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GENERALIZED RIEMANN PROBLEM ON THE DECAY OF A DISCONTINUITY WITH ADDITIONAL CONDITIONS AT THE BOUNDARY AND ITS APPLICATION FOR CONSTRUCTING COMPUTATIONAL ALGORITHMS

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Abstract. We construct an approximation of the fundamental solution of a problem for a hyperbolic system of first-order linear differential equations with constant coefficients. We propose an algorithm for an approximate solution of the generalized Riemann problem on the decay of a discontinuity under additional conditions at the boundaries, which allows one to reduce the problem of finding the values of variables on both sides of the discontinuity surface of the initial data to the solution of a system of algebraic equations. We construct a computational algorithm for an approximate solution of the initial-boundary-value problem for a hyperbolic system of first-order linear differential equations. The algorithm is implemented for a system of equations of elastic dynamics; it is used for solving some applied problems associated with oil production.

***Keywords and phrases*:** decay of a discontinuity, conjugation condition, hyperbolic system, generalized function, Cauchy problem, Green matrix-function, characteristic, Riemann invariant, equation of elastic dynamics.

***AMS Subject Classification*:** 35L40, 35L67, 35L45, 35L50

1. Statement of the problem. This work is devoted to the study of the generalized Riemann problem on the decay of a discontinuity with additional conditions on the boundary. This mathematical problem arose from attempts to solve a specific applied problem discussed below.

Numerous industrial experiments in oil fields show that the installation of a vibration source on the surface and its prolonged operation for several months leads to a significant increase in oil recovery of the oil reservoir; in some cases the effect reaches 40%. The mechanisms and processes leading to such an increase in oil recovery remain unclear today. In particular, it is not clear how the energy of elastic waves generated by a vibration source reaches significant depths (1 km or more) avoiding significant scattering. A vibration source with a characteristic power of 30 kW and a contact patch with the rock of the order of 1 m^2 generates an elastic wave. If the rock is homogeneous, then the energy of the vibration source is scattered over the hemisphere and at depths of the order of 1 km the energy density of elastic waves decreases 10^6 times even in the absence of absorption in the rock. It is doubtful that an elastic wave of such a low energy density will cause any significant processes in the oil-bearing reservoir. Since the effect of increased oil recovery as a result of long-term operation of the surface vibration source has been repeatedly recorded, it follows that under definite conditions, elastic waves propagate in the geological formation avoiding significant scattering. We tried to use mathematical methods to investigate the question of whether the presence of fractures in the geological rock, in which the contacting parts can move relative to each other, lead to the fact that the elastic wave generated by the vibration source does not scatter over the hemisphere, but propagates as a sufficiently narrow beam so that even at significant depths its energy density remains significant in order to cause certain physicochemical processes.

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The state of the geological rock, in which the elastic wave propagates, is described by the vector field of displacement velocity and the stress tensor. Since the fracture may displace parts of the rock relative to each other, when crossing the boundary, the components of the displacement velocity vector and the stress tensor may have discontinuities. Of course, these discontinuities cannot be arbitrary, but must satisfy certain conditions reflecting the physical conditions at the boundary of the contacting media. Since parts of the geological rock can be displaced relative to each other only along the fracture, the components of the displacement velocity vector directed along the normal to the boundary must be continuous when crossing the boundary. Also, by Newton's third law, the force acting from one part of the rock to the other is equal and oppositely directed to the force acting from the second part of the rock to the first; thus, the normal components of the stress tensor on both sides of the boundary must be equal. If slipping is frictionless, then the tangential components of the stress tensor on both sides of the boundary should be zero.

On the other hand, everywhere, except for the crack dividing parts of the rock, the geological formation is continuous. Therefore, the displacement vector must be continuous everywhere (except for the fracture). Also, Newton's third law in any section leads to the continuity of the stress tensor. Thus, everywhere, except for a fracture, the displacement vector and the stress tensor are continuous.

Based on the above, we consider the following setting of the Riemann problem on the decay of a discontinuity. Find a solution of the Cauchy problem for the following system of first-order linear differential equations with constant coefficients:

$$\frac{\partial \boldsymbol{u}(t,\boldsymbol{x})}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{u}(t,\boldsymbol{x})}{\partial x_i} = \boldsymbol{0}, \quad \boldsymbol{x} \in \mathbb{R}^N; \tag{1}$$

the initial data

$$\boldsymbol{u}(t=0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \tag{2}$$

are continuous everywhere except for the hyperplane $\Gamma : x_1 = 0$. A solution must be continuous everywhere except for the hyperplane Γ. Moreover, the following relations must be fulfilled:

$$\boldsymbol{L}\boldsymbol{u}\Big(t,\ x_1=-0,\ x_2,\ \ldots,\ x_N\Big) + \boldsymbol{P}\boldsymbol{u}\Big(t,\ x_1=+0,\ x_2,\ \ldots,\ x_N\Big) = \boldsymbol{0}. \tag{3}$$

These formulas are the so-called conjugation conditions, which relate the values of the variables on both sides of the hyperplane Γ.

Following [5], we call this problem the *generalized Riemann problem* on the decay of a discontinuity with conjugation conditions on the boundaries. The difference between the generalized and classical Riemann problems is as follows: in the classical problem, the initial data are assumed to be constant on both sides of the hyperplane, whereas in the generalized problem, the initial data can be arbitrary smooth functions satisfying the conjugation conditions.

In the case of one spatial variable, many authors proposed various methods for solving the Riemann problem (see [5–8]). In fact, all these methods are associated with the presence of characteristics of hyperbolic systems. In the case of several spatial variables, methods based on the presence of characteristics no longer work, and the Riemann problem is most often solved under the assumption that near the discontinuity the solution is a plane wave moving along the normal to the discontinuity surface (see [4–6]). It is clear that such an approach is not justified in all cases.

In this paper, we discuss the generalized Riemann problem on the decay of a discontinuity with conjugation conditions on the boundaries for hyperbolic systems of first-order linear differential equations with an arbitrary number of spatial variables and propose an algorithm for constructing its solution. This algorithm is based on the fundamental solution of the operator determining the problem. Therefore, in the following sections, we recall the basic concepts of the theory of generalized vector-valued functions and construct an approximation of the fundamental solution of the operator of the problem. The solution of the generalized Riemann problem constructed according to the method proposed serves as a basis of a computational algorithm for finding an approximate solution to the initial-boundary-value problems examined in this paper.

2. Generalized vector-valued functions. In the further presentation, we will use the concepts and statements of the theory of generalized functions (see, e.g., [1–3, 9]).

Introduce the space of test vector-valued functions $\boldsymbol{S}(\mathbb{R}^N)$. Elements of this space are M-dimensional vector-valued functions $\boldsymbol{\varphi} = \big(\varphi_1, \dots, \varphi_M\big)$ whose components $\varphi_1(\boldsymbol{y}), \dots, \varphi_M(\boldsymbol{y})$ belong to the space $S(\mathbb{R}^N)$ consisting of functions of the class $C^\infty(\mathbb{R}^N)$ that decrease together with all their derivatives faster than any degree of $|y|^{-1}$ as $|y| \to \infty$.

