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NUMERICAL-ANALYTIC DETERMINATION OF THE STATIC THERMOELASTIC STATE OF PLANE MULTILAYER THERMOSENSITIVE STRUCTURES

R. M. Kushnir, I. M. Makhorkin, and M. I. Makhorkin

UDC 539.3

We propose a numerical-analytic method for the determination of one-dimensional static thermoelastic states of plane multilayer structures with arbitrary types of temperature dependences of the physical and mechanical characteristics of the materials of their components. The proposed method is based on the use of the theory of generalized functions, approximation of temperature dependences of the physical and mechanical characteristics of materials by piecewise-constant functions, and introduction of an analog of the Kirchhoff function. The method is verified by analyzing the static thermoelastic states of two- and three-layer plates.

Keywords: multilayer plate, temperature-dependent characteristics, numerical-analytic solution, thermoelastic state.

Introduction

As a rule, structural elements of contemporary commercial equipment operate under the conditions of intense thermal and force loads and can be most often regarded as inhomogeneous functionally graded plane bodies or their fragments. It is clear that the adequacy of theoretical investigations of the thermoelastic behavior of these structural elements is strongly affected by neglecting inhomogeneities, including, in particular, their functionally graded nature (the presence of layers in elements) and the temperature dependences (thermal sensitivity) of physicomechanical characteristics (PMC) of materials. The analysis of the state of investigations of the thermomechanical behavior of bodies of this kind shows that taking into account the entire collection of important factors (including the geometry of body, inhomogeneity of its structure, thermal sensitivity of the PMC of materials, and thermal and force actions) leads to mathematical problems whose solution, even with the help of numerical methods, is connected with significant difficulties and requires specially developed algorithms or substantial adaptation of the existing procedures [2, 4, 5, 12, 13, 15]. However, in the engineering practice, it is preferable to use relatively simple numerical-analytic relations and algorithms, which enable one to study the thermomechanical behavior of the object with predicted reliability.

The methods aimed at finding one-dimensional stationary temperature fields and stresses caused by these fields in layered bodies for linear, quadratic, and cubic temperature dependences of the thermal conductivities of the materials of layers were proposed in [3, 6, 10, 13]. Depending on the conditions of heat exchange, for any number of layers and any given form of temperature dependences of their thermal conductivities, these methods reduce the analyzed problem by means of the Kirchhoff transformation (change) [4] to the solution either of a single equation or of a system of two nonlinear algebraic equations. It is recommended to find the solutions of these equations by using numerical methods (of successive approximations, perturbations, etc.). In this case, the

Pidstryhach Institute for Applied Problems in Mechanics and Mathematics, National Academy of Sciences of Ukraine, Lviv, Ukraine; e-mail: mahorkin@ukr.net.

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choice of the initial approximation and the analysis of the problem of existence and uniqueness of the solution require additional investigations. An integral representation of the solution of a one-dimensional stationary problem of heat conduction that does not impose restrictions on the character of temperature dependences of the thermal conductivities was proposed in the appendix of [10] for a special case of heating of multilayer bodies of simple geometric shapes whose inner surface is subjected to the action of a heat flux $q_0 = \text{const}$ and the outer surface is kept at a temperature $t_0 = \text{const}$.

In the present paper, by using an example of plane multilayer plate, we illustrate the application of the procedure of numerical-analytic solution of problems of this kind. The procedure is based on the use of the theory of generalized functions, approximation of the temperature dependences of the PMC of materials by piecewise-constant functions, and introduction of an analog of the Kirchhoff function. The application of this procedure enables one to investigate the thermal state of layered bodies (and the thermal stressed state caused by this state) for any type of temperature dependences of the PMC of materials of their components and any conditions of heat exchange without clarifying whether the solution of the nonlinear problem of heat conduction exists and is unique.

1. Statement of the Problem

Consider a multilayer plate (Fig. 1) referred to a Cartesian coordinate system x, y, z. On the conjugation surfaces $z = z_i = \text{const}$, $i = 1, \ldots, n-1$, of the layers, we impose the conditions of perfect thermomechanical contact. It is assumed that the thermal state caused by a thermal load acting upon the boundary surfaces $z = z_i$, $i = 0, n$, free of force loads is characterized by a one-dimensional stationary temperature field $t(z)$. We also suppose that, in the absence of bulk forces, the end surfaces of the plate are thermally insulated and loaded by a system of forces whose resultant vector and moment are equal to zero.

