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ANALYTIC DETECTION IN HOMOTOPY GROUPS OF SMOOTH MANIFOLDS

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Abstract. In this paper, for the mapping of a sphere into a compact orientable manifold $S^n \to M$, $n \geq 1$, we solve the problem of determining whether it represents a nontrivial element in the homotopy group of the manifold $\pi_n(M)$. For this purpose, we consistently use the theory of iterated integrals developed by Chen. It should be noted that the iterated integrals as repeated integration were previously meaningfully used by Lappo-Danilevsky to represent solutions of systems of linear differential equations and by Whitehead for the analytical description of the Hopf invariant for mappings $f : S^{2n-1} \to S^n$, $n \geq 2$.

We give a brief description of Chen's theory representing Whitehead's and Haefliger's formulas for the Hopf invariant and generalized Hopf invariant. Examples of calculating these invariants using the technique of iterated integrals are given. Further, it is shown how one can detect any element of the fundamental group of a Riemann surface using iterated integrals of holomorphic forms. This required to prove that the intersection of the terms of the lower central series of the fundamental group of a Riemann surface is a unit group.

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1. Introduction

In [14], the iterated integrating of differential forms for determining homotopy classes of maps is used for the first time; multidimensional spheres are mapped. Further, in the beginning of 1950s, Chen interprets the Whitehead relation in the framework of his theory of iterated integrals. For the first time, this theory is developed for mappings of circles into manifolds (see [3, 4]). The Whitehead methods provide a possibility to convert Hopf topological constructions into a differential-analytical form; in the beginning of 1930s, he constructs a homotopically nontrivial map (i.e., a map nonhomotopic to a constant map) S^3 in S^2 such that its Hopf invariant is equal to 1. In [3, 4], iterated integrals are provided such that their values on a given mapping of a surface provide a possibility to find whether this map is homotopically nontrivial. It is shown that for almost each mapping of a circle by means of iterated integrals, its homotopic nontriviality can be determined.

During last 10 to 15 years, these methods are applied in the algebraic geometry study in the number theory. In [11], a generalized Dedekind symbol is defined as an iterated integral along paths determined by geodesics in the upper complex half-plane. In Marin papers (see, e.g., [12]), braids and their central lower series are studied. To obtain his results, one can use methods of iterated integrals as well (see [10]). In [6, 7], this method is used to describe fundamental properties of spaces of modules

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of algebraic curves and to resolve the generalized Riemann–Hilbert problem on complex manifolds. In [9], the method of repeated integrals, equivalent to the method of iterated integrals, is used in the analytic theory of differential equations.

In the present paper, we briefly describe the Chen theory. In the framework of this theory, we present the Whitehead and Haefliger relations (see [14] and [5], respectively) for the Hopf invariant and generalized Hopf invariant. We provide examples of the computation of these invariants by means of the technique of iterated integrals (see [3, 5]). Further, we show how to detect each element of the fundamental group of a Riemannian surface using iterated integrals of holomorphic forms (see [15]). This requires to prove that the intersection of terms of the lower central row of the fundamental group of each Riemannian surface is a unit group.

2. Iterated Integrals and Their Properties

Let X be a differential space (see [3, 4]). A *differential form* on such a differential space is a family of forms ω_U on convex sets $\{U\}$ contained in $\mathbb{R}^n$ (for all $n \geq 0$ in the general case) together with maps

$$\varphi_U : U \to X$$

such that the relation

$$\theta^* \omega_V = \omega_U$$

is satisfied and the map $\theta : U \to V$ makes the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi_U} & X \\ \theta \downarrow & & \| \\ V & \xrightarrow{\varphi_V} & X \end{array}$$

commutative provided that the sets U and V are convex.

Further, for the differential space, we take

$$P_{x_0}(M) = \{\gamma : I = [0,\, 1] \to M,\, \gamma(0) = x_0 \in M\},$$

i.e., the space of piecewise smooth paths on a smooth manifold M starting at a point x_0 (see [3, 4]).

Let $\omega_1, \omega_2, \ldots, \omega_r$ be differential forms on the manifold M of powers $p_1, p_2, \ldots, p_r$, respectively. *The iterated integral* $\int \omega_1\omega_2 \cdots \omega_r$ is the differential form on the differential space $P_{x_0}(M)$ defined as follows. Let $U \subset \mathbb{R}^n$ be a convex set contained in $\mathbb{R}^n$. Let $\varphi_U : U \to P_{x_0}(M)$. Define the superstructure $S(\varphi_U) : U \times I \to M$ as follows:

$$S(\varphi_U)(\xi, t) = \varphi_U(\xi)(t) \in M,$$

where $\xi \in U \subset \mathbb{R}^n$ and $t \in I = [0, 1]$. By the form collection $\omega_1, \omega_2, \ldots, \omega_r$, define the differential form $\int \omega_1\omega_2 \cdots \omega_r$ on U. Considering induced forms on the product $U \times I$, we obtain

$$(S(\varphi_U))^*\omega_1,\ (S(\varphi_U))^*\omega_2,\ \ldots,\ (S(\varphi_U))^*\omega_r.$$

