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INTERACTION PROBLEMS ON PERIODIC HYPERSURFACES FOR DIRAC OPERATORS ON $\mathbb{R}^n$

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Abstract

We consider the Dirac operators with singular potentials

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma} = \mathfrak{D}_{\boldsymbol{A},\Phi,m} + \Gamma\delta_\Sigma \tag{1}$$

where

$$\mathfrak{D}_{\boldsymbol{A},\Phi,m} = \sum_{j=1}^{n} \alpha_j\left(-i\partial_{x_j} + A_j\right) + \alpha_{n+1}m + \Phi I_N \tag{2}$$

is a Dirac operator on $\mathbb{R}^n$ with variable magnetic and electrostatic potentials $\boldsymbol{A} = (A_1, ..., A_n), \Phi$, and the variable mass m. In formula (2), α_j are the $N \times N$ Dirac matrices, that is $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N$, I_N is the unit $N \times N$ matrix, $N = 2^{[(n+1)/2]}$, $\Gamma\delta_\Sigma$ is a singular delta-potential supported on C^2–hypersurface $\Sigma \subset \mathbb{R}^n$ periodic with respect to the action of a lattice $\mathbb{G}$ on $\mathbb{R}^n$. We consider the self-adjointnes and discretness of the spectrum of unbounded in $L^2(\mathbb{T}, \mathbb{C}^N)$ operators associated with the formal Dirac operator (1) on the torus $\mathbb{T} = \mathbb{R}^N \diagup \mathbb{G}$. We study the band-gap structure of the spectrum of self-adjoint operators $\mathcal{D}$ in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ associated with the formal Dirac operator (1) on $\mathbb{R}^n$ with $\mathbb{G}$-periodic regular and singular potentials. We also consider the Fredholm property and the essential spectrum of unbounded operators associated with non-periodic regular and singular potentials supported on $\mathbb{G}$-periodic smooth hypersurfaces in $\mathbb{R}^n$.

Keywords Dirac operators · Singular potential · Delta-interactions · Self-adjointness · Essential spectrum · Floquet theory

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On the occasion of the 80-th anniversary of the birth of Professor Nikolai Karapetiants.

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Introduction

We consider the formal Dirac operators with singular potentials

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma} = \mathfrak{D}_{\boldsymbol{A},\Phi,m} + \Gamma\delta_\Sigma \tag{3}$$

where $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ is a Dirac operator on $\mathbb{R}^n$

$$\begin{aligned}\mathfrak{D}_{\boldsymbol{A},\Phi,m} &= \alpha\cdot(D+\boldsymbol{A}) + \alpha_{n+1}m + \Phi I_N \\ &= \sum_{j=1}^{n}\alpha_j(D_{x_j}+A_j) + \alpha_{n+1}m + \Phi I_N, D_{x_j} = -i\partial_{x_j},\end{aligned} \tag{4}$$

with magnetic and electrostatic potentials $\boldsymbol{A} = (A_1,\dots,A_n), \Phi$, and the variable mass m, such that $A_j,\Phi,m \in L^\infty(\mathbb{R}^n)$. In formula (2), α_j are the $N\times N$ Dirac matrices, that is

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N,$$

I_N is the unit $N\times N$ matrix, $N = 2^{[(n+1)/2]}$ (see [18, 24]), and $\Gamma\delta_\Sigma$ is a singular delta-type potential supported on a C^2- hypersurface $\Sigma\subset\mathbb{R}^n$ periodic with respect to the action of a lattice $\mathbb{G}\subset\mathbb{R}^n$. More exactly, we assume that

$$\Sigma = \bigcup_{g\in\mathbb{G}}\Sigma_g \tag{5}$$

, where $\Sigma_g = \Sigma_0 + g, \Sigma_0$ is a closed C^2-hypersurface which is a boundary of the open bounded set Ω_0. We assume that $\Sigma_{g_1}\cap\Sigma_{g_2} = \emptyset$ if $g_1\neq g_2$. Let $\Omega_+ = \bigcup_{g\in\mathbb{G}}\Omega_g, \Omega_g = \Omega_0 + g$, and $\Omega_- = \mathbb{R}^n\setminus\overline{\Omega_+}$, that is Σ is a common boundary of the sets Ω_+ and Ω_-.

Such Dirac operators arise as approximation of Hamiltonians of interactions of relativistic quantum particles with potentials localized in thin tubular neighborhoods of the supports of singular potentials (see for instance [15, 30, 31]). In physical statements such problems describe the transitions of relativistic particles through obstacles generated by the potentials supported on the mentioned domains in $\mathbb{R}^n$ (see for instance [9, 14, 16, 22, 23, 28]).

The formal Dirac operators with singular potentials are realized as unbounded operators $\mathcal{D}$ in Hilbert spaces with domains described by interaction conditions on the sets carrying the singular potentials. Recently, many papers devoted to their spectral properties for the dimensions $n = 2, 3$ have appeared; see, for instance, [4, 7, 8, 11–13, 15, 21, 29–31, 37, 38].

In the paper [39], it was considered the self-adjointness of the unbounded operators in $L^2(\mathbb{R}^n,\mathbb{C}^N)$ associated with the operators $D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}$ for Σ belonging to the class of so-called uniformly regular C^2-hypersurfaces which contain all closed C^2-hypersurfaces and a wide set of unbounded C^2-hypersurfaces, in particular, $\mathbb{G}-$periodic C^2-hypersurfaces described by formula (5).

Let $H^1(\Omega_\pm,\mathbb{C}^N)$ be the Sobolev spaces of distributions on $\Omega_\pm$ with values in $\mathbb{C}^N$ and we set

$$H^1(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N) = H^1(\Omega_+,\mathbb{C}^N)\oplus H^1(\Omega_-,\mathbb{C}^N).$$

We associate with the formal Dirac operator $D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}$ the unbounded in $L^2(\mathbb{R}^n,\mathbb{C}^N)$ operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ with the domain

$$\begin{aligned}&H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N)\\ &= \left\{u\in H^1(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N) : \mathfrak{B}_\Sigma u(s) = a_+(s)\gamma_\Sigma^+ u(s)(s) + a_-(s)\gamma_\Sigma^- u(s) = 0, s\in\Sigma\right\}\end{aligned} \tag{6}$$

where $\gamma_\Sigma^\pm : H^1(\Omega_\pm\mathbb{C}^N)\to H^{1/2}(\Sigma,\mathbb{C}^N)$ are the trace operators, and

$$a_\pm(s) = \frac{1}{2}\Gamma(s)\mp i\alpha\cdot\nu(s),\ \ \alpha\cdot\nu(s) = \sum_{j=1}^{n}\alpha_j\nu_j(s), s\in\Sigma, \tag{7}$$

$\nu(s) = (\nu_1(s)..., \nu_n(s))$, $s \in \Sigma$ is the field of unit normal vectors to Σ pointed into Ω_-. We also associate the operator $\mathbb{D}_{A,\Phi,m,\mathfrak{B}}$ of the interaction (transmission) problem with the formal Dirac operator $D_{A,\Phi,m,\Gamma\delta_\Sigma}$

$$\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} u = \begin{cases} \mathfrak{D}_{A,\Phi,m} u \text{ on } \mathbb{R}^n \setminus \Sigma \\ \mathfrak{B}_\Sigma u = 0 \text{ on } \Sigma \end{cases}. \tag{8}$$

acting from $H^1(\mathbb{R}^n \setminus \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)$.
The following problems are considered in the paper.

