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ON *J*-UNITARY MATRIX POLYNOMIALS

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Abstract
An efficient method for construction of J-unitary matrix polynomials is proposed, associated with companion matrix functions the last row of which is a polynomial in $1/t$. The method relies on Wiener-Hopf factorization theory and stems from recently developed J-spectral factorization algorithm for certain Hermitian matrix functions.

Keywords J-unitary matrices · Wiener-Hopf factorization

Introduction

Matrix spectral factorization method developed in [3, 7, 8] reveals important properties of unitary matrix functions of the following special structure:

$$U(t)=\begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1m}(t)\\ u_{21}(t) & u_{22}(t) & \cdots & u_{2m}(t)\\ \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m}(t)\\ \overline{u_{m1}(t)} & \overline{u_{m2}(t)} & \cdots & \overline{u_{mm}(t)} \end{pmatrix}, \quad u_{ij}\in\mathcal{P}_N^+, \tag{1.1}$$

which play a crucial role in the method. Here, $\mathcal{P}_N^+ := \{\sum_{k=0}^N \alpha_k t^k : \alpha_k\in\mathbb{C}\}$ is the set of polynomials of degree $\le N$. The matrix functions (1.1) we consider are unitary on the unit circle $\mathbb{T} := \{t\in\mathbb{C} : |t|=1\}$,

$$U(t)U^*(t)=I_m, \quad t\in\mathbb{T},$$

and they have the determinant equal to 1,

$$\det U(t)=1. \tag{1.2}$$

In memory of Nikolai Karapetiants, an outstanding mathematician and a noble person, on the 80th anniversary of his birthday.

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It turns out that these matrices are closely related to the so-called wavelet matrices [9], and complete parameterization of such matrices in terms of coordinates in the Euclidian space $\mathbb{C}^{(m-1)N}$ is given in [2]. In particular, it is proved in [2] that there is one-to-one correspondence between the unitary matrix functions (1.1) satisfying (1.2),

$$U(1) = I_m, \tag{1.3}$$

and

$$\sum_{k=1}^{m} |u_{mk}(0)| > 0, \tag{1.4}$$

and companion matrix functions

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & 1 \end{pmatrix}, \quad \zeta_k \in \mathcal{P}_N^-, \tag{1.5}$$

where $\mathcal{P}_N^- := \{\sum_{k=1}^N \alpha_k t^{-k} : \alpha_k \in \mathbb{C}\}$. This bijective correspondence is reflected in the fact that

$$F^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\zeta_1(t) & -\zeta_2(t) & -\zeta_3(t) & \cdots & -\zeta_{m-1}(t) & 1 \end{pmatrix} \tag{1.6}$$

is the first factor in the right Wiener-Hopf factorization of (1.1). Namely,

$$F(t)U(t) = \Phi^+(t) \Longleftrightarrow U(t) = F^{-1}(t)\Phi^+(t),$$

where $\Phi^+ \in (\mathcal{P}_N^+)^{m\times m}$, i.e., the entries of $m \times m$ matrix Φ^+ are from $\mathcal{P}_N^+$.
It is explicitly described in [2] how to construct $U(t)$ for the given matrix function (1.5) and how to construct $F(t)$ for the given unitary matrix function (1.1). If we multiply the last row of (1.1) by t^N, we get a unitary (for $|z| = 1$) matrix polynomial

$$\mathbf{A}(z) = \mathrm{diag}(1, 1, \dots, 1, z^N)\mathbf{U}(z) = \sum_{k=0}^{N} A_k z^k, \quad A_k \in \mathbb{C}^{m\times m}, \quad z \in \mathbb{C}, \tag{1.7}$$

of rank m, order N, and degree N. Since

$$\mathbf{A}(z)\widetilde{\mathbf{A}}(z) = I_m \quad \text{for each } z \in \mathbb{C}\backslash\{0\}, \tag{1.8}$$

where $\widetilde{\mathbf{A}}(z) = \sum_{k=0}^N A_k^* z^{-k}$, the polynomial $\det \mathbf{A}(z)$ is the monomial cz^d, where $|c| = 1$, and d is called the degree of A. Since we have a natural bijection

$$\mathbf{A}(z) \longleftrightarrow U(t), \tag{1.9}$$

the abovementioned one-to-one map $F(t) \longleftrightarrow U(t)$ parametrizes also the class of unitary matrix polynomials (1.7) such that $\det \mathbf{A}(z) = z^N$, $A(1) = I_m$, and at least one entry in the last row of A_N does not vanish.
In [4], the first steps have been made towards the generalization of the abovementioned matrix spectral factorization method to the J-spectral factorization of Hermitian matrices. This generalization is achieved by observing that J-unitary matrix functions of structure (1.1) play a similar important role in the proposed J-spectral factorization method.

Throughout the paper, J is a diagonal matrix

$$J = \mathrm{diag}(j_1, j_2, \dots, j_{m-1}, 1) \tag{1.10}$$

where each j_k is either positive or negative 1 (without loss of generality, we assume that j_m is always equal to 1). A matrix U is called J-unitary if $UJU^* = J$, and a matrix function $U(t)$ is called J-unitary if

$$U(t)JU^*(t) = J, \ t \in \mathbb{T}.$$

In the present paper, we prove the following:

Theorem 1.1 *For any matrix function $F(t)$ of the form (1.5), there exists (generically) a J-unitary matrix function $U(t)$ of the form (1.1), satisfying (1.2), and such that*

$$F(t)U(t) \in (\mathcal{P}_N^+)^{m\times m} \tag{1.11}$$

holds. The matrix function $U(t)$ satisfies (1.4) and it is unique if we require in addition (1.3).

Conversely, for each J-unitary matrix function $U(t)$ of the form (1.1) which satisfies (1.2) and (1.4), there exists a unique matrix function $F(t)$ of the form (1.5) such that (1.11) holds.

