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ON ONE CLASS OF SINGULAR NOWHERE MONOTONE FUNCTIONS

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We construct a continuous nowhere monotone function that depends on infinitely many parameters such that the derivative of this function is equal to zero almost everywhere (in a sense of the Lebesgue measure). It is shown that this function is well-defined and nowhere monotone. Its differential properties are analyzed, the massiveness of the level sets is studied, and the set of maxima and minima of the function and its structural and variational properties are determined.

1. Introduction

It is known that each function of real variable of bounded variation is either a jump function, or an absolutely continuous function (improper integral of its derivative), or a singular function (whose derivative is equal to zero almost everywhere in a sense of Lebesgue measure), or a linear combination of the indicated functions (a composition of functions of the first three types).

Singular functions form a family of poorly studied functions of pure Lebesgue type [1, 2]. Significant interest in the investigation of these functions (more precisely, monotone singular functions) appeared in the probability theory [3], namely, in the distribution theory, despite the fact that just these function were completely ignored by these theories in the past [4]. In this field, singular functions are regarded as probability distribution functions [3].

The theory of fractals and the ideas of scaling (generalizations of self-similarity and self-affinity) simulated the creation of efficient methods for the theoretical (analytic) development of the theory of singular functions (statement, analysis, interpretation, and applications). Another class of poorly studied functions is formed by continuous nowhere monotone functions [5, 6]. The fact of existence of singular nowhere monotone functions is a nontrivial result and, moreover, the determination of massiveness of this class of functions in different spaces requires special investigations. The structure of functions from this class was determined in [7]. Later, it was discovered that several works dealing with these functions have been published earlier [8]. In the present paper, we construct a continual family of these functions and study their properties.

2. Basic Notions and Facts

In what follows, $2 < s$ is a fixed natural number, $A_s \equiv \{0, 1, \dots, s-1\}$ is an alphabet (collection of digits), and $L \equiv A \times A \times A \times \dots$ is the space of sequences of elements of the alphabet.

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In order to define the main object of our investigations, we use an s-symbol polybase Q_s^*-representation specified by an infinite stochastic matrix $Q_s^* = \|q_{ik}\|$, where $i = \overline{0, s-1}$, $k \in N$, with the following properties:

(i) $q_{ik} > 0, q_{0k} + q_{1k} + \ldots + q_{s-1,k} = 1, k \in N$;

(ii) $\prod_{k=1}^{\infty} \max_i \{q_{ik}\} = 0$.

It is based on the following statement:

Theorem 1 [9]. *For any* $x \in [0;1]$, *there exist at most two sequences* (α_n) *from the space* L *such that the equality*

$$x = \beta_{\alpha_1 1} + \sum_{k=2}^{\infty} \left(\beta_{\alpha_k k} \prod_{j=1}^{k-1} q_{\alpha_j j} \right) \equiv \Delta^{Q_s^*}_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}$$

holds for $\beta_{ik} \equiv q_{0k} + \ldots + q_{i-1,k}$.

The last symbolic notation $\Delta^{Q_s^*}_{\alpha_1 \alpha_2 \ldots \alpha_k \ldots}$ is called the Q_s^*-representation of a number x. Moreover, $\alpha_n = \alpha_n(x)$ is called the nth digit of this representation.

The properties of Q_s^*-representation of numbers and its geometry were well studied in [10]. A countable set of numbers has, in addition, two periodic representations

$$\Delta^{Q_s^*}_{c_1 \ldots c_m (0)} \qquad \text{and} \qquad \Delta^{Q_s^*}_{c_1 \ldots c_{m-1} [c_m - 1] (s-1)}.$$

These numbers are called Q_s^**-binary*, whereas the other numbers admit a unique Q_s^*-representation and are called Q_s^**-unary*.

A Q_s^**-cylinder* of rank m with base $c_1 c_2 \ldots c_m$ is defined as the set $\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m}$ of all numbers from the segment $[0;1]$ whose Q_s^*-representations are such that $\alpha_k(x) = c_k$, $k = \overline{1, m}$.

A Q_s^*-cylinder $\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m}$ is a segment with the ends $\Delta^{Q_s^*}_{c_1 \ldots c_m (0)}$ and $\Delta^{Q_s^*}_{c_1 \ldots c_m (s-1)}$ whose length is given by the formula

$$\left| \Delta^{Q_s^*}_{c_1 c_2 \ldots c_m} \right| = \prod_{i=1}^{m} q_{c_i i}.$$

It is clear that $\Delta^{Q_s^*}_{c_1 \ldots c_m c} \subset \Delta^{Q_s^*}_{c_1 \ldots c_m}$.

For any sequence $(c_n) \in L$, the following equality is true:

$$\bigcap_{m=1}^{\infty} \Delta^{Q_s^*}_{c_1 c_2 \ldots c_m} = \Delta^{Q_s^*}_{c_1 c_2 \ldots c_m \ldots}.$$

If $q_{ik} = q_i = \text{const}$ for any $k \in N$, $i \in A_s$, then a Q_s^*-representation is called a Q_s-representation. For

$$q_i = \frac{1}{s}, \quad i = \overline{0, s-1},$$

a Q_s-representation turns into the classical s*-ary representation* of numbers.