2. Determination of the Thermal State

According to the theory of nonlinear heat conduction of inhomogeneous bodies [4, 7, 9], the mathematical model of thermal behavior of these structures has the form of a nonlinear boundary-value problem of stationary heat conduction aimed at finding the temperature field $t(z)$ from the heat-conduction equation

$$\frac{d}{dz}\left(\lambda_t(t,z)\frac{dt}{dz}\right) = -w_t(z), \tag{1}$$

with conditions of perfect thermal contact on the interfaces of materials of the layers

$$t\big|_{z_i-0} = t\big|_{z_i+0},$$
$$\left(\lambda_t(t,z)\frac{dt}{dz}\right)\bigg|_{z_i-0} = \left(\lambda_t(t,z)\frac{dt}{dz}\right)\bigg|_{z_i+0}, \tag{2}$$

under certain boundary conditions simulating the external thermal load:

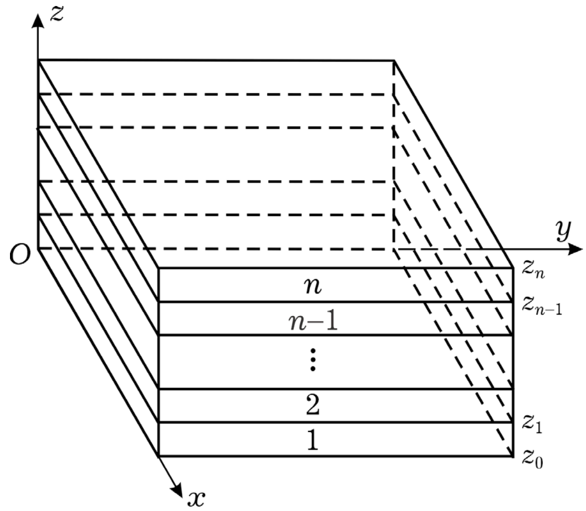

Fig. 1

$$\left(a_i(t)\frac{dt}{dz}+b_i(t)\right)\bigg|_{z_i} = 0, \quad i = 0, n. \tag{3}$$

Here, the functions $a_i(t)$ and $b_i(t)$ are chosen according to the procedure of heating and $w_t(z)$ is the intensity of internal heat sources (sinks). The temperature-and-coordinate dependence of the thermal conductivity $\lambda_t(t,z)$ is chosen in the form

$$\lambda_t(t,z) = \lambda_t^{(1)}(t) + \sum_{i=1}^{n-1}\left(\lambda_t^{(i+1)}(t)-\lambda_t^{(i)}(t)\right)S_+(z-z_i), \tag{4}$$

where $\lambda_t^{(i)}(t)$ is the temperature-dependent thermal conductivity of the material of the ith layer.

3. Procedure of Numerical-Analytic Determination of the Thermal State

The numerical-analytic solution of the boundary-value problem (1)–(3) is reduced to the approximation of the temperature dependences of thermal conductivities of the materials of layers $\lambda_t^{(i)}(t)$ by piecewise-constant functions of temperature of the form [8]

$$\lambda_t^{(i)}(t) \approx \Lambda^{(i)}(t) = \Lambda_1^{(i)} + \sum_{j=1}^{m}\left(\Lambda_{j+1}^{(i)}-\Lambda_j^{(i)}\right)S_+(t-t_j),$$
$$t_{\text{p}} = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = t_{\text{k}}, \tag{5}$$

with introduction of the following analog of the Kirchhoff function [14]:

$$\vartheta(t,z) = \int_0^t \sum_{i=1}^n \Lambda^{(i)}(\xi) N_i(z)\, d\xi, \tag{6}$$

where $N_i(z) = S_+(z - z_{i-1}) - S_+(z - z_i)$, $i = 0, \ldots, n$; $S_+(z - z_0) = 1$; $[t_\mathrm{p}, t_\mathrm{k}]$ is the joint interval of evaluation of $\lambda_t^{(i)}(t)$, $i = 1, \ldots, n$; t_j are the nodes of approximation; $\Lambda_j^{(i)}$ are the approximating coefficients whose numerical values correspond (with required accuracy) to the values of $\lambda_t^{(i)}(t)$ in the temperature ranges $t_{j-1} < t < t_j$; z_0, z_n are the coordinates of the boundary surfaces; z_i, $i = 1, \ldots, n-1$, are the coordinates of the conjugation (contact) surfaces of the ith and $(i+1)$th layers, and

$$S_+(\zeta - \zeta_i) = \begin{cases} 1, & \zeta > \zeta_i, \\ 0, & \zeta \le \zeta_i. \end{cases}$$

