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A VARIATION ON THE $p(x)$-LAPLACE EQUATION

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For an elliptic equation with the $p(x)$-Laplacian, where the exponent $p(\cdot)$ is a bounded measurable function, we find conditions guaranteeing the continuity of the solution at a point. Bibliography: 17 *titles.*

Dedicated to Vladimir Gilelevich Maz'ya

1 Introduction

In a bounded domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, we consider the equation

$$Lu = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = 0 \tag{1.1}$$

with a measurable exponent $p(x)$ such that

$$1 < p_1 \leqslant p(x) \leqslant p_2 < +\infty \quad \text{for a.e. } x \in D. \tag{1.2}$$

To define a solution to Equation (1.1), we introduce the class of functions

$$W(D) = \{u \in W^{1,1}(D) \,:\, |\nabla u|^{p(x)} \in L^1(D)\}.$$

We say that a sequence $u_j \in W(D)$ *converges* to a function $u \in W(D)$ if

$$\int_D |u_j - u|^{p_1}\,dx + \int_D |\nabla u_j - \nabla u|^{p(x)}\,dx \to 0, \quad j \to \infty.$$

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The class $W_0(D)$ is the closure of the set of functions in $W(D)$ with compact support in D with respect to the introduced convergence. We denote by $H(D)$ the closure of $C^\infty(D) \cap W(D)$ in $W(D)$ and by $H_0(D)$ the closure of $C_0^\infty(D)$ in $W(D)$.

We say that a function $u \in W(D)$ ($u \in H(D)$) is a *W-solution* (an *H-solution*) to Equation (1.1) in D if the integral identity

$$\int_D |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \psi \, dx = 0 \tag{1.3}$$

holds for all $\psi \in W_0(D)$ ($\psi \in H_0(D)$). In what follows, W- and H-solutions to Equation (1.1) are referred to *solutions*.

This paper is devoted to the study of the continuity of solutions to Equation (1.1) at an internal point $x_0 \in D$. The systematic study of equations with the nonstandard coercivity and growth conditions was initiated by Zhikov [1]–[3]. A review of results on equations of the form (1.1) and related questions of the theory of functions can be found in [4]–[6].

We denote by $B_r^{x_0}$ an open ball with radius r and center at a point $x_0 \in \mathbb{R}^n$. For a measurable set F we denote by $|F|$ its n-dimensional Lebesgue measure. For a measurable set $F \subset \mathbb{R}^n$ and a function $g \in L^1(F)$ we set

$$\fint_F g \, dx = \frac{1}{|F|} \int_F g \, dx.$$

We assume that $B_{R_0}^{x_0} \subset D$, $R_0 \in (0, 1/4)$, and for a measurable set $E \subset D$ there exists $p_0 \in [p_1, p_2]$ such that

$$|p(x) - p_0| \leqslant \omega(|x - x_0|) \quad \text{for a.e. } x \in B_{R_0}^{x_0} \setminus E. \tag{1.4}$$

Here, ω is a continuous nondecreasing function on $[0, R_0]$ such that $\omega(0) = 0$. Let $\theta(r)$ be a nonincreasing function such that

$$r^{-2\omega(r)} \leqslant \theta(r). \tag{1.5}$$

If the logarithmic condition due to Zhikov [3] holds, when $\omega(r) = O((\ln(1/r))^{-1})$ as $r \to 0$, we can assume that θ is a constant.

On the set E itself, we only require that the exponent p satisfies the condition (1.2) and

$$|B_r^{x_0} \cap E| \leqslant (\chi(r))^{2n} r^{n+2\alpha n}, \quad 0 < r \leqslant R_0, \tag{1.6}$$

where

$$\alpha = (p_2 - p_1) \max(1, 1/(p_1 - 1)), \tag{1.7}$$

whereas the function χ is nonincreasing and satisfies the inequality $\chi(r) \leqslant C(n) r^{-\alpha}$.

We find sufficient conditions on the functions θ and χ guaranteeing the continuity of the solution at the point x_0.

This paper generalizes and combines the results of [7]–[11]. We also note the results of [8, 10] in other directions were generalized to a wider class of equations in [12, 13] and to parabolic equations in [14]. In the case $p = p_0 + \omega(|x - x_0|)$, where ω is not increasing, all bounded H- and W-solutions to Equation (1.1) are Hölder at the point x_0 (cf. [15]).

The main result of this paper is formulated as follows.

Theorem 1.1. *Assume that u is a bounded solution to Equation* (1.1) *in a domain D, $B^{x_0}_{R_0} \subset D$, $\operatorname*{ess\,osc}_{B^{x_0}_{R_0}} u \leqslant M$, and the conditions* (1.2), (1.4), (1.6) *hold. Then there is a positive constant C_0 that depends only on n, p_1, p_2, M and for all $R \leqslant R_0/4$*

$$\operatorname*{ess\,osc}_{B^{x_0}_{R}} u \leqslant (\operatorname*{ess\,osc}_{B^{x_0}_{R_0}} u + 2R_0) \exp\left(-\int\limits_{4R}^{R_0} \exp\left(-C_0(\theta(r)+\chi(r))^{2n+1}\right)\frac{dr}{r}\right).$$

Corollary 1.1. *If, under the assumptions of Theorem* 1.1, *$\theta + \chi = o(h)$ as $r \to 0$, where the nonincreasing function h is such that*

$$\int\limits_{0} \exp\left(-(h(r))^{2n+1}\right)\frac{dr}{r} = +\infty,$$

then any solution to Equation (1.1) *is continuous at the point x_0*; *moreover, for sufficiently small R_0 and $R \leqslant R_0/4$ the following estimate holds*:

$$\operatorname*{ess\,osc}_{B^{x_0}_{R}} u \leqslant (\operatorname*{ess\,osc}_{B^{x_0}_{R_0}} u + 2R_0) \exp\left(-\int\limits_{4R}^{R_0} \exp\left(-(h(r))^{2n+1}\right)\frac{dr}{r}\right).$$

Example 1.1. We give an example of applying Theorem 1.1. Assume that $L < 1/(4n+2)$, $A > 0$, and

$$\omega(r) \leqslant \frac{L \ln\ln\ln(1/r)}{\ln(1/r)}, \quad \chi(r) \leqslant A(\ln\ln(1/r))^{2L}, \quad 0 < r < R_0 < 1/27.$$

Then any solution to Equation (1.1) is continuous at the point x_0; moreover, for any $\delta \in (0,1)$ and sufficiently small $R < R_0$

$$\operatorname*{ess\,osc}_{B^{x_0}_{R}} u \leqslant C \exp\left(-\left(\ln\frac{1}{R}\right)^{\delta}\right), \tag{1.8}$$

where $C = C(n, p_1, p_2, M, \delta, L, A, R_0)$. Indeed, in this case,

$$(\theta(r)+\chi(r))^{2n+1} \leqslant (A+1)^{2n+1}(\ln\ln(1/r))^{1-\varepsilon}, \quad \varepsilon = 1-(4n+2)L.$$

Denoting $C_1 = (A+1)^{2n+1}C_0$, we have

$$\begin{aligned}
&\int\limits_{4R}^{R_0} \exp\left(-C_0(\theta(r)+\chi(r))^{2n+1}\right)\frac{dr}{r} \\
&\geqslant \int\limits_{4R}^{R_0} \exp\left(-C_1(\ln\ln(1/r))^{1-\varepsilon}\right)\frac{dr}{r} = \int\limits_{\ln(1/R_0)}^{\ln(1/4R)} \exp\left(-C_1(\ln t)^{1-\varepsilon}\right)dt \\
&\geqslant (\ln(4R)^{-1})\exp\left(-C_1(\ln\ln(1/(4R)))^{1-\varepsilon}\right) - (\ln(1/R_0))\exp\left(-C_1(\ln\ln(1/(R_0)))^{1-\varepsilon}\right),
\end{aligned}$$

where the last inequality is obtained by integrating by parts. Consequently,

$$\operatorname*{ess\,osc}_{B_R^{x_0}} u \leqslant (\operatorname*{ess\,osc}_{B_{R_0}^{x_0}} u + 2R_0)\frac{\gamma(4R)}{\gamma(R_0)},$$

$$\gamma(t) = \exp\left(-\exp\left(\ln\ln t^{-1} - C_1(\ln\ln t^{-1})^{1-\varepsilon}\right)\right).$$

For arbitrary $\delta \in (0,1)$ and sufficiently small $t > 0$ we have $\gamma(t) \leqslant \exp\left(-(\ln t^{-1})^{\delta}\right)$, which implies the estimate (1.8).

Furthermore, in this paper, we establish a weak type Harnack inequality. We say that a function $u \in W(D)$ ($u \in H(D)$) is a *W-supersolution* (an *H-supersolution*) to Equation (1.1) in D if the integral inequality

$$\int_D |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\psi\, dx \geqslant 0 \tag{1.9}$$

holds for all nonnegative functions $\psi \in W_0(D)$ ($\psi \in H_0(D)$). We say that a function $u \in W(D)$ ($u \in H(D)$) is a *W-subsolution* (an *H-subsolution*) to Equation (1.1) if the integral inequality

$$\int_D |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\psi\, dx \leqslant 0 \tag{1.10}$$

holds for all nonnegative functions $\psi \in W_0(D)$ ($\psi \in H_0(D)$). As in the case of solutions, W- and H-supersolutions are called *supersolutions* and W- and H-subsolutions are referred to as *subsolutions*.

Below, as in Theorem 1.1, we assume that $R \leqslant R_0/4$ and $B_{4R}^{x_0} \subset D$.

Theorem 1.2. *Assume that u is a bounded nonnegative supersolution to Equation* (1.1) *in a domain D, $\operatorname*{ess\,sup}_{B_{4R}^{x_0}} u \leqslant M$, the conditions* (1.2), (1.4), *and* (1.6) *are satisfied, and $s = \operatorname*{ess\,inf}_{B_{4R}^{x_0}\setminus E} p$. Then for any $1/4 \leqslant \tau_1, \tau_2 \leqslant 3$ and $0 < q < n(s-1)/(n-1)$*

$$\left(\fint_{B_{\tau_2 R}^{x_0}} (u+R)^q\, dx\right)^{1/q} \leqslant \exp\left(C(\theta(4R) + \chi(4R))^{4n+2}\right)\operatorname*{ess\,inf}_{B_{\tau_1 R}^{x_0}} (u+R), \tag{1.11}$$

where $C = C(n, p_1, p_2, q, M)$.