The explicit algorithms which construct $U(t)$ from $F(t)$ and $F(t)$ from $U(t)$ are described in the following sections, where the exact meaning of the word "generic" used in the theorem is also specified.
Due to the one-to-one correspondence (1.9), Theorem 1.1 gives a description of the class of J-unitary matrix polynomials (1.7) in terms of the coordinates of the Euclidian space $\mathbb{C}^{(m-1)N}$. Such description can be used to apply convex optimization methods for searching specific J-unitary matrix polynomials with additional properties required, e.g., for construction of quasi-tight framelets [1].

Notation and Preliminary Observations

For a matrix $A = [A_{ij}]$, its transpose is denoted by A^T and $A^* = \overline{A}^T = [\overline{A_{ji}}]$ stands for the Hermitian conjugate.
The set of J-unitary matrices, $\mathcal{U}_J$ (defined in the "Introduction") is a group. Furthermore, $U \in \mathcal{U}_J \Longrightarrow U^T \in \mathcal{U}_J$, since $AJB = J \Longrightarrow BJA = J$. Indeed, $AJB = J \Longrightarrow BJAJB = B \Longrightarrow BJA = J$. Note that $\mathcal{U}_J = \mathcal{U}_{-J}$, which is the reason why we can assume without loss of generality that $j_m = 1$ in (1.10). Obviously, every diagonal matrix D with unimodular diagonal entries belongs to $\mathcal{U}_J$. Also, every permutation matrix $\mathbb{P}$ belongs to $\mathcal{U}_J$,

$$\mathbb{P}\, J\, \mathbb{P}^* = J. \tag{2.1}$$

If A is a square matrix (function), then $\mathrm{Cof}(A)$ denotes the cofactor matrix of A (so that $A^{-1} = \frac{1}{\det(A)}\mathrm{Cof}(A)^T$ for non-singular matrices) and $\mathrm{cof}A_{ij}$ denotes the cofactor of A_{ij} in the matrix A, i.e., $\mathrm{cof}A_{ij} = (\mathrm{Cof}A)_{ij}$.
Let $\mathbb{T}$ be the unit circle in the complex plane and $\mathbb{T}_+ = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}_- = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$.We assign the variable t as an argument to functions with domain $\mathbb{T}$, and the variable z to functions with domain $\mathbb{C}$ (or $\mathbb{C}\backslash\{0\}$). Slightly abusing the notation, we denote by $f(t)$ not only the value of function f at t, but the function itself (with corresponding domain). The exact meaning of the notation will be clear from the context. Also let A^c_{ij} be the submatrix of A obtained by deleting its ith row and jth column.
For a set $\mathcal{S}$, let $\mathcal{S}^{m\times m}$ stand for the set of $m\times m$ matrices with entries in $\mathcal{S}$.
The set of Laurent polynomials with complex coefficients is denoted by $\mathcal{P}$. We consider the following subsets of $\mathcal{P}$: for $M, N \geq 0$, let $\mathcal{P}_{\{-M,N\}} := \{\sum_{k=-M}^N \alpha_k z^k : \alpha_k \in \mathbb{C}\}$; for $N \geq 1$, $\mathcal{P}_N := \mathcal{P}_{\{-N,N\}}$; $\mathcal{P}_N^+ := \mathcal{P}_{\{0,N\}}$; $\mathcal{P}_N^- := \mathcal{P}_{\{-N,-1\}}$; $\mathcal{P}_{N,0}^- := \mathcal{P}_{\{-N,0\}}$; $\mathcal{P}^+ := \cup_{N\geq 1}\mathcal{P}_N^+$, $\mathcal{P}^- := \cup_{N\geq 1}\mathcal{P}_N^-$, $\mathcal{P}_0^- := \cup_{N\geq 1}\mathcal{P}^-_{\{N,0\}}$. Note that $\mathcal{P}_N^+ \cap \mathcal{P}_N^- = \{0\}$ according to our notation.
For $p(z) = \sum_{k=-M}^N \alpha_k z^k \in \mathcal{P}_{\{-M,N\}}$, let

$$\widetilde{p}(z) := \sum_{k=-M}^{N} \overline{\alpha_k} z^{-k}.$$

Note that $\widetilde{p}(z) = \overline{p(z)}$ for $z \in \mathbb{T}$. If

$$P(z) = \sum_{k=-M}^{N} A_k z^k = [p_{ij}(z)] \in (\mathcal{P}_{\{-M,N\}})^{m\times m}, \quad \text{where } A_k \in \mathbb{C}^{m\times m},$$

then

$$\widetilde{P}(z) := \sum_{k=-M}^{N} A_k^* z^{-k} = [\widetilde{p}_{ji}(z)].$$

Note that $\widetilde{P}(z) = P^*(z)$ for $z \in \mathbb{T}$. Furthermore, $\widetilde{PQ} = \widetilde{Q}\widetilde{P}$ and if $P^{-1}(z) \in (\mathcal{P}_{\{-M,N\}})^{m\times m}$, then $(\widetilde{P}(z))^{-1} = \widetilde{P^{-1}}(z)$.
The matrix J was defined in the "Introduction" (see (1.10)). A matrix polynomial $\mathbf{U}(z) \in (\mathcal{P}_N^+)^{m\times m}$ is called J-unitary if

$$\mathbf{A}(z)\,J\,\widetilde{\mathbf{A}}(z) = \widetilde{\mathbf{A}}(z)\,J\,\mathbf{A}(z) = J \text{ for each } z \in \mathbb{C}\backslash\{0\}. \tag{2.2}$$

For

$$\mathbf{A}(z) = \sum_{k=0}^{N} A_k z^k, \quad A_k \in \mathbb{C}^{m\times m}, \tag{2.3}$$

condition (2.2) can be as well expressed in terms of the coefficients:

$$\sum_{k=n}^{N} A_k\,J\,A_{k-n}^* = \delta_{n0} J, \quad n = 0, 1, \dots, N, \tag{2.4}$$

where δ stands for the Kronecker delta.
If $A_N \neq 0$ in (2.3), then we say that the order of A is equal to N.
Note that (2.2) implies that $\det \mathbf{A}(z) = cz^d, |c| = 1$, for some integer d. The latter is called the *degree* of the J-unitary matrix.

