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ON A SUFFICIENT CONDITION FOR A DIFFUSION PROCESS WILL NEVER REACH BOUNDARIES OF SOME INTERVAL

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A one-dimensional diffusion semi-Markov process on some interval of its values is considered. Semi-Markov transition functions of the process satisfy a second order differential equation with coefficients admitting possibility that the process stops inside this interval. In terms of coefficients of this equation, some sufficient conditions are proved for the process will neither reach the left or right boundaries of this interval. Bibliography: 7 titles.

Introduction. In [6], the case of inaccessibility of the boundaries of the range of values of the Markov diffusion process $X(t)$ without a break corresponding to the equation

$$\frac{1}{2}\,y'' + A(x)y' = 0 \tag{1}$$

was considered. The process $X(t)$ in this case is a process with domain $[0,\infty)$ and value domain (a,b), where $-\infty < a < b < \infty$.

The factor 1/2 in this expression is usually kept as a reminder of Kolmogorov's equation about the diffusion Markov process (see, for example, [1, p. 215]). The inaccessibility of the boundaries of the interval (a,b) was achieved by increasing the absolute value of the coefficient A as the value of the process approaches these boundaries (to the left boundary with plus sign and to the right boundary with minus sign). Some results on the unattainability of interval boundaries are contained in the book by Gikhman and Skorokhod [2, p. 164], in which the problem of the first exit time from an interval and the first exit point from an interval was studied in terms of stochastic differential equations. The book by Cherny and Engelbert [7], where the second chapter is devoted to the problem of the inaccessibility of boundaries by a diffusion process, summed up the research in this area, in which the behavior of the process in one-sided neighborhoods of singular points of stochastic differential equations is studied.

In [6], the same problem was studied from the point of view of diffusive semi-Markov processes. In the theory of these processes, the solution of equation (1) with certain boundary conditions is equal to the probabilities of the first exit of the process to the right or left boundary of the interval, which led to a simple solution of the problem of unreachable boundaries, in fact, to the derivation of necessary and sufficient conditions for the inaccessibility of boundaries in terms of the coefficient A.

As was noted in [6], adding the term $-B(x)y$ with nonnegative (not identically equal to zero) function B to the left-hand side removes the process from the class of Markov processes for which this coefficient characterizes a break, i.e., the exit from the interval (a,b) to some point $\delta \notin (-\infty,\infty)$, and therefore does not belong to the problem of unreachable boundaries of a segment. Adding this term leads to an equation for which there is no final solution. It is this equation that is studied in the present paper.

Thus, in the present paper, we consider the equation

$$\frac{1}{2}\,y'' + A(x)\,y' - B(x)\,y = 0, \tag{2}$$

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where the function B is continuous, nonnegative on the interval (a,b), and is not identically zero on this interval. Such an assumption admits the possibility of an infinite stop of the process $X(t)$ within this interval. In this case, the moment of the first exit does not exist. It remains to assume that $X(t)$ is not a Markov process (see, for example, [5]), and that the infinite stop makes sense for it and is defined in terms of semi-Markov transition functions. Note that the moment of the beginning of an infinite interval is not a Markov moment. For a Markov diffusion process associated with such equation, it is usually assumed (see, for example, [1]) that such a process terminates at some random Markov time $\tau_{(a,b)} < \infty$. The very formulation of the problem of unattainability, for example, of the left boundary of an interval, in the Markov case is not without some contradiction. It is assumed that the probability of unattainability of the left boundary is equal to the probability that $\tau_{(a,b)}$ on the set $X(\tau_{(a,b)}) = a$ is equal to infinity, and in order to give this concept some sense, a condition is added: if $\tau_{(a,b)} < \infty$, then $X(\tau_{(a,b)}) = b$. The behavior of the process on the left boundary is determined by its behavior on the right boundary. At the same time, the unattainability of any of the boundaries is a local property of the probabilities near these boundaries.