As a result, the problem of determination of the stationary thermal state of a multilayer plate is reduced to finding the temperature field $t(z)$ from the relation

$$\vartheta(z) = t \sum_{i=1}^n \Lambda^{(i)}(t) N_i(z) - \sum_{i=1}^n \left[\sum_{j=1}^m \left(\Lambda_{j+1}^{(i)} - \Lambda_j^{(i)} \right) t_j\, S_+(t - t_j) \right] N_i(z) \tag{7}$$

according to the solution of the partially degenerate differential equation

$$\frac{d}{dz}\left[\frac{d\vartheta}{dz} - \sum_{i=1}^{n-1} \left(K_i\, \vartheta\big|_{z_i} + Q_i \right) \delta_+(z - z_i) \right] = -w_t(z), \tag{8}$$

obtained from the heat-conduction equation (1) with the help of generalized functions by means of statement of the corresponding generalized problem of conjugation [1, 9, 11, 12]. Here, d/dz is a generalized derivative,

$$K_\ell = \left. \left(\frac{\Lambda^{(\ell+1)}(\vartheta)}{\Lambda^{(\ell)}(\vartheta)} - 1 \right) \right|_{z_\ell}, \quad \text{and} \quad Q_\ell = \left. \left(\frac{\Lambda^{(\ell+1)}(\vartheta)}{\Lambda^{(\ell)}(\vartheta)} F_\ell(\vartheta) - F_{\ell+1}(\vartheta) \right) \right|_{z_\ell}.$$

According to relation (6), there exists a one-to-one correspondence between ϑ and t for $z_k < \alpha < z_{k+1}$, $k = 0, \ldots, n$. Therefore,

$$S_+(t - t_i) = S_+(\vartheta - \vartheta_i), \tag{9}$$

and, hence, $\Lambda^{(i)}(t) = \Lambda^{(i)}(\vartheta)$. As a result, relation (7) takes the form

$$\vartheta(z) = t \sum_{i=1}^n \Lambda^{(i)}(\vartheta) N_i(z) - \sum_{i=1}^n \left[\sum_{j=1}^m \left(\Lambda_{j+1}^{(i)} - \Lambda_j^{(i)} \right) t_j S_+(\vartheta - \vartheta_j) \right] N_i(z).$$

This yields

$$t = \frac{\vartheta + \sum_{i=1}^{n} F_i(\vartheta) N_i(z)}{\sum_{i=1}^{n} \Lambda^{(i)}(\vartheta) N_i(z)}, \tag{10}$$

where

$$F_i(\vartheta) = \sum_{j=1}^{m} \left(\Lambda_{j+1}^{(i)} - \Lambda_j^{(i)}\right) t_j \, S_+(\vartheta - \vartheta_j),$$

$$\vartheta_\ell = \vartheta_\ell(z) = \sum_{i=1}^{n} \left[t_\ell \Lambda^{(i)}(t_\ell) - \sum_{j=1}^{m} \left(\Lambda_{j+1}^{(i)} - \Lambda_j^{(i)}\right) t_j \, S_+(t_\ell - t_j) \right] N_i(z).$$

As a result, the solution $\vartheta \equiv \vartheta(z)$ of Eq. (8) is represented in the form

$$\vartheta = C_1 z + C_2 + \sum_{i=1}^{n-1} \left(K_i \, \vartheta|_{z_i} + Q_i \right) S_+(z - z_i) - W_t(z), \tag{11}$$

where

$$W_t(z) = \int_0^z \int_0^\eta w_t(\zeta) \, d\zeta \, d\eta.$$

We now express $\vartheta|_{z_i}$ in the form

$$\vartheta|_{z_i} = \tilde{K}_1^{(i)} C_1 + \tilde{K}_2^{(i)} C_2 + \tilde{K}_3^{(i)}.$$

As a result, from (11), we obtain the following recurrence relations for $\tilde{K}_j^{(i)}$, $j = 1, 2, 3$:

$$\tilde{K}_1^{(i)} = z_i + \sum_{j=1}^{i-1} K_j \tilde{K}_1^{(j)},$$

$$\tilde{K}_2^{(i)} = 1 + \sum_{j=1}^{i-1} K_j \tilde{K}_2^{(j)}, \tag{12}$$

$$\tilde{K}_3^{(i)} = \sum_{j=1}^{i-1} \left(K_j \tilde{K}_3^{(j)} + Q_j \right) - W(z_i).$$