Lemma 2.1 (cf.[2], Lemma 1) *If (2.3) is a J-unitary matrix polynomial of order N and degree d, then*:

a) $d \geq N$;
b) $d = N$ *if and only if* $\operatorname{rank}(A_0) = m - 1$.

Proof Due to (2.2), we have

$$\widetilde{\mathbf{A}}(z) = J\,\mathbf{A}^{-1}(z)\,J.$$

Therefore,

$$\sum_{k=0}^{N} A_k^* z^{-k} = J\,\frac{1}{\det \mathbf{A}(z)}\left(\operatorname{Cof}\mathbf{A}(z)\right)^T J = cz^{-d}\,J\left(\operatorname{Cof}\mathbf{A}(z)\right)^T J. \tag{2.5}$$

Since $J\left(\operatorname{Cof}\mathbf{A}(z)\right)^T J \in (\mathcal{P}^+)^{m\times m}$ and $A_N^* \neq 0$ the relation (a) follows.

In order to prove (b), note that $\operatorname{rank} A_0 \neq m$ since $A_0(JA_N^*) = 0$ due to (2.4) for $n = N$. In addition, we have $rank\left(A_0\right) < m - 1 \Leftrightarrow \operatorname{Cof}\left(A_0\right) = \mathbf{0}$ and $\operatorname{Cof}\mathbf{A}(z)|_{z=0} = \operatorname{Cof}(A_0)$. Therefore, it follows from (2.5) that $rank\left(A_0\right) < m - 1$ is equivalent to $d > N$. Hence, (b) follows.

The class of J-unitary matrix polynomials (2.3) which have the order and the degree equal to N is denoted by $\mathcal{A}_J(m, N)$. Since $\mathbf{A}(z) \in \mathcal{A}_J(m, N) \Longrightarrow \mathbb{P}\mathbf{A}(z) \in \mathcal{A}_J(m, N)$ for any permutation matrix $\mathbb{P}$ (see (2.1)), and $A_N \neq \mathbf{0}$, we can rearrange the rows, if necessary, and assume without loss of generality that the last row of A_N is nonzero. The subclass of functions (2.3) from $\mathcal{A}_J(m,N)$ for which the last row of A_N is not 0 will be denoted by $\mathcal{A}_J^0(m,N)$.

Wiener-Hopf Factorization

In this section, we briefly review some well-known facts from Wiener-Hopf factorization theory which are used in what follows.
Every $P \in \mathcal{P}^{m\times m}$ without zeros of its determinant and poles of its entries on $\mathbb{T}$ can be factorized as (see, e.g., [6])

$$P(z) = P_+(z)D(z)P_-(z), \tag{3.1}$$

where $P_\pm \in (\mathcal{P}^\pm)^{m\times m}$ with $\det P_\pm(z) \neq 0$ for $z \in \mathbb{T}_\pm$ and $D(z) = \operatorname{diag}(z^{\varkappa_1}, z^{\varkappa_2}, \dots, z^{\varkappa_m})$ is a diagonal matrix function. The factorization (3.1) is called a *left* factorization since the analytic in $\mathbb{T}_+$ factor P_+ stands on the left. The integers $\varkappa_1, \varkappa_2, \dots, \varkappa_m$ are called the (left) partial indices of P and they are determined uniquely if we assume $\varkappa_1 \geq \varkappa_2 \geq \dots \geq \varkappa_m$. If all partial indices are equal to 0, i.e., $D = I_m$, the factorization has the form

$$P(z) = P_+(z)P_-(z) \tag{3.2}$$

and it is called a *canonical* factorization. The necessary and sufficient condition for the existence of canonical factorization is that (see, e.g., [6]) $\det P$ has the winding number equal to 0 and

$$Pu \notin (\mathcal{P}^+)^{m\times 1} \text{ for each } u \in (\mathcal{P}^-)^{m\times 1}.$$

If $P \in \mathcal{P}_N^{m\times m}$ has a canonical factorization (3.2), then $P_+ \in (\mathcal{P}_N^+)^{m\times m}$ and $P_- \in (\mathcal{P}_{N,0}^-)^{m\times m}$ (see [5]). If, in addition, P is symmetric $P(z) = \widetilde{P}(z)$, then its Wiener-Hopf factorization can be represented in J-factorization form (see [10])

$$P(z) = P_+(z)J\widetilde{P_+}(z), \tag{3.3}$$

and it has a special name: J-spectral factorization. Factorization (3.3) is unique up to a constant right J-unitary factor. Indeed if $P_+(z)J\widetilde{P_+}(z) = Q_+(z)J\widetilde{Q_+}(z)$, then the continuous (on $\mathbb{T}$) function $Q_+^{-1}(t)P_+(t) = J\widetilde{Q_+}(t)\widetilde{P_+^{-1}}(t)J$ can be extended analytically on $\mathbb{C} \cup \{\infty\}$ and, therefore, it is constant.

Proof of the Main Result

In this section, we prove Theorem 1.1, which for convenience of exposition is split into two statements.

Theorem 4.1 *Let matrix function (1.5) be such that $F(z)\,J\,\widetilde{F}(z)$ has a left J-spectral factorization*

$$F(z)\,J\,\widetilde{F}(z) = \Phi_+(z)\,J\,\widetilde{\Phi_+}(z). \tag{4.1}$$

Then there exists a J-unitary matrix function

$$\mathbf{U}(z) = \begin{pmatrix} u_{11}(z) & u_{12}(z) & \cdots & u_{1m}(z) \\ u_{21}(z) & u_{22}(z) & \cdots & u_{2m}(z) \\ \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(z) & u_{m-1,2}(z) & \cdots & u_{m-1,m}(z) \\ \widetilde{u_{m1}}(z) & \widetilde{u_{m2}}(z) & \cdots & \widetilde{u_{mm}}(z) \end{pmatrix}, \quad u_{ij} \in \mathcal{P}_N^+, \tag{4.2}$$

with the determinant 1,

$$\det \mathbf{U}(z) = 1, \quad \text{for } z \in \mathbb{C}\backslash\{0\}, \tag{4.3}$$

and such that

$$F\mathbf{U} \in (\mathcal{P}_N^+)^{m\times m}. \tag{4.4}$$

Furthermore, this matrix function is unique under the additional restriction (1.3), *and the relation* (1.4) *is also fulfilled.*

Remark 4.1 The algorithm for construction of (4.2) will be provided in Section "5."

