathrm{Lin}(a, b, ab)^\perp$ and $n(c) = -1$. By Lemma 5.19, there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that

$$\phi\left(\frac{a}{\sqrt{-n(a)}}\right) = \ell, \quad \phi(b) = \ell i, \quad \phi(c) = \ell j.$$

Then

$$\phi(a) = \sqrt{-n(a)}\,\ell.$$

(3) If $n(a) = 0$, then $a = a_1 + a_2$, where $a_1 \in A_1$, $a_2 \in A_2$, $n(a_1) + n(a_2) = n(a) = 0$. Since $a \neq 0$, we have $n(a_1) = -n(a_2) = c^2/4$ for some $c \in \mathbb{R}$, $c \neq 0$. Clearly, $\langle a_1, a_2 \rangle = 0$. We now choose $b \in A_1$ such that $b \in \mathrm{Lin}(a_1, a_1 a_2)^\perp$ and $n(b) = 1$. Then $\langle b, a_2 \rangle = 0$ automatically. Moreover, Lemma 4.2 for $x = a_2$, $y = a_1$, $z = b$ implies $\langle a_2, a_1 b \rangle = \langle \bar{a}_1 a_2, b \rangle = -\langle a_1 a_2, b \rangle = 0$. By Lemma 5.18, there exists $\phi_1 \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that

$$\phi_1\left(\frac{a_1}{c/2}\right) = i, \quad \phi_1(b) = j, \quad \phi_1\left(\frac{a_2}{c/2}\right) = \ell.$$

We have

$$\phi_1(a) = \phi_1(a_1) + \phi_1(a_2) = \frac{ci + c\ell}{2}.$$

Note that

$$ci + c\ell = \left(\frac{1+c^2}{2c}i - \frac{1-c^2}{2c}\ell\right) + \left(\frac{1+c^2}{2c}\ell - \frac{1-c^2}{2c}i\right),$$

$$n\left(\frac{1+c^2}{2c}i - \frac{1-c^2}{2c}\ell\right) = \left(\frac{1+c^2}{2c}\right)^2 - \left(\frac{1-c^2}{2c}\right)^2 = 1,$$

$$n\left(\frac{1+c^2}{2c}\ell - \frac{1-c^2}{2c}i\right) = \left(\frac{1-c^2}{2c}\right)^2 - \left(\frac{1+c^2}{2c}\right)^2 = -1,$$

$$\left\langle \frac{1+c^2}{2c}i - \frac{1-c^2}{2c}\ell, \frac{1+c^2}{2c}\ell - \frac{1-c^2}{2c}i \right\rangle = \frac{1+c^2}{2c}\frac{1-c^2}{2c} - \frac{1-c^2}{2c}\frac{1+c^2}{2c} = 0.$$

By Lemma 5.18, there exists $\phi_2 \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that

$$\phi_2\left(\frac{1+c^2}{2c}i - \frac{1-c^2}{2c}\ell\right) = i, \quad \phi_2\left(\frac{1+c^2}{2c}\ell - \frac{1-c^2}{2c}i\right) = \ell, \quad \phi_2(j) = j.$$

Then

$$\phi_2\left(\frac{ci + c\ell}{2}\right) = \frac{i + \ell}{2}.$$

Next, by Lemma 5.19, there exists $\phi_3 \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that

$$\phi_3(-\ell i) = \ell, \quad \phi_3(\ell) = \ell i, \quad \phi_3(\ell j) = \ell j.$$

Then

$$\phi_3(i) = \phi_3\big((-\ell i) \cdot \ell\big) = \phi_3(-\ell i) \cdot \phi_3(\ell) = \ell \cdot (\ell i) = i,$$

so

$$\phi_3\left(\frac{i+\ell}{2}\right) = \frac{i+\ell i}{2}.$$

Thus, $\phi = \phi_3 \circ \phi_2 \circ \phi_1$ is the desired automorphism. □

Lemma 5.21 ([15, p. 274])**.** *Let $\phi \in \mathrm{Aut}_{\mathbb{R}}(\mathcal{A}_n)$. Then for any $a \in \mathcal{A}_n$ we have $\mathfrak{Re}\big(\phi(a)\big) = \phi\big(\mathfrak{Re}(a)\big) = \mathfrak{Re}(a)$ and $\mathfrak{Im}\big(\phi(a)\big) = \phi\big(\mathfrak{Im}(a)\big)$, thus, $\overline{\phi(a)} = \phi(\bar{a})$ and $n\big(\phi(a)\big) = n(a)$.*