1. We study the Dirac operators on n–dimensional torus $\mathbb{T}$ with singular potentials $\Gamma\delta_\Sigma$ where Σ is $(n-1)$–dimensional C^2–submanifold of $\mathbb{T}$, $\Gamma = (\Gamma_{ij})_{i,j=1}^N$ is the matrix with elements $\Gamma_{ij} \in C^1(\Sigma)$. As above, we associate with the formal Dirac operator with singular potential an unbounded $L^2(\mathbb{T}, \mathbb{C}^n)$ operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ and the interaction operator $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ bounded from $H^1(\mathbb{T} \setminus \Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)$. We study the Fredholm properties of $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$, the self-adjointness and discreetness of the spectrum of the operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ on the torus $\mathbb{T}$.
2. We consider the Floquet theory for the formal Dirac operator (3) where $\Sigma \subset \mathbb{R}^n$ is a $\mathbb{G}$–periodic C^2–hypersurface, Γ is a $\mathbb{G}$– periodic matrix, and the potentials A, Φ, m are $\mathbb{G}$– periodic. We describe the band-gap structure of the spectrum for the self-adjoint operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$.
3. We consider the Fredholm property of $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ and essential spectrum of $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ in the case if Σ is $\mathbb{G}$–periodic hypersurface in $\mathbb{R}^n$ but the matrix Γ, and potentials A, Φ, m are not periodic. Our approach to the investigation of the Fredholm property of $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ and the essential spectrum of the operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is based on the limit operator method (see [32–34]). We associate the sets of the limit operators with the operators $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ and $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$

$$\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma} = \mathbb{D}_{A^h,\Phi^h,m^h,\mathfrak{B}^h_\Sigma}, \mathcal{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma} = \mathcal{D}_{A^h,\Phi^h,m^h,\mathfrak{B}^h_\Sigma}$$

defined by the sequences $\mathbb{G} \ni h_k \to \infty$, where $A^h(x), \Phi^h(x), m^h(x)$ are the limits of the sequences $A(x+h_k), \Phi(x+h_k), m(x+h_k)$ in the sense of uniform convergence on compact sets in $\mathbb{R}^n$, and

$$\mathfrak{B}^h_\Sigma u = a^h_+ \gamma^+_\Sigma u + a^h_- \gamma^-_\Sigma u,$$

where $a^h_\pm(x) = \lim_{k\to\infty} a_\pm(x+h_k)$ in the sense of uniform convergence on compact sets in Σ. We denote by $Lim(\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma})$ $Lim(\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma})$ the set of all limit operators of $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}, \mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$. Applying the limit operators approach, we obtain the following result.

Theorem 1 *Let Σ be $\mathbb{G}$–periodic C^2-hypersurface, $A_j, \Phi, m \in C^1_b(\Omega)$, $\Gamma = (\Gamma_{jl})_{j,l=1}^N$ be an Hermitian matrix defined on Σ such that $\Gamma_{jl} \in C^1_b(\Sigma)$, $k, l = 1, ..., N$. We assume that the Lopatinsky-Shapiro condition*

$$\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2}\right) \neq 0, \xi_x \in T^*_x(\Sigma) : |\xi_x| = 1,$$

holds at every point $x \in \Sigma$ where $T^*_x(\Sigma)$ is the cotangent space to Σ at the point x. Then:

(*i*) $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} : H^1(\mathbb{R}^n \setminus \Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ *is a Fredholm operator if and only if all limit operators* $\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma} : H^1(\mathbb{R}^n \setminus \Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$ *are invertible*;

(*ii*) *The operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is closed and*

$$sp_{ess}\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} = \bigcup_{\mathcal{D}^h \in Lim(\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma})} sp\mathcal{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$$

As an example, we consider the essential spectrum of operators which are perturbations of periodic operators $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ by slowly oscillating at infinity potentials.

Notations and auxiliary material

Notations

- If $\boldsymbol{X}, \boldsymbol{Y}$ are Banach spaces, then we denote by $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ the space of bounded linear operators acting from $\boldsymbol{X}$ into $\boldsymbol{Y}$ with the uniform operator topology, and by $\mathcal{K}(\boldsymbol{X}, \boldsymbol{Y})$ the subspace of $\mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ of all compact operators. In the case $\boldsymbol{X} = \boldsymbol{Y}$, we write shortly $\mathcal{B}(\boldsymbol{X})$ and $\mathcal{K}(\boldsymbol{X})$.
- An operator $A \in \mathcal{B}(\boldsymbol{X}, \boldsymbol{Y})$ is called a Fredholm operator if $kerA$, and $cokerA = \boldsymbol{Y}/\mathrm{Im}A$ are finite dimensional spaces. Let $\mathcal{A}$ be a closed unbounded operator in a Hilbert space $\mathcal{H}$ with a dense in $\mathcal{H}$ domain $dom\mathcal{A}$. Then $\mathcal{A}$ is called a Fredholm operator if $ker\mathcal{A} = \{u \in dom\mathcal{A} : \mathcal{A}u = 0\}$ and $coker\mathcal{A} = \mathcal{H}/\mathrm{Im}\mathcal{A}$ where $\mathrm{Im}\mathcal{A}=\left\{\boldsymbol{w} \in \mathcal{H} : \boldsymbol{w} = \mathcal{A}u, u \in \mathcal{D}_{\mathcal{A}}\right\}$ are the finite-dimensional spaces. Note that $\mathcal{A}$ is a Fredholm operator as the unbounded operator in $\mathcal{H}$ if and only if $\mathcal{A} : dom\mathcal{A} \to \mathcal{H}$ is a Fredholm operator as the bounded operator where $dom\mathcal{A}$ is equipped by the graph norm
$$\|u\|_{dom\mathcal{A}} = \left(\|u\|^2_{\mathcal{H}} + \|\mathcal{A}u\|^2_{\mathcal{H}}\right)^{1/2}, u \in dom\mathcal{A}$$
(see for instance [1]).
- The essential spectrum $sp_{ess}\mathcal{A}$ of an unbounded operator $\mathcal{A}$ is a set of $\lambda \in \mathbb{C}$ such that $\mathcal{A} - \lambda I$ is not the Fredholm operator as the unbounded operator, and the discrete spectrum $sp_{dis}\mathcal{A}$ of $\mathcal{A}$ is a set of isolated eigenvalues of finite multiplicity. It is well known that if $\mathcal{A}$ is a self-adjoint operator, then $sp_{dis}\mathcal{A} = sp\mathcal{A} \setminus sp_{ess}\mathcal{A}$.
- We denote by $L^2(\mathbb{R}^n, \mathbb{C}^N)$ the Hilbert space of $N-$dimensional vector functions $u(x) = (u^1(x), ..., u^N(x)), x \in \mathbb{R}^n$ with the scalar product
$$\langle u, v\rangle = \int_{\mathbb{R}^n} u(x) \cdot v(x)dx,$$
where $u \cdot v = \sum_{j=1}^n u_j\bar{v}_j$.
- We denote by $H^s(\mathbb{R}^n, \mathbb{C}^N)$ the Sobolev space of vector-valued distributions $u \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N)$ such that
$$\|u\|_{H^s(\mathbb{R}^n,\mathbb{C}^N)} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s \|\hat{\boldsymbol{u}}(\xi)\|^2_{\mathbb{C}^N} d\xi\right)^{1/2} < \infty, s \in \mathbb{R},$$
where $\hat{\boldsymbol{u}}$ is the Fourier transform of u. If Ω is a domain in $\mathbb{R}^n$ then $H^s(\Omega, \mathbb{C}^N)$ is the space of restrictions of $u \in H^s(\mathbb{R}^n, \mathbb{C}^N)$ on Ω with the norm
$$\|u\|_{H^s(\Omega,\mathbb{C}^N)} = \inf_{lu\in H^s(\mathbb{R}^n,\mathbb{C}^N)} \|lu\|_{H^s(\mathbb{R}^n,\mathbb{C}^N)},$$
where lu is an extension of u on $\mathbb{R}^n$. If Σ is a smooth enough hypersurface in $\mathbb{R}^n$, we denote by $H^{s-1/2}(\Sigma, \mathbb{C}^N)$ the space of restrictions on Σ the distributions in $H^s(\mathbb{R}^n, \mathbb{C}^N), s > 1/2$.
- We denote by $C_b(\mathbb{R}^n)$ the class of bounded continuous functions on $\mathbb{R}^n$, $C_b^m(\mathbb{R}^n)$ the class of functions a on $\mathbb{R}^n$ such that $\partial^\alpha a \in C_b(\mathbb{R}^n)$ for all multi-indices $\alpha : |\alpha| \leq m$. We denote by $C_b^1(\Sigma)$ the class of differentiable on Σ functions that are bounded with their first derivatives, and $C_b^\infty(\mathbb{R}^n) = \cap_{m\geq 0} C_b^m(\mathbb{R}^n)$.
- Let a C^2-hypersurface $\Sigma \subset \mathbb{R}^n, n \geq 2$ be the common boundary of the domains $\Omega_\pm$. We say that Σ is *uniformly regular* (see for instance [3, 19]) if: (i) there exists $r > 0$ such that for every point $x_0 \in \Sigma$ there exists a ball $B_r(x_0) = \left\{x \in \mathbb{R}^n : \left|x - x_0\right| < r\right\}$ and the diffeomorphism $\varphi_{x_0} : B_r(x_0) \to B_1(0)$ such that
$$\begin{aligned} \varphi_{x_0}\left(B_r(x_0) \cap \Omega_\pm\right) =& B_1(0) \cap \mathbb{R}^n_\pm, \mathbb{R}^n_\pm \\ &= \left\{ y = (y', y_n) \in \mathbb{R}^{n-1}_{y'} \times \mathbb{R}_{y_n} : y_n \gtrless 0 \right\}, \\ \varphi_{x_0}\left(B_r(x_0) \cap \Sigma\right) =& B_1(0) \cap \mathbb{R}^{n-1}_{y'}; \end{aligned}$$
(ii) Let $\varphi^i_{x_0}, \psi^i_{x_0}, i = 1, ..., n$ be the coordinate functions of the mappings $\varphi_{x_0}, \varphi^{-1}_{x_0}$. Then