Proof As it was mentioned in Section "3," the existence of factorization (4.1) is equivalent to the condition that $F(z)\,J\,\widetilde{F}(z)$ possesses the canonical factorization, and

$$\Phi_+ \in (\mathcal{P}_N^+)^{m\times m}. \tag{4.5}$$

Since $|\det \Phi_+(z)|^2 = 1$ for $z \in \mathbb{T}$, because of (4.1), $\det \Phi_+ \in \mathcal{P}^+$ and $\det \Phi_+(z) \neq 0$ for $z \in \mathbb{T}_+$, we have that $\det \Phi_+(z) = \text{Const}$. Without loss of generality, we can assume that

$$\det \Phi_+(z) = 1. \tag{4.6}$$

Suppose

$$\mathbf{U}(z) = F^{-1}(z)\Phi_+(z), \tag{4.7}$$

where F^{-1} is determined by (1.6). Then $\mathbf{U}$ is J-unitary since

$$\mathbf{U}(z)\,J\,\widetilde{\mathbf{U}}(z) = F^{-1}(z)\Phi_+(z)\,J\,\widetilde{\Phi_+}(z)\widetilde{F^{-1}}(z) = F^{-1}(z)F(z)\,J\,\widetilde{F}(z)\widetilde{F^{-1}}(z) = J. \tag{4.8}$$

Equation (4.3) follows from (4.7), (1.6) and (4.6).

It follows from Eqs. (1.6) and (4.7) that the first $m-1$ rows of $\mathbf{U}$ and Φ_+ coincide. Therefore, by virtue of (4.5),

$$u_{ij} = \Phi_{ij}^+ \in \mathcal{P}_N^+, \quad \text{for } 1 \le i \le m-1,\ 1 \le j \le m. \tag{4.9}$$

It follows from (4.8) and (4.3) that

$$\widetilde{\mathbf{U}} = (J\mathbf{U}J)^{-1} = \operatorname{Cof}(J\mathbf{U}J)^T, \tag{4.10}$$

which, taking into account (4.9), implies that

$$\widetilde{\mathbf{U}}_{mj} \in \mathcal{P}^+.$$

Since we know from the beginning that (4.7) belongs to $(\mathcal{P}_N)^{m\times m}$, we conclude that

$$\widetilde{\mathbf{U}}_{mj} \in \mathcal{P}_N^+.$$

Therefore, matrix (4.7) has the structure (4.2). In particular, due to (4.10),

$$u_{mj} = \operatorname{Cof}(J\mathbf{U}J)_{mj}(z), \ \ j = 1, 2, \ldots, m, \tag{4.11}$$

in matrix (4.2).

The relation (1.3) will be satisfied if we consider the matrix $\mathbf{U}(z)\mathbf{U}^{-1}(1)$ instead of $\mathbf{U}(z)$, which does not change the structure of (4.2), and the uniqueness of $\mathbf{U}$ follows from the uniqueness of J-spectral factorization (4.1).

Because of (4.6), we have

$$\sum_{k=1}^{m} \Phi^+_{mk}(z)\mathrm{Cof}\,\Phi^+_{mk}(z) = 1,$$

which implies that

$$\sum_{k=1}^{m} |\mathrm{Cof}\,\Phi^+_{mk}(0)| > 0. \tag{4.12}$$

However, because of (4.9),

$$\mathrm{Cof}\,\mathbf{U}_{mk}(z) = \mathrm{Cof}\,\Phi^+_{mk}, \quad k = 1, 2, \dots, m, \tag{4.13}$$

and

$$|\mathrm{Cof}\,(J\mathbf{U}J)_{mk}(z)| = |\mathrm{Cof}\,\mathbf{U}_{mk}(z)| \tag{4.14}$$

since the signs in matrices $\mathbf{U}$, $J\mathbf{U}$, and $J\mathbf{U}J$ can differ only across entire rows and columns. Therefore, by virtue of (4.11), (4.14), (4.13), and (4.12), the relation (1.4) holds.

Theorem 4.2 *For any J-unitary matrix function $\mathbf{U}$ of structure (4.2) which satisfies (4.3) and (1.4), there exists a unique matrix function F of structure (1.5) such that (4.4) holds.*

Proof Because of (1.4), there exists $n \le m$ such that $u_{mn}(0) \neq 0$. Define functions

$$\zeta_i(z) := j_i \left[\frac{\widetilde{u}_{in}(z)}{u_{mn}(z)} \right]^-, \qquad i = 1, 2, \dots, m-1, \tag{4.15}$$

where $[\,\cdot\,]^-$ stands for the projection operator:

$$\left[\sum_{k=-N}^{\infty} c_k z^k \right]^- = \sum_{k=-N}^{-1} c_k z^k,$$

and $\frac{1}{u_{mn}(z)}$ is understood as its formal series expansion in a neighborhood of 0. Note that we need only the first $N+1$ coefficients of this expansion in order to compute (4.15).

Define the matrix function F by (1.5) and let us prove that (4.4) holds. To this end, we need only to check that the entries of the last row of the product $F\mathbf{U}$,

$$\sum_{i=1}^{m-1} \zeta_i(z) u_{ij}(z) + \widetilde{u}_{mj}(z), \quad j = 1, 2, \dots, m, \tag{4.16}$$

belong to $\mathcal{P}^+_N$. Because of the definition (4.15), we know that the functions in (4.16) belong to $\mathcal{P}_N$. In addition, for $1 \le j \le m$, we have

$$\sum_{i=1}^{m-1} \zeta_i(z) u_{ij}(z) + \widetilde{u}_{mj}(z) = \sum_{i=1}^{m-1} j_i \left(\frac{\widetilde{u}_{in}(z)}{u_{mn}(z)} - \left[\frac{\widetilde{u}_{in}(z)}{u_{mn}(z)} \right]^+ \right) u_{ij}(z) + \widetilde{u}_{mj}(z) =$$
$$\frac{1}{u_{mn}(z)} \left(\sum_{i=1}^{m-1} j_i \widetilde{u}_{in}(z) u_{ij}(z) + u_{mn}(z)\widetilde{u}_{mj}(z) \right) - \sum_{i=1}^{m-1} j_i \left[\frac{\widetilde{u}_{in}(z)}{u_{mn}(z)} \right]^+ u_{ij}(z) =$$
$$\frac{j_n \delta_{nj}}{u_{mn}(z)} - \sum_{i=1}^{m-1} j_i \left[\frac{\widetilde{u}_{in}(z)}{u_{mn}(z)} \right]^+ u_{ij}(z).$$