One-dimensional diffusion semi-Markov process. We consider a one-dimensional diffusion semi-Markov process $X(t)$ defined on the half-line $[0,\infty)$ with values in some domain contained in $\mathbb{R} \equiv (-\infty,\infty)$. The process is homogeneous in time and is completely determined by the set of all its semi-Markov transition functions $Y_{(a,b)}(T,\, a\,|\,x)$ and $Y_{(a,b)}(T,\ ,b\,|\,x)$, where $x \in (a,b)$ and $T \subset (0,\infty)$. The expression $Y_{(a,b)}(T,\,\gamma\,|\,x)$ means the conditional probability (with respect to the condition $X(0) = x$) of the event {the moment of the first exit from the interval (a,b) belongs to T, γ is the point of the first exit from the interval (a,b)}, where γ takes one of the two values: $\gamma = a$ or $\gamma = b$. For a one-dimensional diffusion semi-Markov process, these conditional probabilities are matched by the two following equations:

$$Y_{(a,b)}((0,t),\, a\,|\,x) = \int_0^t Y_{(c,d)}(dy,\, c\,|\,x)\, Y_{(a,b)}((0,t-y),\, a\,|\,c) + \int_0^t Y_{(c,d)}(dy,\, d\,|\,x)\, Y_{(a,b)}((0,t-y),\, a\,|\,d),$$

$$Y_{(a,b)}((0,t),\, b\,|\,x) = \int_0^t Y_{(c,d)}(dy,\, c\,|\,x)\, Y_{(a,b)}((0,t-y),\, b\,|\,c) + \int_0^t Y_{(c,d)}(dy,\, d\,|\,x)\, Y_{(a,b)}((0,t-y),\, b\,|\,d),$$

where $(c,d) \subset (a,b)$. Put

$$g_{(a,b)}(x) \equiv Y_{(a,b)}((0,\infty),\, a\,|\,x), \quad h_{(a,b)}(x) \equiv Y_{(a,b)}((0,\infty),\, b\,|\,x).$$

In addition, having defined these functions at the points $x \notin (a,b)$ in the intuitive way, namely, 1) $g_{(a,b)}(b) = 0$, 2) $g_{(a,b)}(a) = 1$, 3) $h_{(a,b)}(a) = 0$, 4) $h_{(a,b)}(b) = 1$, we obtain the following matching conditions for the family of these functions:

$$g_{(a,b)}(x) = g_{(c,d)}(x)\, g_{(a,b)}(c) + h_{(c,d)}(x)\, g_{(a,b)}(d), \tag{3}$$

$$h_{(a,b)}(x) = g_{(c,d)}(x)\, h_{(a,b)}(c) + h_{(c,d)}(x)\, h_{(a,b)}(d). \tag{4}$$

In particular, if $(c,d)=(a,b)$, then

$$g_{(a,b)}(x)=g_{(a,b)}(x)\,g_{(a,b)}(a)+h_{(a,b)}(x)\,g_{(a,b)}(b),$$
$$h_{(a,b)}(x)=g_{(a,b)}(x)\,h_{(a,b)}(a)+h_{(a,b)}(x)\,h_{(a,b)}(b),$$

and hence, if the functions $g_{(a,b)}(x)$ and $h_{(a,b)}(x)$ are positive inside the interval, then the following statements hold:

12) $g_{(a,b)}(b)=0$ if and only if $g_{(a,b)}(a)=1$,

34) $h_{(a,b)}(b)=1$ if and only if $h_{(a,b)}(a)=0$.

In addition, the values of both these functions are not greater than one, and therefore, for $a=c<x<d<b$, we have

$$h_{(a,b)}(x)=h_{(a,d)}(x)\,h_{(a,b)}(d)\le h_{(a,b)}(d),$$

i.e., $h_{(a,b)}(x)$ does not decrease. And if $a<c<x<d=b$, then

$$g_{(a,b)}(x)=g_{(c,b)}(x)\,g_{(a,b)}(c)\le g_{(a,b)}(c),$$

i.e., $g_{(a,b)}(x)$ does not increase. In what follows, we assume that both these functions are right-continuous on the given interval.