Definition 2.1. Generalized vector-valued functions $\boldsymbol{f} = \big(f_1, \dots, f_M\big) \in \boldsymbol{S'}(\mathbb{R}^N)$ are linear continuous functionals on the vector space of test functions $\boldsymbol{S}(\mathbb{R}^N)$. A functional $\boldsymbol{f}$ acts on a test vector-valued function $\boldsymbol{\varphi} = \big(\varphi_1, \dots, \varphi_M\big)$ by the formula

$$\big(\boldsymbol{f}, \boldsymbol{\varphi}\big) = \big(f_1, \varphi_1\big) + \dots + \big(f_M, \varphi_M\big).$$

Definition 2.2. A generalized solution of the system of equations

$$\frac{\partial \boldsymbol{u}(t,\boldsymbol{x})}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{u}(t,\boldsymbol{x})}{\partial x_i} = \boldsymbol{f}(t,\boldsymbol{x}) \tag{4}$$

is a generalized function $\boldsymbol{u}(t,\boldsymbol{x}) \in \boldsymbol{S'}(\mathbb{R}^{N+1})$ satisfying this equation in the generalized sense: for an arbitrary test function $\boldsymbol{\varphi}(t,\boldsymbol{x}) \in \boldsymbol{S}(\mathbb{R}^{N+1})$, the following equality holds:

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\varphi}\right) + \sum_{i=1}^{N} \left(\boldsymbol{A}_i \frac{\partial \boldsymbol{u}}{\partial x_i}, \boldsymbol{\varphi}\right) = (\boldsymbol{f}, \boldsymbol{\varphi})\,;$$

here $\boldsymbol{A}_i$ are the $(M \times M)$-matrices of the coefficients of the system (4).

In what follows, we assume that each of the all matrices $\boldsymbol{A}_i$ has a complete set of left eigenvectors and, therefore, can be represented in the form

$$\boldsymbol{A}_i = \boldsymbol{R}_i \boldsymbol{\Lambda}_i \boldsymbol{\Omega}_i, \tag{5}$$

where $\boldsymbol{\Lambda}_i$ is the diagonal matrix whose diagonal elements are the eigenvalues of the matrix $\boldsymbol{A}_i$ arranged in the nondescending order, $\boldsymbol{\Omega}_i$ is the matrix whose rows are left eigenvectors of the matrix $\boldsymbol{A}_i$ corresponding to the eigenvalues $\boldsymbol{\Lambda}_i$, and $\boldsymbol{R}_i = \boldsymbol{\Omega}_i^{-1}$ is the matrix whose columns are right eigenvectors of the matrix $\boldsymbol{A}_i$.

Definition 2.3. A fundamental solution of the operator of the problem (4), or the Green matrix-function is a generalized matrix-valued function $\boldsymbol{G}(t,\boldsymbol{x}) \in \boldsymbol{S'}(\mathbb{R}^{N+1})$ satisfying the equation

$$\frac{\partial \boldsymbol{G}}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{G}}{\partial x_i} = \boldsymbol{I}\delta(t,\boldsymbol{x}), \tag{6}$$

where $\boldsymbol{I}$ is the identity $(M \times M)$-matrix.

Definition 2.4. The convolution $\boldsymbol{G} * \boldsymbol{f}$ of a generalized matrix-valued function $\boldsymbol{G} = G_{ij} \in \boldsymbol{S'}$ and a generalized vector-valued function $\boldsymbol{f} = f_j \in \boldsymbol{S'}$ is a generalized vector-valued function $\boldsymbol{u} = u_i \in \boldsymbol{S'}$ such that

$$u_i = \sum_{j=1}^{M} G_{i,j} * f_j,$$

where $G_{i,j} * f_j$ is the convolution of $G_{i,j}$ and f_j considered as generalized functions from S'.

Lemm 2.1. *Let $\boldsymbol{f}(t,\boldsymbol{x}) \in \boldsymbol{S'}$ be such that the convolution $\boldsymbol{G} * \boldsymbol{f}$ exists in $\boldsymbol{S'}$. Then there exists a solution of Eq.* (4) *in $\boldsymbol{S'}$; it can be represented by the formula*

$$\boldsymbol{u} = \boldsymbol{G} * \boldsymbol{f}. \tag{7}$$

This solution is unique in the class of functions from $\boldsymbol{S'}$ for which the convolution with $\boldsymbol{G}$ exists.

Proof. Using the formula of differentiating convolutions, we obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{u}}{\partial x_i} = \frac{\partial(\boldsymbol{G} * \boldsymbol{f})}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial(\boldsymbol{G} * \boldsymbol{f})}{\partial x_i} = \left(\frac{\partial \boldsymbol{G}}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{G}}{\partial x_i}\right) * \boldsymbol{f} = \delta(t,\boldsymbol{x}) * \boldsymbol{f} = \boldsymbol{f}.$$

Therefore, the formula (7) really gives a solution of Eq. (4). Prove the uniqueness of the solution of Eq. (4) in the class of generalized functions from $\boldsymbol{S'}$ for which the convolution with $\boldsymbol{G}$ exists in $\boldsymbol{S'}$. It suffices to verify that the corresponding homogeneous equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{u}}{\partial x_i} = \boldsymbol{0}$$

has only trivial (zero) solution in this class. Indeed, we have

$$\boldsymbol{u} = \delta(t,\boldsymbol{x})\boldsymbol{I} * \boldsymbol{u} = \left(\frac{\partial \boldsymbol{G}}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{G}}{\partial x_i}\right) * \boldsymbol{u} = \boldsymbol{G} * \left(\frac{\partial \boldsymbol{u}}{\partial t} + \sum_{i=1}^{N} \boldsymbol{A}_i \frac{\partial \boldsymbol{u}}{\partial x_i}\right) = \boldsymbol{0}. \qquad \square$$

The following two lemmas will be needed below.

Lemm 2.2. *Let $u(\boldsymbol{x})$ be a locally integrable function in $\mathbb{R}^N$. Then*

$$\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * u(\boldsymbol{x})\delta(t) = \theta(t)u(\boldsymbol{x} - \boldsymbol{a}t).$$