As a consequence, for nonnegative solutions to Equation (1.1) we obtain an analog of the classical Harnack inequality

Theorem 1.3. *Assume that u is a bounded nonnegative solution to Equation* (1.1) *in a domain D, $\operatorname*{ess\,sup}_{B_{4R}^{x_0}} u \leqslant M$, and the conditions* (1.2), (1.4), *and* (1.6) *are satisfied. Then*

$$\operatorname*{ess\,sup}_{B_R^{x_0}} u \leqslant \exp\left(C(\theta(4R) + \chi(4R))^{4n+2}\right)\operatorname*{ess\,inf}_{B_R^{x_0}} (u+R),$$

where $C = C(n, p_1, p_2, M)$.

The paper is organized as follows. In Section 2, we obtain a variant of Moser type estimates. In Section 3, we prove a series of integral estimates for supersolutions. In Section 4, using the obtained estimates, we prove a lemma on oscillations which is used in Section 5 to estimate the modulus of continuity for solutions at a point in the proof of Theorem 1.1. Section 6 is devoted to a weak type Harnack inequality for nonnegative supersolutions (Theorem 1.2) and an analog of the Harnack inequality for nonnegative solutions to the equation under consideration (Theorem 1.3). The method of proof is based on a modification of the tools applied in [9]–[11]. The scheme of the proof mainly remains unchanged.

We denote by C various positive constants appearing in proofs. Unless otherwise stated, such constants depend only on parameters fixed in the corresponding assertion. For the sake of brevity, we set

$$\rho(r) = \theta(r) + \chi(r), \tag{1.12}$$

$$k = n/(n-1), \quad \varkappa = 2n/(2n-1), \quad \varkappa' = 2n. \tag{1.13}$$

2 Moser Type Estimates

We first formulate an interpolation lemma.

Lemma 2.1. *Let a bounded sequence $Y_j \geqslant 0$ be such that $Y_j \leqslant Cb^j Y_{j+1}^{1-r}$, where $0 < r < 1$. Then $Y_0 \leqslant (2C)^{1/r} b^{(1-r)/r^2}$.*

The following assertion is based on the classical Moser iteration technique [16].

Lemma 2.2. *Let (1.6) hold, and let for a bounded nonnegative function $v \in W^{1,1}(B_{4R}^{x_0})$ and a number $s > 1$ the following inequality*

$$\int\limits_{B_{4R}^{x_0}} R|\nabla v| v^{\beta+s-2}\eta^{p_2}\,dx \leqslant \widetilde{C}\theta(4R) \int\limits_{B_{4R}^{x_0}\setminus E} v^{\beta+s-1}((R|\nabla\eta|)^{p_2} + \eta^{p_2})\,dx$$
$$+ \widetilde{C}R^{-\alpha} \int\limits_{B_{4R}^{x_0}\cap E} v^{\beta+s-1}((R|\nabla\eta|)^{p_2} + \eta^{p_2})\,dx \tag{2.1}$$

hold for all $\beta \geqslant 1$, $\eta \in C_0^\infty(B_{4R}^{x_0})$, $0 \leqslant \eta \leqslant 1$. Then for any $q > 0$ and $1/4 \leqslant \tau < t \leqslant 4$

$$\operatorname*{ess\,sup}_{B_{\tau R}^{x_0}} v \leqslant C(\rho(4R))^{2n/q}(t-\tau)^{-2p_2 n/q}\Bigg(\;\fint\limits_{B_{tR}^{x_0}} v^q\,dx\Bigg)^{1/q}, \tag{2.2}$$

where $C = C(n, p_2, q, \widetilde{C})$.

Proof. The inequality (2.1) implies

$$\fint\limits_{B_{4R}^{x_0}} R|\nabla(v^{\beta+s-1}\eta^{p_2})|\,dx$$
$$\leqslant C(n,p_2)(\widetilde{C}(\beta+s-1)+1)\,\theta(4R)R^{-n} \int\limits_{B_{4R}^{x_0}\setminus E} v^{\beta+s-1}((R|\nabla\eta|)^{p_2} + \eta^{p_2})\,dx$$

$$+ C(n,p_2)(\widetilde{C}(\beta+s-1)+1)R^{-n-\alpha} \int\limits_{B_{4R}^{x_0}\cap E} v^{\beta+s-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx. \qquad (2.3)$$

By the Hölder inequality and the condition (1.6),

$$R^{-n-\alpha} \int\limits_{B_{4R}^{x_0}\cap E} v^{\beta+s-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx$$

$$\leqslant \Bigg(R^{-n} \int\limits_{B_{4R}^{x_0}\cap E} v^{(\beta+s-1)\varkappa}((R|\nabla\eta|)^{p_2\varkappa}+\eta^{p_2\varkappa})\,dx\Bigg)^{1/\varkappa} (R^{-n-\alpha\varkappa'}|B_{4R}^{x_0}\cap E|)^{1/\varkappa'}$$

$$\leqslant C(p_1,p_2)\chi(4R)\Bigg(R^{-n} \int\limits_{B_{4R}^{x_0}\cap E} v^{(\beta+s-1)\varkappa}((R|\nabla\eta|)^{p_2\varkappa}+\eta^{p_2\varkappa})\,dx\Bigg)^{1/\varkappa}.$$

Therefore, since $\beta+s-1\geqslant 1$, from (2.3) we find

$$\fint\limits_{B_{4R}^{x_0}} R|\nabla(v^{\beta+s-1}\eta^{p_2})|\,dx$$

$$\leqslant C(n,p_2)\widetilde{C}(\beta+s-1)\theta(4R)R^{-n} \int\limits_{B_{4R}^{x_0}\setminus E} v^{\beta+s-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx$$

$$+ C(n,p_1,p_2)\widetilde{C}(\beta+s-1)\chi(4R)\Bigg(R^{-n} \int\limits_{B_{4R}^{x_0}\cap E} v^{(\beta+s-1)\varkappa}((R|\nabla\eta|)^{p_2\varkappa}+\eta^{p_2\varkappa})\,dx\Bigg)^{1/\varkappa}. \qquad (2.4)$$

Extending the integration domain on the right-hand side of (2.4) to the whole ball $B_{4R}^{x_0}$ and estimating the first term on the right-hand side with the help of the Hölder inequality, we get

$$\fint\limits_{B_{4R}^{x_0}} R|\nabla(v^{\beta+s-1}\eta^{p_2})|\,dx \leqslant C(\beta+s-1)\theta(4R) \fint\limits_{B_{4R}^{x_0}} v^{\beta+s-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx$$

$$+ C(\beta+s-1)\chi(4R)\Bigg(\fint\limits_{B_{4R}^{x_0}} v^{(\beta+s-1)\varkappa}((R|\nabla\eta|)^{p_2\varkappa}+\eta^{p_2\varkappa})\,dx\Bigg)^{1/\varkappa}$$

$$\leqslant C(\beta+s-1)\rho(4R)\Bigg(\fint\limits_{B_{4R}^{x_0}} v^{(\beta+s-1)\varkappa}((R|\nabla\eta|)^{p_2\varkappa}+\eta^{p_2\varkappa})\,dx\Bigg)^{1/\varkappa}.$$

By the Sobolev embedding theorem,

$$\Bigg(\fint\limits_{B_{4R}^{x_0}} v^{(\beta+s-1)k}\eta^{p_2 k}\,dx\Bigg)^{1/k}$$

$$\leqslant C(\beta+s-1)\rho(4R)\Bigg(\fint_{B^{x_0}_{4R}} v^{(\beta+s-1)\varkappa}((R|\nabla\eta|)^{p_2\varkappa}+\eta^{p_2\varkappa})\,dx\Bigg)^{1/\varkappa}. \qquad (2.5)$$

We introduce sequences β_j and $\widetilde{\beta}_j$, where $j=0,1,2,\ldots$, by

$$\beta_{j+1}+s-1=(\beta_j+s-1)k/\varkappa,$$

$$\widetilde{\beta}_{j+1}+s-1=(\beta_j+s-1)/\varkappa.$$

For $1/4\leqslant\widetilde{\tau}<\widetilde{t}\leqslant 4$ we set $\tau_j=\widetilde{\tau}+2^{-j}(\widetilde{t}-\widetilde{\tau})$ and introduce cut-off functions η_j such that $0\leqslant\eta_j\leqslant 1$, $\eta_j=1$ in $B^{x_0}_{\tau_jR}$, $\eta_j\in C_0^\infty(B^{x_0}_{\tau_{j-1}R})$, $R|\nabla\eta_j|\leqslant 2^{j+3}(\widetilde{t}-\widetilde{\tau})^{-1}$. We set

$$Y_j=\Bigg(\fint_{B^{x_0}_{\tau_jR}} v^{\beta_j+s-1}\,dx\Bigg)^{1/(\beta_j+s-1)}.$$

Taking $\eta=\eta_j$ and $\beta=\widetilde{\beta}_{j+1}$ in the estimate (2.5), under the condition $\widetilde{\beta}_j\geqslant 1$, we find

$$Y_{j+1}\leqslant(C(\beta_j+s-1)\rho(4R)2^{p_2j}(\widetilde{t}-\widetilde{\tau})^{-p_2})^{\varkappa/(\beta_j+s-1)}Y_j.$$

Iterating this estimate, we get

$$\sup_{B^{x_0}_{\widetilde{\tau}R}} v\leqslant\lim_{j\to\infty}Y_j\leqslant Y_0\prod_{j=0}^{\infty}(C(\beta_j+s-1)\rho(4R)2^{p_2j}(\widetilde{t}-\widetilde{\tau})^{-p_2})^{\varkappa/(\beta_j+s-1)}$$

$$\leqslant(C_*(\beta_0+s-1)^{2n}(\rho(4R))^{2n}(\widetilde{t}-\widetilde{\tau})^{-2p_2n}\fint_{B^{x_0}_{\widetilde{t}R}} v^{\beta_0+s-1}\,dx)^{1/(\beta_0+s-1)}, \qquad (2.6)$$

where $C_*=C_*(n,p_1,p_2,\widetilde{C})$. Here, we used the relations

$$\sum_{j=0}^{\infty}\frac{\varkappa}{\beta_j+s-1}=\frac{2n}{\beta_0+s-1},$$

$$\sum_{j=0}^{\infty}\frac{\varkappa j}{\beta_j+s-1}=\frac{4n(n-1)}{\beta_0+s-1}.$$

We first assume that $q\geqslant\varkappa p_2$ and set $\beta_0=q-s+1$. Then the condition $\widetilde{\beta}_0\geqslant 1$ is satisfied and (2.6) implies

$$\operatorname*{ess\,sup}_{B^{x_0}_{\widetilde{\tau}R}}\, v\leqslant q^{2n/q}C_*^{1/q}(\rho(4R))^{2n/q}(\widetilde{t}-\widetilde{\tau})^{-2p_2n/q}\Bigg(\fint_{B^{x_0}_{\widetilde{t}R}} v^q\,dx\Bigg)^{1/q}. \qquad (2.7)$$

This is the required estimate (2.2) with $\widetilde{t}=t$ and $\widetilde{\tau}=\tau$.