Represent each form $(S(\varphi_U))^*\omega_i$ as follows:

$$(S(\varphi_U))^*\omega_i = f_i(\xi, t_i)dt_i \wedge \omega_i' + \omega_i'',$$

where ω_i' and ω_i'' contain no differential dt_i inside. Omit ω_i'' and consider the product

$$(f_1(\xi, t_1)dt_1 \wedge \omega_1') \wedge (f_2(\xi, t_2)dt_2 \wedge \omega_2') \wedge \cdots \wedge (f_r(\xi, t_r)dt_r \wedge \omega_r').$$

Integrate this product over the simplex Δ_r:

$$\left(\int \omega_1\omega_2 \cdots \omega_r \right)_U = \left(\int\limits_{\Delta_r} \prod_{i=1}^{r} f(t_i) dt_1 \wedge \cdots \wedge dt_r \right) \omega_1' \wedge \omega_2' \wedge \cdots \wedge \omega_r',$$

where

$$\Delta_r = \{(t_1, t_2, \ldots, t_r) \in \mathbb{R}^r |\ 0 \leq t_1 \leq t_2 \leq \cdots \leq t_r \leq 1\}.$$

Thus, we define the differential form of degree $p_1 + p_2 + \cdots + p_r - r$ on U. In particular, if all forms $\omega_1, \ldots, \omega_r$ have degree one, then the iterated integral is a function on U, i.e., a form of degree zero.

In [3, 4], it is shown that the coordination condition

$$\theta^* \left(\int \omega_1 \cdots \omega_r \right)_V = \left(\int \omega_1 \cdots \omega_r \right)_U$$

is satisfied provided that U and V are different convex sets and $\theta : U \to V$. Thus, the differential form $\int \omega_1 \cdots \omega_r$ is defined on the differential space $P_{x_0}(M)$ of paths. This form is called the *iterated integral* of the differential forms $\omega_1, \ldots, \omega_r$ on the manifold M.

For each map $f : N^m \to P_{x_0}(M)$ and each manifold of dimension $p_1 + \cdots + p_r - r$, the integration $\left\langle \int \omega_1 \cdots \omega_r,\, N^m \right\rangle$ of the form $\int \omega_1 \cdots \omega_r$ can be defined (see [3]). In this case, the relation

$$\left\langle \int \omega, N^{p-1} \right\rangle = \int\limits_{S(f)} \omega,$$

where $S(f)$ is the superstructure over the map f defined above, is satisfied for the 1-iterated integral of the form of degree p. Note that the right-hand side of the relation is the classical integral of a p-degree differential form over a p-dimensional manifold.

In particular, if $m = 0$, then N^m is a point. Denote it by pt. Then, for differential 1-forms, the value $\left\langle \int \omega_1 \cdots \omega_r,\, pt \right\rangle$ is a number. Usually, it is denoted by

$$\int\limits_{\gamma = f(pt)} \omega_1 \cdots \omega_r.$$

2.1. Properties of iterated integrals of differential 1-forms. Consider properties of iterated integrals of differential 1-forms (Properties 1–6 provided below are contained in [2]).

Property 1.

Theorem. *The product of iterated integrals of orders k and l is equal to the following sum of $(k+l)$-order iterated integrals*:

$$\int\limits_{\gamma} \omega_1 \cdots \omega_k \cdot \int\limits_{\gamma} \omega_{k+1} \cdots \omega_{k+l} = \sum_{\sigma \in S_{k,l}} \int\limits_{\gamma} \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)},$$

where the sum is taken over all shuffles (k, l) in the permutation group S_{n+k}.

Property 2. Consider differential forms $\omega_1, \ldots, \omega_r$ defined on M^n and a path $\gamma : [0, 1] \to M^n$. Introduce the notation $\gamma'(\tau) = \gamma(t(\tau))$, where $t(\tau) : [0, 1] \to [0, 1]$ is a change of the variable.

If the function $t(\tau)$ monotonously increases, then the equivalence class of paths (up to such a change) is called an *oriented curve*. The following invariance property takes place if the change is differentiable and monotonously increasing:

$$\int\limits_{\gamma'} \omega_1 \cdots \omega_r = \int\limits_{\gamma} \omega_1 \cdots \omega_r.$$

Property 3.

Definition. Recall the definition of the *product* $\alpha \cdot \beta : [0, 1] \to M^n$ of two paths. Let two paths such that the end of the first one coincides with the beginning of the second one be given:

$$\alpha : [0, 1] \to M^n \quad \text{is the first path,}$$

$$\beta : [0, 1] \to M^n \quad \text{is the second path,}$$

and

$$\alpha(1) = \beta(0).$$

The product is defined by the relations

$$(\alpha \cdot \beta)(t) = \alpha(2t), \qquad 0 \le t \le \frac{1}{2},$$

and

$$(\alpha \cdot \beta)(t) = \beta(2t-1), \qquad \frac{1}{2} \le t \le 1.$$

Note that if the product of paths is defined, then the multiplication on equivalence classes of paths is an associative operation.