Proof. By the definition of the convolution of generalized functions (see [9]), for an arbitrary test function $\varphi(\boldsymbol{x},t) \in \boldsymbol{S}(\mathbb{R}^{N+1})$ and an arbitrary sequence of functions $\eta_k(\boldsymbol{x},\boldsymbol{y},t,\tau) \in \boldsymbol{S}(\mathbb{R}^{2N+2})$ converging to 1 in $\mathbb{R}^{2N+2}$, the following chain of equalities holds:

$$\begin{aligned}
&\Big(\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * u(\boldsymbol{x})\delta(t),\ \varphi(\boldsymbol{x},t)\Big) \\
&\quad\overset{\text{def}}{=} \lim_{k\to\infty} \Big(\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t)u(\boldsymbol{y})\delta(\tau),\ \eta_k(\boldsymbol{x},\boldsymbol{y},t,\tau)\,\varphi(\boldsymbol{x}+\boldsymbol{y},t+\tau)\Big) \\
&\quad= \lim_{k\to\infty} \Big(\theta(t)u(\boldsymbol{y})\delta(\tau),\ \eta_k(\boldsymbol{a}t,\boldsymbol{y},t,\tau)\,\varphi(\boldsymbol{a}t+\boldsymbol{y},t+\tau)\Big) \\
&= \lim_{k\to\infty} \int\limits_{\Gamma:\tau=0} \theta(t)u(\boldsymbol{y})\eta_k(\boldsymbol{a}t,\boldsymbol{y},t,0)\,\varphi(\boldsymbol{a}t+\boldsymbol{y},t)\,d\Gamma = \int_{-\infty}^{+\infty} \theta(t)u(\boldsymbol{y})\,\varphi(\boldsymbol{a}t+\boldsymbol{y},t)\,d\boldsymbol{y}dt \\
&\quad= \int_{-\infty}^{+\infty} \theta(t)u(\boldsymbol{x}-\boldsymbol{a}t)\,\varphi(\boldsymbol{x},t)\,d\boldsymbol{x}dt = \Big(\theta(t)u(\boldsymbol{x}-\boldsymbol{a}t),\ \varphi(\boldsymbol{x},t)\Big). \qquad \square
\end{aligned}$$

Remark 2.1. Lemma 2.2 implies that the value of the convolution $\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * u(\boldsymbol{x})\delta(t)$ at a point $(\boldsymbol{x},t)$ is equal to the value of the function $u(\boldsymbol{x})$ at the point of intersection of the straight line $d\boldsymbol{x}/dt = \boldsymbol{a}$ passing through the point $(\boldsymbol{x},t)$ with the hyperplane $t = 0$.

Lemm 2.3. *Let $v(t,\boldsymbol{x})$ be a locally integrable function in $\mathbb{R}^{N+1}$ and $v(t,\boldsymbol{x}) = 0$ for $t \le 0$. If $a_1 \neq 0$, then*

$$\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * v(t,\boldsymbol{x})\delta(x_1) = \frac{1}{|a_1|}\theta\left(\frac{x_1}{a_1}\right) v\left(t - \frac{x_1}{a_1},\ \boldsymbol{x} - \frac{x_1}{a_1}\boldsymbol{a}\right);$$

if $a_1 = 0$, *then*

$$\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * v(t, \boldsymbol{x})\delta(x_1) = 0.$$

Proof. If $a_1 \neq 0$, then for an arbitrary test function $\varphi(\boldsymbol{x}, t) \in S(\mathbb{R}^{N+1})$ and an arbitrary sequence of functions $\eta_k(\boldsymbol{x}, \boldsymbol{y}, t, \tau) \in S(\mathbb{R}^{2N+2})$ converging to 1 in $\mathbb{R}^{2N+2}$, the following chain of equalities holds:

$$\Big(\theta(t)\delta\big(\boldsymbol{x} - \boldsymbol{a}t\big) * v(t, \boldsymbol{x})\delta(x_1),\ \varphi(\boldsymbol{x}, t)\Big)\ \dots$$

$$\stackrel{\text{def}}{=} \lim_{k\to\infty} \Big(\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t)v(\tau, \boldsymbol{y})\delta(y_1),\ \eta_k(\boldsymbol{x}, \boldsymbol{y}, t, \tau)\varphi(\boldsymbol{x} + \boldsymbol{y}, t + \tau)\Big)$$

$$= \lim_{k\to\infty} \Big(\theta(t)v(\tau, \boldsymbol{y})\delta(y_1), \eta_k(\boldsymbol{a}t, \boldsymbol{y}, t, \tau)\varphi(\boldsymbol{a}t + \boldsymbol{y}, t + \tau)\Big)$$

$$= \int\limits_{\boldsymbol{\Gamma}: y_1 = 0} \theta(t')v(\tau', \boldsymbol{y})\, \varphi(\boldsymbol{a}t' + \boldsymbol{y}, t' + \tau')\, dy_2 \dots dy_N\, dt'\, d\tau'$$

$$= \frac{1}{|a_1|} \int\limits_{-\infty}^{+\infty} \theta\left(\frac{x_1}{a_1}\right) v\left(t - \frac{x_1}{a_1},\ \boldsymbol{x} - \frac{x_1}{a_1}\boldsymbol{a}\right) \varphi(\boldsymbol{x}, t)\, d\boldsymbol{x}\, dt$$

$$= \frac{1}{|a_1|}\theta\left(\frac{x_1}{a_1}\right) v\left(t - \frac{x_1}{a_1},\ \boldsymbol{x} - \frac{x_1}{a_1}\boldsymbol{a}\right),\ \varphi(\boldsymbol{x}, t).$$

Thus, we have prove the lemma for $a_1 \neq 0$.

Now let $v(t, \boldsymbol{x}) = 0$ for $t \leq 0$.

If $t > 0$, $x_1 < 0$, and $a_1 > 0$, then $\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * v(t, \boldsymbol{x})\delta(x_1) = 0$.

If $t > 0$, $x_1 < 0$, and $x_1/t \leq a_1 < 0$, then $\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * v(t, \boldsymbol{x})\delta(x_1) = 0$.

Due to the continuity of the convolution for $t > 0$ and $x_1 < 0$, we have

$$\theta(t)\delta(x_1)\delta(x_2 - a_2 t)\dots\delta(x_N - a_N t) * v(t, \boldsymbol{x})\delta(x_1) = \lim_{a_1\to 0} \Big(\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * v(t, \boldsymbol{x})\delta(x_1)\Big) = 0.$$

Similarly, for $t > 0$ and $x_1 > 0$ we have

$$\theta(t)\delta(x_1)\delta(x_2 - a_2 t)\dots\delta(x_N - a_N t) * v(t, \boldsymbol{x})\delta(x_1) = \lim_{a_1\to 0} \Big(\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * v(t, \boldsymbol{x})\delta(x_1)\Big) = 0.$$

This implies the assertion for arbitrary $\boldsymbol{a}$. □

Remark 2.2. We draw a straight line $d\boldsymbol{x}/dt = \boldsymbol{a}$ through the point $(t, \boldsymbol{x})$, $t > 0$. This line intersects the hyperplane $x_1 = 0$ at the time moment $t^* = t - x_1/a_1$. If this moment lies outside the interval $0 \leq t^* \leq t$, then at the point $(t, \boldsymbol{x})$ we have

$$\theta(t)\delta(\boldsymbol{x} - \boldsymbol{a}t) * \theta(t)v(t, \boldsymbol{x})\delta(x_1) = 0.$$