In view of (12), expression (11) for ϑ takes the form

$$\vartheta = C_1\left[z + \sum_{j=1}^{n-1} K_j \tilde{K}_1^{(j)} S_+(z-z_j)\right]$$

$$+ C_2\left[1 + \sum_{j=1}^{n-1} K_j \tilde{K}_2^{(j)} S_+(z-z_j)\right]$$

$$- \left[W_t(z) - \sum_{j=1}^{n-1}\left(K_j \tilde{K}_3^{(i)} + Q_j\right) S_+(z-z_j)\right]. \tag{13}$$

In the general case, the integration constants C_1 and C_2 are determined from the system of two nonlinear algebraic equations obtained as a result of the substitution of expression (10), with regard for (13), in the boundary conditions (3). Note that, in the case where the conditions of heat exchange of the first or second kind are imposed on one of the boundary surfaces $z = z_i$, $i = 0, n$, the procedure of determination of C_1 and C_2 is reduced to the solution of a single nonlinear algebraic equation. It is convenient to seek the solutions of the indicated nonlinear algebraic equations by the method of simple iterations (successive approximations). Moreover, as the initial approximation, it is reasonable to take the values of these solutions obtained for the case of constant thermal conductivities.

In the case where the conditions of heat transfer of the first kind are given on the surface $z = z_0$ and conditions of the second kind are specified on the surface $z = z_n$, we obtain a closed analytic solution.

4. Determination of the Thermal Stressed State

The thermal stressed state of the analyzed structure under a given thermal and force load is described by the relations [4, 10]

$$\sigma_{xx} = \sigma_{yy} = \sigma_0(z), \qquad \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0.$$

In this case, the equilibrium equations are satisfied identically, while the equations of compatibility of strains in stresses are true under the condition

$$\frac{\partial^2}{\partial z^2}\left[\frac{1-\nu(t,z)}{E(t,z)}\sigma_0 + \Phi(t,z)\right] = 0, \tag{14}$$

where

$$\{\nu(t,z),\, E(t,z),\, \alpha_t(t,z), \Phi(t,z)\} \sim F(t,z) = \sum_{i=1}^{n} F_i(t) N_i(z),$$

$\{\nu_i(t),\, E_i(t),\, \alpha_{ti}(t)\} \sim F_i(t)$, $\nu_i(t)$, $E_i(t)$, $\alpha_{ti}(t)$ are, respectively, the temperature-dependent Poisson's ratio, modulus of elasticity, and the temperature coefficient of linear expansion of the ith layer, and

$$\Phi_i(t) = \int_0^t \alpha_{ti}(\zeta)\, d\zeta$$

is pure thermal deformation.

The general integral of Eq. (14) has the form

$$\sigma_0 = \frac{E(t,z)}{1-\nu(t,z)}\left[A_1 + zA_2 - \Phi(t,z)\right]. \tag{15}$$

Under the condition that the resultant vector and resultant moment of stresses σ_{xx}, σ_{yy} are equal to zero:

$$\int_0^{z_n} \sigma_{xx}\, dz = \int_0^{z_n} \sigma_{yy}\, dz = 0, \qquad \int_0^{z_n} z\,\sigma_{xx}\, dz = \int_0^{z_n} z\,\sigma_{yy}\, dz = 0,$$

we determine the constants of integration A_1 and A_2 by the formulas

$$A_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}^2}, \qquad A_2 = \frac{b_2 a_{11} - b_1 a_{12}}{a_{11}a_{22} - a_{12}^2},$$

where

$$a_{11} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{E_i(t)}{1-\nu_i(t)} dz, \qquad a_{12} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{zE_i(t)}{1-\nu_i(t)} dz,$$

$$a_{22} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{z^2 E_i(t)}{1-\nu_i(t)} dz, \tag{16}$$

$$b_1 = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{E_i(t)\,\Phi_i(t)}{1-\nu_i(t)} dz, \qquad b_2 = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \frac{zE_i(t)\,\Phi_i(t)}{1-\nu_i(t)} dz.$$

Approximating the temperature dependences of the PMC of materials of the components by piecewise-constant functions of temperature of the form (5) and taking into account relation (4) and the one-to-one correspondence between ϑ and t (9), we can represent the temperature-coordinate dependences of the PMC for a stack of layers regarded as a single whole in the following form:

$$\{\lambda_t, E, \nu, \alpha_t\} \sim p(t,z)$$

$$\approx \tilde{p}(\vartheta, z) = \sum_{i=1}^{n}\left[\tilde{p}_{i1} + \sum_{j=1}^{m} (\tilde{p}_{ij+1} - \tilde{p}_{ij}) S_+(\vartheta - \vartheta_{ij})\right] N_i(z), \tag{17}$$

where

$$\vartheta_{ij} = \int_0^{t_j} \lambda_{ti}(\zeta)\, d\xi \approx \sum_{i=1}^{n} \left[t_j \tilde{\lambda}_{ti}(t_j) - \sum_{\ell=1}^{m-1} \left(\tilde{\lambda}_{ti}^{(\ell+1)} - \tilde{\lambda}_{ti}^{(\ell)} \right) t_j S_+(t_j - t_\ell) \right] N_i(z),$$