It is known [11] that the set $E = E[Q_s; q_0, \ldots, q_{s-1}]$ of numbers $x \in [0; 1]$ such that all digits of the alphabet A_s are used in their Q_s-representations and, moreover, the digit i is encountered with a frequency

$$\nu_i(x) \equiv \lim_{k\to\infty} \frac{N_i(x,k)}{k} = q_i,$$

where $N_i(x,k)$ is the number of digits i among $\alpha_1(x), \alpha_2(x), \ldots, \alpha_k(x)$, is a set of full Lebesgue measure.

3. Object of Investigations

Assume that a given matrix $G_s^* \equiv \|g_{ik}\|$, $i \in A_s$, $k \in N$, has the following properties:

(i) $|g_{ik}| < 1$, $g_{0k} + g_1 + \ldots + g_{[s-1]k} = 1$;

(ii) $\delta_{0k} \equiv 0$, $0 < \delta_{ik} \equiv \sum_{j=0}^{i-1} g_{jk} < 1$, $i = \overline{1, s-1}$, $\delta_{sk} \equiv 1$;

(iii) $\prod_{k=1}^{\infty} g_{i_k k} = 0$ for any sequence $(i_k) \in L$.

Conditions (i)–(iii) imposed on the matrix $G_s^* = \|g_{ik}\|$ are called initial.

Note that, for any $m \in N$, the matrix $G_s^*(m) \equiv \|g_{ik}\|$, where $0 \le i < s$, $m \le k \in N$, satisfies all initial conditions.

Lemma 1. *For any $i \in A_s \backslash \{0\}$ and $m \in N$, the equality*

$$\delta_{i-1,m} + g_{i-1,m}\left[\delta_{s-1,m+1} + \sum_{k=2}^{\infty}\left(\delta_{s-1,m+k}\prod_{j=1}^{k-1} g_{s-1,m+j}\right)\right] = \delta_{i,m} \tag{1}$$

is true.

Proof. We now show that the difference between the right- and left-hand sides of equality (1) is equal to 0. To this end, we note that $1 - \delta_{s-1,m+j} = g_{s-1,m+j}$ and consider the differences obtained as a result of term-by-term transfer of the left-hand side into the right-hand side:

$$\delta_{i,m} - \delta_{i-1,m} = g_{i-1,m},$$

$$g_{i-1,m} - \delta_{s-1,m+1} g_{i-1,m} = g_{i-1,m}(1 - \delta_{s-1,m+1}) = g_{i-1,m} g_{s-1,m+1},$$

$$g_{i-1,m} g_{s-1,m+1} - \delta_{s-1,m+2} g_{i-1,m} g_{s-1,m+1} = g_{i-1,m} q_{s-1,m+1} q_{s-1,m+2},$$

. . . .

$$g_{i-1,m}\prod_{j=1}^{k-1} q_{s-1,m+j} - \delta_{s-1,m+k+1} q_{i-1,m}\prod_{j=1}^{k-1} g_{s-1,m+j} = g_{i-1,m}\prod_{j=1}^{k} g_{s-1,m+j}.$$

In view of the fact that the last product tends to 0 as $k \to \infty$ (this is one of the initial conditions for the matrix $\|g_{ik}\|$), we can prove equality (1).

Corollary 1. *The equality*

$$\delta_{s-1,m+1} + \sum_{k=2}^{\infty}\left(\delta_{s-1,m+k}\prod_{j=1}^{k-1} g_{s-1,m+j}\right) = 1$$

is true.

Definition 1. *The function f is defined by the equality*

$$f\left(x = \Delta^{Q_s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) \equiv \delta_{\alpha_1 1} + \sum_{k=2}^{\infty}\left(\delta_{\alpha_k k}\prod_{j=1}^{k-1} g_{\alpha_j j}\right) \equiv \Delta^{G_s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}. \tag{2}$$

Remark 1. A similar function was considered in [12] but under the additional condition for the matrix $\|g_{ik}\|$, which yields property (ii). The elimination of this condition requires the revision of the main steps of substantiation of the well-posedness of definition of the functions, etc.

Theorem 2. *The definition of the function f by equality (2) is well-posed, the function is continuous on the segment* $[0; 1]$, *and its range coincides with the segment* $[0; 1]$.

Proof. To check the well-posedness of the definition of a function, it is sufficient to show that the expression specifying the function gives identical values for different representations of Q_s^*-binary numbers, i.e.,

$$f\left(\Delta^{Q_s^*}_{c_1\ldots c_m(0)}\right) = f\left(\Delta^{Q_s^*}_{c_1\ldots c_{m-1}[c_m-1](s-1)}\right),$$

which is a consequence of the previous lemma and the assertion that series (2) is convergent for any sequence $(\alpha_n) \in L$.