In the case $q < \varkappa p_2$, we set $t_j = t - (t-\tau)2^{-j}$, $M_j = \operatorname*{ess\,sup}_{B^{x_0}_{t_j R}} v$. Applying the estimate (2.7) with $\widetilde{\tau} = t_j$, $\widetilde{t} = t_{j+1}$, $q = \varkappa p_2$, and further using the inequality $v^{\varkappa p_2} \leqslant M^{\varkappa p_2 - q} v^q$, we find

$$M_j \leqslant (\varkappa s)^{2n/(\varkappa p_2)} C_*^{1/(\varkappa p_2)} (\rho(4R))^{2n/(\varkappa p_2)} (t-\tau)^{-2p_2 n/(\varkappa p_2)} 2^{j2p_2 n/(\varkappa p_2)} \Bigg(\fint_{B^{x_0}_{t_{j+1}R}} v^{\varkappa p_2}\, dx \Bigg)^{1/(\varkappa p_2)}$$

$$\leqslant (\varkappa p_2)^{2nr/q} C_*^{r/q} (\rho(4R))^{2nr/q} (t-\tau)^{-2p_2 nr/q} 2^{j2p_2 nr/q} M_{j+1}^{1-r} \Bigg(\fint_{B^{x_0}_{tR}} v^q\, dx \Bigg)^{r/q},$$

where $r = q/(\varkappa p_2)$. Now, from Lemma 2.1 it follows that

$$\operatorname*{ess\,sup}_{B^{x_0}_{\tau R}} v = M_0 \leqslant 2^{4n(p_2/q)^2} (\varkappa p_2)^{2n/q} C_*^{1/q} (\rho(4R))^{2n/q} (t-\tau)^{-2p_2 n/q} \Bigg(\fint_{B^{x_0}_{tR}} v^q\, dx \Bigg)^{1/q},$$

which completes the proof of Lemma 2.2. □

Lemma 2.3. *Let the conditions* (1.2), (1.4), *and* (1.6) *hold. Then for a bounded nonnegative subsolution to Equation* (1.1) *in the ball* $B^{x_0}_{4R}$, $0 < R \leqslant R_0/4$, *such that* $\operatorname*{ess\,sup}_{B^{x_0}_{4R}} u \leqslant M$ *and any* $q > 0$ *and* $1/4 \leqslant \tau < t \leqslant 4$ *the following estimate holds*:

$$\operatorname*{ess\,sup}_{B^{x_0}_{\tau R}} (u+R) \leqslant C(\rho(4R))^{2n/q} (t-\tau)^{-2p_2 n/q} \Bigg(\fint_{B^{x_0}_{tR}} (u+R)^q\, dx \Bigg)^{1/q}, \tag{2.8}$$

where $C = C(n, q, p_1, p_2, M)$.

Proof. Choosing the test function $\psi = (u+R)^\beta \eta^{p_2}$ with $\eta \in C_0^\infty(B^{x_0}_{4R})$, $0 \leqslant \eta \leqslant 1$, and $\beta \geqslant 1$ in (1.10), we obtain the relation

$$\beta \int_{B^{x_0}_{4R}} |\nabla u|^{p(x)} (u+R)^{\beta-1} \eta^{p_2}\, dx \leqslant p_2 \int_{B^{x_0}_{4R}} |\nabla u|^{p(x)-1} (u+R)^\beta \eta^{p_2-1} |\nabla \eta|\, dx.$$

Applying the Young inequality to the integrand on the right-hand side, we find

$$\int_{B^{x_0}_{4R}} |\nabla u|^{p(x)} (u+R)^{\beta-1} \eta^{p_2}\, dx \leqslant C(p_2) \int_{B^{x_0}_{4R}} (u+R)^{p(x)+\beta-1} |\nabla \eta|^{p(x)} \eta^{p_2-p(x)}\, dx. \tag{2.9}$$

Setting

$$s := \operatorname*{ess\,inf}_{B^{x_0}_{4R} \setminus E} p \tag{2.10}$$

and using the Young inequality, we get

$$\int_{B^{x_0}_{4R}} (R|\nabla u|)(u+R)^{\beta+s-2} \eta^{p_2}\, dx$$

$$\leqslant R^s \int_{B^{x_0}_{4R}} |\nabla u|^{p(x)} (u+R)^{\beta-1} \eta^{p_2-p(x)}\, dx + \int_{B^{x_0}_{4R}} R^{(p(x)-s)/(p(x)-1)} (u+R)^{\beta-1+(s-1)p'(x)} \eta^{p_2}\, dx.$$

The last inequality and (2.9) imply

$$\int\limits_{B_{4R}^{x_0}} (R|\nabla u|)(u+R)^{\beta+s-2}\eta^{p_2}\,dx \leqslant C(p_2)R^s \int\limits_{B_{4R}^{x_0}} (u+R)^{p(x)+\beta-1}|\nabla\eta|^{p(x)}\eta^{p_2-p(x)}\,dx + \int\limits_{B_{4R}^{x_0}} (u+R)^{\beta+s-1}(R/(u+R))^{(p(x)-s)/(p(x)-1)}\,dx. \tag{2.11}$$

By (1.2) and (1.4),

$$(u+R)^{p(x)-s}R^s|\nabla\eta|^{p(x)} = (u+R)^{p(x)-s}R^{s-p(x)}(R|\nabla\eta|)^{p(x)} \leqslant (R|\nabla\eta|)^{p(x)}\cdot\begin{cases}\theta(4R)(M+1)^{p_2-p_1}, & x\in B_{4R}^{x_0}\setminus E,\\ (M+1)^{p_2-p_1}R^{p_1-p_2}, & x\in B_{4R}^{x_0}\cap E\cap\{p>s\},\\ 1, & x\in B_{4R}^{x_0}\cap E\cap\{p\leqslant s\},\end{cases}$$

and

$$(R/(u+R))^{(p(x)-s)/(p(x)-1)} \leqslant \begin{cases}1, & x\in B_{4R}^{x_0}\setminus E,\\ 1, & x\in B_{4R}^{x_0}\cap E\cap\{p\geqslant s\},\\ (M+1)^{\frac{p_2-p_1}{p_1-1}}R^{\frac{p_1-p_2}{p_1-1}}, & x\in B_{4R}^{x_0}\cap E\cap\{p<s\}.\end{cases}$$

To obtain these estimates, we used the inequality $(u+R)^{p(x)-s}R^{s-p(x)}\leqslant 1$ for $x\in B_{4R}^{x_0}\cap E$ such that $p(x)<s$.

Applying the obtained estimate to the right-hand side of (2.11), we arrive at the inequality

$$\begin{aligned}\int\limits_{B_{4R}^{x_0}} &(R|\nabla u|)(u+R)^{\beta+s-2}\eta^{p_2}\,dx\\ &\leqslant C(p_1,p_2,M)\,\theta(4R)\int\limits_{B_{4R}^{x_0}\setminus E}(u+R)^{\beta+s-1}(R|\nabla\eta|)^{p(x)}\eta^{p_2-p(x)}\,dx\\ &+C(p_1,p_2,M)R^{p_1-p_2}\int\limits_{B_{4R}^{x_0}\cap E}(u+R)^{\beta+s-1}(R|\nabla\eta|)^{p(x)}\eta^{p_2-p(x)}\,dx\\ &+\int\limits_{B_{4R}^{x_0}\setminus E}(u+R)^{\beta+s-1}\eta^{p_2}\,dx + C(M)R^{(p_1-p_2)/(p_1-1)}\int\limits_{B_{4R}^{x_0}\cap E}(u+R)^{\beta+s-1}\eta^{p_2}\,dx.\end{aligned} \tag{2.12}$$

By the Young inequality,

$$(R|\nabla\eta|)^{p(x)}\eta^{p_2-p(x)} \leqslant \eta^{p_2} + (R|\nabla\eta|)^{p_2}, \tag{2.13}$$

from (2.12) we have

$$\int\limits_{B_{4R}^{x_0}} (R|\nabla u|)(u+R)^{\beta+s-2}\eta^{p_2}\,dx \leqslant C(p_1,p_2,M)\,\theta(4R) \int\limits_{B_{4R}^{x_0}\setminus E} (u+R)^{\beta+s-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx$$

$$+\,C(p_1,p_2,M)R^{-\alpha} \int\limits_{B_{4R}^{x_0}\cap E} (u+R)^{\beta+s-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx,$$

where α is defined in (1.7). Applying Lemma 2.2, we obtain the required estimate (2.8). □

3 Integral Estimates for Supersolutions

Throughout the section, we assume that u is a nonnegative bounded supersolution to Equation (1.1) in the ball $B_{4R}^{x_0}$, $0 < R \leqslant R_0/4$, the conditions (1.2), (1.4), and (1.6) are satisfied, and $\operatorname*{ess\,sup}_{B_{4R}^{x_0}} u \leqslant M$. We recall that k and $\varkappa$ are defined in (1.13), whereas the function ρ is given by (1.12). The quantity s has the same meaning as in (2.10).