The latter expression is analytic in a neighborhood of 0 which yields that (4.16) belongs to $\mathcal{P}_N \cap \mathcal{P}^+ = \mathcal{P}_N^+$. Thus, (4.4) holds.

Let us now show the uniqueness of the desired $F(z)$. Suppose

$$F(z)\mathbf{U}(z) = \Phi^+(z) \in (\mathcal{P}_N^+)^{m\times m} \text{ and } F_1(z)\mathbf{U}(z) = \Phi_1^+(z) \in (\mathcal{P}_N^+)^{m\times m}, \tag{4.17}$$

where $F_1(z)$ has the same form (1.5) with the last row $[\zeta_1'(z), \zeta_2'(z), \dots, \zeta_{m-1}'(z), 1]$, $\zeta_i' \in \mathcal{P}_N^-$, $i = 1, 2, \dots, m-1$. Since $\det \Phi^+(z) = 1$, we have

$$(\Phi^+)^{-1}(z) \in (\mathcal{P}^+)^{m\times m}. \tag{4.18}$$

It follows from (4.17) and (4.18) that

$$F_1(z)F^{-1}(z) = \Phi_1^+(z)(\Phi^+)^{-1}(z) \in (\mathcal{P}^+)^{m\times m}.$$

Hence, the last row of the above product $[\zeta_1' - \zeta_1, \zeta_2' - \zeta_2, \dots, \zeta_{m-1}' - \zeta_{m-1}, 1] \in (\mathcal{P}^+)^{1\times m}$, which implies that $\zeta_i' - \zeta_i = 0$, $i = 1, 2, \dots, m-1$ since these functions belong to $\mathcal{P}_N^-$ which has a trivial intersection with $\mathcal{P}^+$.

Remark 4.2 Theorems 4.1 and 4.2 deliver one-to-one correspondence between the matrix functions (1.5) for which J-spectral factorization (4.1) exists and the class $\mathcal{A}_J^0(m, N)$ defined in the end of Section "2".

The Algorithm for Constructing *J*-unitary Matrix Polynomials

In this section, we provide an algorithm for constructing J-unitary matrix function (4.3) for a given matrix function (1.5). Consider the following system of conditions:

$$\begin{cases} \zeta_1 x_m - j_1 \cdot \widetilde{x_1} \in \mathcal{P}^+, \\ \zeta_2 x_m - j_2 \cdot \widetilde{x_2} \in \mathcal{P}^+, \\ \cdots \\ \zeta_{m-1} x_m - j_{m-1} \cdot \widetilde{x_{m-1}} \in \mathcal{P}^+, \\ \zeta_1 x_1 + \zeta_2 x_2 + \dots + \zeta_{m-1} x_{m-1} + \widetilde{x_m} \in \mathcal{P}^+, \end{cases} \tag{5.1}$$

where $\zeta_i \in \mathcal{P}_N^-$, $i = 1, 2, \dots, m-1$, are the entries of F in (1.5).

Definition 5.1 We say that a vector function

$$\mathbf{u} = \left(u_1, u_2, \dots, u_{m-1}, \widetilde{u_m}\right)^T, \text{ where } u_i \in \mathcal{P}_N^+ \text{ for each } i = 1, 2, \dots, m,$$

is a solution of (5.1) if and only if all the conditions in (5.1) are satisfied whenever $x_i = u_k$, $i = 1, 2, \dots, m$.

Remark 5.1 Note that the set of solutions of (5.1) is a linear subspace of $(\mathcal{P}_N)^{m\times 1}$.

Lemma 5.1 *Let*

$$\mathbf{u} = \left(u_1, u_2, \dots, \widetilde{u_m}\right)^T \text{ and } \mathbf{v} = \left(v_1, v_2, \dots, \widetilde{v_m}\right)^T$$

be two (possibly identical) solutions of the system (5.1). Then

$$\sum_{k=1}^{m-1} j_k u_k \widetilde{v_k} + \widetilde{u_m} v_m = \text{const}. \tag{5.2}$$

Proof Substituting the functions v in the first $m-1$ conditions and the functions u in the last condition of (5.1), and then multiplying the first $m-1$ conditions by u and the last condition by v_m, we get

$$\begin{cases} \zeta_1 v_m u_1 - j_1 \cdot \widetilde{v_1} u_1 \in \mathcal{P}^+, \\ \zeta_2 v_m u_2 - j_2 \cdot \widetilde{v_2} u_2 \in \mathcal{P}^+, \\ \dots \\ \zeta_{m-1} v_m u_{m-1} - j_{m-1} \cdot \widetilde{v_{m-1}} u_{m-1} \in \mathcal{P}^+, \\ \zeta_1 u_1 v_m + \ldots + \zeta_{m-1} u_{m-1} v_m + j_m \widetilde{u_m} v_m \in \mathcal{P}^+. \end{cases}$$

Subtracting the first $m-1$ conditions from the last condition in the latter system, we get

$$\left(\sum_{k=1}^{m-1} j_k u_k \widetilde{v_k} + \widetilde{u_m} v_m \right) \in \mathcal{P}^+. \tag{5.3}$$

We can interchange the roles of u and v in the above discussion to derive in a similar manner that

$$\sum_{k=1}^{m-1} j_k v_k \widetilde{u_k} + \widetilde{v_m} u_m \in \mathcal{P}^+.$$

Consequently, the function in (5.2) belongs to $\mathcal{P}^+ \cap \mathcal{P}_0^-$, which implies (5.2).