A one-dimensional continuous semi-Markov process is said to be diffusion in (a,b) if in the neighborhood of any point x that belongs to (a,b) together with some interval $(x-r,x+r)$, there are functions $A(x)$ which is continuously differentiable and $B(x)$ which is nonnegative continuous, such that as $r>0$, $r\to 0$, we have

$$g_{(x-r,x+r)}(x)=\frac{1}{2}(1-A(x)r-B(x)r^2)+o(r^2), \tag{5}$$

$$h_{(x-r,x+r)}(x)=\frac{1}{2}(1+A(x)r-B(x)r^2)+o(r^2). \tag{6}$$

It is easy to verify the following assertion, which follows from our assumptions: the functions $g_{(a,b)}(x)$ and $h_{(a,b)}(x)$ are continuously differentiable. If they are also twice differentiable on this interval, then they satisfy the differential equation

$$\frac{1}{2}y''+A(x)y'-B(x)y=0. \tag{7}$$

Note that the limits on the right and left at the ends of the interval satisfy the inequalities

$$g_{(a,b)}(a+)\le 1,\quad h_{(a,b)}(a+)\ge 0,\quad g_{(a,b)}(b-)\ge 0,\quad h_{(a,b)}(b-)\le 1,$$

and, in accordance with conditions 1)–4), these functions can have discontinuities at the ends of the interval (the function $g_{(a,b)}(x)$ has negative jumps, and the function $h_{(a,b)}(x)$ has positive jumps). As will be shown below, these discontinuities are related to the unattainability of the ends of the interval (a,b).

To simplify the notation, we will further study the equation of the form

$$f''+A(x)f'-B(x)f=0$$

(without the factor 1/2) and the corresponding functions $g_{(a,b)}(x)$ and $h_{(a,b)}(x)$, taking into account this factor only in the final formulation of the results.

Inaccessibility of the right boundary. A boundary b is said to be unattainable if $h_{(a,b)}(x_0) = 0$ for any point $a < x_0 < b$.

Let us consider the formula

$$h_{(a,b)}(x) = h_{(a,d)}(x) h_{(a,b)}(d). \tag{8}$$

Let $(x_n)_0^\infty$ be a strictly increasing sequence of points with $x_0 \in (a,b)$ and also $x_n \to b$. From here and (8), we obtain the equality

$$h_{(a,x_n)}(x_0) = \prod_{k=1}^{n} h_{(a,x_k)}(x_{k-1}), \quad x_k > x_{k-1},$$

which can be continued as follows:

$$h_{(a,b)}(x_0) = \prod_{k=1}^{\infty} h_{(a,x_k)}(x_{k-1}).$$

Then the unattainability of this boundary is equivalent to the equality of the corresponding infinite product to zero (the product diverges). This, in turn, is equivalent to the divergence of the series (see, for example, [4, p. 357]),

$$\sum_{k=1}^{\infty} (1 - h_{(a,x_k)}(x_{k-1})) = \infty.$$

According to [5, p.167], we have

$$h_{(c,d)}(x) = \frac{x-c}{d-c} + \int_c^d K_{(c,d)}(x,t)\, h_{(c,d)}(t)\, dt,$$

where

$$K_{(c,d)}(x,t) = \begin{cases} \frac{d-x}{d-c}[-A(t) - (B(t) + A'(t))(t-c)] & \text{if} \quad t < x, \\ \frac{x-c}{d-c}[A(t) - (B(t) + A'(t))(d-t)] & \text{if} \quad t > x. \end{cases}$$