Notation 5.22. Let us denote $\mathbf{e}_1 = (1,0,0)^t$, $\mathbf{e}_2 = (0,1,0)^t$, $\mathbf{e}_3 = (0,0,1)^t \in \mathbb{R}^3$.

The following corollary is completely analogous to Lemma 5.4.

Corollary 5.23. *Let $A \in \hat{\mathbb{O}} \setminus \mathbb{R}I$, $\mathrm{dis}(A) = \big(\mathrm{tr}(A)\big)^2 - 4\det(A)$ is introduced in Definition 3.4. Then there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that*

(1) *if $\mathrm{dis}(A) > 0$, i.e., $\mathrm{dis}(A) = d^2$ for some $d \neq 0$, then*

$$\phi(A) = \frac{\mathrm{tr}(A) + d\ell}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{tr}(A) + d & \mathbf{0} \\ \mathbf{0} & \mathrm{tr}(A) - d \end{pmatrix};$$

(2) *if $\mathrm{dis}(A) < 0$, i.e., $\mathrm{dis}(A) = -d^2$ for some $d \neq 0$, then*

$$\phi(A) = \frac{\mathrm{tr}(A) + di}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{tr}(A) & d\mathbf{e}_1 \\ -d\mathbf{e}_1 & \mathrm{tr}(A) \end{pmatrix};$$

(3) *if $\mathrm{dis}(A) = 0$, then*

$$\phi(A) = \frac{\mathrm{tr}(A) + i + \ell i}{2} = \frac{1}{2}\begin{pmatrix} \mathrm{tr}(A) & 2\mathbf{e}_1 \\ \mathbf{0} & \mathrm{tr}(A) \end{pmatrix}.$$

Proof. By Lemma 3.12, $\mathrm{dis}(A) = -4n\big(\mathfrak{Im}(A)\big)$; so the proof follows immediately from Lemmas 5.20 and 5.21. □

5.3.2. Orthogonality.

Notation 5.24. Let us denote

$$E_{11} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix},\ E_{12} = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{pmatrix} \in \hat{\mathbb{O}}.$$

Lemma 5.25. *Let $A \in Z(\hat{\mathbb{O}})$. Then there exists $\phi \in \mathrm{Aut}_{\mathbb{R}}(\hat{\mathbb{O}})$ such that*

(1) *if $\mathrm{tr}(A) \neq 0$, then $\phi(A) = \mathrm{tr}(A)E_{11}$;*
(2) *if $\mathrm{tr}(A) = 0$, then $\phi(A) = E_{12}$.*

Proof. By Corollary 4.12, $A \in Z(\hat{\mathbb{O}})$ implies $\det(A) = 0$, whence $\mathrm{dis}(A) = \big(\mathrm{tr}(A)\big)^2 - 4\det(A) = \big(\mathrm{tr}(A)\big)^2$. Thus, both cases follow immediately from Corollary 5.23. □

Lemma 5.26. *Let $A \in Z_{\mathfrak{Im}}(\hat{\mathbb{O}})$. Then $\dim\big(O_{\hat{\mathbb{O}}}(A)\big) = 3$.*

Proof. It follows immediately from Lemma 4.21. □

Lemma 5.27. *Let*

$$A = \begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{0} & 0 \end{pmatrix} \in \hat{\mathbb{O}}, \quad \mathbf{v} \neq \mathbf{0}.$$

Then

$$O_{\hat{\mathbb{O}}}(A) = \mathbb{R}A \oplus \left\{ \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{w} & 0 \end{pmatrix} \,\middle|\, \mathbf{w}\cdot\mathbf{v} = 0 \right\} = \left\{ \begin{pmatrix} 0 & \mu\mathbf{v} \\ \mathbf{w} & 0 \end{pmatrix} \,\middle|\, \mu \in \mathbb{R},\ \mathbf{w}\cdot\mathbf{v} = 0 \right\}.$$