3. Fundamental solution. Now we construct the fundamental solution of the operator of the problem (4). Denote by $\boldsymbol{V}(t, \boldsymbol{\xi}) = F_{\boldsymbol{x}}[\boldsymbol{G}]$ the Fourier transform $\boldsymbol{G}(t, \boldsymbol{x})$ with respect to the spatial variables. We perform the Fourier transform of Eqs. (6) with respect to the spatial variable. Taking into account the fact that $F_{\boldsymbol{x}}[\boldsymbol{G}] = -i\xi_j F_{\boldsymbol{x}}[\boldsymbol{G}]$, for the generalized function $\boldsymbol{V}(t, \boldsymbol{\xi})$ we obtain the equation

$$\frac{\partial \boldsymbol{V}}{\partial t} - i\sum_{j=1}^{N} \xi_j \boldsymbol{A}_j \boldsymbol{V} = \boldsymbol{I}\,\delta(t). \tag{8}$$

The solution of Eq. (8) has the form

$$\boldsymbol{V}(t, \boldsymbol{\xi}) = \theta(t)\exp\left(i\sum_{j=1}^{N} \xi_j \boldsymbol{A}_j t\right),$$

where $\theta(t)$ is the Heaviside function:

$$\theta(t) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

By the definition of the matrix exponent,

$$\exp\left(i\sum_{j=1}^{N}\xi_j \boldsymbol{A}_j t\right) = \prod_{j=1}^{N}\exp\big(i\xi_j\boldsymbol{A}_j t\big) + \sum_{|\boldsymbol{\alpha}|\ge 2} t^{|\boldsymbol{\alpha}|}\boldsymbol{B}_{\boldsymbol{\alpha}}\prod_{j=1}^{N}\big(-i\xi_j\big)^{\alpha_j}.$$

Here $\boldsymbol{\alpha} = \big(\alpha_1, \alpha_2, \dots, \alpha_N\big)$ is an integer-valued vector with nonnegative components α_j (multi-index), $|\boldsymbol{\alpha}| = \alpha_1 + \dots + \alpha_N$, $\boldsymbol{B}_{\boldsymbol{\alpha}}$ are $(M \times M)$-matrices, which are polynomials of the matrices $\boldsymbol{A}_j$ of degree $|\boldsymbol{\alpha}|$.

Taking into account (5), we obtain

$$\exp\big(i\xi_j\boldsymbol{A}_j t\big) = \boldsymbol{R}_j\exp\big(i\xi_j\boldsymbol{\Lambda}_j t\big)\boldsymbol{\Omega}_j.$$

Therefore,

$$\exp\left(i\sum_{j=1}^{N}\xi_j \boldsymbol{A}_j t\right) = \prod_{j=1}^{N}\boldsymbol{R}_j\exp\left(i\xi_j\boldsymbol{\Lambda}_j t\right)\boldsymbol{\Omega}_j + \sum_{|\boldsymbol{\alpha}|\ge 2} t^{|\boldsymbol{\alpha}|}\boldsymbol{B}_{\boldsymbol{\alpha}}\prod_{j=1}^{N}\big(-i\xi_j\big)^{\alpha_j}.$$

Performing the inverse Fourier transform, we obtain the Green matrix-function:

$$\boldsymbol{G}(t,\boldsymbol{x}) = \theta(t)\left(\prod_{j=1}^{N}\boldsymbol{R}_j\delta\big(\boldsymbol{I}x_j - \boldsymbol{\Lambda}_j t\big)\boldsymbol{\Omega}_j + \sum_{|\boldsymbol{\alpha}|\ge 2} t^{|\boldsymbol{\alpha}|}\boldsymbol{B}_{\boldsymbol{\alpha}}D^{\boldsymbol{\alpha}}\delta(\boldsymbol{x})\right);$$

here $\delta(\boldsymbol{I}x_j - \boldsymbol{\Lambda}_j t)$ are diagonal matrices whose kth rows contain the generalized function $\delta(x_j - \lambda_j^k t)$, λ_j^k is the kth eigenvalue of the matrix $\boldsymbol{A}_j$, $D^{\boldsymbol{\alpha}} = \dfrac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_N^{\alpha_N}}$ is the differentiation operator with respect to the spatial variables.

Consider the factor $\boldsymbol{R}_j\delta(\boldsymbol{I}x_j - \boldsymbol{\Lambda}_j t)\boldsymbol{\Omega}_j$. Denote by $\boldsymbol{D}^k$ the square $(M \times M)$-matrix all of whose elements are equal to 0 except for the kth element of the principal diagonal, which is equal to 1. Then

$$\boldsymbol{R}_j\delta(\boldsymbol{I}x_j - \boldsymbol{\Lambda}_j t)\boldsymbol{\Omega}_j = \sum_{k=1}^{M}\boldsymbol{R}_j\boldsymbol{D}^k\boldsymbol{\Omega}_j\delta(x_j - \lambda_j^k t) = \sum_{k=1}^{M}\boldsymbol{C}_j^k\delta(x_j - \lambda_j^k t).$$

Therefore,

$$\prod_{j=1}^{N}\boldsymbol{R}_j\delta(\boldsymbol{I}x_j - \boldsymbol{\Lambda}_j t)\boldsymbol{\Omega}_j = \sum_{k_1=1}^{M}\sum_{k_2=1}^{M}\cdots\sum_{k_N=1}^{M}\boldsymbol{C}_1^{k_1}\boldsymbol{C}_2^{k_2}\dots\boldsymbol{C}_N^{k_N}\delta(x_1 - \lambda_1^{k_1}t)\delta(x_2 - \lambda_2^{k_2}t)\dots\delta(x_N - \lambda_N^{k_N}t).$$

Consider the multi-index $\boldsymbol{k} = (k_1, k_2, \dots, k_N)$ with integer-valued components $k_j = 1, \dots, M$ and introduce the notation $\boldsymbol{C}^{\boldsymbol{k}} = \boldsymbol{C}_1^{k_1}\boldsymbol{C}_2^{k_2}\dots\boldsymbol{C}_N^{k_N}$ and $\boldsymbol{\lambda}^{\boldsymbol{k}} = \big(\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_N^{k_N}\big)$. We have

$$\prod_{j=1}^{N}\boldsymbol{R}_j\delta(\boldsymbol{I}x_j - \boldsymbol{\Lambda}_j t)\boldsymbol{\Omega}_j = \sum_{\boldsymbol{k}}\boldsymbol{C}^{\boldsymbol{k}}\delta(\boldsymbol{x} - \boldsymbol{\lambda}^{\boldsymbol{k}}t);$$

then

$$\boldsymbol{G}(t,\boldsymbol{x}) = \theta(t)\sum_{\boldsymbol{k}}\boldsymbol{C}^{\boldsymbol{k}}\delta\left(\boldsymbol{x} - \boldsymbol{\lambda}^{\boldsymbol{k}}t\right) + O(t^2).$$

In the case of two spatial variables

$$\boldsymbol{G}(t,\boldsymbol{x}) = \theta(t)\sum_{\boldsymbol{k}}\boldsymbol{C}_1^{k_1}\boldsymbol{C}_2^{k_2}\delta(\boldsymbol{x} - \boldsymbol{\lambda}^{\boldsymbol{k}}t) + \frac{\theta(t)}{2}t^2\big(\boldsymbol{A}_2\boldsymbol{A}_1 - \boldsymbol{A}_1\boldsymbol{A}_2\big)\frac{\partial^2\delta(\boldsymbol{x})}{\partial x_1\partial x_2} + O(t^3). \tag{9}$$