We denote $f_+ = \max(f,0)$, in particular,

$$\ln{}_+(x) = \max(\ln x, 0).$$

We will use the estimate

$$x \ln{}_+ \frac{\mu}{x} \leqslant \frac{\mu}{e}, \quad x,\mu > 0. \tag{3.1}$$

Lemma 3.1. *For the function*

$$v = \ln{}_+ \frac{\mu}{u+R}, \quad R \leqslant \mu \leqslant 2M+2R, \tag{3.2}$$

and any $q > 0$ and $1/4 \leqslant \tau < t \leqslant 4$ the following inequality holds:

$$\operatorname*{ess\,sup}_{B_{\tau R}^{x_0}} v \leqslant C(\rho(4R))^{2n/q}(t-\tau)^{-2p_2 n/q}\Bigg(1 + \fint\limits_{B_{tR}^{x_0}} v^q\,dx\Bigg)^{1/q}, \tag{3.3}$$

where $C = C(n,q,p_1,p_2,M)$.

Proof. Choosing the test function $\psi = v^\beta (u+R)^{1-s}\eta^{p_2}$, $\beta \geqslant 1$, $\eta \in C_0^\infty(B_{4R}^{x_0})$, $0 \leqslant \eta \leqslant 1$, in the integral inequality (1.9), we get

$$\beta \int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{-s}v^{\beta-1}\eta^{p_2}\,dx + (s-1)\int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{-s}v^{\beta}\eta^{p_2}\,dx$$

$$\leqslant p_2 \int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)-1}(u+R)^{1-s}v^{\beta}\eta^{p_2-1}|\nabla\eta|\,dx.$$

Removing the second term on the left-hand side of this inequality, applying the Young inequality to the integrand on the right-hand side, and taking into account the inequality $|\nabla v| \leqslant (u+R)^{-1}|\nabla u|$, we find

$$\int\limits_{B_{4R}^{x_0}} |\nabla v|^{p(x)}(u+R)^{p(x)-s}v^{\beta-1}\eta^{p_2}\,dx \leqslant C(p_2)\int\limits_{B_{4R}^{x_0}} v^{\beta+s-1}(v(u+R))^{p(x)-s}\eta^{p_2-p(x)}|\nabla\eta|^{p(x)}\,dx. \tag{3.4}$$

Denote

$$w = (u+R)v = (u+R)\ln{}_{+}\frac{\mu}{u+R}.$$

By the Young inequality,

$$\int\limits_{B_{4R}^{x_0}} (R|\nabla v|)v^{\beta+s-2}\eta^{p_2}\,dx \leqslant \int\limits_{B_{4R}^{x_0}} R^s|\nabla v|^{p(x)}(u+R)^{p(x)-s}v^{\beta-1}\eta^{p_2}\,dx + \int\limits_{B_{4R}^{x_0}} (R/w)^{(p(x)-s)/(p(x)-1)}v^{\beta+s-1}\eta^{p_2}\,dx. \tag{3.5}$$

Combining the estimates (3.4) and (3.5), we arrive at the inequality

$$\int\limits_{B_{4R}^{x_0}} (R|\nabla v|)v^{\beta+s-2}\eta^{p_2}\,dx \leqslant C(p_2)\int\limits_{B_{4R}^{x_0}} v^{\beta+s-1}(w/R)^{p(x)-s}\eta^{p_2-p(x)}(R|\nabla\eta|)^{p(x)}\,dx + \int\limits_{B_{4R}^{x_0}} (R/w)^{(p(x)-s)/(p(x)-1)}v^{\beta+s-1}\eta^{p_2}\,dx. \tag{3.6}$$

By (3.1), we have $w \leqslant M+R$. Using this inequality and the conditions (1.2), (1.4), we obtain the estimate

$$v^{\beta+s-1}(w/R)^{p(x)-s} \leqslant \begin{cases} v^{\beta+s-1}\theta(4R)(M+1)^{p_2-p_1}, & x \in B_{4R}^{x_0}\setminus E,\\ v^{\beta+s-1}(M+1)^{p_2-p_1}R^{p_1-p_2}, & x \in B_{4R}^{x_0}\cap E\cap\{p>s\},\\ v^{\beta+s-1}+v^{\beta-1}, & x\in B_{4R}^{x_0}\cap E\cap\{p\leqslant s\}. \end{cases} \tag{3.7}$$

Here, for estimating on the set $B_{4R}^{x_0}\cap\{p(x)\leqslant s\}$, we used the relation

$$v^{\beta+s-1}(w/R)^{p(x)-s} = v^{\beta+p(x)-1}(R/(u+R))^{s-p(x)} \leqslant v^{\beta+s-1}+v^{\beta-1}$$

Similarly,

$$(R/w)^{(p(x)-s)/(p(x)-1)}v^{\beta+s-1} \leqslant \begin{cases} v^{\beta+s-1}+v^{\beta-1}, & x\in B_{4R}^{x_0}\setminus E,\\ v^{\beta+s-1}+v^{\beta-1}, & x\in B_{4R}^{x_0}\cap E\cap\{p\geqslant s\},\\ v^{\beta+s-1}\Big(\dfrac{M+1}{R}\Big)^{(p_2-p_1)/(p_1-1)}, & x\in B_{4R}^{x_0}\cap E\cap\{p<s\}, \end{cases} \tag{3.8}$$

where for points in $B_{4R}^{x_0} \cap \{p(x) \geqslant s\}$ we used the estimate

$$
\begin{aligned}
&(R/w)^{(p(x)-s)/(p(x)-1)} v^{\beta+s-1} \\
&= (R/(u+R))^{(p(x)-s)/(p(x)-1)} v^{\beta-1+(s-1)p'(x)} \leqslant v^{\beta+s-1} + v^{\beta-1}.
\end{aligned}
$$

The last inequality is valid since $p(x) \geqslant s$ implies $(s-1)p'(x) \leqslant s$.

Taking into account the inequality (3.7), (3.8), from (3.6) we have

$$
\begin{aligned}
\fint_{B_{4R}^{x_0}} R|\nabla v| v^{\beta+s-2} \eta^{p_2}\, dx \leqslant C\theta(4R) &\fint_{B_{4R}^{x_0}\setminus E} (v^{\beta+s-1} + v^{\beta-1}) \eta^{p_2 - p(x)} (R|\nabla \eta|)^{p(x)} + \eta^{p_2})\, dx \\
&+ CR^{-\alpha} \fint_{B_{4R}^{x_0}\cap E} (v^{\beta+s-1} + v^{\beta-1}) \eta^{p_2 - p(x)} (R|\nabla \eta|)^{p(x)} + \eta^{p_2})\, dx,
\end{aligned}
$$

where $C = C(p_1, p_2, M)$. Using (2.13) and the inequalities

$$
\begin{aligned}
&v^{\beta-1} + v^{\beta+s-1} \leqslant 2\max\{v,1\}^{\beta+s-1}, \\
&|\nabla \max\{v,1\}| \max\{v,1\}^{\beta+s-2} \leqslant |\nabla v| v^{\beta+s-2},
\end{aligned}
$$

we find

$$
\begin{aligned}
\fint_{B_{4R}^{x_0}} R|\nabla \max\{v,1\}| \max\{v,1\}^{\beta+s-2} \eta^{p_2}\, dx \leqslant C\theta(4R) &\fint_{B_{4R}^{x_0}\setminus E} \max\{v,1\}^{\beta+s-1} (\eta^{p_2} + (R|\nabla \eta|)^{p_2})\, dx \\
&+ CR^{-\alpha} \fint_{B_{4R}^{x_0}\cap E} \max\{v,1\}^{\beta+s-1} (\eta^{p_2} + (R|\nabla \eta|)^{p_2})\, dx.
\end{aligned}
$$

Now, the required assertion follows from Lemma 2.2. □

To estimate the integral on the right-hand side of (3.3), we need the following assertion.

Lemma 3.2. *For $1/4 < t < 4$ the following estimate holds*:

$$
\fint_{B_{tR}^{x_0}} R|\nabla \ln(u+R)|\, dx \leqslant C(p_1, p_2, M)\rho(4R)(4-t)^{-p_2}. \tag{3.9}
$$

Proof. We set $w = \ln(u+R)$. Let $\eta \in C_0^\infty(B_{4R}^{x_0})$, $0 \leqslant \eta \leqslant 1$. Choosing the test function $\psi = (u+R)^{1-s}\eta^{p_2}$ in the integral inequality (1.9), we obtain the inequality

$$
(s-1) \int_{B_{4R}^{x_0}} |\nabla u|^{p(x)} (u+R)^{-s} \eta^{p_2}\, dx \leqslant p_2 \int_{B_{4R}^{x_0}} |\nabla u|^{p(x)-1} (u+R)^{1-s} \eta^{p_2-1} |\nabla \eta|\, dx
$$

which implies

$$
\int_{B_{4R}^{x_0}} |\nabla w|^{p(x)} (u+R)^{p(x)-s} \eta^{p_2}\, dx \leqslant C(p_1, p_2) \int_{B_{4R}^{x_0}} |\nabla w|^{p(x)-1} (u+R)^{p(x)-s} \eta^{p_2-1} |\nabla \eta|\, dx.
$$

By the Young inequality applied to the integrand on the right-hand side and the last relation,

$$\int\limits_{B^{x_0}_{4R}} |\nabla w|^{p(x)}(u+R)^{p(x)-s}\eta^{p_2}\,dx \leqslant C(p_1,p_2)\int\limits_{B^{x_0}_{4R}} (u+R)^{p(x)-s}\eta^{p_2-p(x)}|\nabla\eta|^{p(x)}\,dx. \tag{3.10}$$