$t_\text{p} = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = t_\text{k}$ are the joint nodes of approximation, $\tilde{p}_{ij}$ are the approximating coefficients of the temperature dependence $p_i(t)$ for the corresponding PMC of the ith layer equal (with required accuracy) to their values within temperature ranges $t_{j-1} < t < t_j$. In this case, according to [9], their algebraic combinations have the same form (17) and relations (15) and (16) can be represented in the following way:

$$\sigma_0 = \tilde{L}(\vartheta, z)\left[A_1 + z A_2 - \tilde{\Phi}(\vartheta, z) \right], \tag{18}$$

$$a_{11} = \int_0^{z_n} \tilde{L}(\vartheta, z)\, dz, \quad a_{12} = \int_0^{z_n} z \tilde{L}(\vartheta, z)\, dz, \quad a_{22} = \int_0^{z_n} z^2 \tilde{L}(\vartheta, z)\, dz,$$
$$b_1 = \int_0^{z_n} \tilde{L}(\vartheta, z)\, \tilde{\Phi}(\vartheta, z)\, dz, \quad b_2 = \int_0^{z_n} z \tilde{L}(\vartheta, z)\, \tilde{\Phi}(\vartheta, z)\, dz. \tag{19}$$

Here,

$$\tilde{L}(\vartheta, z) = \sum_{i=1}^{n} \left[\frac{\tilde{E}_{i1}}{1 - \tilde{\nu}_{i1}} + \sum_{j=1}^{m} \left(\frac{\tilde{E}_{ij+1}}{1 - \tilde{\nu}_{ij+1}} - \frac{\tilde{E}_{ij}}{1 - \tilde{\nu}_{ij}} \right) S_+(\vartheta - \vartheta_{ij}) \right] N_i(z),$$

$$\tilde{\Phi}(\vartheta, z) = \sum_{i=1}^{n} \left[\frac{\tilde{\alpha}_{ti}^{(1)}}{\tilde{\lambda}_{ti}^{(1)}} \vartheta + \sum_{j=1}^{m} \left(\frac{\tilde{\alpha}_{ti}^{(j1)}}{\tilde{\lambda}_{ti}^{(j+1)}} - \frac{\tilde{\alpha}_{ti}^{(j)}}{\tilde{\lambda}_{ti}^{(j)}} \right) (\vartheta - \vartheta_{ij})\, S_+(\vartheta - \vartheta_{ij}) \right] N_i(z).$$

Relations (18) and (19) enable one to directly study the thermoelastic behavior of a plane layered stack according to the solution of the corresponding boundary-value problem for Eq. (8) by determining a function of the Kirchhoff type (6) and avoiding the necessity of establishing the unique solvability of the corresponding nonlinear heat-conduction problem.

5. Results of Numerical Verification

The verification of the proposed numerical-analytic approach is carried by using an example of numerical analysis of the stationary thermal state of a three-layer plate and the static thermoelastic state caused by this thermal state. The boundary surface of the plate $z_0 = 0$ is kept at a constant temperature t_0, while the boundary surface $z_3 = h_3$ is exposed to a heat flux q.

The temperature field of the analyzed three-layer structure, according to relations (10) and (13), is determined as follows:

$$t = \frac{\vartheta + F_1(\vartheta)\,S_-(h_1 - z) + F_2(\vartheta)\,N_2(z) + F_3(\vartheta)\,S_+(z - h_2)}{\Lambda^{(1)}(\vartheta)\,S_-(h_1 - z) + \Lambda^{(2)}(\vartheta)\,N_2(z) + \Lambda^{(3)}(\vartheta)\,S_+(z - h_2)}, \qquad (20)$$

where

$$\vartheta = C_1\big[z + K_1 h_1 S_+(z - h_1) + (h_2 + h_1 K_1) K_2 S_+(z - h_2)\big]$$