We construct m linearly independent solutions of (5.1). Namely, we rewrite (5.1) in equivalent form of a linear system of equations. Let in (5.1)

$$\zeta_i(z) = \sum_{n=1}^{N} \gamma_{in} z^{-n}, \quad i = 1, 2, \ldots, m-1 \tag{5.4}$$

and

$$x_i(z) = \sum_{n=0}^{N} a_{in} z^n, \quad i = 1, 2, \ldots, m, \tag{5.5}$$

Equating all the coefficients of the non-positive powers of z of the functions in the left-hand side of (5.1) with 0, except for the free term of the qth function which we set equal to 1, we arrive at the following system of algebraic equations in the block matrix form, which we denote by $\mathbb{S}_q$:

$$\mathbb{S}_q := \begin{cases} \Gamma_1 X_m - j_1 \overline{X_1} = \mathbf{0}, \\ \Gamma_2 X_m - j_2 \overline{X_2} = \mathbf{0}, \\ \\ \Gamma_q X_m - j_q \overline{X_q} = \mathbf{1}, \\ \\ \Gamma_{m-1} X_m - j_{m-1} \overline{X_{m-1}} = \mathbf{0}, \\ \Gamma_1 X_1 + \ldots + \Gamma_{m-1} X_{m-1} + \overline{X_m} = \mathbf{0} \end{cases} \tag{5.6}$$

Here the following notation is used:

$$\Gamma_i = \begin{pmatrix} \gamma_{i0} & \gamma_{i1} & \gamma_{i2} & \cdots & \gamma_{i,N-1} & \gamma_{iN} \\ \gamma_{i1} & \gamma_{i2} & \gamma_{i3} & \cdots & \gamma_{iN} & 0 \\ \gamma_{i2} & \gamma_{i3} & \gamma_{i4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \gamma_{iN} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, 2, \dots, m-1,$$
$$X_i = (a_{i0}, a_{i1}, \dots, a_{iN})^T, \quad i = 1, 2, \dots, m \quad \text{(see (5.5))},$$
$$\mathbf{0} = (0, 0, \dots, 0)^T \in \mathbb{C}^{(N+1)\times 1}, \text{ and } \mathbf{1} = (1, 0, 0, \dots, 0)^T \in \mathbb{C}^{(N+1)\times 1}. \tag{5.7}$$

Determining X_i, $i = 1, 2, \dots, m-1$, from the first $m-1$ equations of (5.6),

$$X_i = j_i\Big(\overline{\Gamma_i}\, \overline{X_m} - \delta_{iq}\, \mathbf{1} \Big), \tag{5.8}$$

$i = 1, 2, \dots, m-1$, and then substituting them in the last equation of (5.6), we get

$$j_1\Gamma_1\ \overline{\Gamma_1}\, \overline{X_m} + j_2\Gamma_2\ \overline{\Gamma_2}\, \overline{X_m} + \dots + j_{m-1}\Gamma_{m-1}\ \overline{\Gamma_{m-1}}\, \overline{X_m} + \overline{X_m} = j_q\Gamma_q\ \mathbf{1} \tag{5.9}$$

(it is assumed that the right-hand side is equal to $\mathbf{1}$ when $q = m$) or, equivalently,

$$(j_1\Gamma_1\, \Gamma_1^* + + \ldots + j_{m-1}\Gamma_{m-1}\, \Gamma_{m-1}^* + j_m I_{N+1})\, \overline{X_m} = j_q\, \Gamma_q\, \mathbf{1} \tag{5.10}$$

(we used Γ^* in place of $\overline{\Gamma}$ because $\Gamma^T = \Gamma$). For each $q = 1, 2, \dots, m$, (5.10) is a linear system of $N+1$ equations with $N+1$ unknowns. This system (5.10), and consequently (5.6), has the unique solution for each $q = 1, 2, \dots, m$ if and only if

$$\det \Delta \neq 0, \tag{5.11}$$

where

$$\Delta = \sum\nolimits_{k=1}^{m-1} j_k\Gamma_k\Gamma_k^* + I_{N+1}\,. \tag{5.12}$$

We will assume that (5.11) holds. Finding $\overline{X_m}$ from (5.10) and then determining $X_1, X_2, \dots, X_{m-1}$ from (5.8), we get the unique solution of $\mathbb{S}_q$. To indicate its dependence on q, we denote the solution of $\mathbb{S}_q$ by $(X_1^q, X_2^q, \dots, X_{m-1}^q, X_m^q)$,

$$X_i^q := (a_{i0}^q, a_{i1}^q, \dots, a_{iN}^q)^T, \quad i = 1, 2, \dots, m, \tag{5.13}$$

so that if we construct a matrix function V,

$$V = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ v_{m-1,1} & v_{m-1,2} & \cdots & v_{m-1,m} \\ \widetilde{v_{m1}} & \widetilde{v_{m2}} & \cdots & \widetilde{v_{mm}} \end{pmatrix}, \tag{5.14}$$

by letting (see (5.13))

$$v_{ij}(z) = \sum_{n=0}^{N} a_{in}^j z^n, \quad 1 \le i, j \le m, \tag{5.15}$$

then the columns of (5.14) are solutions of the system (5.1). Hence, due to the last equation in (5.1),

$$FV \in (\mathcal{P}_N^+)^{m\times m} \tag{5.16}$$

and, by virtue of Lemma 5.1,

$$\widetilde{V}(z)\,J\,V(z) = C, \tag{5.17}$$

where C is a constant Hermitian matrix with signature J.
Let us show also that the determinant of V is constant.
It follows from (1.5) and (5.16) that

$$\det V \in \mathcal{P}^{+}. \tag{5.18}$$

Since the columns of (5.14) are solutions of (5.1), the direct computations show that

$$(F^{-1})^T J\widetilde{V} = \begin{pmatrix} \psi_{11}^{+} & \psi_{12}^{+} & \cdots & \psi_{1m}^{+} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{m-1,1}^{+} & \psi_{m-1,2}^{+} & \cdots & \psi_{m-1,m}^{+} \\ v_{m1} & v_{m2} & \cdots & v_{mm} \end{pmatrix}, \tag{5.19}$$

where $\psi_{ik}^{+} = -\zeta_i v_{mk} + j_i \widetilde{v_{ik}}$. In addition, due to the choice of the right-hand side vector in (5.6), we have

$$\psi_{ik}^{+}(0) = -\delta_{ik}, \quad 1 \le i \le m-1, \quad 1 \le k \le m. \tag{5.20}$$