We again use the Young inequality to obtain the auxiliary estimate

$$\int\limits_{B^{x_0}_{4R}} R|\nabla w|\eta^{p_2}\,dx \leqslant \int\limits_{B^{x_0}_{4R}} R^{s}|\nabla w|^{p(x)}(u+R)^{p(x)-s}\eta^{p_2}\,dx + \int\limits_{B^{x_0}_{4R}} R^{(p(x)-s)/(p(x)-1)}(u+R)^{(s-p(x))/(p(x)-1)}\eta^{p_2}\,dx. \tag{3.11}$$

Now, (3.10) and (3.11) imply

$$\int\limits_{B^{x_0}_{4R}} R|\nabla w|\eta^{p_2}\,dx \leqslant C(p_1,p_2)\int\limits_{B^{x_0}_{4R}} ((u+R)/R)^{p(x)-s}\eta^{p_2-p(x)}(R|\nabla\eta|)^{p(x)}\,dx + \int\limits_{B^{x_0}_{4R}} ((u+R)/R)^{(s-p(x))/(p(x)-1)}\eta^{p_2}\,dx. \tag{3.12}$$

By (1.2) and (1.4),

$$((u+R)/R)^{p(x)-s} \leqslant \begin{cases} \theta(4R)(M+1)^{p_2-p_1}, & x\in B^{x_0}_{4R}\setminus E,\\ R^{p_1-p_2}(M+1)^{p_2-p_1}, & x\in B^{x_0}_{4R}\cap E\cap\{p\geqslant s\},\\ 1, & x\in B^{x_0}_{4R}\cap E\cap\{p\leqslant s\}, \end{cases} \tag{3.13}$$

and

$$((u+R)/R)^{(s-p(x))/(p(x)-1)} \leqslant \begin{cases} 1, & x\in B^{x_0}_{4R}\setminus E,\\ 1, & x\in B^{x_0}_{4R}\cap E\cap\{p\geqslant s\},\\ \Big(\dfrac{M+1}{R}\Big)^{(p_2-p_1)/(p_1-1)}, & x\in B^{x_0}_{4R}\cap E\cap\{p\leqslant s\}. \end{cases} \tag{3.14}$$

Applying these estimates to the right-hand side of (3.12), we find

$$\int\limits_{B^{x_0}_{4R}} R|\nabla w|\eta^{p_2}\,dx \leqslant C(p_1,p_2,M)\,\theta(4R)\int\limits_{B^{x_0}_{4R}\setminus E} (\eta^{p_2}+(R|\nabla\eta|)^{p_2})\,dx + C(p_1,p_2,M)R^{-\alpha}\int\limits_{B^{x_0}_{4R}\cap E} (\eta^{p_2}+(R|\nabla\eta|)^{p_2})\,dx. \tag{3.15}$$

We choose a cut-off function η such that $\eta=1$ on $B^{x_0}_{tR}$ and $|\nabla\eta|\leqslant 4(4-t)^{-1}$. Now, using (3.15), (1.6), and the inequality $(\chi(4R))^{2n}(4R)^{\alpha(2n-1)}\leqslant C(n)\chi(4R)$, we arrive at (3.9). □

Lemma 3.3. *For all $1/4 \leqslant \tau < t \leqslant 7/2$ the following estimate holds:*

$$\operatorname*{ess\,inf}_{B^{x_0}_{\tau R}} (u+R) \geqslant \mu \cdot \exp\left(-C(\rho(4R))^{2n+1}(t-\tau)^{-2p_2 n}\right), \tag{3.16}$$

where

$$\mu = \exp \fint_{B^{x_0}_{tR}} \ln(u+R)\,dx$$

and $C = C(n, p_1, p_2, M)$.

Proof. By the Jensen inequality,

$$\mu \leqslant \fint_{B^{x_0}_{tR}} (u+R)\,dx \leqslant M + R.$$

By the Poincaré inequality and Lemma 3.2, for the function v defined in (3.2) we have

$$\fint_{B^{x_0}_{tR}} v\,dx \leqslant \fint_{B^{x_0}_{tR}} \left|\ln \frac{\mu}{u+R}\right| dx \leqslant C(n)R \fint_{B^{x_0}_{tR}} \left|\nabla \ln \frac{\mu}{u+R}\right| dx \leqslant C\rho(4R).$$

From the last inequality and the estimate in Lemma 3.1 it follows that

$$\operatorname*{ess\,sup}_{B^{x_0}_{\tau R}} v \leqslant C(\rho(4R))^{2n}(t-\tau)^{-2p_2 n}\left(1 + \fint_{B^{x_0}_{tR}} v\,dx\right) \leqslant C(t-\tau)^{-2p_2 n}(\rho(4R))^{2n+1}.$$

Recalling the definition v and μ, we obtain the required estimate (3.16). □

The following assertion is based on the technique of [17], which was adapted in [9] to the $p(x)$-Laplacian with logarithmic modulus of continuity at a point x_0, and further generalized in [10] and [11].

Lemma 3.4. *There is a positive constant $C = C(n, p_1, p_2, M)$ such that for all $1/4 \leqslant \sigma < t \leqslant 7/2$*

$$\left(\fint_{B^{x_0}_{\sigma R}} (u+R)^{\delta_0}\,dx\right)^{1/\delta_0} \leqslant \mu \cdot 2^{1/\delta_0}, \tag{3.17}$$

where

$$\delta_0 = C(t-\sigma)^{2np_2}(\rho(4R))^{-2n}, \quad \mu = \exp \fint_{B^{x_0}_{tR}} \ln(u+R)\,dx. \tag{3.18}$$

Proof. We set $w = \ln_+(u+R)/\mu$. Assume that $\gamma \geqslant 1$ and $c_0 = 2/(s-1)$. Choosing the test function $\psi = z^{\gamma-1}(u+R)^{1-s}\eta^{p_2}$, $z = \max(w, c_0\gamma)$, where $\eta \in C_0^\infty(B^{x_0}_{tR})$, $0 \leqslant \eta \leqslant 1$, in the integral inequality (1.9), we have

$$(s-1)\int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{-s}z^{\gamma-1}\eta^{p_2}\,dx \leqslant (\gamma-1)\int\limits_{B_{4R}^{x_0}\cap\{w>c_0\gamma\}} |\nabla u|^{p(x)}(u+R)^{-s}z^{\gamma-2}\eta^{p_2}\,dx$$

$$+\,p_2\int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)-2}(u+R)^{1-s}z^{\gamma-1}\eta^{p_2-1}\nabla u\cdot\nabla\eta\,dx.$$

Using the relation $(\gamma-1)z^{-1}=(\gamma-1)/\max(w,c_0\gamma)\leqslant 1/c_0=(s-1)/2$, we find

$$\int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{-s}z^{\gamma-1}\eta^{p_2}\,dx \leqslant \frac{2p_2}{s-1}\int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)-1}(u+R)^{1-s}z^{\gamma-1}\eta^{p_2-1}|\nabla\eta|\,dx.$$

By the Young inequality,

$$\int\limits_{B_{4R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{-s}z^{\gamma-1}\eta^{p_2}\,dx \leqslant C(p_1,p_2)\int\limits_{B_{4R}^{x_0}} (u+R)^{p(x)-s}z^{\gamma-1}\eta^{p_2-p(x)}|\nabla\eta|^{p(x)}\,dx.$$

Since $|\nabla w|\leqslant|\nabla u|(u+R)^{-1}$, we obtain the estimate

$$\int\limits_{B_{4R}^{x_0}} |\nabla w|^{p(x)}(u+R)^{p(x)-s}z^{\gamma-1}\eta^{p_2}\,dx$$

$$\leqslant C(p_1,p_2)\int\limits_{B_{4R}^{x_0}} (u+R)^{p(x)-s}z^{\gamma-1}\eta^{p_2-p(x)}|\nabla\eta|^{p(x)}\,dx. \tag{3.19}$$

By the Young inequality,

$$R|\nabla w|z^{\gamma-1}\eta^{p_2} \leqslant R^s|\nabla w|^{p(x)}(u+R)^{p(x)-s}z^{\gamma-1}\eta^{p_2}$$

$$+\,((u+R)/R)^{(s-p(x))/(p(x)-1)}z^{\gamma-1}\eta^{p_2}. \tag{3.20}$$

From (3.20), (3.19), and the inequality $|\nabla z|\leqslant|\nabla w|$ we get

$$\int\limits_{B_{4R}^{x_0}} R|\nabla z|z^{\gamma-1}\eta^{p_2}\,dx \leqslant C(p_1,p_2)\int\limits_{B_{4R}^{x_0}} ((u+R)/R)^{p(x)-s}z^{\gamma-1}\eta^{p_2-p(x)}(R|\nabla\eta|)^{p(x)}\,dx$$

$$+\int\limits_{B_{4R}^{x_0}} ((u+R)/R)^{(s-p(x))/(p(x)-1)}z^{\gamma-1}\eta^{p_2}\,dx.$$

We apply the inequality (3.13) to estimate the first integral on the right-hand side and the inequality (3.14) for estimating the second one. Taking into account (1.4) and (2.13), we find

$$\int\limits_{B_{4R}^{x_0}} R|\nabla z|z^{\gamma-1}\eta^{p_2}\,dx \leqslant C(p_1,p_2,M)\,\theta(4R)\int\limits_{B_{4R}^{x_0}\setminus E} z^{\gamma-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx$$

$$+\,C(p_1,p_2,M)R^{-\alpha}\int\limits_{B_{4R}^{x_0}\cap E} z^{\gamma-1}((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx,$$

where α is defined in (1.7). Multiplying both sides of the last estimate by $R^{-n}\gamma$ and using the relations

$$\gamma z^{-1} = \gamma / \max(w, c_0\gamma) \leqslant 1/c_0 \leqslant (p_2 - 1)/2,$$
$$R|\nabla(z^\gamma \eta^{p_2})| \leqslant \gamma R|\nabla z| z^{\gamma-1}\eta^{p_2} + p_2 z^\gamma ((R|\nabla\eta|)^{p_2} + \eta^{p_2}),$$

we find

$$R^{-n} \int\limits_{B_{4R}^{x_0}} R|\nabla(z^\gamma \eta^{p_2})|\, dx \leqslant C(p_1, p_2, M)\, \theta(4R) R^{-n} \int\limits_{B_{4R}^{x_0}\setminus E} z^\gamma ((R|\nabla\eta|)^{p_2} + \eta^{p_2})\, dx$$
$$+ C(p_1, p_2, M) R^{-n-\alpha} \int\limits_{B_{4R}^{x_0}\cap E} z^\gamma ((R|\nabla\eta|)^{p_2} + \eta^{p_2})\, dx.$$