Theorem. *Let α and β be paths such that the product $\gamma = \alpha \cdot \beta : [0,1] \to M^n$ is defined for them. The following relation for the value of the iterated integral on the product of the paths is valid*:

$$\int\limits_{\gamma=\alpha\cdot\beta} \omega_1 \cdots \omega_r = \int\limits_{\alpha} \omega_1 \cdots \omega_r + \sum_{k=1}^{r-1} \int\limits_{\alpha} \omega_1 \cdots \omega_k \int\limits_{\beta} \omega_{k+1} \cdots \omega_r + \int\limits_{\beta} \omega_1 \cdots \omega_r.$$

Property 4.

Definition. The *path* γ^{-1} is defined as follows:

$$\gamma^{-1}(t) = \gamma(1-t), \quad t \in [0,1].$$

Theorem. *The following relation for the value of the r-iterated integral on the inverse path, generalizing Property* 2, *holds*:

$$\int\limits_{\gamma^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int\limits_{\gamma} \omega_r \cdots \omega_1.$$

Property 5.

Definition. Each path of the kind $\alpha = \gamma\gamma^{-1}$ is called a *spike*. A *spike insertion* is the presentation of a path by a product $\alpha = \beta_1\gamma\gamma^{-1}\beta_2$. A *spike deletion* is the elimination of a factor $\gamma\gamma^{-1}$.

Theorem. *No iterated integral depends on insertions or deletions of spikes.*

Property 6.

Definition. Each path such that its beginning and end coincide with each other, $\gamma(0) = \gamma(1)$, is called a *loop*.

Let $\Omega_{x_0}(M)$ denote the space of paths beginning from the point x_0. If loops are considered up to spike insertions or deletions, then it is easy to realize that we deal with the equivalence relation on the space of loops.

The set of equivalence classes is a topological space with respect to the quotient topology of the original compact-open topology on the space of loops. Denote the obtained quotient space by $\overline{\Omega_{x_0}(M)}$.

This space is a topological group with respect to the operation corresponding to the product of loops in the original space of loops.

The connected component of the unit in the group $\overline{\Omega_{x_0}(M)}$ is a normal subgroup in it. The quotient group of the group $\overline{\Omega_{x_0}(M)}$ with respect to this normal subgroup is a group isomorphic to the fundamental group of the manifold M^n.

Property 5 shows that iterated integrals are continuous functions on the group $\overline{\Omega_{x_0}(M)}$. Moreover, they are differentiable and belong to the same smoothness class as the considered space of differential 1-forms and the space of loops.

Property 7. We provide and prove the relation for the value of the 2-iterated integral of closed differential 1-forms on commutators of loops. This relation is generalized for values of 2-iterated integrals of forms of degrees exceeding one on Whitehead products in homotopy groups (see Sec. 4).

Proposition. *The value of the* 2*-iterated integral* $\int \omega_1\omega_2$ *on the commutator* $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ *of two loops* α *and* β *is computed by means of* 1*-iterated integrals as follows*:

$$\int\limits_{[\alpha,\beta]} \omega_1\omega_2 = \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 - \int\limits_{\beta}\omega_1\int\limits_{\alpha}\omega_2 = \begin{vmatrix} \int\limits_{\alpha_1}\omega_1 & \int\limits_{\alpha_1}\omega_2 \\ \int\limits_{\beta_1}\omega_1 & \int\limits_{\beta_1}\omega_2 \end{vmatrix}.$$

Proof. To prove this proposition, place brackets in the commutator of the loops α and β as follows:

$$[\alpha,\beta] = (\alpha\beta)(\alpha^{-1}\beta^{-1}) = (\alpha\beta)((\beta\alpha)^{-1}).$$

Earlier, we note that no iterated integral depends on the bracket locations in products of loops. We prove the required relation directly, using the considered properties of iterated integrals:

$$\int\limits_{[\alpha,\beta]}\omega_1\omega_2 = \int\limits_{(\alpha\beta)((\beta\alpha)^{-1})}\omega_1\omega_2 = \int\limits_{\alpha\beta}\omega_1\omega_2 + \int\limits_{(\beta\alpha)^{-1}}\omega_1\omega_2 + \int\limits_{\alpha\beta}\omega_1\int\limits_{(\beta\alpha)^{-1}}\omega_2$$
$$= \int\limits_{\alpha}\omega_1\omega_2 + \int\limits_{\beta}\omega_1\omega_2 + \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 + \int\limits_{\beta\alpha}\omega_2\omega_1$$
$$-\Bigg(\int\limits_{\alpha}\omega_1 + \int\limits_{\beta}\omega_1\Bigg)\Bigg(\int\limits_{\alpha}\omega_2 + \int\limits_{\beta}\omega_2\Bigg) = \int\limits_{\alpha}\omega_1\omega_2 + \int\limits_{\beta}\omega_1\omega_2 + \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 + \int\limits_{\beta}\omega_2\omega_1$$
$$+\int\limits_{\alpha}\omega_2\omega_1 + \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 - \Bigg(\int\limits_{\alpha}\omega_1 + \int\limits_{\beta}\omega_1\Bigg)\Bigg(\int\limits_{\alpha}\omega_2 + \int\limits_{\beta}\omega_2\Bigg)$$
$$= \Bigg(\int\limits_{\alpha}\omega_1\omega_2 + \int\limits_{\alpha}\omega_2\omega_1\Bigg) + \Bigg(\int\limits_{\beta}\omega_1\omega_2 + \int\limits_{\beta}\omega_2\omega_1\Bigg)$$
$$+\int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 + \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 - \Bigg(\int\limits_{\alpha}\omega_1 + \int\limits_{\beta}\omega_1\Bigg)\Bigg(\int\limits_{\alpha}\omega_2 + \int\limits_{\beta}\omega_2\Bigg)$$
$$= \int\limits_{\alpha}\omega_1\int\limits_{\alpha}\omega_2 + \int\limits_{\beta}\omega_1\int\limits_{\beta}\omega_2 + \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 + \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2$$
$$-\Bigg(\int\limits_{\alpha}\omega_1 + \int\limits_{\beta}\omega_1\Bigg)\Bigg(\int\limits_{\alpha}\omega_2 + \int\limits_{\beta}\omega_2\Bigg) = \int\limits_{\alpha}\omega_1\int\limits_{\beta}\omega_2 - \int\limits_{\beta}\omega_1\int\limits_{\alpha}\omega_2.$$

This completes the proof of the property. ∎

Property 8. The value of the 2-iterated integral $\int \omega_1\omega_2$ for the product

$$\gamma = \prod_{i=1}^{m}[\alpha_i,\beta_i]$$

of several loops is equal to the sum of values of the 2-iterated integrals on commutator factors, i.e.,

$$\int\limits_{\prod_{i=1}^{m}[\alpha_i,\beta_i]} \omega_1\omega_2 = \sum_{i=1}^{m} \int\limits_{[\alpha_i,\beta_i]} \omega_1\omega_2.$$

Using the previous property, we can obtain that the 2-iterated integral is equal to a sum of commutator-type expressions:

$$\int\limits_{\prod_{i=1}^{m}[\alpha_i,\beta_i]} \omega_1\omega_2 = \sum_{i=1}^{m} \biggl(\int\limits_{\alpha} \omega_1 \int\limits_{\beta} \omega_2 - \int\limits_{\beta} \omega_1 \int\limits_{\alpha} \omega_2 \biggr).$$

Property 9. A very important property of iterated integrals is related to the differentiation. Iterated integrals and their differentials obey the Stokes relation playing an important role in the proof of the homotopic invariance of iterated integrals. Iterated integrals of 1-forms are functions on the space $\Omega_{x_0}(M^n)$ of loops, where the initial point x_0 is given on the manifold M^n. In other words, they are differential 0-forms on $\Omega_{x_0}(M^n)$. Thus, the following assertion is valid.

Proposition. *Iterated integrals of* 1*-forms on the space* $\Omega_{x_0}M = P_{x_0}^{x_0}(M)$ *of loops satisfy the differentiating relation*

$$d\int \omega_1\cdots\omega_q = -\sum_{i=1}^{q} \int \omega_1\cdots\omega_{i-1} d\omega_i \omega_{i+1}\ldots\omega_q$$
$$-\sum_{i=1}^{q-1}(-1)^i \int \omega_1\cdots\omega_{i-1}(\omega_i\wedge\omega_{i+1})\omega_{i+2}\cdots\omega_q \qquad (*)$$

(see [3, 7]) *and the Stokes relation*

$$\int\limits_{C} \biggl(d\int \omega_1\cdots\omega_q \biggr) = \int\limits_{\partial C} \biggl(\int \omega_1\cdots\omega_q \biggr) = \int\limits_{C(1)} \omega_1\cdots\omega_q - \int\limits_{C(0)} \omega_1\cdots\omega_q,$$

where the path $C : [0;1] \to P_{x_0}^{x_1}(M)$ *is a singular simplex in the space* $P_{x_0}^{x_1}(M)$*. This simplex defines a homotopy between the paths* γ_1 *and* γ_2*. In the space* $P_{x_0}^{x_1}(X_n)$ *of paths, the relations* $C(0) = \gamma_1$ *and* $C(1) = \gamma_2$ *are valid.*

Definition. If the differential of a linear combination of iterated integrals is equal to zero, then this combination is called a *homotopic period*.