Since the right-hand side of (5.19) belongs to $(\mathcal{P}^{+})^{m\times m}$, we have

$$\det \widetilde{V} \in \mathcal{P}^{+}. \tag{5.21}$$

Relations (5.18) and (5.21) imply that $\det V \in \mathcal{P}^{+} \cap \mathcal{P}_0^{-}$. Hence, it is constant:

$$\det V(z) = c. \tag{5.22}$$

If we denote the right-hand matrix in (5.19) by Ψ^{+}, then it follows from (5.22) that $\det \Psi^{+}(z) = \mathrm{Const}$ and hence

$$\det \Psi(z) = \det \Psi(0).$$

It now follows from (5.20) that

$$\det \Psi^{+}(0) = (-1)^{m-1} v_{mm}(0).$$

Recall from (5.9)–(5.12) that $\overline{v_{mm}(0)}$ is the 0th term of the solution of the equation

$$\Delta X = \mathbf{1}, \tag{5.23}$$

where $\mathbf{1} = (1, 0, 0, \dots, 0)^T \in \mathbb{C}^{(N+1)\times 1}$, i.e., if the solution of (5.23) is $X = (x_0, x_1, \dots, x_N)^T$, then $x_0 = \overline{v_{mm}(0)}$.
Since $x_0 = \Delta_0/\Delta$, where Δ_0 is the $N \times N$ submatrix of Δ obtained by deleteing its first row and column,

$$\Delta_0 = \Delta_{11}^{c}, \tag{5.24}$$

we get

$$\det V(z) = c \neq 0 \iff \det \Delta_0 \neq 0. \tag{5.25}$$

The following simple example shows that (5.25) does not always hold.

Example If $m = 2$, $J = \mathrm{diag}(-1, 1)$, and

$$\Gamma = \begin{pmatrix} 0 & \sqrt{\alpha} & \sqrt{\alpha} \\ \sqrt{\alpha} & \sqrt{\alpha} & 0 \\ \sqrt{\alpha} & 0 & 0 \end{pmatrix}, \tag{5.26}$$

where α is a root of the equation $x^2 - 3x + 1 = 0$, then $\det \Delta \neq 0$ and $\det \Delta_0 = 0$, where Δ is defined by (5.12) and Δ_0 is defined by (5.24). Indeed

$$-\Gamma\Gamma^* + I_3 = -\begin{pmatrix} 2\alpha-1 & \alpha & 0 \\ \alpha & 2\alpha-1 & \alpha \\ 0 & \alpha & \alpha-1 \end{pmatrix}$$

and $\det \Delta_0 = \begin{vmatrix} 2\alpha-1 & \alpha \\ \alpha & \alpha-1 \end{vmatrix} = \alpha^2 - 3\alpha + 1 = 0$. Note that corresponding to (5.26) matrix function F is

$$F(z) = \begin{pmatrix} 1 & 0 \\ \sqrt{\alpha}(z^{-1}+z^{-2}) & 1 \end{pmatrix}.$$

We proceed with the construction of $\mathbf{U}$ by the algorithm under the additional restriction (5.25), since, as it is proved in Theorem 5.1 below, the desired J-unitary matrix $\mathbf{U}$ does not otherwise exist.

Suppose

$$\mathbf{U}(z) = V(z)V^{-1}(1) \tag{5.27}$$

(the matrix $V^{-1}(1)$ exists since $V^*(1)JV(1) = C$ and $|\det C| = |\det V(1)|^2 = |c|^2 \neq 0$ because of (5.25)). Then (5.27) is J-unitary since $\widetilde{\mathbf{U}}(z)J\mathbf{U}(z)$ is equal to

$$\widetilde{V^{-1}}(1)\widetilde{V}(z)JV(z)V^{-1}(1) = (\widetilde{V}(1))^{-1}CV^{-1}(1) = (\widetilde{V}(1))^{-1}\widetilde{V}(1)JV(1)V^{-1}(1) = J.$$

Of course, (4.4) holds because of (5.16) and (5.27). The structure of the matrix function (5.14) is preserved, and $\det \mathbf{U}(z) = \det \mathbf{U}(1) = 1$, so that (4.3) holds. Thus, $\mathbf{U}$ is the desired matrix because of the uniqueness of J-spectral factorization presented in Section "3."

Theorem 5.1 *Let F be a matrix function* (1.5), *and define matrices Γ_i $i = 1, 2, \dots, m-1$, Δ, and Δ_0 according to* (5.4), (5.7), (5.12), *and* (5.24). *Suppose* $\det \Delta \neq 0$ *and* $\det \Delta_0 = 0$. *Then the canonical J-spectral factorization of F, i.e., the representation* (4.1), *does not exist.*

First, we prove the following:

Lemma 5.2 *Let F and $\mathbf{U}$ be as in Theorem 4.1. Then the columns of* $\mathbf{U}$,

$$\mathbf{U}_k = (u_{1k}, u_{2k}, \dots, u_{m-1,k}, \widetilde{u_{mk}})^T, \quad k = 1, 2, \dots, m, \tag{5.28}$$

are solutions of the system (5.1).

Proof The last equation in (5.1) holds automatically because of (4.4).

