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CONTACT INTERACTION OF A PRESTRAINED THICK PLATE WITH PARABOLIC PUNCH

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Within the framework of linearized statement of the problem of elasticity theory, we study the stress-strain state of a prestrained thick plate modeled by a prestressed half space in the case of its smooth contact interaction with a rigid axisymmetric parabolic punch. The dual integral equations of this problem are solved by representing the required functions in the form of partial sums of series of Bessel functions with unknown coefficients. For their determination, we deduce finite systems of linear algebraic equations. We also analyze the influence of initial strains and the shape of the punch on the level and character of the contact stresses and vertical displacements of the boundary plane of the plate for the cases of compressible and incompressible solids. The accumulated results are illustrated for a plate with the Bartenev–Khazanovich elastic potential and with the harmonic-type potential.

Key words: contact stresses, initial strains, parabolic punch, thick plate, elastic half space.

Introduction

The numerical analysis of the strength of structural elements and mechanisms is one of the important stages of their design. To estimate the strength of contacting bodies, it is necessary to find the contact stresses and strains. The problem of minimization of the errors of these calculations requires the analysis of the maximal possible number of factors affecting the contact interaction of the bodies and, in particular, of the presence of residual (initial) stresses or strains.

The influence of initial stresses on the contact interaction of bodies for various specific forms of the elastic potential was studied by numerous domestic and foreign researchers. In general, for the solution of problems of this kind, it is necessary to involve the tools of nonlinear elasticity theory. However, for sufficiently high levels of initial strains, it is possible to restrict ourselves to the linearized statement. In particular, for the linearized statement of problems of the elasticity theory, a three-dimensional finite-element model aimed at the investigation of microstrains in the joints strengthened by isotropic and anisotropic fibers was proposed in [5, 6]. In [1], one can find a detailed survey of the literature devoted to the analysis of contact problems for bodies with initial stresses or strains. On the basis of this survey, we can make, in particular, a conclusion that the problem of analysis of the interaction of punches of complex shapes with prestressed thick plates has been studied quite poorly.

The aim of the present paper is to propose a procedure for the evaluation of the axisymmetric stress-strain state of a preliminary stressed thick plate in its contact interaction with a rigid punch and study the influence of initial strains and the shape of the punch on the distribution of contact stresses and vertical displacements in the boundary plane of the plate.

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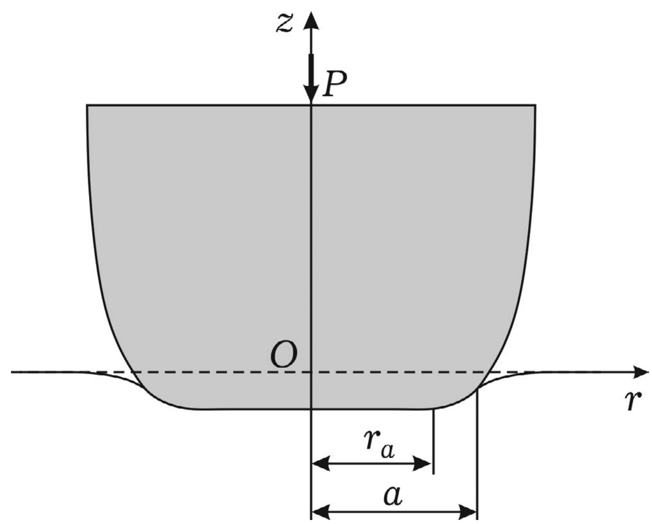

Fig. 1. Schematic diagram of contact interaction.

1. Statement of the Problem

Consider a problem of indentation of a rigid punch of complex shape with a constant force P into a pre-strained thick plate modeled by an elastic half space. We introduce a cylindrical coordinate system $Or\theta z$ such that the coordinate plane $Or\theta$ coincides with the boundary plane of the half space, and the Oz-axis coincides with the direction of action of the force P (Fig. 1). It is assumed that the punch is formed by the rotation of the line

$$W(r) = \begin{cases} 0, & 0 \le r \le r_a, \\ (r-r_a)^2/(2R), & r_a < r, \end{cases}$$

about the Oz-axis. Here, R is the focal parameter of parabola and r_a is the boundary of the plane section of the punch footing.