Let $B_s(M)$ denote the vector space of iterated integrals over M such that no their length exceeds s. Let η_x denote the constant path at the point x on M (i.e., $\eta_x(t) = x$ for all t). If $r \geq 1$, then

$$\biggl\langle \int \omega_1\ldots\omega_r, \eta_x \biggr\rangle = 0$$

for all $x \in M$. Thus, the value on the constant path η_x determines the linear functional

$$\varepsilon : B_s(M) \to \mathbb{R},$$
$$I \to \langle I, \eta_x \rangle,$$

independent of x. If

$$I = \lambda + \sum a_i \int \omega_i + \sum a_{ij} \int \omega_i\omega_j + \cdots,$$

then $\varepsilon(I) = \lambda$. Denote the kernel of the map ε by $\overline{B}_\varepsilon(M)$. We have iterated integrals of length $\leq s$ with the zero constant. The natural inclusion $i : \mathbb{R} \to B_s(M)$ such that $\varepsilon \circ i = id$ takes place. Thus, we have the natural expansion of the direct sum:

$$B_s(M) \cong \mathbb{R} \oplus \overline{B}_s(M).$$

For loops $\alpha, \beta \in P(M)$ at the point x, one can consider the commutator

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}.$$

Frequently, the constant loop η_x on x is denoted by 1.

For the paths $\alpha\eta_x$ and $\eta_x\alpha$ and the iterated integral I, the following relation holds:

$$I(\alpha) = I(\alpha\eta_x) = I(\eta_x\alpha).$$

Recall that the classical line integral satisfies the following conditions:

$$\left\langle \int \omega, [\alpha, \beta] \right\rangle = 0,$$

where α and β are loops at the point x. The *detection* is the finding of an iterated integral such that it does not vanish on the loop presenting a nontrivial element of the fundamental group. This element is not trivial, i.e., homotopic to a constant loop. The integral of each commutator is equal to zero. Therefore, one cannot use 1-iterated integrals to detect nontrivial commutators in fundamental groups. If I is an iterated integral of order r and $r < s$, then

$$\langle I, [\alpha_1[\alpha_2[\ldots[\alpha_s]\ldots]]]\rangle = 0.$$

No iterated integral of order $r < s$ detects commutators if their orders are not lower than s.

2.2. Properties of iterated integrals of differential forms of arbitrary degrees. Suppose that $\omega_1, \omega_2, \ldots, \omega_r$ are differential forms of degrees $\deg \omega_i = p_i$ on a compact closed manifold M^n. Then the iterated integral $\int \omega_1 \cdots \omega_r$ is a differential form of degree $p_1 + p_2 + \cdots + p_r - r$ on the space $P(M)$ of paths. If the initial point x_0 and the final point x_1 are fixed, then the space of paths is denoted by $P_{x_0}^{x_1}(M)$. If $x_0 = x_1$, then we have the space $\Omega_{x_0}(M)$ of loops, which is a subset of the space of paths: $\Omega_{x_0}(M) \subset P(M)$.

Proposition. *Iterated integrals of differential forms of arbitrary degrees satisfy the following differentiation relation generalizing the differentiation relation for iterated integrals of differential* 1*-forms*:

$$d \int \omega_1\omega_2 \cdots \omega_r = \sum_{i=1}^{r} (-1)^i \int J\omega_1 \cdots J\omega_{i-1} d\omega_i \omega_{i+1} \cdots \omega_r - \sum_{i=1}^{r-1} J\omega_1 \cdots J\omega_{i-1}(J\omega_i \wedge \omega_{i+1})\omega_{i+2} \cdots \omega_r,$$

where

$$J\omega_i = (-1)^{\deg \omega_i} \cdot \omega_i$$

(see [3, 7]).

In this general case, an analog of the Stokes relation (see [3, 7]) *holds for iterated integrals on the space $\Omega_{x_0}(M)$ of loops. The Stokes relation is as follows*:

$$< d \int \omega_1\omega_2 \cdots \omega_r, C > = < \int \omega_1\omega_2 \cdots \omega_r, \partial C >,$$

where $C = N^m$ is a manifold with boundary and $m = p_1 + p_2 + \cdots + p_r - r + 1$.

3. Whitehead Relations for Hopf Invariants

In this section, the Whitehead relation for the Hopf invariant is presented in terms of iterated integrals defined in the previous section. To do that, we pass from the space $P_{x_0}(M)$ of paths to the space

$$\Omega_{x_0}(M) = P_{x_0}^{x_0}(M) \subset P_{x_0}(M)$$

of loops. Let M be a smooth oriented n-dimensional manifold,

$$f : S^{2n-1} \to M^n$$

be a smooth map, and ω^n is the highest-degree form determining the orientation on the manifold. Then the Whitehead relation for the Hopf invariant of the map f has the following form (see [14]):

$$h(f) = \int\limits_{S^{2n-1}} f^*\omega^n \wedge \psi,$$

where $d\psi = \omega^n$.