$$\sup_{x_0\in\Sigma}\sup_{|\alpha|\le 2, x\in B_r(x_0)}\left|\partial^\alpha\varphi^i_{x_0}(x)\right| < \infty, i = 1, ..., n;$$
$$\sup_{x_0\in\Sigma}\sup_{|\alpha|\le 2, y\in B_1(0)}\left|\partial^\alpha\psi^i_{x_0}(y)\right| < \infty, i = 1, ..., n.$$

Note that each closed C^2–hypersurface is uniformly regular.

Auxiliary material

Dirac operators on $\mathbb{R}^n$ with singular potentials ([39]).

- Let

$$\mathfrak{D}_{\boldsymbol{A},\Phi,m,\Gamma\delta}u(x) = \left(\mathfrak{D}_{\boldsymbol{A},\Phi,m} + \Gamma\delta_\Sigma\right)u(x), x \in \mathbb{R}^n$$

be the formal Dirac operator defined by formulas (3), (5). We assume that Σ is the uniformly regular C^2–hypersurface in $\mathbb{R}^n, A_j, \Phi, m \in L^\infty(\mathbb{R}^n), \Gamma = \left(\Gamma_{i,j}\right)_{i,j=1}^N, \Gamma_{i,j} \in C_b^1(\Sigma)$. We define the product $\Gamma\delta_\Sigma u$ where $u \in H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N)$ as a distribution in $\mathcal{D}'(\mathbb{R}^n, \mathbb{C}^N) = \mathcal{D}'(\mathbb{R}^n)\otimes\mathbb{C}^N$ acting on the test functions $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^N)$ as

$$\left(\Gamma\delta_\Sigma u\right)(\boldsymbol{\varphi}) = \frac{1}{2}\int_\Sigma \Gamma(s)\left(\gamma_\Sigma^+ u(s) + \gamma_\Sigma^- u(s)\right)\cdot\varphi(s)ds. \tag{9}$$

Integrating by parts and taking into account (9), we obtain that

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}u = \mathfrak{D}_{\boldsymbol{A},\Phi,m}u - \left[i\alpha\cdot\nu\left(\gamma_\Sigma^+u - \gamma_\Sigma^-u\right) + \frac{1}{2}\Gamma\left(\gamma_\Sigma^+u + \gamma_\Sigma^-u\right)\right]\delta_\Sigma, \tag{10}$$

where $\gamma_\Sigma^\pm : H^1(\Omega_\pm, \mathbb{C}^N) \to H^{1/2}(\Omega_\pm, \mathbb{C}^N)$ are the trace operators, $\nu(s) = (\nu_1(s), ..., \nu_n(s))$ is the field of unit normal vectors pointed to Ω_-. Formula (10) yields that in the distribution sense

$$D_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}u = \mathfrak{D}_{\boldsymbol{A},\Phi,m}u - \left[i\alpha\cdot\nu\left(\gamma_\Sigma^+u - \gamma_\Sigma^-u\right) + \frac{1}{2}\Gamma\left(\gamma_\Sigma^+u + \gamma_\Sigma^-u\right)\right]\delta_\Sigma, \tag{11}$$

where $\mathfrak{D}_{\boldsymbol{A},\Phi,m}u$ is the regular distribution given by the function $\mathfrak{D}_{\boldsymbol{A},\Phi,m}u \in L^2(\mathbb{R}^n, \mathbb{C}^N)$. Formula (11) yields that $\mathfrak{D}_{\boldsymbol{A},\Phi,m,\Gamma\delta_\Sigma}u \in L^2(\mathbb{R}^n, \mathbb{C}^N)$ if and only if

$$-i\alpha\cdot\nu\left(\gamma_\Sigma^+u - \gamma_\Sigma^-u\right) + \frac{1}{2}\Gamma\left(\gamma_\Sigma^+u + \gamma_\Sigma^-u\right) = 0 \text{ on } \Sigma. \tag{12}$$

Condition (12) can be written in the form

$$\mathfrak{B}_\Sigma u = a_+\gamma_\Sigma^+u + a_-\gamma_\Sigma^-u = \boldsymbol{0} \text{ on } \Sigma \tag{13}$$

where $a_\pm$ are $N\times N$ matrices:

$$a_\pm = \frac{1}{2}\Gamma \mp i\alpha\cdot\nu \text{ on } \Sigma. \tag{14}$$

We associate with the formal Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,\Gamma\delta_\Sigma}$ the unbounded in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,m}$ with the domain

$$\begin{aligned} dom\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} =& H^1_{\mathfrak{B}_\Sigma}(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N) \\ =& \left\{u \in H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N) : \mathfrak{B}_\Sigma u = 0 \text{ on } \Sigma\right\}, \end{aligned} \tag{15}$$

and the bounded operator of the interaction (transmission) problem

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}u = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m}u \text{ on } \mathbb{R}^n\setminus\Sigma, \\ \mathfrak{B}_\Sigma u = a_+\gamma_\Sigma^+ u + a_-\gamma_\Sigma^- u = 0 \text{ on } \Sigma \end{cases} \tag{16}$$

acting from $H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)$.