Suppose that the radius $a \ge r_a$ of the contact zone is known. Then the applied force is determined from the condition

$$P = -2\pi \int_0^a r\sigma_{zz}(r,0)\,dr \tag{1}$$

and the boundary conditions of the problem take the form

$$\sigma_{rz}(r) = 0, \quad 0 \le r < \infty, \tag{2}$$

$$\sigma_{zz}(r) = 0, \quad a < r, \tag{3}$$

$$u_z(r) = f(r), \quad 0 \le r \le a. \tag{4}$$

The function $f(r)$ corresponds to the shape of boundary surface of the punch. We choose it in the form $f(r) = u_z(a) + \omega(r)$. As a result, condition (4) takes the form

$$u_z(r) - u_z(a) = \omega(r), \quad 0 \le r \le a, \tag{5}$$

where

$$\omega(r) = \begin{cases} -(r_a - a)^2/(2R), & 0 \le r \le r_a, \\ \left((r_a - r)^2 - (r_a - a)^2\right)/(2R), & r_a < r \le a. \end{cases}$$

2. Construction of the Solution

We assume that the residual stresses formed in the half space are homogeneous. Then the expressions for the components of the stress tensor and displacement vector take the form [4]:

$$\sigma_{rz}(r,z) = -\frac{c_{44}(1+m_1)}{\sqrt{n_1}} \int_0^\infty \alpha^3 \Big((A_1 + A_2(s_0 + \alpha z))\, e^{\alpha z}$$

$$+ (B_1 + B_2(s_0 - \alpha z))\, e^{-\alpha z} \Big) J_0(\alpha r)\, d\alpha,$$

$$\sigma_{zz}(r,z) = c_{44}(1+m_1)\, \ell_1 \int_0^\infty \alpha^3 \Big((A_1 + A_2(s + \alpha z))\, e^{\alpha z}$$

$$+ (B_1 + B_2(s - \alpha z))\, e^{-\alpha z} \Big) J_0(\alpha r)\, d\alpha,$$

$$u_r(r,z) = -\int_0^\infty \alpha^2 \Big((A_1 + A_2(1 + \alpha z))\, e^{\alpha z}$$

$$+ (B_1 + B_2(1 + \alpha z))\, e^{-\alpha z} \Big) J_1(\alpha r)\, d\alpha,$$

$$u_z(r,z) = \frac{m_1}{\sqrt{n_1}} \int_0^\infty \alpha^2 \Big((A_1 + A_2(s_1 + \alpha z))\, e^{\alpha z}$$

$$+ (B_1 + B_2(s_1 - \alpha z))\, e^{-\alpha z} \Big) J_0(\alpha r)\, d\alpha. \tag{6}$$

Here, c_{44}, m_1, n_1, ℓ_1, s, s_0, and s_1 are constants depending on the elastic potential [4] and A_i and B_i are unknown functions that can be found from the boundary conditions.

On the boundary plane of the half space $z=0$, with the use of notation $F_j = A_j + B_j$, $j=1,2$, from expressions (6), we obtain

$$\sigma_{rz}(r,0) = \frac{c_{44}(1+m_1)}{\sqrt{n_1}} \int_0^\infty \alpha^3 (F_1 + s_0 F_2) J_0(\alpha r)\, d\alpha, \tag{7}$$

$$\sigma_{zz}(r,0) = c_{44}(1+m_1)\,\ell_1 \int_0^\infty \alpha^3 (F_1 + s F_2) J_0(\alpha r)\, d\alpha, \tag{8}$$

$$u_z(r,0) = \frac{m_1}{\sqrt{n_1}} \int_0^\infty \alpha^2 (F_1 + s_1 F_2) J_0(\alpha r)\, d\alpha. \tag{9}$$