Theorem. *In terms of iterated integrals, the right-hand side of the Whitehead relation for the Hopf invariant can be represented as*

$$\left\langle \int \omega^n\omega^n, S^{2n-2} \right\rangle = \int\limits_{S^{2n-1}} f^*\omega^n \wedge \psi,$$

where the left-hand side is defined by means of the map $g : S^{2n-2} \to \Omega_{x_0}M^n$. Its superstructure $S(g)$ coincides with f, i.e., $S(g) = f$. Hence, the Hopf invariant is determined by the value of the iterated integral

$$h(f) = \left\langle \int \omega^n\omega^n, S^{2n-2} \right\rangle.$$

The map g is a delooping of the map f, i.e., for $x \in S^{2n-2}$, the value of the loop $g(x)(t)$ at the point t is determined by the relation $g(x)(t) = f(x,t)$. Thus, $g(x) \in \Omega_{x_0}M^n$.

4. Whitehead Product and Haefliger Theorem

Define the Whitehead product. Let D^n be a collection of vectors in $\mathbb{R}^n$ such that the length of each one does not exceed one. The disk boundary is a sphere of dimension $n-1 : \partial D^n = S^{n-1}$. Let X be a topological space with a marked point x_0.

Definition. Take two continuous maps

$$f_i : (D^{p_i}, \partial D^{p_i}) \to (X, x_0), \quad i = 1, 2.$$

The *Whitehead product* of the maps f_1 and f_2 is the map $[f_1, f_2]$ of the boundary of the product $\partial(D^{p1} \times D^{p_2})$ of disks into the topological space X as follows:

$$[f_1, f_2](x_1, x_2) = \begin{cases} [f_1, f_2](x_1, x_2) = f_1(x_1) & \text{as} \quad x_2 \in \partial D^{p_2}, \\ [f_1, f_2](x_1, x_2) = f_2(x_2) & \text{as} \quad x_1 \in \partial D^{p_1}. \end{cases}$$

The maps f_i represent elements φ_i of the homotopy groups $\pi_{p_i}(M, x_0)$. Note that the homotopy class

$$[\varphi_1, \varphi_2] \in \pi_{p_1+p_2-1}(M, x_0)[f_1, f_2]$$

depends only on φ_1 and φ_2 and is called the *Whitehead product* of φ_1 and φ_2. If the spheres are one-dimensional, then the maps f_1 and f_2 are elements φ_1 and φ_2 in the fundamental group of the manifold. In this case, the Whitehead product $[\varphi_1, \varphi_2]$ treated as a homotopic class of the manifold coincides with the commutator of these elements in the fundamental group:

$$[\varphi_1, \varphi_2] = \varphi_1\varphi_2\varphi_1^{-1}\varphi_2^{-1}.$$

Differential forms can be used to detect the Whitehead product. Let ω_1 and ω_2 be forms on a differentiable manifold M, their degrees be equal to p_1 and p_2, respectively, and each one be greater than one. Assume that $d\omega_1 = d\omega_2 = 0$ and $\omega_1 \wedge \omega_2 = 0$.

If $f : S^{p_1+p_2-1} \to M$ is a smooth map and a form $f^*\omega_1$ has degree $p_1 < p_1 + p_2 - 1$, then there exists a form α_1 such that $d\alpha_1 = f^*\omega_1$.

Define the generalized Hopf invariant for the sphere $S^{p_1+p_2-1}$ into the smooth manifold M:

$$h_f(\omega_1, \omega_2) = \int\limits_{S^{p_1+p_2-1}} \alpha_1 \wedge f^*\omega_2 = \left\langle \int \omega_1\omega_2, S^{p_1+p_2-2} \right\rangle.$$

The last expression represents the generalized Hopf invariant $h_f(\omega_1, \omega_2)$ in terms of 2-iterated Chen integrals. Note that the generalized Hopf invariant does not depend on the choice of the differential form α_1 converting ω_1 into an exact form; it depends only on the homotopy class of the map f. This number determines the homomorphism of $\pi_{p_1+p_2-1}(M)$ into $\mathbb{R}$.

If $h_f(\omega_1, \omega_2) \neq 0$, then the map f represents a nonzero element of the homotopy group $\pi_{p_1+p_2-1}(M)$.

Below, the Haefliger theorem about the value of the generalized Hopf invariant on the Whitehead product of two spheres (see [5]) is formulated in terms of iterated integrals.

Let $f = [f_1, f_2]$ be the Whitehead product of two maps of the spheres S^{p_i}, $p_i > 1$, $i = 1, 2$.