$$+ C_2\big[1 + K_1 S_+(z - h_1) + (1 + K_1) K_2 S_+(z - h_2)\big]$$

$$+ Q_1 S_+(z - h_1) + (Q_1 K_2 + Q_2)\,S_+(z - h_2),$$

$$K_k = \left(\frac{\Lambda^{(k+1)}(\vartheta)}{\Lambda^{(k)}(\vartheta)} - 1\right)\Bigg|_{h_k}, \qquad Q_k = \left(\frac{\Lambda^{(k+1)}(\vartheta)}{\Lambda^{(k)}(\vartheta)} F_k - F_{k+1}(\vartheta)\right)\Bigg|_{h_k}, \qquad k = 1, 2,$$

$$S_-(h_1 - z) = 1 - S_+(z - h_1),$$

$$F_k(\vartheta) = \sum_{j=1}^{m} \left(\Lambda_{j+1}^{(k)} - \Lambda_j^{(k)}\right) t_j S_+(\vartheta - \vartheta_j),$$

$$\vartheta_\ell = \vartheta_\ell(z) = \sum_{i=1}^{3} \left[t_\ell \Lambda^{(i)}(t_\ell) - \sum_{j=1}^{m} \left(\Lambda_{j+1}^{(i)} - \Lambda_j^{(i)}\right) t_j S_+(t_\ell - t_j) \right] N_i(z).$$

The integration constants C_1 and C_2 are determined from the boundary conditions as follows:

$$C_1 = q \quad \text{and} \quad C_2 = \vartheta(t_0, z_0) = \int_0^{t_0} \lambda_{t1}(\xi)\,d\xi.$$

If we assume that the PMC of materials of the specific neighboring layers are identical, then relations (15), (18), and (20) describe the thermal stressed state of a homogeneous or two-layer thermosensitive plate.

In order to verify our numerical results, we find the static thermoelastic state caused by the temperature field (20) by using both relation (15) and relation (18).

Our numerical analyses were carried out for plates made of aluminum oxide (engineering ceramics) and Ti-6Al-4V titanium alloy. The temperature dependences of the PMC within the temperature range of their definition $[t_p = 273°K,\ t_k = 873°K]$ were chosen in the form

– for aluminum oxide (engineering ceramics) [4, 16]:

$$\lambda_t(t) = \left(1.5828 \cdot 10^4 t^{-1} - 14.087 + 8.772 \cdot 10^{-2} t\right) \text{ [W/(m·K)]},$$

$$E(t) = \left(3.4955 \cdot 10^{11} - 1.3468 \cdot 10^{8} t + 1.4076 \cdot 10^{5} t^{2}\right) \text{ [Pa]},$$

$$\alpha_t(t) = \left(6.8269 \cdot 10^{-6} + 1.2548 \cdot 10^{-9} t\right) \text{ [K}^{-1}\text{]}, \tag{21}$$

$$\nu(t) = 236.283 \cdot 10^{-3} - 31.7412 \cdot 10^{-6} t + 71.2602 \cdot 10^{-9} t^{2};$$

– for Ti-6Al-4V titanium alloy [16]:

$$\lambda_t(t) = \left(1 + 1.704 \cdot 10^{-2} t\right) \text{ [W/(m·K)]},$$

$$E(t) = \left(122.56 \cdot 10^{9} - 56.206 \cdot 10^{9} t\right) \text{ [Pa]},$$

$$\alpha_t(t) = \left(7.5788 \cdot 10^{-6} + 5.03081 \cdot 10^{-9} t - 23.8505 \cdot 10^{-12} t^{2}\right) \text{ [K}^{-1}\text{]}, \tag{22}$$

$$\nu(t) = 288.4 \cdot 10^{-3} + 32.3296 \cdot 10^{-6} t.$$

The temperature dependences of the PMC of materials are approximated within the temperature range of their definition by expressions of the form (17), where the approximating coefficients $\tilde{p}_{ij}$ and the nodes of approximation t_j are specified as follows:

$$\tilde{p}_{ij} = p_i(t_j^*), \qquad t_j = t_\text{p} + j\,\frac{t_\text{k} - t_\text{p}}{m+1}, \qquad t_j^* = t_j - \frac{t_\text{k} - t_\text{p}}{2(m+1)}, \tag{23}$$

and m is the number of approximation nodes.

Typical results of numerical investigations are shown in the form of plots in Figs. 2–7.

The distributions of temperature $t(\tilde{z})$ and stresses $\sigma(\tilde{z})$ over the thickness $\tilde{z} = z/h_3$ for different numbers of approximation nodes in a two-layer ceramics–Ti-6Al-4V-alloy plate with $h_1/h_3 < h_2/h_3 = 0.8$, $t_0 = 273°\text{K}$, and $q = 20$ kW/m^2 are illustrated in Figs. 2–4. The results obtained for a three-layer plate made of engineering ceramics and reinforced with a layer of Ti-6Al-4V alloy with $h_1/h_3 = 0$, $h_2/h_3 = 0.6$, $t_0 = 273°\text{K}$, and $q = 20$ kW/m^2 are shown in Figs. 5–7.