Estimating the second term on the right-hand side by using the Hölder inequality and taking into account (1.6), we get

$$R^{-n-\alpha} \int\limits_{B_{4R}^{x_0}\cap E} z^\gamma ((R|\nabla\eta|)^{p_2} + \eta^{p_2})\, dx$$
$$\leqslant \Bigg(R^{-n} \int\limits_{B_{4R}^{x_0}\cap E} z^{\gamma\varkappa} ((R|\nabla\eta|)^{p_2} + \eta^{p_2})^\varkappa\, dx \Bigg)^{1/\varkappa} (|B_{4R}^{x_0} \cap E| \cdot R^{-n-2\alpha n})^{1/2n}$$
$$\leqslant C(p_1, p_2)\chi(4R) \Bigg(\fint\limits_{B_{4R}^{x_0}} z^{\gamma\varkappa} ((R|\nabla\eta|)^{p_2} + \eta^{p_2})^\varkappa\, dx \Bigg)^{1/\varkappa}.$$

Consequently,

$$\fint\limits_{B_{4R}^{x_0}} R|\nabla(z^\gamma \eta^{p_2})|\, dx \leqslant C(n, p_1, p_2, M)\, \theta(4R) \fint\limits_{B_{4R}^{x_0}} z^\gamma ((R|\nabla\eta|)^{p_2} + \eta^{p_2})\, dx$$
$$+ C(n, p_1, p_2, M)\chi(4R) \Bigg(\fint\limits_{B_{4R}^{x_0}} z^{\gamma\varkappa} ((R|\nabla\eta|)^{p_2} + \eta^{p_2})^\varkappa\, dx \Bigg)^{1/\varkappa}.$$

Hence, again by the Hölder inequality, we have

$$\fint\limits_{B_{4R}^{x_0}} R|\nabla(z^\gamma \eta^{p_2})|\, dx \leqslant C\rho(4R) \Bigg(\fint\limits_{B_{4R}^{x_0}} z^{\gamma\varkappa} ((R|\nabla\eta|)^{p_2} + \eta^{p_2})^\varkappa\, dx \Bigg)^{1/\varkappa}.$$

Hereinafter in the proof, $C = C(n, p_1, p_2, M)$ unless otherwise stated. By the Sobolev embedding theorem and the definition of $z = \max(w, c_0\gamma)$, we arrive at the inequality

$$\fint\limits_{B_{4R}^{x_0}} \max(w, c_0\gamma)^{\gamma k} \eta^{p_2 k}\, dx \leqslant C(\rho(4R))^k \Bigg(\fint\limits_{B_{4R}^{x_0}} \max(w, c_0\gamma)^{\gamma\varkappa} (\eta^{p_2} + (R|\nabla\eta|)^{p_2\varkappa})\, dx \Bigg)^{k/\varkappa}. \quad (3.21)$$

Let $\sigma \in (\tau, t)$. For $j = 0, 1, 2, \dots$ we set $r_j = \sigma + (t-\sigma)2^{-j}$ and introduce cut-off functions $\eta = \eta_j$ as follows: $\eta_j \in C_0^\infty(B_{r_j}^{x_0})$, $0 \leqslant \eta_j \leqslant 1$, $\eta_j = 1$ in $B_{r_{j+1}}^{x_0}$, and $|\nabla \eta_j| \leqslant 2^{j+3}(t-\sigma)^{-1}$. We introduce sequences γ_j and $\widetilde{\gamma}_j$ by $\gamma_j = \varkappa(k/\varkappa)^j$, $j \in \mathbb{N} \cup \{0\}$, $\widetilde{\gamma}_j = (k/\varkappa)^{j-1}$, $j \in \mathbb{N}$. Now, from the inequality (3.21) with $\gamma = \widetilde{\gamma}_{j+1}$, $\eta = \eta_j$ it follows that

$$\fint_{B_{r_{j+1}}^{x_0}} (\max(w, c_0\gamma_{j+1}))^{\gamma_{j+1}}\, dx \leqslant (k/\varkappa)^{\gamma_{j+1}} \fint_{B_{r_{j+1}}^{x_0}} (\max(w, c_0\gamma_j))^{\gamma_{j+1}}\, dx$$

$$\leqslant C(\rho(4R))^k \cdot (k/\varkappa)^{\gamma_{j+1}} (t-\sigma)^{-p_2 k} 2^{jp_2 k} \Bigg(\fint_{B_{r_j}^{x_0}} (\max(w, c_0\gamma_j))^{\gamma_j}\, dx \Bigg)^{k/\varkappa}. \tag{3.22}$$

From the estimate (3.9) and the Sobolev–Poincaré inequality it follows that

$$\fint_{B_{tR}^{x_0}} w^{\gamma_0}\, dx = \fint_{B_{tR}^{x_0}} w^{\varkappa}\, dx \leqslant \fint_{B_{tR}^{x_0}} |\ln((u+R)/\mu)|^{\varkappa}\, dx$$

$$\leqslant \fint_{B_{tR}^{x_0}} R|\nabla \ln((u+R)/\mu)|\, dx \leqslant C\rho(4R). \tag{3.23}$$

Setting

$$Y_j = \fint_{B_{r_j}^{x_0}} (\max(w, c_0\gamma_j))^{\gamma_j}\, dx,$$

we write (3.22) and (3.23) in the form

$$Y_j \leqslant K_j Y_{j-1}^{k/\varkappa}, \quad Y_0 \leqslant C\rho(4R), \quad K_j = C(\rho(4R))^k (t-\sigma)^{-p_2 k} (k/\varkappa)^{\gamma_j} 2^{p_2 k j}.$$

Iterating this inequality, we find

$$Y_j \leqslant \prod_{m=0}^{j-1} (K_{j-m})^{(k/\varkappa)^m} (C\rho(4R))^{(k/\varkappa)^j}.$$

Since $\gamma_j = \varkappa(k/\varkappa)^j$, for $j \geqslant 1$ we use the equality

$$\prod_{m=0}^{j-1} (K_{j-m})^{(k/\varkappa)^m} = (C(\rho(4R))^k (t-\sigma)^{-p_2 k})^{\Sigma_1} (k/\varkappa)^{\varkappa j (k/\varkappa)^j} 2^{p_2 k \Sigma_2},$$

$$\Sigma_1 = \sum_{m=0}^{j-1} (k/\varkappa)^m, \quad \Sigma_2 = \sum_{m=0}^{j-1} (j-m)(k/\varkappa)^m.$$

Further, we estimate

$$\Sigma_1 \leqslant (k/\varkappa)^j \varkappa/(k-\varkappa) = \gamma_j/(k-\varkappa),$$

$$\Sigma_2 = (k/\varkappa)^j \sum_{l=1}^{j} l(k/\varkappa)^{-l} \leqslant (k/\varkappa)^{j-1} \sum_{l=0}^{\infty} l(k/\varkappa)^{1-l}$$

$$= (k/\varkappa)^{j-1}(1-(k/\varkappa)^{-1})^{-2} = \gamma_j \frac{k}{(k-\varkappa)^2}.$$

Thus,

$$Y_j \leqslant (C\rho(4R))^{(1/\varkappa)+k/(k-\varkappa)}(t-\sigma)^{-p_2k/(k-\varkappa)})^{\gamma_j}(k/\varkappa)^{j\gamma_j}.$$

Taking into account that $k/(k-\varkappa) = 2n-1$, $1/\varkappa < 1$, we find

$$\left(\fint_{B^{x_0}_{r_j}} (\max(w, c_0\gamma_j))^{\gamma_j}\, dx \right)^{1/\gamma_j} \leqslant C(\rho(4R))^{2n}(t-\sigma)^{-2np_2}(k/\varkappa)^j. \tag{3.24}$$

Let m be an integer in $[\gamma_{l-1}, \gamma_l]$, $l \in \mathbb{N}$. It is clear that $(k/\varkappa)^l \leqslant (k/\varkappa)m/\varkappa$. Using the Hölder inequality and the Stirling estimate ($m! \geqslant \sqrt{2\pi m}(m/e)^m$), we find

$$\fint_{B^{x_0}_{\sigma R}} \frac{w^m}{m!}\, dx \leqslant \frac{1}{m!}\left(\fint_{B^{x_0}_{\sigma R}} w^{\gamma_l}\, dx \right)^{m/\gamma_l} \leqslant C^m(\rho(4R))^{2nm}(t-\sigma)^{-2p_2nm}(k/\varkappa)^{lm}/m!$$

$$\leqslant (C_1(t-\sigma)^{-2p_2n})^m(\rho(4R))^{2nm}. \tag{3.25}$$

Consequently, for $\delta_0 \leqslant (t-\sigma)^{2p_2n}(2C_1)^{-1}(\rho(4R))^{-2n}$

$$\fint_{B^{x_0}_{\sigma R}} \frac{(w\delta_0)^m}{m!}\, dx \leqslant 2^{-m}.$$

For $m = 1$ from (3.9) and the Poincaré inequality we find

$$\fint_{B^{x_0}_{\sigma R}} \frac{w\delta_0}{1!}\, dx \leqslant C_2\delta_0.$$

Now, setting $\delta_0 = \min((2C_1)^{-1}, (2C_2)^{-1})(t-\sigma)^{2p_2n}(\rho(4R))^{-2n}$, we get

$$\fint_{B^{x_0}_{\sigma R}} e^{\delta_0 w}\, dx = \sum_{m=0}^{\infty} \fint_{B^{x_0}_{\sigma R}} \frac{(\delta_0 w)^m}{m!}\, dx \leqslant 2.$$

Therefore, since $(u+R)^{\delta_0} \leqslant \mu^{\delta_0}\exp(\delta_0 w)$ by the definition of w, we find

$$\left(\fint_{B^{x_0}_{\sigma R}} (u+R)^{\delta_0}\, dx \right)^{1/\delta_0} \leqslant \mu \cdot 2^{1/\delta_0} = \mu \cdot \exp\left(C(t-\sigma)^{-2np_2}(\rho(4R))^{2n}\right).$$

Lemma 3.4 is proved. □

Lemma 3.5. *Assume that* $0 < \delta_0 < q < n(s-1)/(n-1)$ *and* $1/4 \leqslant \tau < \sigma \leqslant 7/2$. *Then*

$$\left(\fint_{B^{x_0}_{\tau R}} (u+R)^q\, dx \right)^{1/q} \leqslant (C(\sigma-\tau)^{-p_2}\gamma_0^{-p_2}\rho(4R))^{3n/\delta_0}\left(\fint_{B^{x_0}_{\sigma R}} (u+R)^{\delta_0}\, dx \right)^{1/\delta_0}, \tag{3.26}$$

where $C = C(n, p_1, p_2, M)$, $\gamma_0 = s - 1 - q(n-1)/n$.