- We consider the parameter-dependent operator

$$\begin{aligned} &\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)u = \left(\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} - i\mu I_N\right)u \\ &= \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m}(i\mu)u = (\mathfrak{D}_{\boldsymbol{A},\Phi,m} - i\mu I_N)u \text{ on } \mathbb{R}^n\setminus\Sigma, \\ \mathfrak{B}_\Sigma u = a_+\gamma_\Sigma^+ u + a_-\gamma_\Sigma^- u = 0 \text{ on } \Sigma \end{cases}, \mu \in \mathbb{R} \end{aligned} \tag{17}$$

acting from $H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N)$ into $L^2(\mathbb{R}^n, \mathbb{C}^N)$.

- Condition

$$\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2} - i\mu I_N\right) \neq 0 \text{ for } (\xi_x, \mu) \in T_x^*(\Sigma) \times \mathbb{R} : \left|\xi_x\right|^2 + \mu^2 = 1 \tag{18}$$

is called the local parameter-dependent Lopatinsky-Shapiro condition where $T_x^*(\Sigma)$ is the cotangent space to Σ at the point $x \in \Sigma$, and the condition

$$\inf_{x\in\Sigma} \inf_{(\xi_x,\mu)\in T_x^*(\Sigma)\times\mathbb{R} : \left|\xi_x\right|^2+\mu^2=1} \left|\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2} - i\mu I_N\right)\right| > 0 \tag{19}$$

is called the **uniform parameter-dependent Lopatinsky-Shapiro condition**.

- Note that if the matrix $\Gamma(x)$ is Hermitian for every $x \in \Gamma$, then condition (18) becomes the local Lopatinsky-Shapiro condition

$$\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2}\right) \neq 0 \text{ for } \xi_x \in T_x^*(\Sigma) : \left|\xi_x\right| = 1 \tag{20}$$

Theorem 2 *Let Σ be the uniformly regular C^2–hypersurface in $\mathbb{R}^n$, $A_j, \Phi, m \in L^\infty(\mathbb{R}^n)$, $\Gamma = \left(\Gamma_{i,j}\right)_{i,j=1}^N, \Gamma_{i,j} \in C_b^1(\Sigma)$, and the uniform parameter-dependent Lopatinsky-Shapiro condition (19) hold. Then there exists $\mu_0 \in \mathbb{R}$ such that the operator*

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu) : H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$$

is invertible for every $\mu \in \mathbb{R} : |\mu| > R$.

Theorem 3 *Let Σ be the uniformly regular C^2–hypersurface in $\mathbb{R}^n$, $A_j, \Phi, m \in L^\infty(\mathbb{R}^n), \Gamma = \left(\Gamma_{i,j}\right)_{i,j=1}^N, \Gamma_{i,j} \in C_b^1(\Sigma), A_j, \Phi, m$ be real-valued functions, $\Gamma(x)$ be Hermitian matrix for every $x \in \Sigma$, and the uniform Lopatinsky-Shapiro condition*

$$\inf_{x\in\Sigma} \inf_{\xi_x\in T_x^*(\Sigma) : \left|\xi_x\right|^2=1} \left|\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2}\right)\right| > 0 \tag{21}$$

hold. Then the operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ is self-adjoint.

Example 4 Let $\Gamma(x) = \eta(x)I_N + \tau(x)\alpha_{n+1}$ where $\eta(x), \tau(x) \in C_b^1(\Sigma)$ be real-valued functions. Then condition

$$\inf_{x\in\mathbb{R}^n} \left|\eta^2(x) - \tau^2(x) - 4\right| > 0 \tag{22}$$

ensures the condition (21), and therefore the invertibility $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}(i\mu)$ for large enough $|\mu|$, and the self-adjointness of $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$.

Note that the singular potential $\Gamma\delta_\Sigma$ describes the electrostatic and Lorentz scalar shell interactions in $\mathbb{R}^n$ (see [11–13].)

Band-dominated operators on $\mathbb{R}^n$ and their local invertibility at infinity ([32–34] ,[26])

Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(x) = 1$ if $|x| \leq 1/2$ and $\psi(x) = 0$ if $|x| \geq 1$, $\chi(x) = 1 - \psi(x)$, $\psi_R(x) = \psi_R(x/R)$, $\chi_R(x) = \chi(x/R)$.

Definition 5 We say that $A \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^N))$ is locally invertible at infinity if there exists $R > 0$ and the operators $\mathcal{L}_R, \mathcal{R}_R \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^N))$ such that

$$\mathcal{L}_R A \chi_R I = \chi_R I, \chi_R A \mathcal{R}_R = \chi_R I.$$

Definition 6 We say that the operator $A \in \mathcal{B}\left(L^2(\mathbb{R}^n, \mathbb{C}^N)\right)$ belongs to the class $\mathcal{A}(\mathbb{R}^n, \mathbb{C}^N)$ of band-dominated operators on $\mathbb{R}^n$ if for every function $\varphi \in C_b^\infty(\mathbb{R}^n)$

$$\lim_{t\to 0} \left\| [\varphi_t I, A] \right\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^N))} = \lim_{t\to 0} \left\| [\varphi_t A - A\varphi_t I] \right\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^N))} = 0$$

where $\varphi_t(x) = \varphi(t_1 x_1, ..., t_n x_n)$, $t = (t_1, ..., t_n) \in \mathbb{R}^n$.

Note that $\mathcal{A}(\mathbb{R}^n, \mathbb{C}^N)$ is an inverse closed C^*−algebra.
We denote by $V_h, h \in \mathbb{G}$ the unitary in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ shift operator $V_h u(x) = u(x - h)$.
Ler $\mathbb{G}$ be the lattice in $\mathbb{R}^n$, that is

$$\mathbb{G} = \left\{ g \in \mathbb{R}^n \, : \, g = \sum_{j=1}^{n} g_j a_j, g_j \in \mathbb{Z} \right\}, \tag{23}$$

where $\left\{a_1, ..., a_n\right\}$ is a linearly independent system of vectors in $\mathbb{R}^n$.

Definition 7 Let the sequence $\mathbb{G} \ni h_k \to \infty$. We say that the operator $A^h \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^N))$ is a limit operator of $A \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^N))$ if for every function $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned} &\lim_{k\to\infty} \left\| \left(V_{-h_k} A V_{h_k} - A^h\right)\varphi I \right\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^N))} \\ &= \lim_{k\to\infty} \left\| \varphi\left(V_{-h_k} A V_{h_k} - A^h\right) \right\|_{\mathcal{B}(L^2(\mathbb{R}^n,\mathbb{C}^N))} = 0. \end{aligned}$$

We say that the operator $A \in \mathcal{B}(L^2(\mathbb{R}^n, \mathbb{C}^N))$ is **rich** if every sequence $\mathbb{G} \ni h_k \to \infty$ has a subsequence h_{k_l} defining the limit operator.