In particular, for the equation $f'' - Bf = 0$ with $A = 0$, we have

$$K_{(c,d)}(x,t) = \begin{cases} \frac{d-x}{d-c}[-B(t)(t-c)] & \text{if} \quad t < x, \\ \frac{x-c}{d-c}[-B(t)(d-t)] & \text{if} \quad t > x, \end{cases}$$

$$h_{(c,d)}(x) = \frac{x-c}{d-c} - \int_c^x \frac{d-x}{d-c} B(t)\,(t-c)\, h_{(c,d)}(t)\, dt - \int_x^d \frac{x-c}{d-c} B(t)\,(d-t)\, h_{(c,d)}(t)\, dt,$$

$$1 - h_{(c,d)}(x) = \frac{d-x}{d-c} + \int_c^x \frac{d-x}{d-c} B(t)\,(t-c)\, h_{(c,d)}(t)\, dt + \int_x^d \frac{x-c}{d-c} B(t)\,(d-t)\, h_{(c,d)}(t)\, dt$$

$$\geq \frac{d-x}{d-c} + h_{(c,d)}(x) \int_x^d \frac{x-c}{d-c} B(t)\,(d-t)\, dt,$$

$$1 - h_{(c,d)}(x) - h_{(c,d)}(x) \int_x^d \frac{x-c}{d-c} B(t)\,(d-t)\, dt \geq \frac{d-x}{d-c},$$

$$\frac{x-c}{d-c} \geq h_{(c,d)}(x)\left[1+\int_x^d \frac{x-c}{d-c}B(t)\,(d-t)\,dt\right],$$

$$h_{(c,d)}(x) \leq \frac{x-c}{d-c}\left[1+\int_x^d \frac{x-c}{d-c}B(t)\,(d-t)\,dt\right]^{-1},$$

$$1-h_{(c,d)}(x) \geq 1-\frac{x-c}{d-c}\left[1+\int_x^d \frac{x-c}{d-c}\,B(t)\,(d-t)\,dt\right]^{-1} \geq 1-\left[1+\int_x^d \frac{x-c}{d-c}B(t)(d-t)\,dt\right]^{-1}$$

$$= \int_x^d \frac{x-c}{d-c}\,B(t)\,(d-t)\,dt\left[1+\int_x^d \frac{x-c}{d-c}\,B(t)\,(d-t)\,dt\right]^{-1}.$$

Taking into account the inequalities $a < x_{k-1} < x_k < b$ and the fact that the function $f(x) \equiv x/(1+x)$ strictly increases, and setting $c \equiv a$, $d \equiv x_k$, $x \equiv x_{k-1}$, we get

$$1-h_{(a,x_k)}(x_{k-1}) \geq \int_{x_{k-1}}^{x_k} \frac{x_{k-1}-a}{x_k-a}\,B(t)\,(x_k-t)\,dt\left[1+\int_{x_{k-1}}^{x_k} \frac{x_{k-1}-a}{x_k-a}\,B(t)\,(x_k-t)\,dt\right]^{-1}$$

$$\geq \int_{x_{k-1}}^{x_k} \frac{x_0-a}{b-a}\,B(t)\,(x_k-t)\,dt\left[1+\int_{x_{k-1}}^{x_k} \frac{x_0-a}{b-a}B(t)\,(x_k-t)\,dt\right]^{-1}.$$

The condition of unattainability of the right boundary of the interval is equivalent to

$$\sum_{k=1}^{\infty}\int_{x_{k-1}}^{x_k} \frac{x_0-a}{b-a}B(t)\,(x_k-t)\,dt\left[1+\int_{x_{k-1}}^{x_k} \frac{x_0-a}{b-a}B(t)\,(x_k-t)\,dt\right]^{-1} = \infty.$$

A sufficient condition for the unattainability of the right boundary of the interval, for example, is: $(\exists\epsilon > 0)$ $(\forall k \geq 1)$

$$\int_{x_{k-1}}^{x_k} \frac{x_0-a}{b-a}\,B(t)\,(x_k-t)\,dt\left[1+\int_{x_{k-1}}^{x_k} \frac{x_0-a}{b-a}\,B(t)\,(x_k-t)\,dt,\right]^{-1} \geq \epsilon,$$

but it is too cumbersome and difficult to check.