Proof. Choosing the test function $\psi = (u+R)^{1-s+\beta}\eta^{p_2}$, $0 < \beta < s-1$, $0 \leqslant \eta \leqslant 1$, $\eta \in C_0^\infty(B_{\sigma R}^{x_0})$, in the integral inequality (1.9), we get

$$(s-1-\beta)\int\limits_{B_{\sigma R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{\beta-s}\eta^{p_2}\,dx \leqslant p_2 \int\limits_{B_{\sigma R}^{x_0}} |\nabla u|^{p(x)-2}(u+R)^{1-s+\beta}\eta^{p_2-1}\nabla u \nabla\eta\,dx$$

$$\leqslant p_2 \int\limits_{B_{\sigma R}^{x_0}} |\nabla u|^{p(x)-1}(u+R)^{1-s+\beta}\eta^{p_2-1}|\nabla\eta|\,dx.$$

By the Young inequality,

$$\int\limits_{B_{\sigma R}^{x_0}} |\nabla u|^{p(x)}(u+R)^{\beta-s}\eta^{p_2}\,dx$$

$$\leqslant C(p_1,p_2)(s-1-\beta)^{-p_2}\int\limits_{B_{\sigma R}^{x_0}} (u+R)^{p(x)-s+\beta}|\nabla\eta|^{p(x)}\eta^{p_2-p(x)}\,dx. \tag{3.27}$$

We again use the Young inequality to derive

$$R|\nabla u|(u+R)^{\beta-1} \leqslant R^s|\nabla u|^{p(x)}(u+R)^{\beta-s} + (u+R)^\beta((u+R)/R)^{(s-p(x))(p(x)-1)}. \tag{3.28}$$

Now, from (3.27) and (3.28) we find

$$\int\limits_{B_{\sigma R}^{x_0}} R|\nabla u|(u+R)^{\beta-1}\eta^{p_2}\,dx$$

$$\leqslant C(p_1,p_2)(s-1-\beta)^{-p_2}\int\limits_{B_{\sigma R}^{x_0}} ((u+R)/R)^{p(x)-s}(u+R)^\beta(R|\nabla\eta|)^{p(x)}\eta^{p_2-p(x)}\,dx$$

$$+\int\limits_{B_{\sigma R}^{x_0}} ((u+R)/R)^{(s-p(x))/(p(x)-1)}(u+R)^\beta\eta^{p_2}\,dx.$$

Using the estimates (3.13) and (3.14) and taking into account (1.4) and (2.13), we have

$$\int\limits_{B_{\sigma R}^{x_0}} R|\nabla u|(u+R)^{\beta-1}\eta^{p_2}\,dx$$

$$\leqslant C(p_1,p_2,M)(s-1-\beta)^{-p_2}\theta(4R)\int\limits_{B_{\sigma R}^{x_0}\setminus E} (u+R)^\beta((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx$$

$$+C(p_1,p_2,M)(s-1-\beta)^{-p_2}R^{-\alpha}\int\limits_{B_{\sigma R}^{x_0}\cap E} (u+R)^\beta((R|\nabla\eta|)^{p_2}+\eta^{p_2})\,dx. \tag{3.29}$$

To estimate the second term on the right-hand side of (3.29), we use the condition (1.6) and the Hölder inequality, which implies

$$R^{-n-\alpha} \int\limits_{B_{\sigma R}^{x_0} \cap E} (u+R)^{\beta}((R|\nabla\eta|)^{p_2} + \eta^{p_2})\, dx$$

$$\leqslant \left(R^{-n} \int\limits_{B_{\sigma R}^{x_0} \cap E} (u+R)^{\beta\varkappa}((R|\nabla\eta|)^{p_2} + \eta^{p_2})^{\varkappa}\, dx \right)^{1/\varkappa} (|B_{\sigma R}^{x_0} \cap E| \cdot R^{-n-2n\alpha})^{1/2n}$$

$$\leqslant C(p_1,p_2)\chi(4R) \left(\fint\limits_{B_{\sigma R}^{x_0}} (u+R)^{\beta\varkappa}((R|\nabla\eta|)^{p_2} + \eta^{p_2})^{\varkappa}\, dx \right)^{1/\varkappa}.$$

Multiplying both sides of (3.29) by R^{-n} and using the last estimate and Hölder inequality, we find

$$\fint\limits_{B_{\sigma R}^{x_0}} R|\nabla((u+R)^{\beta}\eta^{p_2})|\, dx \leqslant C(s-1-\beta)^{-p_2}\rho(4R) \left(\fint\limits_{B_{\sigma R}^{x_0}} (u+R)^{\beta\varkappa}((R|\nabla\eta|)^{p_2} + \eta^{p_2})^{\varkappa}\, dx \right)^{1/\varkappa}.$$

By the Sobolev embedding theorem,

$$\left(\fint\limits_{B_{\sigma R}^{x_0}} (u+R)^{\beta k}\eta^{p_2 k}\, dx \right)^{1/(\beta k)}$$
$$\leqslant (C(s-1-\beta)^{-p_2\varkappa}(\rho(4R))^{\varkappa} \fint\limits_{B_{\sigma R}^{x_0}} (u+R)^{\beta\varkappa}((R|\nabla\eta|)^{p_2} + \eta^{p_2})^{\varkappa}\, dx)^{1/(\beta\varkappa)}. \tag{3.30}$$

Let j_0 be the minimal natural number such that $q \leqslant \delta_0(k/\varkappa)^{j_0}$. For $j = 0,1,\dots,j_0$ we set $\delta_1 = q(k/\varkappa)^{-j_0}$, $\beta_j = \delta_1(k/\varkappa)^j$, $\widetilde{\beta}_{j+1} = \beta_j/\varkappa$, $r_j = \sigma - (\sigma-\tau)(1-2^{-j})$. By the condition $q < k(s-1)$ for all $j = 0,1,\dots,j_0-1$, we have

$$s - 1 - \widetilde{\beta}_{j+1} \geqslant s - 1 - \widetilde{\beta}_{j_0+1} = s - 1 - \frac{q}{k} = \gamma_0 > 0. \tag{3.31}$$

Assume that $\eta_j \in C_0^{\infty}(B_{r_j}^{x_0})$, $\eta_j = 1$ on $B_{r_{j+1}}^{x_0}$, $0 \leqslant \eta_j \leqslant 1$, $|\nabla\eta_j| \leqslant 2^{j+2}/(\sigma-\tau)$. Successively writing down the inequality (3.30) for $\eta = \eta_j$, $\beta = \widetilde{\beta}_{j+1}$, $j = 0,1,\dots,j_0-1$, and using the estimates

$$\varkappa \sum_{j=0}^{j_0-1} \frac{1}{\beta_j} \leqslant \varkappa \sum_{j=0}^{\infty} \frac{1}{\beta_j} = \frac{2n}{\delta_1} \leqslant \frac{3n}{\delta_0},$$

$$\sum_{j=0}^{j_0-1} \frac{j}{\beta_j} \leqslant \frac{1}{\delta_1} \sum_{j=0}^{\infty} j(\varkappa/k)^j = \frac{1}{\delta_1} \frac{\varkappa/k}{(1-(\varkappa/k)^2)} \leqslant \frac{4n^2}{\delta_0},$$

we find

$$
\begin{aligned}
\left(\fint_{B^{x_0}_{\tau R}} (u+R)^q \, dx \right)^{1/q} &= \left(\fint_{B^{x_0}_{r_{j_0}}} (u+R)^{\beta_{j_0}} \, dx \right)^{1/\beta_{j_0}} \\
&\leqslant \prod_{j=0}^{j_0-1} (C(\sigma-\tau)^{-1} 2^j \gamma_0^{-1})^{p_2 \varkappa/\beta_j} (\rho(4R))^{\varkappa/\beta_j} \left(\fint_{B^{x_0}_{\sigma R}} (u+R)^{\delta_1} \, dx \right)^{1/\delta_1} \\
&\leqslant (C(\sigma-\tau)^{-1} \gamma_0^{-1})^{3p_2 n/\delta_0} (\rho(4R))^{3n/\delta_0} \left(\fint_{B^{x_0}_{\sigma R}} (u+R)^{\delta_0} \, dx \right)^{1/\delta_0},
\end{aligned}
$$

which means the estimate (3.26). At the last step, we applied the Hölder inequality. Lemma 3.5 is proved. □

Lemma 3.6. *For any* $1/4 \leqslant \tau < t \leqslant 7/2$ *and all* $0 < q < n(s-1)/(n-1)$

$$
\left(\fint_{B^{x_0}_{\tau R}} (u+R)^q \, dx \right)^{1/q} \leqslant \exp\left(C\xi \ln (C \gamma_0^{-2p_2 n} \xi) + \fint_{B^{x_0}_{tR}} \ln (u+R) \, dx\right), \tag{3.32}
$$

where $C = C(n, p_1, p_2, M)$, $\gamma_0 = s - 1 - q(n-1)/n$, $\xi = (t-\tau)^{-2np_2} (\rho(4R))^{2n}$.