It is clear that

$$f(0) = f\left(\Delta^{Q_s^*}_{(0)}\right) = 0 \qquad \text{and} \qquad f(1) = \left(\Delta^{Q_s^*}_{(s-1)}\right) = 1.$$

We represent the function $f(x)$ in the form

$$f(x) = S_m(x) + \left(\prod_{j=1}^{m} g_{\alpha_j(x)j}\right)\left(\delta_{\alpha_{m+1}(x)[m+1]} + \sum_{k=m+2}^{\infty}\left(\delta_{\alpha_k(x)k}\prod_{j=m+1}^{k-1} g_{\alpha_j(x)j}\right)\right),$$

where

$$S_m(x) = \delta_{\alpha_1(x)1} + \sum_{k=1}^{m}\left(\delta_{\alpha_k(x)k}\prod_{j=1}^{k-1} g_{\alpha_j(x)j}\right)$$

is a partial sum of series (2). By induction, we can prove that $0 \le S_m < 1$ for any $m \in N$ and a collection of digits $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

For $m = 1$, it is clear that $S_1 = \delta_{\alpha_1 1} \in [0, 1)$ according to the conditions imposed on the matrix $\|g_{ik}\|$; moreover, $S_1 = 0$ only in the case where $\alpha_1 = 0$.

Consider $S_2 = \delta_{\alpha_1 1} + \delta_{\alpha_2 2} g_{\alpha_1 1}$. If $\alpha_1 = 0$, then

$$S_2 = \delta_{01} + \delta_{\alpha_2 2} g_{01} = \delta_{\alpha_2 2} g_{01}$$

and

$$0 < S_2 = \delta_{\alpha_2 2} g_{01} < \delta_{\alpha_2 2} < 1.$$

Let $\alpha_1 > 0$. If $g_{\alpha_1 1} > 0$, then

$$0 \leq \delta_{\alpha_1 1} < S_2 = \delta_{\alpha_1 1} + \delta_{\alpha_2 2} g_{\alpha_1 1} < \delta_{\alpha_1 1} + g_{\alpha_1 1} = \delta_{[\alpha_1+1]1} < 1.$$

Further, if $g_{\alpha_1 1} < 0$, then

$$0 \leq \delta_{[\alpha_1+1]1} = \delta_{\alpha_1 1} + g_{\alpha_1 1} < S_2 = \delta_{\alpha_1 1} + \delta_{\alpha_2 2} g_{\alpha_1 1} < \delta_{\alpha_1 1} < 1.$$

Thus, $0 \leq S_2 < 1$.

We now assume that $0 \leq S_k < 1$ for any sequence $(\alpha_k) \in L$ and consider S_{k+1}

$$S_{k+1} = \delta_{\alpha_1 1} + g_{\alpha_1 1}\left(S'_{k-1} + \delta_{\alpha_{k+1}[k+1]} \prod_{j=2}^{k} g_{\alpha_j j} \right) = \delta_{\alpha_1 1} + g_{\alpha_1 1} S''_k,$$

where S'_k and S''_k are partial sums of series (2). Since, by assumption, $0 \leq S''_k < 1$, we conclude that

$$0 \leq \delta_{\alpha_1 1} < S_{k+1} < \delta_{\alpha_1 1} + g_{\alpha_1 1} = \delta_{[\alpha_1+1]1} < 1$$

for $g_{\alpha_1} > 0$ and

$$0 \leq \delta_{[\alpha_1+1]1} = \delta_{\alpha_1 1} + g_{\alpha_1 1} < S_{k+1} < \delta_{\alpha_1 1} < 1$$

for $g_{\alpha_1} < 0$.

Hence, $0 \leq S_{k+1} < 1$ for any sequence (α_n).

Thus, for any $x \in [0, 1]$ and natural m, we get $0 \leq S_m < 1$ and, therefore,

$$0 \leq f(x) = \lim_{m\to\infty} S_m \leq 1.$$

We now prove the continuity of the function. Let x_0 be an arbitrary Q_2^*-unary point of the interval (0; 1). If $x \neq x_0$, then there exists $m \in N$ such that $\alpha_m(x) \neq \alpha_n(x_0)$ but $\alpha_i(x) \neq \alpha_i(x_0)$ for $i < m$. Moreover, the fact that $x \to x_0$ is equivalent to $m \to \infty$. Consider the modulus of the difference

$$|f(x) - f(x_0)| = \left| \prod_{i=1}^{m-1} g_{\alpha_i(x_0)i} \right| |M|,$$

$$M = \delta_{\alpha_m(x)m} - \delta_{\alpha_m(x_0)m} + \sum_{k=m+1}^{\infty} \left(\delta_{\alpha_k(x)k} \prod_{j=m+1}^{k-1} g_{\alpha_j(x)j} \right) - \sum_{k=m+1}^{\infty} \left(\delta_{\alpha_k(x_0)k} \prod_{j=m+1}^{k-1} g_{\alpha_j(x_0)j} \right).$$

Since $|M|$ is the number that does not exceed 1 (as the modulus of the difference of numbers from [0; 1]) and the first factor approaches zero as $m \to \infty$, we conclude that

$$|f(x) - f(x_0)| \to 0 \qquad (x \to x_0).$$

This implies that the function f is continuous at the point x_0.