Changing the notation of the spatial variables, we obtain

$$\boldsymbol{G}(t,\boldsymbol{x}) = \theta(t)\sum_{\boldsymbol{k}} \boldsymbol{C}_2^{k_2}\boldsymbol{C}_1^{k_1}\delta(\boldsymbol{x}-\boldsymbol{\lambda}^{\boldsymbol{k}}t) - \frac{\theta(t)}{2}t^2\big(\boldsymbol{A}_2\boldsymbol{A}_1 - \boldsymbol{A}_1\boldsymbol{A}_2\big)\frac{\partial^2\delta(\boldsymbol{x})}{\partial x_1\partial x_2} + O(t^3). \tag{10}$$

Comparing (9) and (10), we have

$$\boldsymbol{G}(t,\boldsymbol{x}) = \theta(t)\sum_{\boldsymbol{k}} \bar{\boldsymbol{C}}^{\boldsymbol{k}}\delta(\boldsymbol{x}-\boldsymbol{\lambda}^{\boldsymbol{k}}t) + O(t^3);$$

here we used the notation

$$\bar{\boldsymbol{C}}^{\boldsymbol{k}} = \frac{1}{2}\left(\boldsymbol{C}_1^{k_1}\boldsymbol{C}_2^{k_2} + \boldsymbol{C}_2^{k_2}\boldsymbol{C}_1^{k_1}\right).$$

Note that

$$\sum_{k_j=1}^{M}\boldsymbol{C}^{k_j} = \boldsymbol{R}_j\left(\sum_{k_j=1}^{M}\boldsymbol{D}^{k_j}\right)\boldsymbol{\Omega}_j = \boldsymbol{R}_j\boldsymbol{I}\boldsymbol{\Omega}_j = \boldsymbol{I} \tag{11}$$

since $\boldsymbol{C}^{k_j} = \boldsymbol{R}_j\boldsymbol{D}^{k_j}\boldsymbol{\Omega}_j$.

4. Riemann problem. Let $\boldsymbol{u}(t,\boldsymbol{x})$ be a solution of the Riemann problem (1)–(3). Introduce the notation

$$\boldsymbol{v}(t,\boldsymbol{x}) = \theta(t)\Big(\boldsymbol{u}\big(t, x_1=+0,\ x_2,\dots,x_N\big) - \boldsymbol{u}\big(t, x_1=-0,\ x_2,\dots,x_N\big)\Big).$$

We show that the function $\boldsymbol{u}(t,\boldsymbol{x})$ considered as a generalized function from $\boldsymbol{S'}$ satisfies the equation

$$\frac{\partial\boldsymbol{u}}{\partial t} + \sum_{i=1}^{N}\boldsymbol{A}_i\frac{\partial\boldsymbol{u}}{\partial x_i} = \boldsymbol{u}_0\delta(t) + \boldsymbol{A}_1\boldsymbol{v}\delta(x_1). \tag{12}$$

Indeed, for all $\boldsymbol{\varphi}(t,\boldsymbol{x}) \in \boldsymbol{S}$ we have the following chain of equalities:

$$\left(\frac{\partial\boldsymbol{u}}{\partial t} + \sum_{i=1}^{N}\boldsymbol{A}_i\frac{\partial\boldsymbol{u}}{\partial x_i},\ \boldsymbol{\varphi}\right) = -\int\left(\frac{\partial\boldsymbol{\varphi}^T}{\partial t}\boldsymbol{u} + \sum_{i=1}^{N}\frac{\partial\boldsymbol{\varphi}^T}{\partial x_i}\boldsymbol{A}_i\boldsymbol{u}\right)dt\,d\boldsymbol{x}$$

$$= \dots = \int\left(\boldsymbol{\varphi}^T\left(\frac{\partial\boldsymbol{u}}{\partial t} + \sum_{i=1}^{N}\boldsymbol{A}_i\frac{\partial\boldsymbol{u}}{\partial x_i}\right)\right)dt\,d\boldsymbol{x} + \int\Big(\boldsymbol{\varphi}^T(0,\boldsymbol{x})\boldsymbol{u}(0,\boldsymbol{x})\Big)d\boldsymbol{x} + \int_{\Gamma}\left(\boldsymbol{\varphi}^T\boldsymbol{A}_1\boldsymbol{v}\right)\,dt\,d\Gamma,$$

which implies Eq. (12).

A solution of Eq. (12) can be represented in the convolution form

$$\boldsymbol{u} = \boldsymbol{G} * \boldsymbol{u}_0\delta(t) + \boldsymbol{G} * \boldsymbol{A}_1\boldsymbol{v}\delta(x_1).$$

Due to Lemmas 2.2 and 2.3, for points $\boldsymbol{x} = (x_1, x_2, \dots, x_N)$ lying in the left half-plane ($x_1 \le 0$), we have with accuracy $O(t^2)$

$$\boldsymbol{u}(t, x_1, x_2, \dots, x_N) = \sum_{\boldsymbol{k}}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0(\boldsymbol{x}-\boldsymbol{\lambda}^{\boldsymbol{k}}t) - \sum_{\boldsymbol{k}:\ x_1>\lambda_-^{k_1}t}\frac{\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{A}_1}{\lambda^{k_1}}\boldsymbol{v}\left(t - \frac{x_1}{\lambda^{k_1}},\ \boldsymbol{x} - \frac{x_1}{\lambda^{k_1}}\boldsymbol{\lambda}^{\boldsymbol{k}}\right). \tag{13}$$

Passing in (13) to the limit as $x_1 \to -0$, we obtain

$$\boldsymbol{u}(t, x_1=-0, x_2, \dots, x_N) = \sum_{\boldsymbol{k}}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2-\boldsymbol{\lambda}^{k_2}t,\ \dots,\ x_N-\boldsymbol{\lambda}^{k_N}t\Big) - \sum_{\boldsymbol{k}:\ \lambda^{k_1}<0}\frac{\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{A}_1}{\lambda^{k_1}}\boldsymbol{v}. \tag{14}$$

Taking into account (11), we have

$$\sum_{\boldsymbol{k}:\ \lambda^{k_1}<0}\frac{1}{\lambda^{k_1}}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{A}_1 = \sum_{k_1:\ \lambda^{k_1}<0}\frac{1}{\lambda^{k_1}}\boldsymbol{C}^{k_1}\boldsymbol{A}_1.$$

Since $\boldsymbol{A}_1 = \boldsymbol{R}_1\boldsymbol{\Lambda}_1\boldsymbol{\Omega}_1$ and $\boldsymbol{C}^{k_1} = \boldsymbol{R}_1\boldsymbol{D}^{k_1}\boldsymbol{\Omega}_1$, we have $(1/\lambda^{k_1})\boldsymbol{C}^{k_1}\boldsymbol{A}_1 = \boldsymbol{C}^{k_1}$. Equality (14) takes the form

$$\boldsymbol{u}(t, x_1 = -0, x_2, \ldots, x_N) = \sum_{\boldsymbol{k}} \boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2 - \boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N - \boldsymbol{\lambda}^{k_N}t\Big) - \sum_{k_1:\ \lambda^{k_1}<0} \boldsymbol{C}^{k_1}\boldsymbol{v}. \quad (15)$$