Satisfying the boundary condition (2), by using (7), we arrive at the following relation for the functions F_1 and F_2:

$$F_1 = -s_0 F_2. \tag{10}$$

In view of (10), we can write the expressions for the normal stresses (8) and vertical displacements (9) in the form

$$\sigma_{zz}(r) = c_{44}(1+m_1)(s-s_0)\ell_1 \int_0^\infty \alpha^3 F_2 J_0(\alpha r)\, d\alpha, \tag{11}$$

$$u_z(r) = \frac{m_1(s_1-s_0)}{\sqrt{n_1}} \int_0^\infty \alpha^2 F_2 J_0(\alpha r)\, d\alpha. \tag{12}$$

Satisfying condition (3) with regard for expression (11), we obtain

$$c_{44}(1+m_1)(s-s_0)\ell_1 \int_0^\infty \alpha^3 F_2 J_0(\alpha r)\, d\alpha = 0, \quad a<r. \tag{13}$$

Further, we introduce an unknown function $x(r)$, $0 \le r \le a$, with the help of which relation (13) can be extended to the interval $0 \le r < \infty$:

$$c_{44}(1+m_1)(s-s_0)\ell_1 \int_0^\infty \alpha^3 F_2 J_0(\alpha r)\, d\alpha = x(r)\eta(a-r), \quad 0 \le r < \infty, \tag{14}$$

where $\eta(r)$ is the unit Heaviside function.

The function $x(r)$ specifies the distribution of contact stresses under the punch. In view of their continuity and equality to zero on the boundary of the contact zone (for $r = a$), we can represent this function as a partial sum of the generalized Fourier series in the functions $J_0(\lambda_n r/a)$:

$$x(r) = \sigma_{zz}(r) = \sum_{n=1}^{N} a_n J_0\left(\frac{\lambda_n r}{a}\right), \quad 0 \le r \le a, \tag{15}$$

where λ_n are positive roots of the Bessel function $J_0(\lambda_n) = 0$, $n = 1, \ldots, N$, and a_n are unknown coefficients.

Applying the formula of inversion for the Hankel integral transformation to relation (14), we arrive at the expression

$$\alpha^2 F_2 = \frac{1}{c_{44}(1+m_1)(s-s_0)\ell_1} \sum_{n=1}^{N} a_n \int_0^a r J_0\left(\frac{\lambda_n}{a} r\right) J_0(\alpha r)\, dr, \qquad 0 \le \alpha < \infty. \tag{16}$$

By using relations (12), (16) and condition (5), we find

$$k_1 \sum_{n=1}^{N} a_n \int_0^\infty \Psi_n(\alpha)\left[J_0(\alpha r) - J_0(\alpha a)\right] d\alpha = \omega(r), \quad 0 \le r \le a, \tag{17}$$

where

$$k_1 = \frac{m_1(s_1 - s_0)}{c_{44}(1+m_1)(s-s_0)\ell_1\sqrt{n_1}}, \qquad \text{and} \qquad \Psi_n(\alpha) = \int_0^a r J_0\left(\frac{\lambda_n}{a} r\right) J_0(\alpha r)\, dr.$$

Multiplying (17) by $r J_0(\lambda_q r/a)$ and integrating the result with respect to r from 0 to a, we get

$$\sum_{n=1}^{N} a_n \int_0^\infty \Psi_n(\alpha)\left(\Psi_q(\alpha) - K_q J_0(\alpha a)\right) d\alpha = \frac{w_q}{k_1}, \quad q = 1, \ldots, N, \tag{18}$$

where

$$K_q = \int_0^a r J_0\left(\frac{\lambda_q}{a} r\right) dr, \qquad \text{and} \qquad w_q = \int_0^a r\omega(r) J_0\left(\frac{\lambda_q}{a} r\right) dr.$$

Relations (18) specify the system of N linear algebraic equations for unknowns a_n.