The curves plotted in Figs. 2–7 illustrate the distributions of temperature $t(\tilde{z})$ and stresses $\sigma(\tilde{z})$ computed for the approximation of all temperature dependences of the PMC of materials according to relations (23) with $m = 3, 6, 12$ (curves *1–3*, respectively) and the mean-integral values

$$p_{ij}^* = \frac{1}{t_\text{k} - t_\text{p}} \int_{t_\text{p}}^{t_\text{k}} p_{ij}(t)\, dt$$

within the interval of their definition (curve *4*), and solely for the mean-integral values of thermal conductivity (curve *5*). The dashed curves correspond to the values of temperature $t(\tilde{z})$ and stresses $\sigma(\tilde{z})$ computed by using the accepted temperature dependences of the PMC (21) and (22).

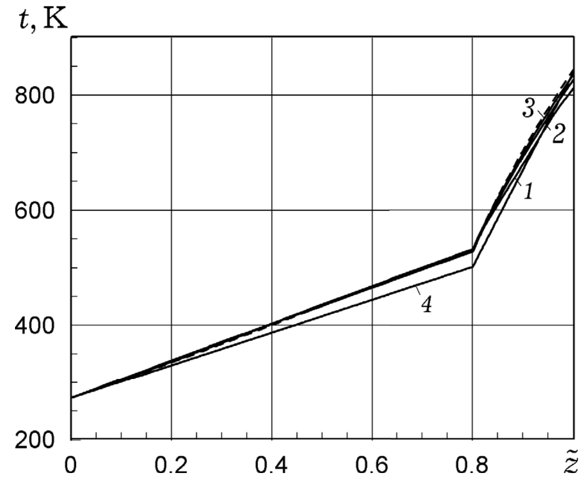

Fig. 2

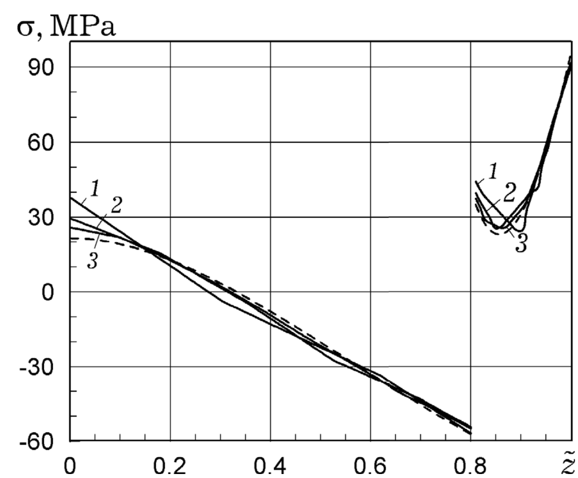

Fig. 3

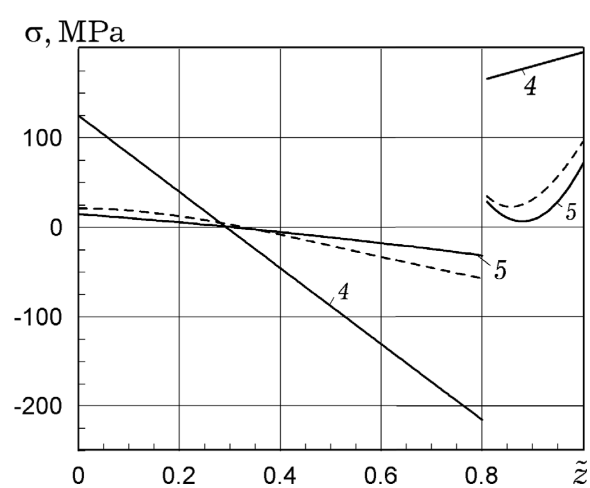

Fig. 4

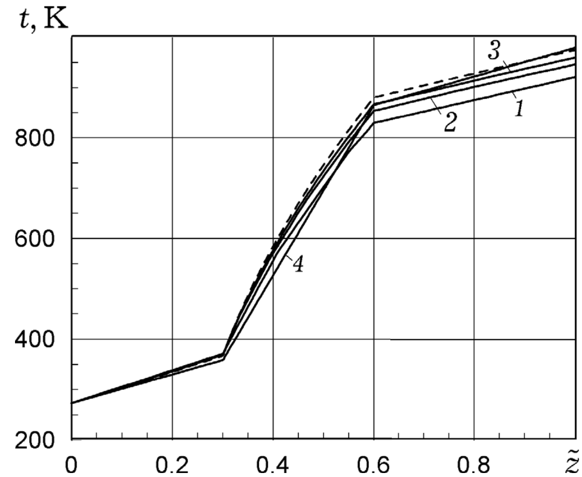

Fig. 5

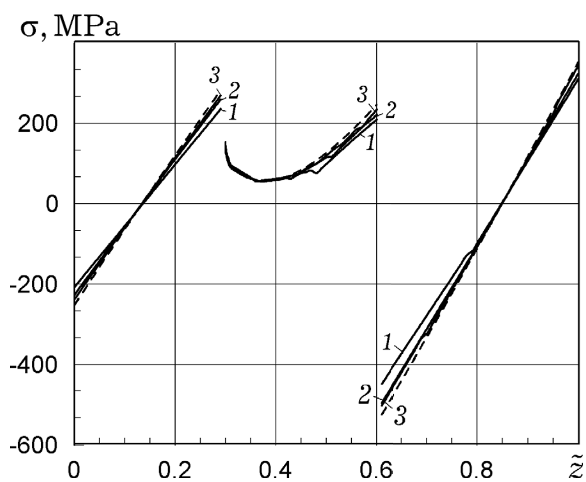

Fig. 6

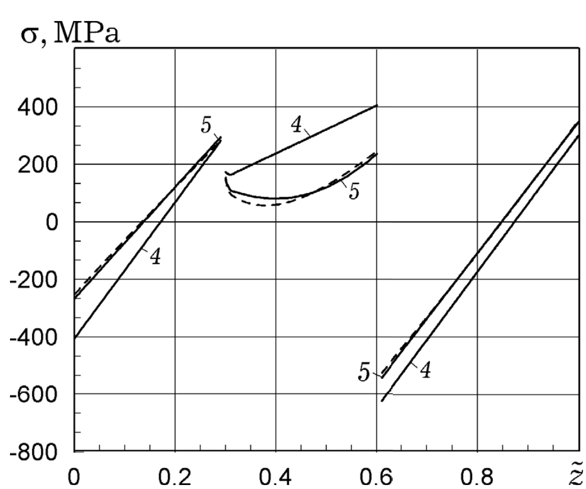

Fig. 7

In Table 1, we illustrate the maximal values of relative error

$$\varepsilon_F = \left| \frac{F - F^*}{\overline{\Delta F}} \right|$$

depending on the number of the nodes of approximation (F and F^* are, respectively, the exact and approximate values of the function and $\overline{\Delta F} = F_{\max} - F_{\min}$).

Table 1

No. of approximation nodes		Two-layer plate		Three-layer plate	
		ε_t, % $\overline{\Delta t} = 800\,°\mathrm{K}$	ε_σ, % $\overline{\Delta\sigma} = 1.53 \cdot 10^8$ Pa	ε_t, % $\overline{\Delta t} = 800\,°\mathrm{K}$	ε_σ, % $\overline{\Delta\sigma} = 1.53 \cdot 10^8$ Pa
3		3.8	10.7	6.8	3.8
6		2.1	5.3	3.6	1.6