Theorem 8 (see [26, 32, 33]). *Let $A \in \mathcal{A}(\mathbb{R}^n, \mathbb{C}^N)$ be a rich operator acting in $L^2(\mathbb{R}^n, \mathbb{C}^N)$. Then the following assertions are equivalent*:

(i) *A is a locally invertible at infinity operator*;
(ii) *The family $Lim(A)$ of all limit operators is uniformly invertible in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ that is every limit operator A^h has inverse $\left(A^h\right)^{-1}$, and*

$$\sup_{A^h \in Lim(A)} \left\| \left(A^h\right)^{-1} \right\| < \infty;$$

(iii) *Each limit operator $A^h \in Lim(A)$ is invertible in $L^2(\mathbb{R}^n, \mathbb{C}^N)$.*

Remark 9 The equivalence of conditions (i) and (ii) has been proved in [32, 33], but the question of the equivalence of conditions (ii) and (iii) has been open for a long time. The affirmative answer to this question has been obtained in [26].

Interaction problems for Dirac operators on the torus

Let $\mathbb{G}$ be the lattice defined by (23). We consider the formal Dirac operator on $\mathbb{R}^n$ given by formulas (3), (5) with the $\mathbb{G}$–periodic potentials A_j, Φ, m, and the $\mathbb{G}$–periodic singular potentials $\Gamma\delta_\Sigma$. Let W be a fundamental domain for the action of the group $\mathbb{G}$ on $\mathbb{R}^n$, and Ω_0 be a domain such that $\overline{\Omega_0} \subset int(W)$. We set

$$\Omega_+ = \bigcup_{g\in G} \Omega_g, \Omega_- = \mathbb{R}^n \diagdown \overline{\Omega}_+,$$

and

$$\Sigma = \bigcup_{g\in\mathbb{G}} \Sigma_g,\ \Sigma_g = \partial\Omega_g$$

is the periodic common boundary of the domains $\Omega_\pm$.
We associate with the periodic formal Dirac operator

$$D_{\boldsymbol{A},\Phi,m,\mathfrak{B},\Gamma\delta_\Sigma} = \mathfrak{D}_{\boldsymbol{A},\Phi,m} + \Gamma\delta_\Sigma$$

where $A_j, \Phi, m \in C^1(\mathbb{R}^n)$ are $\mathbb{G}$–periodic function on $\mathbb{R}^n$, $\Gamma = \left(\Gamma_{kl}\right)_{k.l=1}^N$ is a periodic matrix with $\Gamma_{kl} \in C^1(\Sigma)$,the interaction operator $\mathbb{D}^{\mathbb{T}}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ on the torus $\mathbb{T} = \mathbb{R}^n \diagup \mathbb{G}$

$$\mathbb{D}^{\mathbb{T}}_{\boldsymbol{A},\Phi,\Psi,\mathfrak{B}_\Sigma} u = \begin{cases} \mathfrak{D}_{\boldsymbol{A},\Phi,m} u \text{ on } \mathbb{T}\diagdown\tilde{\Sigma} \\ \mathfrak{B}_{\tilde{\Sigma}} u \ = a_+\gamma^+_{\Sigma_0} u + a_-\gamma^-_{\Sigma_0} u = 0 \text{ on } \tilde{\Sigma} \end{cases}, \tag{24}$$

where $a_\pm = \frac{\Gamma}{2} \mp i\alpha \cdot \nu$, $\tilde{\Sigma}$ is a C^2 manifold on $\mathbb{T}$ of dimension $(n-1)$ which is the natural projection on $\mathbb{T}$ by the hypersurface $\Sigma \subset \mathbb{R}^n$, and $\tilde{\Sigma}$ is the common boundary of the domain $\tilde{\Omega}_\pm \subset \mathbb{T}$, which are the projections of $\Omega_\pm$ on $\mathbb{T}$, $\nu(s)$ is the unit normal vector to $\tilde{\Sigma}$ pointed to $\tilde{\Omega}_-$.
Let

$$H^1(\mathbb{T}\diagdown\tilde{\Sigma}, \mathbb{C}^N) = H^1(\tilde{\Omega}_+, \mathbb{C}^N) \oplus H^1(\tilde{\Omega}_-, \mathbb{C}^N),$$

$H^1(\tilde{\Omega}_\pm, \mathbb{C}^N)$ are Sobolev spaces on domains $\tilde{\Omega}_\pm \subset \mathbb{T}$. We consider $\mathbb{D}^{\mathbb{T}}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ as a bounded operator from $H^1(\mathbb{T}\diagdown\tilde{\Sigma}, \mathbb{C}^N)$ into $L^2(\mathbb{T}, \mathbb{C}^N)$. We denoted by $\mathcal{D}^{\mathbb{T}}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ the unbounded operator in $L^2(\mathbb{T}, \mathbb{C}^N)$ generated by the Dirac operator $\mathfrak{D}_{\boldsymbol{A},\Phi,\Psi}$ on the torus $\mathbb{T}$ with the domain

$$dom\mathcal{D}^{\mathbb{T}}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} = \left\{u \in H^1(\mathbb{T}\diagdown\tilde{\Sigma}, \mathbb{C}^N) : \mathfrak{B}_{\tilde{\Sigma}} u = 0 \text{ on } \tilde{\Sigma}\right\}.$$

Theorem 10 (i) *Let $\tilde{\Sigma} \subset \mathbb{T}$ be a C^2-submanifold of the dimension $(n-1)$, $A_j, \Phi, m \in C^1(\mathbb{T})$, the matrix $\Gamma = \left(\Gamma_{ij}\right)_{i,j=1}^N$ be defined on $\tilde{\Sigma}$ and such that $\Gamma_{ij} \in C^1(\tilde{\Sigma})$. Moreover, let for every point $x \in \tilde{\Sigma}$ the Lopatinsky-Shapiro condition*

$$\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2}\right) \neq 0, \text{ for each } \xi_x \in T^*_x(\tilde{\Sigma}) : \left|\xi_x\right| = 1 \tag{25}$$

hold, where $T^*_x(\tilde{\Sigma})$ is the cotangent space to the manifold $\tilde{\Sigma}$ at the point x. Then, $\mathbb{D}_{\boldsymbol{A},\Phi,\Psi,\mathfrak{B}\tilde{\Sigma}} : H^1(\mathbb{T}, \mathbb{C}^N) \to L^2(\mathbb{T}, \mathbb{C}^N)$ is the Fredholm operator.

(*ii*) *Let in addition to the above conditions the matrix $\Gamma(x)$ be Hermitian for each $x \in \tilde{\Sigma}$. Then $ind(\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}) = 0$, and the operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}} - \lambda I : H^1(\mathbb{T}\diagdown\tilde{\Sigma}, \mathbb{C}^N) \to L^2(\mathbb{T}, \mathbb{C}^N)$ is invertible for each $\lambda \in \mathbb{C}\diagdown\Pi$ where Π is a discrete set in $\mathbb{C}$ with a unique limit point ∞. The unbounded operator $\mathcal{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}$ is closed and has the discrete spectrum only.*

Proof (i) As in the paper [39], one can prove that the Lopatinsky-Shapiro condition (25) is sufficient for the local Fredholmness of the operator $\mathbb{D}_{A,\Phi,\Psi,\mathfrak{B}_{\tilde{\Sigma}}}$ at the point $x \in \tilde{\Sigma}$. Since the operator $\mathfrak{D}_{A,\Phi,m}$ is elliptic at every point $x \in \mathbb{T}$ the local principle of the elliptic theory [1] yields that the operator $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}$ is Fredholm if condition (25) holds at every point $x \in \tilde{\Sigma}$;

(ii) It follows from (i) the operator $\mathbb{D}_{A,\Phi,\Psi,\mathfrak{B}_{\tilde{\Sigma}}} - i\mu I_N$ is the Fredholm operator for every $\mu \in \mathbb{C}$. Hence $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}} - i\mu I_N$ is the analytical family of the Fredholm operators. Moreover, since $\Gamma(x)$ is a Hermitian matrix for every $x \in \mathbb{R}^n$ the parameter-dependent Lopatinsky-Shapiro condition holds for every $\mu \in \mathbb{R}$. By Theorem 2, the operator $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}} - i\mu I_N$ is invertible for $\mu \in \mathbb{R}$ with $|\mu|$ is large enough. Hence, by the Analytic Fredholm Theorem (see [10, 20]), the operator

$$\begin{gathered}\mathbb{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}} - \lambda I : H^1(\mathbb{T}\setminus\tilde{\Sigma}, \mathbb{C}^N) \to L^2(\mathbb{T}, \mathbb{C}^N) \\ \text{is invertible for each } \lambda \in \mathbb{C}\setminus\Pi\end{gathered} \tag{26}$$

where Π is a discrete set with a possible limit point ∞.