The relationship between the focal parameter of the parabola R and the applied force follows from relation (1), namely,

$$R = -\frac{\pi}{2k_1 P}\sum_{n=1}^{N} a_n^* K_n. \tag{19}$$

Here,

$$a_n^* = a_n/(2k_1 R). \tag{20}$$

By using (19), with the help of (15) and (20), we establish the law of distribution of contact stresses under the punch:

$$\sigma_{zz}(r) = -\frac{P}{2\pi}\left(\sum_{n=1}^{N} a_n^* K_n\right)^{-1}\sum_{n=1}^{N} a_n^* J_0\left(\frac{\lambda_n}{a} r\right). \tag{21}$$

Moreover, in view of relations (12), (16), and (19), we get the following formula for the vertical displacements of points of the boundary plane of the half space:

$$u_z(r) = -\frac{k_1 P}{2\pi}\left(\sum_{n=1}^{N} a_n^* K_n\right)^{-1}\sum_{n=1}^{N} a_n^* \int_0^{\infty} \Psi_n(\alpha) J_0(\alpha r)\, d\alpha. \tag{22}$$

4. Examples of Numerical Analyses

To estimate the efficiency of the proposed procedure, we compare the obtained approximate solution with the exact solution of the problem of contact of a parabolic punch with an isotropic half space [2]:

$$\sigma_{zz}(r) = -\frac{3P}{2\pi a^3}\sqrt{a^2 - r^2}. \tag{23}$$

In Fig. 2, the solid line corresponds to function (23), while the dashed line corresponds to function (21) with $a = 1$ and $r_a = 0$ (there is no plane domain at the footing of the punch). As follows from this figure, the deviation of the approximate solution from the exact solution does not exceed 2%.

The coefficient k_1 characterizes the influence of initial strains on the stresses and displacements (6) and depends on the structure of elastic potential of the preliminary stressed plate. In particular, for the Bartenev–Khazanovich potential [3], we have

$$k_1 = 2\frac{1+\nu}{E}\frac{\lambda_1^{7/2}}{3\lambda_1^3 - 1},$$

where ν is Poisson's ratio, and E is Young's modulus of the material of half space. As follows from the presented relation, $k_1 \to \infty$ as $\lambda_1^3 \to 1/3$, i.e., as $\lambda_1 \to \lambda_{kp} \approx 0.693$. The value of λ_{kp} corresponds to the surface instability under uniform biaxial compression. In this case, as follows from relations (19), (21), and (22),

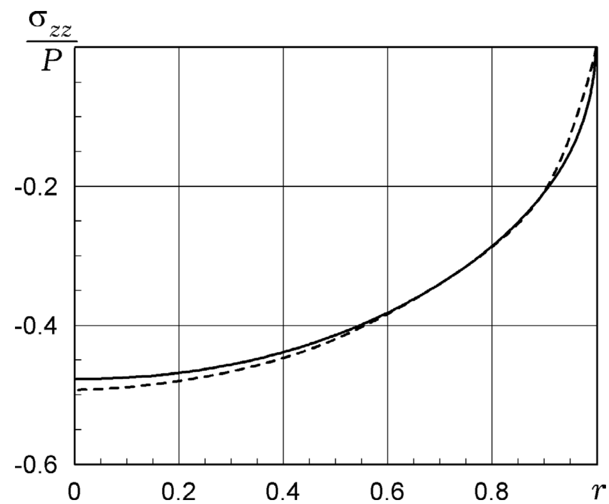

Fig. 2. Comparison of the exact (23) and approximate (21) solutions.

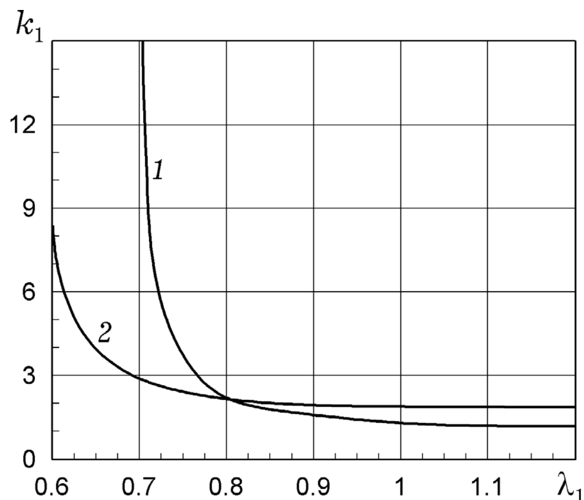

Fig. 3. Dependence of k_1 on λ_1.

the vertical displacements of points of the boundary plane of the half space are infinitely increasing and the contact stresses are absent. Hence, we observe the following mechanical effect: As λ_1 approaches the critical value λ_{kp}, the phenomena of "resonance nature" discovered by O. M. Guz' for the problems of brittle fracture of materials with initial stresses appear in the half space [1].