Proof. Let μ and δ_0 be defined by (3.18). Combining the inequalities (3.18) and (3.26) for $\sigma = (t+\tau)/2$, we have

$$
\begin{aligned}
\left(\fint_{B^{x_0}_{\tau R}} (u+R)^q \, dx \right)^{1/q} &\leqslant \mu 2^{1/\delta_0} (C(\sigma-\tau)^{-1} \gamma_0^{-1})^{3p_2 n/\delta_0} (\rho(4R))^{3n/\delta_0} \\
&\leqslant \mu \cdot \exp\left(C\xi \ln (C \gamma_0^{-2p_2 n} \xi)\right),
\end{aligned}
$$

which means the estimate (3.32). Lemma 3.6 is proved. □

4 Lemma on Oscillation Decrease

For $0 < r < R_0$ we set

$$
M_r = \operatorname*{ess\,sup}_{B^{x_0}_r} u, \quad m_r = \operatorname*{ess\,inf}_{B^{x_0}_r} u. \tag{4.1}
$$

Lemma 4.1. *Assume that* u *is a bounded solution to Equation* (1.1) *in the ball* $B^{x_0}_{4R}$, $\operatorname*{ess\,osc}_{B^{x_0}_{4R}} u \leqslant M$, *and the conditions* (1.2), (1.4), *and* (1.6) *hold. Then*

$$
\begin{aligned}
&M_R - m_R + 2R \leqslant (1-\delta)(M_{4R} - m_{4R} + 8R), \\
&\delta = \exp\left(-C_0 (\rho(4R))^{2n+1}\right), \quad C_0 = C_0(n, p_1, p_2, M).
\end{aligned} \tag{4.2}
$$

Proof. We first assume that

$$|\{u > (M_{4R}+m_{4R})/2\} \cap B^{x_0}_{2R}| > \frac{1}{2}|B^{x_0}_{2R}|. \tag{4.3}$$

In Lemma 3.1, we set $\mu = M_{4R} - m_{4R} + 2R$ and replace u with $\widetilde{u} = u - m_{4R}$. Then for the function

$$\widetilde{v} = \ln{}_+ \frac{\mu}{\widetilde{u}+R} = \ln \frac{M_{4R}-m_{4R}+2R}{u-m_{4R}+R}$$

we have the estimate

$$\operatorname*{ess\,sup}_{B^{x_0}_R} \widetilde{v} \leqslant C(\rho(4R))^{2n}\Bigg(1 + \fint_{B^{x_0}_{2R}} \widetilde{v}\,dx\Bigg), \quad C = C(n,p_1,p_2,M), \tag{4.4}$$

in view of (3.3) with parameters $q = 1$, $\tau = 1$, $t = 2$. On the set of those points where $u > (M_{4R}+m_{4R})/2$, we have $\widetilde{v} < \ln 2$. Thus,

$$|\{\widetilde{v} \leqslant \ln 2\} \cap B^{x_0}_{2R}| \geqslant \frac{1}{2}|B^{x_0}_{2R}|.$$

Using the Poincaré inequality and the estimate (3.9) with $t = 2$, we find

$$\fint_{B^{x_0}_{2R}} (\widetilde{v} - \ln 2)_+\,dx \leqslant \frac{C(n)}{|\{\widetilde{v} \leqslant \ln 2\} \cap B^{x_0}_{2R}|} \int_{B^{x_0}_{2R}} R|\nabla \widetilde{v}|\,dx \leqslant C(n,p_1,p_2,M)\rho(4R).$$

Hence (4.4) implies

$$\operatorname*{ess\,sup}_{B^{x_0}_R} \widetilde{v} \leqslant C_0(\rho(4R))^{2n+1}, \quad C_0 = C_0(n,p_1,p_2,M),$$

which implies

$$\operatorname*{ess\,inf}_{B^{x_0}_R} u \geqslant m_{4R} + (M_{4R} - m_{4R} + 2R)\exp\big(-C_0(\rho(4R))^{2n+1}\big) - R. \tag{4.5}$$

If (4.3) fails, then

$$|\{u \leqslant (M_{4R}+m_{4R})/2\} \cap B^{x_0}_{2R}| \geqslant \frac{1}{2}|B^{x_0}_{2R}|.$$

In this case, we consider the functions

$$\widehat{u} = M_{4R} - u, \quad \widehat{v} = \ln{}_+ \frac{\mu}{\widehat{u}+R} = \ln \frac{M_{4R}-m_{4R}+2R}{M_{4R}-u+R}.$$

Then

$$|\{\widehat{v} \leqslant \ln 2\} \cap B^{x_0}_{2R}| = |\{\widehat{u} \geqslant (M_{4R}+m_{4R})/2\} \cap B^{x_0}_{2R}| \geqslant \frac{1}{2}|B^{x_0}_{2R}|.$$

Arguing as above with $\widetilde{u}$ and $\widetilde{v}$ replaced by $\widehat{u}$ and $\widehat{v}$, we obtain the estimate

$$\operatorname*{ess\,sup}_{B^{x_0}_R} u \leqslant M_{4R} - (M_{4R} - m_{4R} + 2R)\exp\big(-C_0(\rho(4R))^{2n+1}\big) + R. \tag{4.6}$$

Thus, either the estimate (4.5) or the estimate (4.6) holds for the solution u. Therefore,

$$M_R - m_R \leqslant (1-\delta)(M_{4R} - m_{4R}) + R, \quad \delta = \exp\big(-C_0(\rho(4R))^{2n+1}\big). \tag{4.7}$$

Without loss of generality we assume that $\delta \leqslant 1/2$. Adding $2R$ to both sides of the inequality (4.7), we get (4.2) in view of the obvious relation $3R \leqslant (1-\delta)8R$. $\square$

5 Proof of Theorem 1.1

We set $\Lambda(r) = M_r - m_r = \operatorname*{ess\,osc}_{B_r^{x_0}} u$, where M_r and m_r are defined by (4.1). Let $R_m = 4^{-m}R_0$, $m \in \mathbb{N}$. Iterating the estimate (4.2), we find

$$\Lambda(R_m) + 2R_m \leqslant (\Lambda(R_0) + 2R_0) \prod_{j=0}^{m-1} (1 - \delta_j), \quad \delta_j = \exp\left(-C_0(\rho(R_j))^{2n}\right).$$

Since the logarithm is concave, we can write

$$\prod_{j=0}^{m-1} (1 - \delta_j) = \exp\left(\sum_{j=0}^{m-1} \ln(1 - \delta_j)\right) \leqslant \exp\left(-\sum_{j=0}^{m-1} \delta_j\right).$$

Thus,

$$\Lambda(R_m) + 2R_m \leqslant (\Lambda(R_0) + 2R_0) \exp\left(-\sum_{j=0}^{m-1} \exp\left(-C_0(\rho(R_j))^{2n+1}\right)\right). \tag{5.1}$$

If $\theta + \chi$ is not monotonically increasing, then

$$\exp\left(-C_0(\rho(R_j))^{2n+1}\right) \geqslant \int_{R_{j+1}}^{R_j} \exp\left(-C_0(\rho(r))^{2n+1}\right) \frac{dr}{r}.$$

Then from (5.1) we find

$$\Lambda(R_j) \leqslant (\Lambda(R_0) + 2R_0) \exp\left(-\int_{R_j}^{R_0} \exp\left(-C_0(\rho(r))^{2n+1}\right) \frac{dr}{r}\right).$$

Respectively, for arbitrary $R \leqslant R_0/4$

$$\operatorname*{ess\,osc}_{B_R^{x_0}} u \leqslant (\operatorname*{ess\,osc}_{B_{R_0}^{x_0}} u + 2R_0) \exp\left(-\int_{4R}^{R_0} \exp\left(-C_0(\rho(r))^{2n+1}\right) \frac{dr}{r}\right).$$

Theorem 1.1 is proved.

6 Harnack Inequality

In this section, we prove a weak type Harnack inequality for supersolutions (Theorem 1.2) and an analog of the classical Harnack inequality for solutions (Theorem 1.3).

Proof of Theorem 1.2. Combining the estimate of Lemma 3.3 with parameters $\tau = \tau_1$, $t = 7/2$ and the estimate of Lemma 3.6 with parameters $\tau = \tau_2$, $t = 7/2$, we find

$$\operatorname*{ess\,inf}_{B_{\tau_1 R}^{x_0}} (u + R) \geqslant \exp\left(-C(\rho(4R))^{2n+1} + \fint_{B_{7R/2}^{x_0}} \ln(u + R)\, dx\right)$$

$$\geqslant \exp\left(-C(\rho(4R))^{4n+1} \ln\left(C\gamma_0^{-2p_2 n}(\rho(4R))^{2n}\right)\right) \left(\fint_{B_{\tau_2 R}^{x_0}} (u + R)^q\, dx\right)^{1/q},$$

where $\gamma_0 = s - 1 - q(n-1)/n$, which immediately implies the estimate (1.11). Theorem 1.2 is proved. □

Proof of Theorem 1.3. From the estimate (1.11) of Theorem 1.2 and the estimate (2.8) of Lemma 2.3 with parameters $q = n(p_1 - 1)/(2(n-1))$, $t = 2$, $\tau = 1$ we find

$$\operatorname*{ess\,sup}_{B_R^{x_0}} u \leqslant C(\rho(4R))^{2n/q} \left(\fint_{B_{2R}^{x_0}} (u+R)^q \, dx \right)^{1/q}$$

$$\leqslant C(\rho(4R))^{2n/q} \exp\left(C(\rho(4R))^{4n+2}\right) \operatorname*{ess\,inf}_{B_R^{x_0}} (u+R) \leqslant \exp\left(C(\rho(4R))^{4n+2}\right) \operatorname*{ess\,inf}_{B_R^{x_0}} (u+R),$$

where the constants C depend only on n, p_1, p_2, M. Theorem 1.3 is proved. □

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