Suppose

$$F(z)\mathbf{U}(z) = \Phi_+(z) \tag{5.29}$$

where $\Phi_+ \in (\mathcal{P}_N^+)^{m\times m}$. Since $\mathbf{U}^{-1}(z) = J\widetilde{\mathbf{U}}(z)J$ and $\det \Phi_+(z) = 1$, the equation (5.29) implies

$$J\widetilde{\mathbf{U}}(z)JF^{-1}(z) = (\mathrm{Cof}\,\Phi_+)^T.$$

Hence,

$$J\widetilde{\mathbf{U}}(z)JF^{-1}(z) \in (\mathcal{P}^+)^{m\times m}. \tag{5.30}$$

Writing the left-hand side product in (5.30) explicitly, we get that

$$\begin{pmatrix} \widetilde{u_{11}} & \widetilde{u_{21}} & \dots & \widetilde{u_{m-1,1}} & u_{m1} \\ \widetilde{u_{12}} & \widetilde{u_{22}} & \dots & \widetilde{u_{m-1,2}} & u_{m2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widetilde{u_{1m}} & \widetilde{u_{2m}} & \dots & \widetilde{u_{m-1,m}} & u_{mm} \end{pmatrix} \begin{pmatrix} j_1 & 0 & \dots & 0 & 0 \\ 0 & j_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & \dots & j_{m-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\zeta_1 & -\zeta_2 & \cdots & -\zeta_{m-1} & 1 \end{pmatrix}$$

belongs to $(\mathcal{P}^+)^{m\times m}$, which shows that (5.28) are solutions of (5.1).

Proof of Theorem 5.1 For the sake of contradiction, let us assume that factorization (4.1) exists. Then, by virtue of Theorem 4.1, there exists (4.2) such that (4.4) holds, and by virtue of Lemma 5.2, the columns (5.28) are the solutions of (5.1). Consequently, for each $q = 1, 2, \dots, m$, the vectors constructed from the coefficients of $\mathbf{U}_q = (u_{1q}, u_{2q}, \dots, u_{m-1,q}, \widetilde{u_{mq}})^T$

$$X_i^q = (\alpha_{i0}^q, \alpha_{i1}^q, \dots, \alpha_{iN}^q)^T, \text{ where } u_{iq} = \sum\nolimits_{k=0}^N \alpha_{ik}^q z^k,$$

will be solutions of the following system of linear equations (cf. (5.6))

$$\mathbb{S}_q' := \begin{cases} \overline{\Gamma_1}\,\overline{X_m} - j_1 X_1 = B_{q1}, \\ \overline{\Gamma_2}\,\overline{X_m} - j_2 X_2 = B_{q2}, \\ \vdots \\ \overline{\Gamma_{m-1}}\,\overline{X_m} - j_{m-1} X_{m-1} = B_{q,m-1}, \\ \Gamma_1 X_1 + \dots + \Gamma_{m-1} X_{m-1} + \overline{X_m} = B_{qm} \end{cases},$$

where each B_{qi} has the form

$$B_{qi} = (b_{qi}, 0, 0, \dots, 0)^T \in \mathbb{C}^{(N+1)\times 1}.$$

We can consider $\mathbb{S}_q$ as $m(N+1) \times m(N+1)$ linear system of equations (with unknown $X = (X_1^T, X_2^T, \dots, X_{m-1}^T, X_m^T)^T$). It is clear that since $\Delta X = b$ has a unique solution for each $b \in \mathbb{C}^{(N+1)\times 1}$, the system $\mathbb{S}_q$ will also be non-singular. Consequently, since the columns of the matrix (5.14) constructed by the algorithm are solutions of the system (5.1), and the corresponding vectors

$$X_i^j = (\alpha_{i0}^j, \alpha_{i1}^j, \dots, \alpha_{iN}^j)^T$$

(see (5.15)) are solutions of (5.6) with the standard right-hand side vectors, we will have

$$\mathbf{U}_k = (u_{1k}, u_{2k}, \dots, u_{m-1,k}, \widetilde{u_{mk}})^T = \sum_{i=1}^N b_{qi} V_i.$$

This implies that $\mathbf{U}(z) = V(z)B$ for some $B \in \mathbb{C}^{m\times m}$ which is impossible since $\det \mathbf{U}(z) = 1$ and $\det V(z) = 0$. We arrive at a contradiction.

Remark 5.2 It is natural to ask if the condition (5.11) is necessary for the existence of J-spectral factorization (4.1). Presently, we do not know the answer.

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REFERENCES

1. C. Diao and B. Han, *Generalized matrix spectral factorization and quasi-tight framelets with a minimum number of generators*, Math. Comp. **89** (2020), no. 326, 2867–2911.
2. L. Ephremidze and E. Lagvilava, *On compact wavelet matrices of rankmand of order and degreeN*, J. Fourier Anal. Appl. **20** (2014), no. 2, 401–420.
3. L. Ephremidze, F. Saied, and I. Spitkovsky, *On the algorithmization of Janashia-Lagvilava matrix spectral factorization method*, IEEE Trans. Inform. Theory **64** (2018), no. 2, 728–737.
4. L. Ephremidze and I. Spitkovsky, *An algorithm for J-spectral factorization of certain matrix functions*, IEEE CDC2021, 60th Conference on Decision and Control.
5. L. Ephremidze and I. Spitkovsky, *A remark on a polynomial matrix factorization theorem*, Georgian Math. J. **19** (2012), 489–495.
6. I. Gohberg, M. A. Kaashoek, and I. M. Spitkovsky, *An overview of matrix factorization theory and operator applications*, Factorization and Integrable Systems (Basel), Birkhäuser Basel, 2003, pp. 1–102.
7. G. Janashia and E. Lagvilava, *A method of approximate factorization of positive definite matrix functions*, Studia Math. **137** (1999), no. 1, 93–100.
8. G. Janashia, E. Lagvilava, and L. Ephremidze, *A new method of matrix spectral factorization*, IEEE Trans. Inform. Theory **57** (2011), no. 4, 2318–2326.
9. G. Janashia, E. Lagvilava, and L. Ephremidze, *Matrix spectral factorization and wavelets*, J. Math. Sci. (N.Y.) **195** (2013), no. 4, 445–454, Translated from Sovrem. Mat. Prilozh., Vol. 83, 2012. MR 3207131
10. A. M. Nikolaichuk and I. Spitkovskii, *The factorization of hermitian matrix functions and its applications to boundary-value problems*, Ukrainian Mathematical Journal **27** (1975), 629–639.