To prove the continuity of the function at the Q_s^*-binary point x_0, we can use the same reasoning. However, in order to establish the left continuity of the function, it suffices to consider the representation of the number x_0 with period $(s-1)$. At the same time, to prove the right continuity of the function, it is sufficient to consider the representation of the number x_0 with period (0).

Theorem 2 is proved.

4. Properties of Monotonicity and Extrema of the Function

Lemma 2. *The increment*

$$\mu_f\left(\Delta^{Q_s^*}_{c_1c_2\ldots c_m}\right) \equiv f\left(\Delta^{Q_s^*}_{c_1c_2\ldots c_m(s-1)}\right) - f\left(\Delta^{Q_s^*}_{c_1c_2\ldots c_m(0)}\right)$$

of the function f on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m}$ is given by the formula

$$\mu_f\left(\Delta^{Q_s^*}_{c_1c_2\ldots c_m}\right) = \prod_{i=1}^{m} g_{c_i i}. \tag{3}$$

Proof. We represent this increment in the form

$$\mu_f\left(\Delta^{Q_s^*}_{c_1c_2\ldots c_m}\right) = \left(\prod_{i=1}^{m} g_{c_i i}\right)\left[\delta_{s-1,m+1} + \sum_{k=2}^{\infty}\left(\delta_{s-1,m+k}\prod_{j=1}^{k-1} g_{s-1,m+j}\right)\right].$$

By Lemma 1, the expression in the square brackets has the form

$$\frac{\delta_{s,m} - \delta_{s-1,m}}{g_{s-1,m}} = \frac{1 - \delta_{s-1,m}}{g_{s-1,m}} = 1.$$

Hence, equality (3) is true.

Theorem 3. *The function f is:*

(i) constant on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m}$ if and only if there exists $g_{c_k k} = 0$ for some $k \leqslant m$;

(ii) nondecreasing if the matrix $\|g_{ik}\|$ does not have negative elements and, moreover, strictly increasing if all elements of the matrix are positive;

(iii) nowhere monotone if the matrix $\|g_{ik}\|$ does not have zeros and there are negative numbers in the infinite number of columns.

Proof. (i) If $g_{c_k k} = 0$, where $k \le m$, then

$$\prod_{i=1}^{m} g_{c_i i} = 0.$$

Hence, for any $x \in \Delta^{Q_s^*}_{c_1 c_2 \ldots c_m}$, we find

$$f(x) = \delta_{c_1 1} + \sum_{i=1}^{k-1} \left(\delta_{c_i i} \prod_{j=1}^{i-1} g_{c_j j} \right).$$

Now let $f(x) = \text{const}$ for any $x \in \Delta^{Q_s^*}_{c_1 c_2 \ldots c_m}$. Then the equality

$$f\left(\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m 1(0)}\right) = f\left(\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m (0)}\right)$$

is true and, therefore,

$$g_{0,m+1} \prod_{i=1}^{m} g_{c_i i} = 0,$$

i.e., $g_{c_k k} = 0$ for some $k \le m$ because $g_{0,m+1} \ne 0$. Assertion (i) of the theorem is proved.

(ii) Let $g_{ik} > 0$ for all $i \in A_s$, $k \in N$. Then the expression for the value of the function is a polybase s-symbolic representation generated by the matrix G_s^*. Hence, in this case, Assertion (ii) of the theorem is evident. In other words, it follows from Assertion (i) and the previous lemma because the increments of the function on all cylinders are nonnegative.

(iii) To prove the nowhere monotonicity of the function under the imposed conditions, it suffices to show that it is not monotone on any cylinder. To this end, we consider an arbitrary Q_s^*-cylinder $\Delta^{Q_s^*}_{c_1 \ldots c_m}$.

Since the matrix $\|g_{ik}\|$ has infinitely many columns with negative elements, we consider its $(m+k)$th column that contains a negative element $g_{i,m+k}$ and the corresponding two cylinders:

$$\mu_f\left(\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m \underbrace{0 \ldots 0}_{k-1} 0}\right) \mu_f\left(\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m \underbrace{0 \ldots 0}_{k-1} i}\right) = \left(\prod_{j=1}^{m} g_{c_j j}\right)^2 \left(\prod_{j=1+m_0}^{m_0+k-1} g_{0j}\right)^2 g_{0,m+k} g_{i,m+k}.$$

Since $g_{0,m+k} > 0$, we get

$$g_{0,m+k} g_{i,m+k} < 0$$

and, hence, the function has a positive increment on one of the cylinders $\left(\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m \underbrace{0 \ldots 0}_{k-1} 0}\right.$ or $\left.\Delta^{Q_s^*}_{c_1 c_2 \ldots c_m \underbrace{0 \ldots 0}_{k-1} i}\right)$ and a negative increment on the other cylinder. Thus, the increments of the function on the indicated cylinders contained in the cylinder $\Delta^{Q_s^*}_{c_1 \ldots c_m}$ have different signs. Therefore, the function f is not monotone on the cylinder $\Delta^{Q_s^*}_{c_1 \ldots c_m}$. Thus, it is nowhere monotone.