Proof. One can verify that the inclusion from right to left holds. Since $A \in Z_{\mathfrak{Im}}(\hat{\mathbb{O}})$, it follows from Lemma 5.26 that $\dim\big(O_{\hat{\mathbb{O}}}(A)\big) = 3$. Thus, we have the equality. □

Lemma 5.28. *Let*

$$A = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{w} & 0 \end{pmatrix} \in \hat{\mathbb{O}}, \quad \mathbf{w} \neq \mathbf{0}.$$

Then

$$O_{\hat{\mathbb{O}}}(A) = \mathbb{R}A \oplus \left\{ \begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{0} & 0 \end{pmatrix} \,\middle|\, \mathbf{v}\cdot\mathbf{w} = 0 \right\} = \left\{ \begin{pmatrix} 0 & \mathbf{v} \\ \lambda\mathbf{w} & 0 \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R},\ \mathbf{v}\cdot\mathbf{w} = 0 \right\}.$$

Proof. One can verify that the inclusion from right to left holds. Since $A \in Z_{\mathfrak{Im}}(\hat{\mathbb{O}})$, it follows from Lemma 5.26 that $\dim\big(O_{\hat{\mathbb{O}}}(A)\big) = 3$. Thus we have the equality. □

Lemma 5.29 ([16, p. 200])**.** *For any* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, *we have* $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{u}\cdot\mathbf{v})$.

Theorem 5.30. $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$ *is connected, and its diameter equals* 3.

Proof. Lemmas 3.28 and 5.25 imply that it is sufficient to prove the following fact. Let

$$A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \in Z_{\mathfrak{Im}}(\hat{\mathbb{O}}).$$

Then $d(E_{12}, A) \le 3$.

We have $\operatorname{tr}(A) = a + b = 0$, whence $b = -a$. Thus, $\det(A) = -a^2 - \mathbf{v}\cdot\mathbf{w} = 0$, whence $a = 0$ if and only if $\mathbf{w} \perp \mathbf{v}$.

If $\mathbf{w} \perp \mathbf{e}_1$ and $\mathbf{w} \perp \mathbf{v}$, then $a = 0$ and there exists $\tilde{\mathbf{w}} \neq \mathbf{0}$ such that $\tilde{\mathbf{w}} \perp \mathbf{e}_1$, $\tilde{\mathbf{w}} \perp \mathbf{v}$, and $\tilde{\mathbf{w}} \parallel \mathbf{w}$, i.e., $\mathbf{w} \times \tilde{\mathbf{w}} = \mathbf{0}$. Thus, by Lemma 5.28, there exists a path of length 2 between E_{12} and A, namely,

$$E_{12} = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{pmatrix} \longleftrightarrow B = \begin{pmatrix} 0 & \mathbf{0} \\ \tilde{\mathbf{w}} & 0 \end{pmatrix} \longleftrightarrow A = \begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{w} & 0 \end{pmatrix}.$$

If $\mathbf{w} \not\perp \mathbf{e}_1$ or $\mathbf{w} \not\perp \mathbf{v}$, then there exists $\tilde{\mathbf{w}} \neq \mathbf{0}$ such that $\tilde{\mathbf{w}} \perp \mathbf{e}_1$, $\tilde{\mathbf{w}} \perp \mathbf{v}$, and $\tilde{\mathbf{w}} \nparallel \mathbf{w}$, i.e., $\mathbf{w} \times \tilde{\mathbf{w}} \neq \mathbf{0}$. Thus, there exists a path of length 3 between E_{12} and A, namely,

$$E_{12} = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{pmatrix} \longleftrightarrow B = \begin{pmatrix} 0 & \mathbf{0} \\ \tilde{\mathbf{w}} & 0 \end{pmatrix} \longleftrightarrow C = \begin{pmatrix} 0 & \mathbf{w} \times \tilde{\mathbf{w}} \\ -a\tilde{\mathbf{w}} & 0 \end{pmatrix} \longleftrightarrow A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & -a \end{pmatrix}.$$