Similarly, for points $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$ lying in the right half-plane ($x_1 \geq 0$), we have with accuracy $O(t^2)$

$$\boldsymbol{u}(t, x_1, x_2, \ldots, x_N) = \sum_{\boldsymbol{k}} \boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0(\boldsymbol{x} - \boldsymbol{\lambda}^{\boldsymbol{k}}t) + \sum_{\boldsymbol{k}:\ x_1<\lambda_-^{k_1}t} \frac{\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{A}_1}{\lambda^{k_1}}\boldsymbol{v}\left(t - \frac{x_1}{\lambda^{k_1}},\ \boldsymbol{x} - \frac{x_1}{\lambda^{k_1}}\boldsymbol{\lambda}^{\boldsymbol{k}}\right). \quad (16)$$

Similarly to (15), passing in (16) to the limit as $x_1 \to +0$, we obtain

$$\boldsymbol{u}(t, x_1 = +0, x_2, \ldots, x_N) = \sum_{\boldsymbol{k}} \boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2 - \boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N - \boldsymbol{\lambda}^{k_N}t\Big) + \sum_{k_1:\ \lambda^{k_1}>0} \boldsymbol{C}^{k_1}\boldsymbol{v}. \quad (17)$$

Substituting the expressions (15) and (17) into the conjugation conditions (3), we obtain a system for the jump of the variables of $\boldsymbol{v}(t, \boldsymbol{x})$ when passing through the hyperplane Γ defined by the equation $x_1 = 0$:

$$\sum_{k_1:\ \lambda^{k_1}>0} \boldsymbol{P}\boldsymbol{C}^{k_1}\boldsymbol{v} - \sum_{k_1:\ \lambda^{k_1}<0} \boldsymbol{L}\boldsymbol{C}^{k_1}\boldsymbol{v} = \boldsymbol{b}^{\boldsymbol{k}}. \quad (18)$$

Here

$$\boldsymbol{b}^{\boldsymbol{k}} = -\sum_{\boldsymbol{k}} \boldsymbol{P}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(+0 - \boldsymbol{\lambda}^{k_1}t,\ x_2 - \boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N - \boldsymbol{\lambda}^{k_N}t\Big)$$
$$- \sum_{\boldsymbol{k}} \boldsymbol{L}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-0 - \boldsymbol{\lambda}^{k_1}t,\ x_2 - \boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N - \boldsymbol{\lambda}^{k_N}t\Big).$$

Subtracting Eq. (15) from Eq. (17), we obtain the following system for the jump of the variables $\boldsymbol{v}(t, \boldsymbol{x})$:

$$\sum_{k_1:\ \lambda^{k_1}=0} \boldsymbol{C}^{k_1}\boldsymbol{v} = \sum_{\boldsymbol{k}:\ \lambda^{k_1}=0} \boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{d}^{\boldsymbol{k}}. \quad (19)$$

Here

$$\boldsymbol{d}^{\boldsymbol{k}} = \boldsymbol{u}_0\Big(+0,\ x_2 - \boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N - \boldsymbol{\lambda}^{k_N}t\Big) - \boldsymbol{u}_0\Big(-0,\ x_2 - \boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N - \boldsymbol{\lambda}^{k_N}t\Big).$$

The number of linearly independent equations in the system (19) is equal to the multiplicity of the zero eigenvalue of the matrix $\boldsymbol{A}_1$. We multiply Eq. (19) by left eigenvectors (rows) of the matrix $\boldsymbol{A}_1$ corresponding to the zero eigenvalue:

$$\boldsymbol{l}^{k_1}\boldsymbol{v} = \sum_{\boldsymbol{k}:\ \lambda^{k_1}=0} \boldsymbol{l}^{k_1}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{d}^{\boldsymbol{k}}, \quad k_1 : \lambda^{k_1} = 0. \quad (20)$$

Combining Eqs. (18) and (20), we obtain a system of linear algebraic equations for the jump of the variables $\boldsymbol{v}(t, \boldsymbol{x})$ when passing through the hyperplane $\Gamma : x_1 = 0$:

$$\begin{cases} \displaystyle\sum_{k_1:\ \lambda^{k_1}>0} \boldsymbol{P}\boldsymbol{C}^{k_1}\boldsymbol{v} - \sum_{k_1:\ \lambda^{k_1}<0} \boldsymbol{L}\boldsymbol{C}^{k_1}\boldsymbol{v} = \boldsymbol{b}^{\boldsymbol{k}}, \\ \boldsymbol{l}^{k_1}\boldsymbol{v} = \displaystyle\sum_{\boldsymbol{k}:\ \lambda^{k_1}=0} \boldsymbol{l}^{k_1}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{d}^{\boldsymbol{k}}, \quad k_1 : \lambda^{k_1} = 0. \end{cases} \quad (21)$$

A solution of the generalized Riemann problem on the decay of the discontinuity with the conjugation condition on the boundary is simple-valued if and only if the system (21) has a unique solution. Solving the system (21), we find $\boldsymbol{v}(t, \boldsymbol{x})$.

The formulas (13) and (16) together with the formulas for $\boldsymbol{v}(t,\boldsymbol{x})$ give a complete solution of the generalized Riemann problem on the decay of the discontinuity in the case of several spatial variables.

The approximate solution of the generalized Riemann problem with additional conjugation conditions on the discontinuity constructed above is an exact solution in the case of one spatial variable. Also, this solution is exact in the case of several spatial variables if the initial data are linear on both sides of the hyperplane $\Gamma : x_1 = 0$.

5. Boundary conditions. Now we consider another problem, which will be used below for constructing a calculational algorithm: In the half-space $x_1 \leq 0$, find a solution of the initial-boundary-value problem for the system of first-order linear differential equations with constant coefficients (1) with the initial data (2). A solution must be continuous in the half-space $x_1 \leq 0$ and satisfy the following boundary conditions on the hyperplane $\Gamma : x_1 = 0$:

$$\boldsymbol{L}\boldsymbol{u}\Big(t,\ x_1 = -0,\ x_2,\ \ldots,\ x_N\Big) = \boldsymbol{0}. \tag{22}$$

We assume that the initial data satisfy the boundary conditions. This problem is called the generalized Riemann problem with boundary conditions.

Let $\boldsymbol{u}(t,\boldsymbol{x})$ be a solution of this problem. We assume that the function $\boldsymbol{u}(t,\boldsymbol{x})$ is equal to zero for $t < 0$ and $t \geq 0$ and $x_1 > 0$. Also, we set $\boldsymbol{u}_0(\boldsymbol{x}) = 0$ for $x_1 > 0$. Introduce the notation $\boldsymbol{v}(t,\boldsymbol{x}) = -\theta(t)\boldsymbol{u}\big(t,\ x_1 = -0,\ x_2,\ \ldots,\ x_N\big)$.