Theorem 4. *(i) If* $g_{i,m+1}g_{i+1,m+1} < 0$ *for some* i*, then a* Q_s^**-binary point of the form* $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i(0)}$ *is a point of extremum of the function* f *and, moreover,*

(i.1) a point of maximum for $D_m g_{i,m+1} > 0$;

(i.2) a point of minimum for $D_m g_{i,m+1} < 0$;

where

$$D_m = \prod_{k=1}^{m} g_{c_k k} \neq 0$$

is the increment of the function on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m}$.

(ii) if $g_{i,m+1}g_{i+1,m+1} \geq 0$, *then any point of the form* $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i(0)}$ *is not a point of extremum of the function* f.

Proof. (i). Let

$$D_m = \prod_{k=1}^{m} g_{c_k k} \neq 0.$$

We consider all possible cases.

(i.1). Let $D_m > 0$. If $g_{i+1,m+1} > 0$, then the function has a positive increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m[i+1]}$ and a negative increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i}$ lying to the left. Hence, the point $x_i \equiv \Delta^{Q_s^*}_{c_1c_2\ldots c_m i(0)}$, which is the common end of these cylinders, is a point of maximum.

If $g_{i+1,m+1} < 0$, then the function has a negative increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m[i+1]}$ and a positive increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i}$. Therefore, the point x_i is a point of minimum.

(i.2). Let $D_m < 0$. If $g_{i+1,m+1} > 0$, then the function has a negative increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m[i+1]}$ and a positive increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i}$. Hence, the point x_i is a point of minimum.

If $g_{i+1,m+1} < 0$, then the function has a positive increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m[i+1]}$ and a negative increment on the cylinder $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i}$. Thus, the point x_i is a point of maximum.

(ii). If $g_{i,m+1}g_{i+1,m+1} = 0$, then, by Theorem 3, the function is constant on at least one of the cylinders $\Delta^{Q_s^*}_{c_1c_2\ldots c_m i}$ or $\Delta^{Q_s^*}_{c_1c_2\ldots c_m[i+1]}$ and, therefore, the point x_i is not a point of extremum.

If $g_{i,m+1}g_{i+1,m+1} > 0$, then the function f has increments of the same sign on both cylinders and, hence, the point x_i is not a point of extremum because it is a point of maximum for one cylinder and a point of minimum for the other cylinder.

5. Variational Properties

Theorem 5 [12]. *For the cylinder* $\Delta^{Q_s^*}_{c_1c_2\ldots c_m} = [u; v]$, *the function* f *takes its maximum and minimum values at the ends. Moreover, if*

$$D_m \equiv \prod_{i=1}^{m} g_{c_i i} \neq 0, \qquad y_m = \delta_{c_1 1} + \sum_{k=2}^{m}\left(\delta_{c_k k}\prod_{i=1}^{k-1} g_{c_i i}\right),$$

then $\max f(x) = f(v) = y_m + D_m$ *and* $\min f(x) = f(u) = y_m$ *for* $D_m > 0$ *or* $\max f(x) = f(u) = y_m$ *and* $\min f(x) = f(v) = y_m + D_m$ *for* $D_m < 0$.

Theorem 6. *The variation of the function* $f(x)$ *is given by the formula*

$$V_0^1(f) = \prod_{n=1}^{\infty}\left(\sum_{i=0}^{s-1}|g_{in}|\right). \tag{4}$$

Proof. In view of the fact that, on each cylinder, the function f takes its maximum and minimum values at the ends, we consider the sums of oscillations of the function on cylinders of the first rank:

$$V_1 = \sum_{i=0}^{s-1}\left|f\left(\Delta^{Q_s^*}_{i+1,(0)}\right) - f\left(\Delta^{Q_s^*}_{i(0)}\right)\right| = \sum_{i=0}^{s-1}|g_{i1}|,$$

of the second rank

$$V_2 = \sum_{j=0}^{s-1}|g_{i_1 1}|\sum_{i=1}^{s-1}\left|f\left(\Delta^{Q_s^*}_{i_1,i_2+1,(0)}\right) - f\left(\Delta^{Q_s^*}_{i_1,i_2(0)}\right)\right| = \left(\sum_{j=0}^{s-1}|g_{i_1 1}|\right)\left(\sum_{i=0}^{s-1}|g_{i_2 2}|\right),$$

and of the nth rank

$$V_n = \left(\sum_{i_1=0}^{s-1}|g_{i_1 1}|\right)\left(\sum_{i_2=0}^{s-1}|g_{i_2 2}|\right)\ldots\left(\sum_{i_n=0}^{s-1}|g_{i_n n}|\right) = \prod_{k=1}^{n}\left(\sum_{i_k=0}^{s-1}|g_{i_k k}|\right).$$

Thus, for any natural n,

$$V_n \le V_0^1(f) \le \limsup_{n\to\infty} V_n.$$

Since the sequence (V_n) is monotone, we get

$$V_0^1(f) = \lim_{n\to\infty} V_n = \prod_{n=1}^{\infty}\left(\sum_{i=0}^{s-1}|g_{in}|\right).$$