Moreover, $ind\left(\mathbb{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}\right) = 0$. The Lopatinsky-Shapiro condition (25) yields the a priori estimate

$$\|u\|_{H^1(\mathbb{T}\setminus\tilde{\Sigma},\mathbb{C}^N)} \le C\left(\left\|\mathfrak{D}_{A,\Phi,m}u\right\|_{L^2(\mathbb{T},\mathbb{C}^N)} + \|u\|_{L^2(\mathbb{T},\mathbb{C}^N)}\right) \tag{27}$$

for every $u \in H^1(\mathbb{T}\setminus\tilde{\Sigma}, \mathbb{C}^N)$ with a constant $C > 0$ independent of u. The a priori estimate (27) implies the closedness of $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}$. Moreover, applying property (26) we obtain that $sp\mathcal{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}$ is discrete.

Theorem 11 *Let conditions* (*i*) *of Theorem* 10 *hold. Moreover*, $A_j\Phi, m$ *are real-valued functions*, *and the matrix* $\Gamma(x)$ *is Hermitian for every* $x \in \tilde{\Sigma}$. *Then the operator* $\mathcal{D}^{\mathbb{T}}_{A,\Phi,m,\mathfrak{B}_{\Sigma}}$ *is self-adjoint in* $L^2(\mathbb{T}, \mathbb{C}^n)$.

Proof We turn to the paper [39] where the similar result was obtined for the unbounded in $L^2(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N)$ opertator $\mathcal{D}_{\mathbf{A}\Phi,m,\mathfrak{B}_{\Sigma}}$.

1^0. Let $u, v \in dom\mathcal{D}_{A,\Phi,m,\mathfrak{B}_{\Sigma}}$. Then integrating by parts we obtain that

$$\begin{aligned}&\left\langle \mathfrak{D}_{A,\Phi,m}u, v\right\rangle_{L^2(\mathbb{T},\mathbb{C}^N)} - \left\langle u, \mathfrak{D}_{A,\Phi,m}v\right\rangle_{L^2(\mathbb{T},\mathbb{C}^N)} \\ &= -\frac{1}{4}\left\langle \Gamma\left(\gamma^+_{\Sigma_0}u + \gamma^-_{\Sigma_0}u\right), \gamma^+_{\Sigma_0}v - \gamma^-_{\Sigma_0}v\right\rangle_{L^2(\tilde{\Sigma},\mathbb{C}^N)} \\ &+ \frac{1}{4}\left\langle \gamma^+_{\Sigma_0}u + \gamma^-_{\Sigma_0}u, \Gamma(\gamma^+_{\Sigma}v - \gamma^-_{\Sigma}v\right\rangle_{L^2(\tilde{\Sigma},\mathbb{C}^N)}.\end{aligned} \tag{28}$$

Since Γ is an Hermitian matrix we obtain that $\mathfrak{D}_{A,\Phi,m}$ is a symmetric operator.

2^0. Let

$$\mathbb{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}(i\mu)u = \begin{cases} \left(\mathfrak{D}_{A,\Phi,m} - i\mu I_N\right)u \text{ on } \mathbb{T}\setminus\tilde{\Sigma} \\ \mathfrak{B}_{\tilde{\Sigma}}u \ = 0 \text{ on } \tilde{\Sigma}\end{cases} \tag{29}$$

be the operator depending on the parameter $\mu \in \mathbb{R}$ acting from $H^1(\mathbb{T}\setminus\tilde{\Sigma}, \mathbb{C}^N)$ into $L^2(\mathbb{T}, \mathbb{C}^N)$, and let the Lopatinsky-Shapiro condition (25) holds at every point $x \in \tilde{\Sigma}$. Then since $\mathfrak{D}_{A,\Phi,\Psi} - i\mu I_N$ is the elliptic with parameter operator on the torus $\mathbb{T}$, and the Lopatinsky-Shapiro condition (25) yields the parameter-dependent Lopatinsky-Shapiro condition

$$\det\left(\alpha\cdot\xi_x + \frac{\Gamma(x)}{2} - i\mu I_N\right) \ne 0, \xi_x \in T^*_x(\tilde{\Sigma}) : \left|\xi_x\right|^2 + \mu^2 = 1 \tag{30a}$$

since Γ is a Hermitian matrix. Condition (30a) yields that the interaction (transmission) operator $\mathbb{D}^{\mathbb{T}}_{A,\Phi,m,\mathfrak{B}_{\Sigma_0}}(i\mu)$ is invertible for the large value of $|\mu|$ (see [1, 2]). Moreover, the invertibility of $\mathbb{D}^{\mathbb{T}}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}(i\mu)$ for large $|\mu|$ implies that $Range\left(\mathfrak{D}_{A,\Phi,m} - i\mu I_N\right) = L^2(\mathbb{T}, \mathbb{C}^N)$ for all $\mu \in \mathbb{R}$ with large enough $|\mu|$. Hence, the deficiency indices of $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}$ are equal to zero, and the operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}$ is self-adjoint.

Floquet theory of interaction problems on periodic hypersurfaces

Let $\mathbb{G}$ be the lattice (23), and $\mathbb{G}^*$ be the reciprocal lattice

$$\mathbb{G}^* = \left\{ k \in \mathbb{R}^n : k = \sum_{j=1}^{n} k_j b_j . k_j \in \mathbb{R} \right\},$$
$$a_j \cdot b_l = 2\pi\delta_{jl}, j, l = 1, ..., n.$$

We fix a connected fundamental domain $W_0 \subset \mathbb{R}^n$ (Wigner–Seitz cell) of the lattice $\mathbb{G}$ in $\mathbb{R}^n$, i.e., a set

$$W_0 = \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{n} t_j a_j, t_j \in [0,1) \right\}.$$

such that $\mathbb{R}^n = \sum_{g\in\mathbb{G}} W_g, W_g = W_0 + g$. We will also fix a connected fundamental domain B_0 (Brillouin zone) of the reciprocal lattice $\mathbb{G}^*$

$$B_0 = \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{n} t_j b_j, t_j \in [0,1) \right\}. \tag{31}$$