Theorem. *Let ω_1 and ω_2 be closed differential forms on a smooth manifold M and $\omega_1 \wedge \omega_2 = 0$. Let $f_i : (S^{p_i}, y_i) \to (M, x_0)$ be smooth maps of spheres of dimensions p_1 and p_2 representing elements of homotopy groups $\pi_{p_i}(M, x_0)$, $i = 1, 2$.*

Then, for the Whitehead product $f = [f_1, f_2]$, the generalized Hopf invariant is computed as follows:

$$h_f(\omega_1, \omega_2) = \omega_1(f_1)\omega_2(f_2) + (-1)^{p_1p_2}\omega_1(f_2)\omega_2(f_1),$$

where

$$\omega_i(f_j) = \left\langle \int \omega_i, S^{p_j-1} \right\rangle.$$

Here, $\int \omega_i$ is the 1-iterated integral of the form ω_i. It is integrated for the map $g_j : S^{p_j-1} \to \Omega_{x_0}(M)$ (where g_j is the delooping of the map f_j).

For the case where $p_1 = p_2 = 1$, this relation is valid as well. Since the Whitehead product coincides with the commutator of loops for the case where $p_1 = p_2 = 1$, it follows that, in this case, the relation from the theorem coincides with the value of the 2-iterated integral on the commutator of loops from Property 7 of iterated integrals (see Sec. 2).

5. Detection of Nontrivial Elements in Homotopy Groups of Smooth Manifolds

5.1. Examples. The following two examples for mappings of multidimensional spheres can be found in [5, 13, 14].

Example 1. Let $M = S^n$ and ω be an n-form on S^n such that $\int\limits_{S^n} \omega = 1$. Let f be a smooth 1-degree map of the disk D^n onto the sphere S^n such that the disk boundary ∂D^n is mapped into a point. Then

$$h_{[f,f]}(\omega, \omega) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 2 & \text{if } n \text{ is even}. \end{cases}$$

This result follows from the Haefliger theorem because

$$\int\limits_{D^n} f^*\omega = \int\limits_{S^n} \omega = 1.$$

Example 2. Let M be the complement to the points $(1,0,0)$ and $(-1,0,0)$ in $\mathbb{R}^3$. Then M is retracted to the union of two spheres S_+ and S_- of radius 1 centered at $(1,0,0)$ and $(-1,0,0)$, respectively.

Let ω'_+ be a 2-form on S_+, its support be contained in a small neighborhood of the point $(2,0,0)$, and

$$\int\limits_{S_+} \omega'_+ = 1.$$

Let ω_+ can be represented as an induced 2-form $\rho_+^*\omega'_+$, where ρ_+^* is the radial retraction of M on S_+.

Hence, ω_+ is a closed 2-form such that its support is contained in the half-plane $\{x_1 > 0\}$. Using the symmetry with respect to the plane $\{x_1 = 0\}$, argue in the same way for ω_-. Since the supports of the forms are disjoint, it follows that $\omega_+ \wedge \omega_- = 0$.

Let i_+ and i_- be two natural embeddings of S^2 into M with images S_+ and S_-, respectively. Let us prove that their Whitehead product $[i_+, i_-]$ is a nontrivial element in the homotopy group $\pi_3(M) = \pi(S^2 \vee S^2)$.

Since

$$\int\limits_{S_+} \omega_+ = \int\limits_{S_-} \omega_- = 1 \quad \text{and} \quad \int\limits_{S_+} \omega_- = 0,$$

it follows that

$$h_{[i_+,i_-]}(\omega_+, \omega_-) = 1.$$

Similar examples can be constructed for the mapping of high-dimensional spheres S^{p_1} and S^{p_2}, $p_i > 1$, into unions (see [5]).

5.2. Detection of nontrivial loops on Riemannian surfaces. In this section, the detection problem is considered for homotopically nontrivial loops on Riemannian surfaces (see [15]). We use only Chen iterated integrals of holomorphic or meromorphic 1-forms on Riemannian surfaces that are homotopic periods.

The homotopic nontriviality of loops on Riemannian surfaces is determined by nonzero values of homotopic periods on these loops. Recall that homotopic periods are iterated integrals of holomorphic or meromorphic forms, depending only on the loop homotopy class. Thus, the homotopic period is well defined on the element of the fundamental group corresponding to the loop. Homotopic periods determine functions on the fundamental group of a Riemannian surface. Consider Riemannian surfaces such that the fundamental group of each one is a group with a finite amount of generatrices and a finite amount of relations (finitely represented groups). We describe several properties of the fundamental group of such Riemannian surfaces.

Proposition. *The intersection of terms of the lower central row of a finitely represented fundamental group of a Riemannian surface is the unit group.*
Proof.