We are looking for a simpler sufficient condition. Let $(x_n)_0^\infty$ be a sequence such that

$$(\forall k \geq 1) \quad x_k = x_0 + (b-x_0)(1-2^{-k}).$$

Then $r_k = (b-x_0)2^{-k}$, where $r_k \equiv x_k - x_{k-1}$. A simpler sufficiency condition is as follows:

$$(\exists\epsilon > 0)\ (\forall k \geq 1) \quad \int_{x_{k-1}}^{x_k} B(t)\,(x_k-t)\,dt \geq \epsilon.$$

Assume that $B(\cdot)$ does not decrease on the interval (x_0, b). In this case, to fulfill the previous condition, it suffices that $(\exists\epsilon > 0)$ $(\forall k \geq 1)$

$$B(x_{k-1})\int_{x_{k-1}}^{x_k} (x_k-t)\,dt \geq \epsilon,$$

which is equivalent to

$$B(x_{k-1}) \geq \epsilon \left[\frac{r_k^2}{2}\right]^{-1}.$$

Consider the extension of the function x_n given for all integers $n \geq 0$ on the interval of all $t \geq 0$,

$$x_t = x_0 + (b - x_0)(1 - 2^{-t}).$$

We have $r_t = (b - x_0)/2^t$. Using the expression

$$2^t = \frac{b - x_0}{b - x_t}$$

for the multiplier 2^t in terms of x_t, we obtain an extension of the function $B(x_n)$ to all $z \in [x_0, b)$ and a sufficiency condition in terms of $B(z)$,

$$B(z) \geq \frac{2\epsilon}{r_{t+1}^2} = \frac{8\epsilon}{(b - z)^2}.$$

Inaccessibility of the left boundary. The boundary a is said to be unreachable if $g_{(a,b)}(y_0) = 0$ for any point $y_0 > a$.

We have

$$g_{(a,b)}(x) = g_{(c,b)}(x) g_{(a,b)}(c). \tag{9}$$

Let $(y_n)_0^\infty$ be a strictly decreasing sequence of points with $y_0 \in (a, b)$ and also $y_n \to a$. From here and (9), we obtain the equality

$$g_{(y_n,b)}(y_0) = \prod_{k=1}^{n} g_{(y_k,b)}(y_{k-1}), \quad y_k < y_{k-1},$$

which can be continued as

$$g_{(a,b)}(y_0) = \prod_{k=1}^{\infty} g_{(y_{k-1},b)}(y_k).$$

From this, in turn, it follows that the inaccessibility of the left boundary is equivalent to the divergence of the series (see, for example, [4, p. 357]),

$$\sum_{k=1}^{\infty} (1 - g_{(a,y_{k-1})}(y_k)) = \infty. \tag{10}$$

The following formula is derived in [5, p. 167]:

$$g_{(c,d)}(x) = \frac{d - x}{d - c} + \int_c^d K_{(c,d)}(x, t) g_{(c,d)}(t)\, dt,$$

where the kernel $K_{(c,d)}(x, t)$ is as above. In particular, if $A \equiv 0$, then

$$K_{(c,d)}(x, t) = \begin{cases} \frac{d-x}{d-c}[-B(t)(t - c)] & \text{if} \quad t < x, \\ \frac{x-c}{d-c}[-B(t)(d - t)] & \text{if} \quad t > x. \end{cases}$$