We also introduce two tori $\mathbb{T} = \mathbb{R}^n / \mathbb{G}$ and $\mathbb{T}^* = \mathbb{R}^n / \mathbb{G}^*$ which can be identified naturally with fundamental domains W_0, B_0, respectively. Let as above

$$\Omega_+ = \sum_{g\in\mathbb{G}} \Omega_g, \text{ where } \bar{\Omega}_0 \subset int(W_0), \Omega_g = \Omega_0 + g, g \in \mathbb{G}, \Omega_- = \mathbb{R}^n \diagdown \overline{\Omega_+}, \tag{32}$$

and

$$\Sigma = \bigcup_{g\in\mathbb{G}} \Sigma_g, \ \Sigma_0 = \partial\Omega_0, \ \Sigma_g = \Sigma_0 + g \tag{33}$$

is the common boundary of the domains $\Omega_\pm$.
We consider here the periodic formal Dirac operator on $\mathbb{R}^n$ with singular potentials of the form

$$D_{\mathbf{A},\Phi,m,\Gamma\delta_\Sigma} = \mathfrak{D}_{\mathbf{A},\Phi,m} + \Gamma\delta_\Sigma \tag{34}$$

where $\mathfrak{D}_{\mathbf{A},\Phi,m}$ is a Dirac operator on $\mathbb{R}^n$ given by formula (4), $A_j, \Phi, m \in C^1(\mathbb{R}^n)$ are real-valued $\mathbb{G}$–periodic functions, and $\Gamma = \left(\Gamma_{i,j}\right)_{i,j=1}^{N}$ is a $\mathbb{G}$-periodic Hermitian matrix with $\Gamma_{i,j} \in C^1(\Sigma)$.
Let $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ be the unbounded operator in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ generated by the Dirac operator $\mathfrak{D}_{\mathbf{A},\Phi,m}$ with the domain

$$dom\left(\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}\right) = \left\{ u \in H^1(\mathbb{R}^n \diagdown \Sigma, \mathbb{C}^N) : \mathfrak{B}_\Sigma u = 0 \right\}$$

associated with the formal Dirac operator $D_{\mathbf{A},\Phi,m,\Gamma\delta_\Sigma}$.
We assume that the local Lopatinsky-Shapiro condition

$$\det\left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2}\right) \neq 0, \xi_x \in T_x^*(\Sigma) : \left|\xi_x\right| = 1 \tag{35}$$

is satisfied at every point $x \in \Sigma$.
By Theorem 3 $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is a self-adjoint operator in $L^2(\mathbb{R}^n, \mathbb{C}^N)$. Since the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ is invariant with respect to the shifts $V_g, g \in \mathbb{G}$

$$sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = sp\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}.$$

We consider the band-gap structure of the spectrum of $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ applying the Floquet transform (see for instance [25, 40]).

Let $f \in S(\mathbb{R}^n, \mathbb{C}^N) = S(\mathbb{R}^n) \otimes \mathbb{C}^N$, where $S(\mathbb{R}^n)$ is the Schwartz space. The Floquet transform of f is defined as

$$\left(\mathcal{F}_{\mathbb{G}} f\right)(x,k) = \sum_{\gamma \in \mathbb{G}} f(x-\gamma) e^{-ik\cdot(x-\gamma)}, k \in B_0,$$

and the inverse Floquet transform is

$$\left(\mathcal{F}_{\mathbb{G}}^{-1} v\right)(x) = \int_{B_0} v(x,k) e^{ix\cdot k} \frac{dk}{vol(B_0)}$$

The operator $\mathcal{F}_{\mathbb{G}}$ is continued from the space $S(\mathbb{R}^n, \mathbb{C}^N)$ to the unitary operator acting from $L^2(\mathbb{R}^n, \mathbb{C}^N)$ into the space $L^2(\mathbb{T} \times \mathbb{T}^*, \mathbb{C}^N)$. Applying the Floquet transform, we obtain that

$$\mathcal{F}_{\mathbb{G}} \mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} \mathcal{F}_{\mathbb{G}}^* = \int_{k \in \mathbb{T}^*}^{\oplus} \mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(k) \frac{dk}{vol(B_0)} \tag{36}$$

where $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(k), k \in \mathbb{T}^*$ is the unbounded operator in $L^2(\mathbb{T}, \mathbb{C}^N)$ generated by the Dirac operator

$$\mathfrak{D}_{\mathbf{A},\Phi,m}(k) = \mathfrak{D}_{\mathbf{A}+k,\Phi,m} = \alpha \cdot (D + A + k) + m\alpha_{n+1} + \Phi I_N \text{ on } \mathbb{T}$$

with domain

$$dom(\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}(k)) = \left\{u \in H^1(\mathbb{T} \setminus \tilde{\Sigma}, \mathbb{C}^N) : \mathfrak{B}_{\tilde{\Sigma}} u = 0\right\}.$$

As follows from Theorem 11 the operator $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}(k)$ has real discrete spectrum

$$sp\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_{\tilde{\Sigma}}}(k) = \left\{\lambda_j(k)\right\}_{j=-\infty}^{\infty}, k \in \mathbb{T}^*$$

where $\lambda_j(k) < \lambda_{j+1}(k)$ for every $j \in \mathbb{Z}$ and $\lambda_j(k)$ are continuous real-valued functions on the torus $\mathbb{T}^*$.
The decomposition of $\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma}$ in the direct integral (36) yields that

$$sp\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = sp_{ess}\mathcal{D}_{\mathbf{A},\Phi,m,\mathfrak{B}_\Sigma} = \bigcup_{j \in \mathbb{Z}} \left[a_j, b_j\right] \tag{37}$$

where

$$\left[a_j, b_j\right] = \left\{\lambda \in \mathbb{R} : \lambda = \lambda_j(k), k \in \mathbb{T}^*\right\}. \tag{38}$$

Fredholm theory and essential spectrum of interaction problems on periodic hypersurfaces in $\mathbb{R}^n$

We consider the interaction problem on the domains $\Omega_\pm$ with the common boundary Σ described in (32) and (33). Hence, the domains $\Omega_\pm$ and the hypersurface Σ are invariant with respect to the shifts on the vectors $h \in \mathbb{G}$. We do not assume the periodicity of the potentials and the matrix Γ with respect to the action of $\mathbb{G}$.
We assume that

$$A_j, \Phi, m \in C_b^1(\mathbb{R}^n), \Gamma_{ij} \in C_b^1(\Sigma). \tag{39}$$

Our approach is based on the limit operators method and Theorem 8. We introduce the limit operators defined by the sequence $\mathbb{G} \ni h_k \to \infty$. We set

$$\boldsymbol{A}^h(x) = \lim_{k\to\infty} \boldsymbol{A}(x+h_k), \Phi^h(x) = \lim_{k\to\infty} \Phi(x+h_k), m^h(x) = \lim_{k\to\infty} m(x+h_k)$$

where the limits are understood in the sense of the uniform converges on compact sets in $\mathbb{R}^n$, and

$$\Gamma^h(x) = \lim_{k\to\infty} \Gamma(x + h_k)$$

is understood in the sense of the converges on the finite unions $\cup_{|g|\le l}\Sigma_g, l \in \mathbb{N}$.
We use the notations $\mathbb{X} = H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N)$ and $\mathbb{Y} = L^2(\mathbb{R}^n, \mathbb{C}^N)$, and