As was shown above, the function $\boldsymbol{u}(t,\boldsymbol{x})$ considered as a generalized function from $\boldsymbol{S'}$ satisfies Eq. (12). A solution of this equation is defined by the formula (13). The vector $\boldsymbol{v}(t,\boldsymbol{x})$ satisfies Eqs. (15) with accuracy $O(t^2)$. We can rewrite Eqs. (15) as follows:

$$-\sum_{k_1:\ \lambda^{k_1}\geq 0} \boldsymbol{C}^{k_1}\boldsymbol{v} = \sum_{\boldsymbol{k}:\ \lambda^{k_1}\geq 0} \boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2-\boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N-\boldsymbol{\lambda}^{k_N}t\Big). \tag{23}$$

The number of linearly independent equations in the system (23) is equal ti the number of linearly independent eigenvectors of the matrix $\boldsymbol{A}_1$ corresponding to nonnegative eigenvalues. We multiply Eq. (23) by left row eigenvectors of the matrix $\boldsymbol{A}_1$ corresponding to nonnegative eigenvalues:

$$-\boldsymbol{l}^{k_1}\boldsymbol{v} = \sum_{\boldsymbol{k}:\ \lambda^{k_1}\geq 0} \boldsymbol{l}^{k_1}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2-\boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N-\boldsymbol{\lambda}^{k_N}t\Big), \quad k_1 : \lambda^{k_1} \geq 0. \tag{24}$$

Combining Eqs. (24) and (22), we obtain the following system of linear algebraic equations for the values of the solution of the problem on the boundary Γ:

$$\begin{cases} -\boldsymbol{l}^{k_1}\boldsymbol{v} = \displaystyle\sum_{\boldsymbol{k}:\ \lambda^{k_1}\geq 0} \boldsymbol{l}^{k_1}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2-\boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N-\boldsymbol{\lambda}^{k_N}t\Big), \quad k_1 : \lambda^{k_1} \geq 0, \\ \boldsymbol{L}\boldsymbol{v} = \boldsymbol{0}. \end{cases} \tag{25}$$

We solve the system (25) and define the value of $\boldsymbol{v}(t,\boldsymbol{x})$ for $t > 0$ on the hyperplane $x_1 = 0$. The formulas (13) together with the formulas for $\boldsymbol{v}(t,\boldsymbol{x})$ yield a complete solution of the generalized Riemann problem with boundary conditions for the case of several spatial variables with accuracy $O(t^2)$. Again, if the initial data $\boldsymbol{u}_0(\boldsymbol{x})$ are linear functions, then the solution obtained is an exact solution of the problem.

In particular, if the boundary conditions have the form

$$\sum_{\boldsymbol{k}:\ \lambda^{-}_{k_1}<0} \boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{v} = \boldsymbol{0}$$

(this means that all waves pass through the boundary without reflection), then these conditions are said to be "transparent." In this case, premultiplying the boundary conditions by left row eigenvectors of the matrix $\boldsymbol{A}_1$, we arrive at the following form of the system (25):

$$\begin{cases} -\boldsymbol{l}^{k_1}\boldsymbol{v} = \displaystyle\sum_{\boldsymbol{k}:\ \lambda^{k_1}\ge 0} \boldsymbol{l}^{k_1}\boldsymbol{C}^{\boldsymbol{k}}\boldsymbol{u}_0\Big(-\boldsymbol{\lambda}^{k_1}t,\ x_2-\boldsymbol{\lambda}^{k_2}t,\ \ldots,\ x_N-\boldsymbol{\lambda}^{k_N}t\Big), & k_1:\lambda^{k_1}\ge 0, \\ \boldsymbol{l}^{k_1}\boldsymbol{v}=\boldsymbol{0}, & k_1:\lambda^{k_1}<0. \end{cases} \tag{26}$$

Since the system (1) is hyperbolic and the matrix $\boldsymbol{A}_1$ possesses a complete set of linearly independent left eigenvectors, the system (26) is solvable, and the generalized Riemann problem with transparent boundary conditions has a unique solution.

6. Propagation of elastic waves in block-fractured media. We demonstrate the construction of a calculational algorithm based on the results presented above by the problem on propagation of elastic waves in an inhomogeneous block-fractured medium.

6.1. Mathematical model. According to [6], we write the system of equations that describe the propagation of elastic waves in the case of two spatial variables as follows:

$$\begin{aligned} \frac{\partial}{\partial t}\sigma_{11} - (\lambda+2\mu)\frac{\partial}{\partial x_1}v_1 - \lambda\frac{\partial}{\partial x_2}v_2 &= 0, \\ \frac{\partial}{\partial t}\sigma_{22} - \lambda\frac{\partial}{\partial x_1}v_1 - (\lambda+2\mu)\frac{\partial}{\partial x_2}v_2 &= 0, \\ \frac{\partial}{\partial t}\sigma_{12} - \mu\frac{\partial}{\partial x_1}v_2 - \mu\frac{\partial}{\partial x_2}v_1 &= 0, \\ \rho\frac{\partial}{\partial t}v_1 - \frac{\partial}{\partial x_1}\sigma_{11} - \frac{\partial}{\partial x_2}\sigma_{12} &= 0, \\ \rho\frac{\partial}{\partial t}v_2 - \frac{\partial}{\partial x_1}\sigma_{12} - \frac{\partial}{\partial x_2}\sigma_{22} &= 0, \end{aligned} \tag{27}$$

where λ and μ are the Lamé coefficients, ρ is the mass density of the medium, σ_{11}, σ_{22}, and σ_{12} are the components of the stress tensor, and v_1 and v_2 are the components of the vector of the displacement rate. Introducing the vector of variables $\boldsymbol{u} = (\sigma_{11}, \sigma_{22}, \sigma_{12}, v_1, v_2)^T$ and the matrices

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 0 & -(\lambda+2\mu) & 0 \\ 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & -\mu \\ -1/\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\rho & 0 & 0 \end{pmatrix}, \quad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -\lambda \\ 0 & 0 & 0 & 0 & -(\lambda+2\mu) \\ 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & -1/\rho & 0 \\ 0 & -1/\rho & 0 & 0 & 0 \end{pmatrix},$$

we rewrite this system in the form (1).

The system (27) is hyperbolic and the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ possess complete sets of linearly independent eigenvectors and can be represented in the form (5).