Indeed, by Lemma 5.28, $E_{12}, C \in O_{\hat{\mathbb{O}}}(B)$. Moreover, by Lemma 5.29,

$$(\mathbf{w} \times \tilde{\mathbf{w}}) \times \mathbf{v} = -\big(\mathbf{v} \times (\mathbf{w} \times \tilde{\mathbf{w}})\big) = -\big(\mathbf{w}(\mathbf{v}\cdot\tilde{\mathbf{w}}) - \tilde{\mathbf{w}}(\mathbf{v}\cdot\mathbf{w})\big) = (\mathbf{v}\cdot\mathbf{w})\tilde{\mathbf{w}} = -a^2\tilde{\mathbf{w}},$$

so

$$CA = \begin{pmatrix} 0\cdot a + (\mathbf{w} \times \tilde{\mathbf{w}})\cdot\mathbf{w} & \mathbf{0} - a(\mathbf{w} \times \tilde{\mathbf{w}}) + (-a\tilde{\mathbf{w}}) \times \mathbf{w} \\ -a^2\tilde{\mathbf{w}} + \mathbf{0} - (\mathbf{w} \times \tilde{\mathbf{w}}) \times \mathbf{v} & 0\cdot(-a) + (-a\tilde{\mathbf{w}})\cdot\mathbf{v} \end{pmatrix} = 0.$$

Then $AC = \bar{A}\bar{C} = \overline{CA} = 0$, i.e., A and C are orthogonal.

It can be easily seen from Lemmas 5.27 and 5.28 that for any $\mathbf{w}$ and $\mathbf{v}$ such that $\mathbf{w} \not\perp \mathbf{v}$ there is no path of length ≤ 2 between

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{w} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{0} & 0 \end{pmatrix}.$$

Therefore, the diameter of $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$ is exactly 3. $\square$

Theorem 5.31. *The maximal cliques in* $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$ *are of the form* $\operatorname{Lin}^*(A, B)$*, where* A *and* B *are orthogonal and linearly independent.*

Proof. Let Q be a maximal clique of $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$, $A \in Q$. By Lemma 5.26, we have $\dim\bigl(O_{\hat{\mathbb{O}}}(A)\bigr) = 3$. Hence the inclusion $Q \subset \operatorname{Lin}^*(A)$ fails, and there exists $B \in Q$ such that A and B are linearly independent. Clearly, $A \in O_{\hat{\mathbb{S}}}(A)$ and $B \in O_{\hat{\mathbb{S}}}(B)$, thus, $\operatorname{Lin}^*(A, B) \subset Q$.

Let $A, B, C \in Z_{\mathfrak{Im}}(\hat{\mathbb{O}})$ be linearly independent. Then A, B, and C do not form a 3-cycle in $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$. We assume the contrary. Then for any $D \in \operatorname{Lin}^*(A, B, C)$, we have $\operatorname{Lin}(A, B, C) \subset O_{\hat{\mathbb{O}}}(D)$ and $\dim\bigl(\operatorname{Lin}(A, B, C)\bigr) = \dim\bigl(O_{\hat{\mathbb{O}}}(D)\bigr) = 3$, and thus $O_{\hat{\mathbb{O}}}(D) = \operatorname{Lin}(A, B, C)$. Therefore, the induced subgraph of $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$ on the vertex set $\operatorname{Lin}^*(A, B, C)$ is a connected component. But it follows from Theorem 5.30 that $\Gamma_O^{\mathfrak{Im}}(\hat{\mathbb{O}})$ is connected. However, the inclusion $Z_{\mathfrak{Im}}(\hat{\mathbb{O}}) \subset \operatorname{Lin}(A, B, C)$ fails for reasons of dimension. So we have a contradiction.

Therefore, $Q = \operatorname{Lin}^*(A, B)$. $\square$

5.3.3. Zero divisors.

Theorem 5.32. *The diameter of* $\Gamma_Z(\hat{\mathbb{O}})$ *equals* 2.

Proof. First, we show that $d\bigl(\Gamma_Z(\hat{\mathbb{O}})\bigr) \le 2$. It follows from Lemmas 3.28 and 5.25 that it is sufficient to prove the following.