1. If the Riemannian surface C is not compact, then its fundamental group is a free group $\pi_1(C, x_0) = F_n,\ n > 0$, with a finite amount of generatrices:

$$\bigcap_{k=1}^{\infty} \Gamma_k F_n = \{e\}.$$

2. If the Riemannian surface is compact, then its fundamental group is either the trivial group $G = \{e\}$ or a group with a finite amount of generatrices and one relation. Delete one point from this Riemannian surface C, i.e., assign $X = C\backslash\{x\}$. The fundamental group of the obtained surface X is a free F_{2g}, where g is the genus of C. The embedding $i : X \to C$ induces the epimorphism

$$\pi_1(X) \to \pi_1(C) \to 1$$

of fundamental groups and the epimorphism

$$\Gamma_k\pi_1(X) \to \Gamma_k\pi_1(C) \to 1, \quad k = 1, 2, \dots$$

of their lower central rows. Since

$$\bigcap_{k=1}^{\infty} \Gamma_k\pi_1(X) = \{e\},$$

it follows that

$$\bigcap_{k=1}^{\infty} \Gamma_k\pi_1(C) = \{e\}.$$

Thus, if the fundamental group of the Riemannian surface $\pi_1(C)$ is finitely represented, then the intersection of terms of the lower central row is trivial. □

Thus, if the Riemannian surface is compact, then the intersection of terms of the lower central row is trivial for the group $\pi_1(C)$, i.e.,

$$\bigcap_{k=1}^{\infty} \Gamma_k\pi_1(C) = \{e\}.$$

Hence, for each nontrivial element g of the fundamental group $\pi_1(C)$ there exists the greatest positive integer r such that g has a nonzero image in the quotient group $\Gamma_r(C)/\Gamma_{r+1}(C)$.

Select a system of canonical loops $a_1, \dots, a_g, b_1, \dots, b_g$ on C. Cut C along these loops and convert this Riemannian surface into a $2g$-gon. It is possible to select a canonical system of loops such that they start at the point $x_0 \in C$ and represent generatrices in the fundamental group $\pi_1(C, x_0)$. Denote them by the same symbols. Each iterated integral $\int \omega_1 \cdots \omega_r$ of length $r \geq 1$ of holomorphic 1-forms on the Riemannian surface is a homotopic period. Indeed, let loops $\gamma_1, \gamma_2 \in \Omega_{x_0}(C)$ be homotopic, i.e., let there exist a map

$$h : [0, 1] \to \Omega_{x_0}(C), \quad h(0) = \gamma_1, \quad h(1) = \gamma_2.$$

Then properties of iterated integrals imply the relations

$$\int_{\gamma_2} \omega_1 \cdots \omega_r - \int_{\gamma_1} \omega_1 \cdots \omega_r = \int_{\partial h} \omega_1 \cdots \omega_r = \int_h d \int \omega_1 \cdots \omega_r$$
$$= -\int_h \sum_{i=1}^{r} \int \omega_1 \cdots d\omega_i \cdots \omega_r - \sum_{i=1}^{r-1} \int \omega_1 \cdots (\omega_i \wedge \omega_{i+1}) \cdots \omega_r = 0.$$

The last relation follows from the fact that $d\omega_i = 0$, $i = 1, \dots, r$, and $\omega_i \wedge \omega_{i+1} = 0$, $i = 1, \dots, r-1$, for holomorphic forms on a Riemannian surface. Hence,

$$\int_{\gamma_1} \omega_1 \cdots \omega_r = \int_{\gamma_2} \omega_1 \cdots \omega_r,$$

i.e., the iterated integral $\int \omega_1 \cdots \omega_r$ of holomorphic forms is a homotopic period.

It is well known that there exists a holomorphic 1-form ω_i on C such that

$$\operatorname{Re} \int_{a_i} \omega_i = 1, \quad \operatorname{Re} \int_{a_j} \omega_i = 0, \quad j \neq i, j = 1, \dots, r,$$

and

$$\operatorname{Re} \int_{b_j} \omega_i = 0, \quad j = 1, \dots, r$$

for b-periods. In the same way, there exists a holomorphic form ω_j such that

$$\mathrm{Re} \int\limits_{b_j} \omega_j = 1$$

and other periods of this form are equal to zero. Such 1-form detects the homotopic nontriviality of the generatrices $a_1, \dots, a_g, b_1, \dots, b_g$.

The Chen theorem establishes an isomorphism between the space of homotopic periods defined by iterated integrals of real-valued differential forms and the space of homomorphisms of the group algebra of the fundamental group $\mathbb{R}[\pi_1(M)]$ into real numbers (see [3, 4]):

$$H^0(B_r, x_0) = \mathrm{Hom}\left(\mathbb{R}[\pi_1(M, x_0)]/J^{r+1}, \mathbb{R}\right),$$

where $H^0(B_r, x_0)$ are homotopic periods from r-iterated integrals. Applying the Chen theorem to Riemannian surfaces and using the proposition formulated above, one can prove the homotopic nontriviality of each element $\pi_1(M, x_0)$ by means of iterated integrals of holomorphic or meromorphic forms because, in our case, the ideal J for Riemannian surfaces is zero.

Now, we can formulate the following theorem.

Theorem. *Let M be a compact Riemannian surface of genus g with k deleted points. Then each element of the fundamental group of this surface is detected by iterated integrals of holomorphic or meromorphic forms on this Riemannian surface.*

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