We represent the variation in the form

$$V_0^1(f) = \prod_{n=1}^{\infty}\left(1 - \left(1 - \sum_{i=0}^{s-1}|g_{in}|\right)\right).$$

Corollary 2. *The function* $f(x)$ *is a function of bounded variation if and only if*

$$W \equiv \sum_{n=1}^{\infty}\left(\sum_{i=0}^{s-1}|g_{in}| - 1\right) < \infty. \tag{5}$$

Corollary 3. *In order that f be a function of unbounded variation, it is necessary and sufficient that $W = \infty$. In particular, if*

$$u_n = \sum_{i=0}^{s-1} |g_{in}| \nrightarrow 1, \qquad n \to \infty,$$

then $V_0^1(f) = \infty$.

If all columns of the matrix G_s^* are identical and, moreover, contain negative elements, then f is a function of unbounded variation.

6. Differential Properties of the Function

Lemma 3. *If the function f has a finite derivative $f'(x_0)$ at the point x_0, then it is given by the formula*

$$f'(x_0) = \prod_{k=1}^{\infty} \frac{g_{\alpha_k(x_0)k}}{q_{\alpha_k(x_0)k}}. \tag{6}$$

Proof. It is known that if the function f has a finite derivative, then it is equal to the cylindrical derivative

$$f'\left(x_0 = \Delta^{Q_s^*}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \lim_{n\to\infty} \frac{f\left(\Delta^{Q_s^*}_{\alpha_1\ldots\alpha_n(s-1)}\right) - f\left(\Delta^{Q_s^*}_{\alpha_1\ldots\alpha_n(0)}\right)}{\left|\Delta^{Q_s^*}_{\alpha_1\ldots\alpha_n}\right|}$$

$$= \lim_{n\to\infty} \frac{\prod_{i=1}^{n} g_{\alpha_i i}}{\prod_{i=1}^{n} q_{\alpha_i i}} = \lim_{n\to\infty} \prod_{i=1}^{n} \frac{g_{\alpha_i i}}{q_{\alpha_i i}},$$

i.e., equality (6) is true.

Theorem 7. *If the function f has a finite variation, i.e., condition (5) is satisfied and, for all possible $x \in [0; 1]$, the limit of the sequence $\left(\frac{g_{\alpha_k(x_0)k}}{q_{\alpha_k(x_0)k}}\right)$ either does not exist or differs from 1, then f is a singular function (its derivative is equal to zero almost everywhere in a sense of Lebesgue measure).*

Proof. It is known that every continuous function of bounded variation is equal to the difference of two monotone functions and, hence, has a finite derivative almost everywhere in the sense of Lebesgue measure.

Let x_0 be an arbitrary point of the set of full Lebesgue measure in which the derivative exists and is finite. Hence, by the previous lemma, it is given by relation (6) but, under the conditions of the theorem, the necessary condition of convergence of the infinite product is not satisfied. Thus, it diverges to zero. Therefore, the function f is singular.

Corollary 4. *If, under the conditions of the theorem, the matrix $\|g_{ik}\|$ does not have zero elements but has infinitely many negative elements, then the function f is a singular nowhere monotone function.*

7. Special Case

We now consider a simple but interesting special case. Consider the function f under the following restrictions imposed on the matrix G_s^*:

1) $s = 2k-1$, $2 \le k \in N$, $0 < b_1 < 1$, $0 < q < 1$, $b_n = b_1 q^{n-1}$;

2) the nth column of the matrix is formed by the elements

$$g_{0n} = g_{s-1,n} = \frac{1+b_n}{2},$$

$$g_{1n} = g_{3n} = \ldots = g_{s-2,n} = -b_n,$$

$$g_{2n} = g_{4n} = \ldots = g_{s-3,n} = b_n.$$

In this case, we have $\delta_{0n} = 0$, $\delta_{1n} = \delta_{3n} = \delta_{2k-1,n} = \frac{1+b_n}{2}$, $\delta_{2n} = \delta_{4n} = \delta_{2k,n} = \frac{1-b_n}{2}$, and $u_n \equiv \sum_{i=0}^{s-1} |g_{in}| - 1 = 2(k-1)b_n$. Then

$$\sum_{n=1}^{\infty} u_n = \frac{2(k-1)b_1}{1-q} < \infty,$$

Thus, according to Corollary 2, the function f has a bounded variation. Moreover, by Theorem 3, this function is nowhere monotone and, by Theorem 7, it is singular. We now focus our attention on its fractal properties, namely, on the properties of the level sets of functions.

Recall that a *level set* y_0 *of the function* f is defined as the set $\{x\colon f(x) = y_0\}$ and denoted by $f^{-1}(y_0)$.