(1) Let

$$A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix}, \quad \det(A) = ab - \mathbf{v} \cdot \mathbf{w} = 0.$$

Then $d(E_{11}, A) \le 2$.

- If $a \ne 0$ or $\mathbf{w} \ne \mathbf{0}$, then

$$E_{11} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \longrightarrow B = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{w} & -a \end{pmatrix} \longrightarrow A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix},$$

$$E_{11}B = \begin{pmatrix} 1 \cdot 0 + \mathbf{0} \cdot \mathbf{w} & \mathbf{0} + \mathbf{0} + \mathbf{0} \times \mathbf{w} \\ \mathbf{0} + \mathbf{0} - \mathbf{0} \times \mathbf{0} & 0 \cdot (-a) + \mathbf{0} \cdot \mathbf{0} \end{pmatrix} = 0,$$

$$BA = \begin{pmatrix} 0 \cdot a + \mathbf{0} \cdot \mathbf{w} & \mathbf{0} + \mathbf{0} + \mathbf{w} \times \mathbf{w} \\ a\mathbf{w} - a\mathbf{w} - \mathbf{0} \times \mathbf{v} & (-a) \cdot b + \mathbf{v} \cdot \mathbf{w} \end{pmatrix} = 0.$$

- If $a = 0$ and $\mathbf{w} = \mathbf{0}$, then either $b \ne 0$ or $\mathbf{v} \ne \mathbf{0}$. Let $\mathbf{e_v} \in \mathbb{R}^3$ be such that $\mathbf{v} = |\mathbf{v}|\mathbf{e_v}$ and $|\mathbf{e_v}| = 1$. Consider the following path:

$$E_{11} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \longrightarrow B = \begin{pmatrix} 0 & \mathbf{0} \\ b\mathbf{e_v} & -|\mathbf{v}| \end{pmatrix} \longrightarrow A = \begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{0} & b \end{pmatrix},$$

$$E_{11}B = \begin{pmatrix} 1 \cdot 0 + \mathbf{0} \cdot (b\mathbf{e_v}) & \mathbf{0} + \mathbf{0} + \mathbf{0} \times (b\mathbf{e_v}) \\ \mathbf{0} + \mathbf{0} - \mathbf{0} \times \mathbf{0} & 0 \cdot (-|\mathbf{v}|) + \mathbf{0} \cdot \mathbf{0} \end{pmatrix} = 0,$$

$$BA = \begin{pmatrix} 0 \cdot 0 + \mathbf{0} \cdot \mathbf{0} & \mathbf{0} + \mathbf{0} + (b\mathbf{e_v}) \times \mathbf{0} \\ \mathbf{0} + \mathbf{0} - (b\mathbf{e_v}) \times \mathbf{v} & (-|\mathbf{v}|) \cdot b + \mathbf{v} \cdot (b\mathbf{e_v}) \end{pmatrix} = 0.$$

Similarly, $d(A, E_{11}) \le 2$.

- If $a \ne 0$ or $\mathbf{v} \ne \mathbf{0}$, then

$$A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix} \longrightarrow B = \begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{0} & -a \end{pmatrix} \longrightarrow E_{11} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}.$$

- If $a = 0$ and $\mathbf{v} = \mathbf{0}$, then either $b \neq 0$ or $\mathbf{w} \neq \mathbf{0}$. Let $\mathbf{e_w} \in \mathbb{R}^3$ be such that $\mathbf{w} = |\mathbf{w}|\mathbf{e_w}$ and $|\mathbf{e_w}| = 1$. The path from A to E_{11} is as follows:

$$A = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{w} & b \end{pmatrix} \longrightarrow B = \begin{pmatrix} 0 & b\mathbf{e_w} \\ \mathbf{0} & -|\mathbf{w}| \end{pmatrix} \longrightarrow E_{11} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}.$$

(2) Let

$$A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix},$$

$\det(A) = 0$ and $\operatorname{tr}(A) = 0$, i.e., $b = -a$ and $ab - \mathbf{v} \cdot \mathbf{w} = 0$. Then $d(E_{12}, A) \leq 2$.