In what follows, all dimensional values are specified SI units. Consider the following problem: In the domain

$$\Omega = \Big[-30 < x_1 < 30,\ -600 < x_2 < 0\Big],$$

find a solution of the initial-boundary-value problem for the system (27). Equations (27) must be fulfilled everywhere in Ω except for inner boundaries Γ_γ, $\gamma = 1, 2$, determined by the conditions $\Gamma_1 : x_1 = -15$ and $\Gamma_2 : x_1 = 15$. On these boundaries, we impose the so-called "conditions of frictionless slip": when passing through these boundaries, the normal components of the displacement vector are continuous, the normal components of the forces from different sides of the boundary have equal magnitudes and opposite directions, and the tangential components of the forces acting on both sides of the boundary are equal to zero:

$$\begin{aligned} v_1\big(t, \boldsymbol{x} \in \Gamma_\gamma^-\big) - v_1\big(t, \boldsymbol{x} \in \Gamma_\gamma^+\big) &= 0, \\ \sigma_{11}\big(t, \boldsymbol{x} \in \Gamma_\gamma^-\big) - \sigma_{11}\big(t, \boldsymbol{x} \in \Gamma_\gamma^+\big) &= 0, \\ \sigma_{12}\big(t, \boldsymbol{x} \in \Gamma_\gamma^-\big) &= 0, \\ \sigma_{12}\big(t, \boldsymbol{x} \in \Gamma_\gamma^+\big) &= 0. \end{aligned} \tag{28}$$

On the outer boundaries $x_1 = -30$, $x_2 = -600$, and $x_1 = 30$, the transparent boundary conditions are imposed.

On the boundary $x_2 = 0$, a source of vibration operates; it acts on the geological medium with force $F_j \sin \omega t$, $j = 1, 2$, which is distributed with the density $P_j(x_1) \sin \omega t$, so that

$$\int P_j \, dx_1 = F_j.$$

Corresponding to third Newton's law, at all points of this boundary, the following conditions hold:

$$\sigma_{12} = -P_1 \sin \omega t, \quad \sigma_{22} = -P_2 \sin \omega t. \tag{29}$$

6.2. Numerical algorithm. The domain Ω is split by inner boundaries into the subdomains

$$\begin{aligned} \Omega_1 &= \Big[-30 < x_1 < -15,\ -600 < x_2 < 0\Big], \\ \Omega_2 &= \Big[-15 < x_1 < 15,\ -600 < x_2 < 0\Big], \\ \Omega_3 &= \Big[15 < x_1 < 30,\ -600 < x_2 < 0\Big]. \end{aligned}$$

In each of the subdomains, we construct a rectangular grid with sides parallel to the coordinate axes so that the nodes lying on the inner boundaries coincide for both adjacent subdomains. We use uniform grids for each of the coordinates; denote by h_j, $j = 1, 2$, the grid steps in the corresponding dimensions. Let p_1, p_2, and P_3 be the numbers of the grid nodes in the first, second, and third subdomains Ω_1, Ω_2, and Ω_3, respectively. Below, we use the common notation p if this does not lead to confusion and assume that this index takes the specific values in each of the subdomains.

We assume that the force generated by the vibration source is directed vertically, i.e., $P_1(x_1) = 0$, and the distribution of $P_2(x_1)$ is a piecewise, linear function, which vanish at all nodes of the boundary $x_2 = 0$ except for the node with the coordinates $x_1 = 0$ and $x_2 = 0$. At this node, P_1 is equal to F_2/h_1.

Introduce a uniform grid in time $t_m = m\tau$, $m = 0 : 1 : M$. The grid step τ must satisfy the condition

$$\tau \le \min\left(\min_k \frac{h_1}{\lambda_1^k},\ \min_k \frac{h_2}{\lambda_2^k}\right). \tag{30}$$

In each of the subdomains, we construct a system of basic polynomials $H_p(\boldsymbol{x})$, each of which is equal to 1 at the node corresponding to the index p, to 0 at all other nodes, and is a bilinear (i.e., linear with respect to each variable) function in each cell of the grid.

We approximate the solution in each of the subdomains by the linear combination

$$\boldsymbol{u}(t, \boldsymbol{x}) = \sum_p H_p(\boldsymbol{x}) \boldsymbol{u}^p(t).$$

Then the construction of an approximate solution of the initial-boundary problem for the system (27) with the conditions (28) on the inner boundaries on each temporal layer is reduced to the search for values at the nodes $\boldsymbol{u}^p(t_{m+1})$ if the values on the previous temporal layers $\boldsymbol{u}^p(t_m)$ are known.

For inner nodes of the subdomains, the values on the subsequent temporal layer are defined by the formulas

$$\boldsymbol{u}\big(t_{m+1}, \boldsymbol{x}\big) = \sum_{\boldsymbol{k}} \boldsymbol{C_k} \boldsymbol{u}\big(t_m, \boldsymbol{x} - \boldsymbol{\lambda_k} \tau\big). \tag{31}$$

The right-hand side of the formula (31) does not involves terms related to the conditions on the inner and outer boundaries. These terms are equal to 0 due to Remark 2.2. Indeed, by the condition (30), the straight line $\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{\lambda}$ passing through the point $(t_{m+1} > 0, \boldsymbol{x})$ intersects the inner and outer boundaries at the moment $t^* < t_m$.

For nodes lying on the outer boundaries, where the "transparent" boundary conditions are imposed, the values on the subsequent temporal layer can be calculated as a solution of the Riemann problem with the "transparent" boundary conditions (26).

For nodes on the outer boundaries, where the "free" boundary conditions are imposed, the values on the subsequent temporal node can be calculated as a solution of the Riemann problem with the boundary conditions (25). As conditions on the boundary, we take Eqs. (29).

Based on the above, we constructed a calculational algorithm and implemented it as a MATLAB software. The results of calculations confirm the high efficiency of this algorithm. Comparing with other computational algorithms, in particular, with the discontinuous Galerkin method, shows that for a given required accuracy, the computation time and the necessary memory requirements are significantly less. Since the solution on the subsequent temporal layer is calculated independently at each grid node, this algorithm admits parallelization.

The results of calculations confirm the hypothesis that the presence of fractures in the geological rock, where the contacting parts move relative to each other, can lead to the fact that elastic waves generated by a vibration source does not scatter over the hemisphere, but propagates as a narrow beam so that at significant depths, its energy density remains significant in order to cause various physicochemical processes. Inner boundaries (fractures) serves as walls of waveguides. Elastic waves practically does not pass through them, and the perturbation reaches considerable depths without scattering. These and other results of computational experiments will be described in detail in subsequent publications.

7. Conclusion. In this paper, for first-order hyperbolic systems of linear differential equations with constant coefficients, we constructed an approximate solution of the generalized Riemann problem with conjugation conditions on the discontinuities. Also, we obtained an approximate solution of the generalized Riemann problem with boundary conditions. For this purpose, a fundamental solution of the operator involved in the problem was constructed. This, in turn, made it possible to reduce the Riemann problem to the solution of a system of algebraic equations whose right-hand side depends on the values of the variables at the initial moment of time at a finite number of points.

Based on these solutions of the Riemann problem, we constructed and implemented a computational algorithm for finding a solution to the initial-boundary-value problem for first-order hyperbolic systems of linear differential equations with constant coefficients. In this case, the statement of the problem admits the existence of inner boundaries, on which solutions can have discontinuities and some conditions connecting the values of the variables on both sides of these boundaries must be fulfilled.

The computational algorithm developed was applied to the study of the propagation of elastic waves generated by a periodically operating vibration source in block-fractured geological media. The existence of fractures was taken into account the model as the presence of inner boundaries on which the "conditions of frictionless slip" are satisfied. Numerical experiments confirmed the high efficiency of this computational algorithm for the study of practical applied problems.

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