Let

$$C\left[Q_s^*; B_n\right] \equiv \left\{x\colon x = \Delta^{Q_s^*}_{\alpha_1(x)\alpha_2(x)\ldots\alpha_n(x)\ldots},\ \ \alpha_n(x) \in B_n \subset A_s,\ \ n \in N\right\},$$

$$V \equiv A_s \setminus \{0, s-1\}, \quad V_0 \equiv \{2, 4, \ldots, s-2\}, \quad V_1 \equiv \{1, 3, \ldots, s-3\}.$$

Lemma 4. *A set* $C\left[Q_s^*; V\right]$ *completely belongs to the set* $f^{-1}\left(\frac{1}{2}\right)$ *of the level*

$$y_0 = \frac{1}{2}.$$

It is a continual perfect nowhere dense set and, depending on the matrix $Q_s^* = \|q_{ik}\|$, *may have either null or positive Lebesgue measure, namely, it has a positive Lebesgue measure if and only if the positive series*

$$\sum_{k=1}^{\infty} \frac{q_{0k} + q_{s-1,k}}{q_{1k} + q_{2k} + \ldots + q_{s-2,k}}$$

is convergent. If the Q_s^*-representation is the Q_s-representation, then the set $C\left[Q_s^*; V\right]$ is a self-similar Cantor-type set whose fractal Hausdorff–Besicovitch dimensionality is a solution of the equation

$$\sum_{i=1}^{s-2} q_i^x = 1, \quad \text{i.e.,} \quad x = \log_{q_1,\ldots,q_{s-2}} 1. \tag{7}$$

Proof. Since the properties of the set $C\left[Q_s^*, V\right]$ are well known [9, 10], it is necessary to prove solely the first part of the lemma.

Note that $f\left(\Delta_{(i)}^{Q_s^*}\right) = \frac{1}{2}$ for $i = 1, 2, \ldots, s-2$. Indeed, we have

$$f\left(\Delta_{(i)}^{Q_s^*}\right) = \frac{1-b_1}{2} + \sum_{k=2}^{\infty}\left(\frac{1-b_k}{2}\prod_{j=1}^{k-1} b_j\right) = \frac{1}{2}$$

for even i and

$$f\left(\Delta_{(i)}^{Q_s^*}\right) = \frac{1+b_1}{2} + \sum_{k=2}^{\infty}\left(\frac{1+b_k}{2}\prod_{j=1}^{k-1} (-b_j)\right) = \frac{1}{2}$$

for odd i.

It is clear that, in the case where the digits i and j have the same parity and differ from 0 and $(s-1)$, we have

$$f\left(\Delta_{\alpha_1\alpha_2\ldots\alpha_{k-1}i\alpha_{k+1}\ldots}^{Q_2^*}\right) = f\left(\Delta_{\alpha_1\alpha_2\ldots\alpha_{k-1}j\alpha_{k+1}\alpha_{k+2}\ldots}^{Q_s^*}\right).$$

Hence, the set $f^{-1}\left(\frac{1}{2}\right)$ of preimages of the number $\frac{1}{2}$ under mapping f includes both sets $C\left[Q_s^*, V_i\right]$, where V_i is the set of all digits of the alphabet A_s of the same parity.

Moreover, if $\alpha_n \in V$, then

$$f\left(x = \Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots}^{Q_s^*}\right) = \frac{1}{2}.$$

Hence,

$$f^{-1}\left(\frac{1}{2}\right) \supset C\left[Q_s^*, V\right].$$

It is known [13] that the Lebesgue measure of the set $C\left[Q_s^*, B_n\right]$ is given by the formula

$$\lambda[C] = \prod_{k=1}^{\infty}\left[1 - \frac{\lambda(\overline{F}_k)}{\lambda(F_{k-1})}\right],$$

where F_k is the inion of cylinders of rank k whose interior points contain points of the set $C\left[Q_s^*, B_n\right]$, $\overline{F}_k \equiv F_{k-1} \setminus F_k$, namely:

$$F_k = \sum_{c_1\in B_1}\cdots\sum_{c_k\in B_k}\left|\Delta_{c_1\ldots c_k}^{Q_s^*}\right|, \qquad \overline{F}_k = \sum_{c_1\in B_1}\cdots\sum_{c_{k-1}\in B_{k-1}}\sum_{c\in A_s\setminus B_k}\left|\Delta_{c_1\ldots c_k}^{Q_s^*}\right|.$$

For the set $C\left[Q_s^*, V\right]$, we get $F_k = q_{1k} + \ldots + q_{s-2,k}$ and $\overline{F}_k = q_{0k} + q_{s-1,k}$. Thus, according to the theorems on relationship between the convergences of infinite products and series, we get

$$\lambda(C) > 0 \quad \Leftrightarrow \quad \sum_{k=1}^{\infty} \frac{q_{0k} + q_{s-1,k}}{q_{1k} + q_{2k} + \ldots + q_{s-2,k}} < \infty. \tag{8}$$

The structure of self-similarity of the set $C = C\left[Q_s^*, V\right]$ has the form

$$C\left[Q_s^*, V\right] = C_1 \cup C_2 \cup \ldots \cup C_{s-2},$$

where the set C is similar to the set $C_i \equiv \Delta_i^{Q_s^*} \cap C$ with coefficient q_i.