12		1.0	2.8	1.8	1.2
24		0.5	1.5	1.2	0.8
For the mean-integral values of	all PMC	4.7	103.6	7.4	9.7
	only $\lambda_t(t)$	4.7	18.3	7.4	1.3

The results obtained in the present work demonstrate that:

– the proposed procedure guarantees rapid convergence of the process of numerical determination of the thermal state and thermal stressed state (caused by this thermal state): the twofold increase in the number of approximation nodes for the analyzed cases leads to an about 1.5–2-fold decrease in the maximum value of the reduced relative error;

– neglecting the character of temperature dependence of $\lambda_t(t)$ for the materials of layers leads, as a rule, to significant errors in the evaluation of thermal and thermal stressed states of the system (both quantitative and qualitative; thus, in the analyzed cases, depending on the structure of the plate, the relative error attained 11.9% for temperature and 32.8% for stresses);

– formal application (in the course of preliminary calculations) of the approximation of temperature dependences of the PMC by constant quantities equal to their mean-integral values would lead, most likely, to inadequate estimates of the thermal stressed state of the analyzed object.

CONCLUSIONS

The proposed numerical-analytic approach to the solution of one-dimensional problems of stationary heat conduction and static problems of thermoelasticity of plane layered structures made of thermally sensitive materials enables one to study the thermal and stressed states for different types of temperature dependences of the PMC of the components of these structures.

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