We define a specific sequence

$$y_n = a + (y_0 - a)2^{-n}$$

and its extension

$$y_t = a + (y_0 - a)2^{-t} \tag{11}$$

to all $t > 0$. Then

$$r_n \equiv y_{n-1} - y_n = (y_0 - a)2^{-n},$$

and also

$$1 - g_{(y_n,b)}(y_{n-1}) \geq \frac{r_n}{b-y_n} + \int\limits_{y_n}^{y_{n-1}} \frac{b-y_{n-1}}{b-y_n} B(z)(z-y_n) g_{(y_n,b)}(z)\, dz$$

$$\geq \frac{r_n}{b-y_n} + g_{(y_n,b)}(y_{n-1}) L_n,$$

where

$$L_n \equiv \frac{b-y_{n-1}}{b-y_n} \int\limits_{y_n}^{y_{n-1}} B(z)(z-y_n)\, dz.$$

It follows that

$$L_n \geq \frac{b-y_0}{b-a} \int\limits_{y_n}^{y_{n-1}} B(z)(z-y_n)\, dz,$$

$$\frac{L_n}{1+L_n} \geq \left(\frac{b-y_0}{b-a} \int\limits_{y_n}^{y_{n-1}} B(z)(z-y_n)\, dz \right) \left[1 + \frac{b-y_0}{b-a} \int\limits_{y_n}^{y_{n-1}} B(z)(z-y_n)\, dz \right]^{-1},$$

and the series (10) diverges if there exists $\epsilon > 0$ such that

$$\int\limits_{y_n}^{y_{n-1}} B(z)(z-y_n)\, dz \geq \epsilon.$$

We assume that the function $B(z)$ does not increase on the interval (a, y_0). Then

$$L_n \geq \frac{b-y_0}{b-a} B(y_n) \int\limits_{y_n}^{y_{n-1}} (z-y_n)\, dz.$$

Therefore the series (10) diverges if there exists $\epsilon > 0$ such that

$$B(y_n) \int\limits_{y_n}^{y_{n-1}} (z-y_n)\, dz \geq \epsilon.$$

It is equivalent to

$$B(y_n) \geq \frac{2\epsilon}{(y_n - y_{n-1})^2}.$$

For this condition, it suffices that

$$B(y_{n-1}) \geq \frac{2\epsilon}{(y_n - y_{n-1})^2},$$

which for the extended sequence (11) is equivalent to the inequality

$$B(y_t) \geq \frac{2\epsilon}{(y_{t+1} - y_t)^2}.$$

Substituting the values of the extended sequence (11) into this inequality, we obtain

$$B(y_t) \geq \frac{8\epsilon}{(y_t - a)^2},$$

which in terms of the interval (a, b) can be written as

$$B(z) \geq \frac{8\epsilon}{(z-a)^2},$$

which gives a sufficient condition for the divergence of the series (10).

Bringing an Equation to Normal Form. We consider the equation

$$y'' + A(x)\,y' - B(x)\,y = 0. \tag{12}$$

With function substitution

$$y(x) \equiv u(x) \exp\left(\frac{1}{2}\int_x^b A(t)\,dt\right)$$

this equation is obviously transformed to the normal form (see, for example, [3, p. 145])

$$u'' - B_1(x)\,u = 0, \tag{13}$$

where

$$B_1 \equiv B + \frac{1}{4}A^2 + \frac{1}{2}A'. \tag{14}$$

For the equation (13) to correspond to a diffusion semi-Markov process, we assume that

$$B_1 \geq 0.$$

Denote by $h^{(u)}_{(a,b)}(x)$ the probability of the first exit of this new process to the right boundary of the interval (a, b), namely,

$$h^{(u)}_{(a,b)}(a) = 0, \quad h^{(u)}_{(a,b)}(b) = 1.$$

According to the change of functions, for the corresponding solution of the equation (12) we have the representation

$$h_{(a,b)}(x) = h^{(u)}_{(a,b)}(x) \exp\left(\frac{1}{2}\int_x^b A(t)\,dt\right).$$

In particular,

$$h_{(a,b)}(a) = h^{(u)}_{(a,b)}(a) \exp\left(\frac{1}{2}\int_a^b A(t)\,dt\right) = 0,$$

$$h_{(a,b)}(b) = h^{(u)}_{(a,b)}(b) \exp\left(\frac{1}{2}\int_b^b A(t)\,dt\right) = 1.$$

According to the comparisons obtained above, for the right boundary of the interval (a, b) to be unattainable by a new process, it suffices that for any $z \in (a, b)$,

$$B_1(z) \geq \frac{8\epsilon}{(b-z)^2}. \tag{15}$$

From the definition of the unattainability of the right boundary, it follows that the old process has this unattainability at the same time as the new one. This means that the condition (15) is sufficient for the old process (with equation (12)) not to reach the right boundary.