Since C satisfies the condition of an open set, its self-similar dimension, which is a solution of Eq. (7), coincides with its Hausdorff–Besicovitch fractal dimension.

Theorem 8. *Every binary rational number y_0 is the image of a continual fractal number set whose Q_s^*-representation does not contain the digits* 0 *and* $(s-1)$ *starting from a certain position.*

Proof. Let $y_0 = \Delta^2_{c_1 \ldots c_m 1(0)}$ be an arbitrary binary rational number. Every number $x = \Delta^{Q_s^*}_{\alpha_1 \ldots \alpha_m \ldots}$, where $\alpha_j \in V$ for $j > m$ and

$$\alpha_i = \begin{cases} 0 & \text{for} \quad c_i = 0, \\ 1 & \text{for} \quad c_i = 1, \quad i = \overline{1, m}, \end{cases}$$

is the preimage of y_0.

Hence, the set $f^{-1}(y_0)$ of preimages of the number y_0 under the mapping f contains the set $C\left[Q_s^*, B_n\right]$, where

$$B_n = \begin{cases} \{0\}, & c_n = 0, \\ \{1\}, & c_n = 1 \end{cases}$$

for $n \le m$ and $B_{m+k} = V$.

Then the Hausdorff–Besicovitch fractal dimension of the set $f^{-1}(y_0)$ is not smaller than the dimension of the set $C\left[Q_s^*; V\right]$.

Corollary 5. *If the matrix $Q_s^* = \|q_{ik}\|$ satisfies condition (8), then all binary rational numbers of the segment* $[0; 1]$ *are atoms of the distribution of values of the random variable $Y = f(X)$, where X is a random variable with uniform distribution on* $[0; 1]$.

Corollary 6. *If the Q_s^*-representation is the classical s-ary representation, then*

$$\int_0^1 f(x)dx = \frac{1}{2}.$$

Furthermore, if y_0 is an s-ary rational number, then the Hausdorff–Besicovitch fractal dimension of the level set y_0 is equal to $\log_s(s-2)$.

REFERENCES

1. M. V. Prats'ovytyi and A. V. Kalashnikov, "Self-affine singular and nowhere monotone functions related to the Q-representation of real numbers," *Ukr. Mat. Zh.,* **65**, No. 3, 405–417 (2013); ***English translation:*** *Ukr. Math. J.,* **65**, No. 3, 448–462 (2013).
2. M. V. Prats'ovytyi and O. V. Svynchuk, "Spread of values of a Cantor-type fractal continuous nonmonotone function," *Nelin. Kolyv.,* **21**, No. 1, 116–130 (2018); ***English translation:*** *J. Math. Sci.,* **240**, No. 3, 342–357 (2019).
3. A. F. Turbin and N. V. Pratsevityi, *Fractal Sets, Functions, and Distributions* [in Russian], Naukova Dumka, Kiev (1992).
4. W. Feller, *An Introduction to Probability Theory and Its Applications,* Vol. 1, Wiley, New York (1970).
5. N. V. Pratsevityi, "Continuous Cantor projectors," in: *Methods for the Investigation of Algebraic and Topological Structures* [in Russian], Kiev Pedagogical Institute, Kiev (1989), pp. 95–105.
6. M. Jarnicki and P. Pflug, *Continuous Nowhere Differentiable Function. The Monsters of Analysis,* Springer, Cham (2015).
7. M. V. Prats'ovytyi "Nowhere monotone singular functions," *Nauk. Chasopys. Nats. Ped. Univ., Ser. 1, Fiz.-Mat. Nauk.,* No 12, 24–36 (2011).
8. A. N. Agadzhanov, "Singular functions that do not have intervals of monotonicity in problems of finite control by distributed systems," *Dokl. Akad. Nauk ,* **454**, Issue 5, 503–506 (2014).
9. G. M. Torbin and N. V. Pratsevityi, "Random variables with independent Q^*-signs," in: *Random Evolutions: Theoretical and Applied Problems* [in Russian], Kiev (1992), pp. 95–104.
10. M. V. Prats'ovytyi, *Fractal Approach in the Investigation of Singular Distributions* [in Ukrainian], National Pedagogic University, Kyiv (1998).
11. I. V. Zamrii and M. V. Prats'ovytyi, "Singularity of the digit inversor for the Q_3-representation of the fractional part of a real number, its fractal and integral properties," *Nelin. Kolyv.,* **18**, No. 1, 55–70 (2015); ***English translation:*** *J. Math. Sci.,* **215**, No. 3, 323–340 (2016).
12. M. V. Prats'ovytyi and O. V. Svynchuk, "Singular nonmonotone functions defined in terms of Q_s^*-representation of argument," *Nauk. Chasopys. Nats. Ped. Univ., Ser. 1, Fiz.-Mat. Nauk.,* **15**, 144–155 (2013).
13. S. Albeverio, M. Pratsiovytyi, and G. Torbin, *Fractal probability distributions and transformations preserving the Hausdorff–Besicovitch dimension*, Ergodic Theory Dynam. Systems, **24**, 1–16 (2004).