- If $a = 0$, $\mathbf{e}_1 \times \mathbf{v} = 0$, and $\mathbf{e}_1 \cdot \mathbf{w} = 0$, then E_{12} and A are adjacent:

$$E_{12} = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{pmatrix} \longrightarrow A = \begin{pmatrix} 0 & \mathbf{v} \\ \mathbf{w} & 0 \end{pmatrix},$$

$$E_{12}A = \begin{pmatrix} 0 \cdot 0 + \mathbf{e}_1 \cdot \mathbf{w} & \mathbf{0} + \mathbf{0} + \mathbf{0} \times \mathbf{w} \\ \mathbf{0} + \mathbf{0} - \mathbf{e}_1 \times \mathbf{v} & 0 \cdot 0 + \mathbf{v} \cdot \mathbf{0} \end{pmatrix} = 0.$$

- Otherwise, there exists the following path of length 2 from E_{12} to A:

$$E_{12} = \begin{pmatrix} 0 & \mathbf{e}_1 \\ \mathbf{0} & 0 \end{pmatrix} \longrightarrow B = \begin{pmatrix} -\mathbf{e}_1 \cdot \mathbf{w} & a\,\mathbf{e}_1 \\ \mathbf{e}_1 \times \mathbf{v} & 0 \end{pmatrix} \longrightarrow A = \begin{pmatrix} a & \mathbf{v} \\ \mathbf{w} & b \end{pmatrix},$$

$$E_{12}B = \begin{pmatrix} 0 \cdot (-\mathbf{e}_1 \cdot \mathbf{w}) + \mathbf{e}_1 \cdot (\mathbf{e}_1 \times \mathbf{v}) & \mathbf{0} + \mathbf{0} + \mathbf{0} \times (\mathbf{e}_1 \times \mathbf{v}) \\ \mathbf{0} + \mathbf{0} - \mathbf{e}_1 \times (a\mathbf{e}_1) & 0 \cdot 0 + (a\mathbf{e}_1) \cdot \mathbf{0} \end{pmatrix} = 0,$$

$$BA = \begin{pmatrix} (-\mathbf{e}_1 \cdot \mathbf{w}) \cdot a + (a\mathbf{e}_1) \cdot \mathbf{w} & (-\mathbf{e}_1 \cdot \mathbf{w})\mathbf{v} + b(a\mathbf{e}_1) + (\mathbf{e}_1 \times \mathbf{v}) \times \mathbf{w} \\ a(\mathbf{e}_1 \times \mathbf{v}) + \mathbf{0} - (a\mathbf{e}_1) \times \mathbf{v} & 0 \cdot b + \mathbf{v} \cdot (\mathbf{e}_1 \times \mathbf{v}) \end{pmatrix} = 0,$$

since by Lemma 5.29 we have

$$(\mathbf{e}_1 \times \mathbf{v}) \times \mathbf{w} = -\mathbf{w} \times (\mathbf{e}_1 \times \mathbf{v}) = -\big(\mathbf{e}_1(\mathbf{w} \cdot \mathbf{v}) - \mathbf{v}(\mathbf{w} \cdot \mathbf{e}_1)\big) = \mathbf{v}(\mathbf{w} \cdot \mathbf{e}_1) - \mathbf{e}_1(ab).$$

Moreover, $d\big(\Gamma_Z(\hat{\mathbb{O}})\big) \geq 2$, as there exists a pair of vertices that are not adjacent (for example, $E_{11} \not\to E_{12}$). Hence $d\big(\Gamma_Z(\hat{\mathbb{O}})\big) = 2$. □

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A. E. Guterman
Lomonosov Moscow State University, Moscow, 119991, Russia;
Moscow Institute of Physics and Technology, Dolgoprudny, 141701, Russia;
Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991, Russia
E-mail: `guterman@list.ru`

S. A. Zhilina
Lomonosov Moscow State University, Moscow, 119991, Russia;
Moscow Center for Fundamental and Applied Mathematics, Moscow, 119991, Russia
E-mail: `s.a.zhilina@gmail.com`