The condition that the left boundary cannot be reached by the old process can be obtained by replacing the function

$$y(x) = v(x) \exp\Big(-\frac{1}{2} \int\limits_a^x A(t)\, dt \Big).$$

In this case, the equation (12) is obviously transformed to the normal form

$$v'' - B_1(x)\, v = 0, \tag{16}$$

where the coefficient $B_1(x)$ is as in (14). As before, we assume that the coefficient $B_1(x)$ is nonnegative everywhere on the interval (a, b). Therefore, there is a diffusion semi-Markov process corresponding to this new equation. For this process, the probability of the first exit to the left boundary of the interval is determined as a function of the initial point of the process $g^v_{(a,b)}(x)$. The left boundary of the interval is said to be unreachable if $g^v_{(a,b)}(x) = 0$ for any initial point x of the process. A sufficient condition for this unattainability is the inequality

$$B_1(z) \geq \frac{8\epsilon}{(z-a)^2} \tag{17}$$

that was proved above. The function $g_{(a,b)}(x)$ of the original process is defined by the condition

$$g_{(a,b)}(x) = g^{(v)}_{(a,b)}(x) \exp\Big(-\frac{1}{2} \int\limits_a^x A(t)\, dt \Big).$$

Therefore, a sufficient condition for the unattainability of the left boundary for the original process is the same condition (17). Thus the following theorem holds true.

Theorem 1. *Let a function B be continuous and nonnegative on an interval (a, b), and a function A continuously differentiable on this interval. In order for the diffusion semi-Markov process corresponding to the equation*

$$\frac{1}{2} f'' + A(x) f' - B(x) f = 0, \tag{18}$$

the right boundary of the interval is unattainable, it is sufficient that the function $B_1 \equiv B + \frac{1}{4}A^2 + \frac{1}{2}A'$ is nonnegative on (a, b) and nondecreasing on some interval (x_0, b) $(x_0 \in (a, b))$, and for any $z \in [x_0, b)$,

$$B_1(z) \geq \frac{4\epsilon}{(b-z)^2}.$$

In order for the diffusion semi-Markov process corresponding to the equation (18) *with values on the interval (a, b) to have the left boundary of the interval unattainable, it is sufficient that the function B_1 is nonnegative on (a, b) and nonincreasing on some interval (a, y_0) $(y_0 \in (a, b))$, and also for any $z \in (a, y_0]$,*

$$B_1(z) \geq \frac{4\epsilon}{(z-a)^2}.$$

Example. The following example illustrates the assumption that the function B_1 is nonnegative for $B \equiv 0$. In this case, the unreachability of one of the boundaries of the interval is achieved due to the “behavior” of the function A on the corresponding boundary.

Let $\frac{1}{4}A^2 + \frac{1}{2}A' = 1$. It is easy to check that one of the solutions of this equation is the function $A \equiv 2 \coth(x + C)$, where C is a constant.

The case $C = -a$ is an example of a nonincreasing function tending to $+\infty$ as $x \in (a, b)$, $x \to a$ (the left boundary of the interval is not reachable).

The case $C = -b$ is an example of a nonincreasing function tending to $-\infty$ as $x \in (a, b)$, $x \to b$ (the right boundary of the interval is not reachable).

This character of the change in the coefficient A at $B \equiv 0$ was fixed in [6] as a necessary condition for the unattainability of the corresponding boundary of the interval.

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