A similar effect is observed in bodies with harmonic-type elastic potential [3] for which the coefficient k_1 takes the form

$$k_1 = 2\frac{1-\nu^2}{E}\frac{\lambda_1^2}{\lambda_1(2+\nu)-1-\nu}.$$

In this case, the critical values λ_{kp} are different for different materials because they depend on ν. In particular, for $\nu = 0.3$, the "resonance" phenomena are observed as $\lambda_1 \to \lambda_{kp} \approx 0.565$.

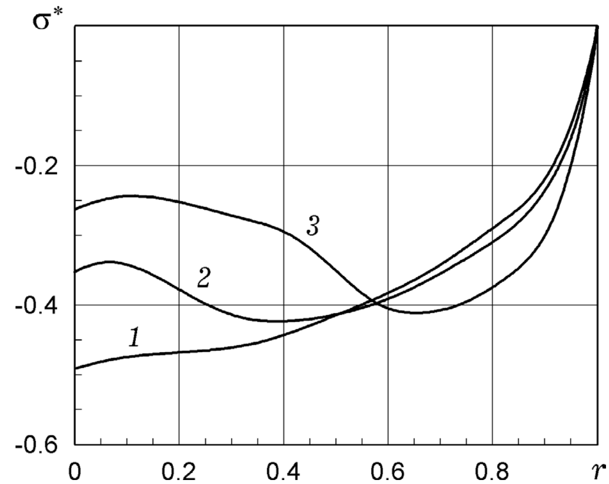

Fig. 4. Distributions of contact stresses for various values of r_a.

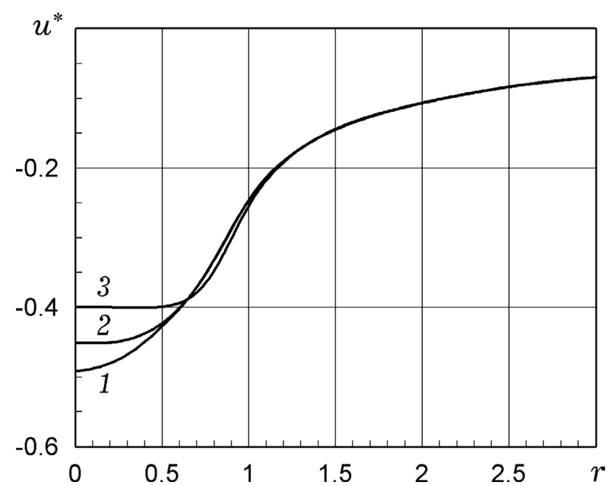

Fig. 5. Distributions of the vertical displacements for various values of r_a.

In Fig. 3, we illustrate the dependences of k_1 on the parameter of linear elongation λ_1 for the Bartenev–Khazanovich elastic potential (curve *1*) and the harmonic-type potential (curve *2*).

We analyzed the influence of presence of the plane domain in the punch footing on the distribution of contact stresses and the character of vertical displacements. In Figs. 4 and 5, we present the plots of the functions $\sigma^* = \sigma_{zz}(r,0)/P$ and $u^* = u_z(r,0)/P$ for the parabolic punch in the absence of residual strains in the half space, constant contact zone $a = 1$, and various values of the parameter r_a: $r_a = 0$ (curve *1*), $r_a = 0.2$ (curve *2*), and $r_a = 0.5$ (curve *3*).

CONCLUSIONS

The performed numerical analysis enables us to conclude that the influence of initial strains on the vertical displacements of compressible and incompressible bodies is much stronger in the presence of preliminary compressive strains $\lambda_1 < 1$ than in the case of preliminary tensile strains, $\lambda_1 > 1$. The shape of the punch strongly affects the level and character of distribution of the contact stresses. In particular, for the parabolic punch whose base does not contain a plane part, the extreme values of contact stresses are detected at the center of the contact zone. The appearance of a plane part leads to a shift of the points of extreme to the edge of contact zone and decreases their absolute value.

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