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} u = (\mathfrak{D}_{\boldsymbol{A},\Phi,m} u, \mathfrak{B}_\Sigma u = 0)$$

is a bounded operator from $\mathbb{X}$ into $\mathbb{Y}$. We introduce the limit operator $\mathbb{D}^h_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ defined by the sequence $\mathbb{G} \ni h_k \to \infty$ as follows:

$$\mathbb{D}^h_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} u = \mathbb{D}_{\boldsymbol{A}^h,\Phi^h,m^h,\mathfrak{B}^h_\Sigma} u = (\mathfrak{D}_{\boldsymbol{A}^h,\Phi^h,m^h} u, \mathfrak{B}^h_\Sigma u = 0)$$

where $\mathfrak{B}^h_\Sigma u = a^h_+ \gamma^+_\Sigma u + a^h_- \gamma^-_\Sigma u, a^\pm = \frac{\Gamma^h}{2} \mp i\alpha \cdot \nu$.
One can see that for every $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\lim_{k\to\infty} \left\| \left(V_{-h_k} \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} V_{-h_k} - \mathbb{D}^h_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \right) \varphi I \right\|_{\mathcal{B}(\mathbb{X},\mathbb{Y})}$$
$$= \lim_{k\to\infty} \left\| \varphi \left(V_{-h_k} \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} V_{-h_k} - \mathbb{D}^h_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \right) \right\|_{\mathcal{B}(\mathbb{X},\mathbb{Y})} = 0.$$

The Arcela-Ascoli Theorem implies that every sequence $\mathbb{G} \ni h_k \to \infty$ has a subsequence defining the limit operator $\mathbb{D}^h_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$.

Definition 12 (i) We say that: (a) the operator

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : \mathbb{X} \to \mathbb{Y}$$

is the locally Fredholm if for every $R > 0$ there exist operators $\mathcal{L}_R, \mathcal{R}_R \in B(\mathbb{Y}, \mathbb{X})$ such that

$$\begin{aligned} \mathcal{L}_R \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \psi_R I_{\mathbb{X}} &= \psi_R I_{\mathbb{X}} + \mathcal{T}^1_R, \\ \psi_R \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \mathcal{R}_R &= \psi_R I_{\mathbb{Y}} + \mathcal{T}^2_Y \end{aligned} \tag{40}$$

where $\mathcal{T}^1_R \in K(\mathbb{X}), \mathcal{T}^2_R \in \mathcal{K}(\mathbb{Y})$;

(ii) The operator $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : \mathbb{X} \to \mathbb{Y}$ is locally invertible at infinity if there exists $R > 0$ and the operators $\mathcal{L}'_R, \mathcal{R}'_R \in B(\mathbb{Y}, \mathbb{X})$ such that

$$\begin{aligned} \mathcal{L}'_R \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \chi_R I_{\mathbb{X}} &= \chi_R I_{\mathbb{X}}, \\ \chi_R \mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} \mathcal{R}'_R &= \chi_R I_{\mathbb{Y}}. \end{aligned} \tag{41}$$

We will use the following simple statement.

Proposition 13 *The operator* $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : \mathbb{X} \to \mathbb{Y}$ *is Fredholm if and only if*:

(*i*) $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is locally Fredholm*; (*ii*) $\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma}$ *is locally invertible at infinity*.

Theorem 14 *Let conditions* (39) *hold, and the Lopatinsky-Shapiro condition*

$$\det \left(\alpha \cdot \xi_x + \frac{\Gamma(x)}{2} \right) \neq 0, \xi_x \in T^*_x(\Sigma_0) : |\xi_x| = 1 \tag{42a}$$

be satisfied at every point $x \in \Sigma$. *Then the operator*

$$\mathbb{D}_{\boldsymbol{A},\Phi,m,\mathfrak{B}_\Sigma} : H^1(\mathbb{R}^n\setminus\Sigma, \mathbb{C}^N) \to L^2(\mathbb{R}^n, \mathbb{C}^N)$$

is Fredholm if and only if all limit operators $\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$ *are invertible.*

Proof The ellipticity of $\mathfrak{D}_{A,\Phi,m}$ and the local Lopatinsky-Shapiro condition imply that the operator $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is locally Fredholm. Hence, Proposition 13 yields that $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is the Fredholm operator if and only if $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity. We reduce the study of local invertibility at infinity of $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ to the application of Proposition 13.

We introduce the operator

$$\mathbb{D}^0_{\mathfrak{B}_\Sigma}(i\mu)u = \begin{cases} (\alpha\cdot D_x - i\mu I_N)u \text{ on } \mathbb{R}^n\setminus\Sigma, \\ \mathfrak{B}_\Sigma u = a^0_+\gamma^+_\Sigma u + a^0_- u\gamma^-_\Sigma = 0 \text{ on } \Sigma \end{cases}$$

where $a^0_\pm = \frac{1}{2}\alpha_{n+1} \mp i\alpha\cdot\nu$ acting from $\mathbb{X}$ into $\mathbb{Y}$. Then according to Example 4 the operator

$$\mathbb{D}^0_{\mathfrak{B}_\Sigma}(i\mu) : H^1(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N) \to L^2(\mathbb{R}^n,\mathbb{C}^N).$$

is invertible for $|\mu|$ large enough. We fix such μ. Let $\Xi(i\mu) = \left(\mathbb{D}^0_{\mathfrak{B}_\Sigma}(i\mu)\right)^{-1}$. We introduce the bounded in $L^2(\mathbb{R}^n,\mathbb{C}^N)$ operator

$$\tilde{\mathbb{D}}_{A,\Phi,m,\mathfrak{B}_\Sigma} = \mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}\Xi(i\mu). \tag{43}$$

It is easy to prove that

$$\lim_{R\to\infty}\left\|\left[\chi_R I,\Xi(i\mu)\right]\right\|_{\mathcal{B}\left(L^2(\mathbb{R}^n,\mathbb{C}^N),H^1(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N)\right)} = 0. \tag{44}$$

Formula (44) implies that the operator $\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity if and only if the operator $\tilde{\mathbb{D}}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity. One can prove that the operator $\tilde{\mathbb{D}}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ belongs to the algebra $\mathcal{A}\left(\mathbb{R}^n,\mathbb{C}^N\right)$. Hence, $\tilde{\mathbb{D}}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity if and only if all limit operators $\tilde{\mathbb{D}}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$ are invertible. Formula

$$\begin{aligned} V_{-h}\tilde{\mathbb{D}}_{A,\Phi,m,\mathfrak{B}_\Sigma}V_h &= \left(V_{-h}\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}V_h\right)\left(V_{-h}\Xi(i\mu)V_h\right) \\ &= \left(V_{-h}\mathbb{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}V_h\right)\Xi(i\mu), h\in\mathbb{G} \end{aligned} \tag{45}$$

implies that $\tilde{\mathbb{D}}^h_{A,\Phi,m,\mathfrak{B}_\Sigma} = \mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}\Xi(i\mu)$.

Hence, $\tilde{\mathbb{D}}^h_{A,\Phi,m,\mathfrak{B}_\Sigma} : L^2(\mathbb{R}^n,\mathbb{C}^N) \to L^2(\mathbb{R}^n,\mathbb{C}^N)$ is invertible if and only if $\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}: H^1(\mathbb{R}^n\setminus\Sigma,\mathbb{C}^N) \to L^2(\mathbb{R}^n,\mathbb{C}^N)$ is invertible, and by Theorem 8 the operator $\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$ is locally invertible at infinity if and only if all limit operators $\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$ are invertible. Hence, the Theorem has been proved.

Corollary 15 *Let conditions of Theorem 14 hold. Then*

$$sp_{ess}\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} = \bigcup_h sp\mathcal{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma} \tag{46}$$

where $\mathcal{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$ *are unbounded operators associated with the operators* $\mathbb{D}^h_{A,\Phi,m,\mathfrak{B}_\Sigma}$ *and the union is taken with respect to all such limit operators.*

Example 16 We consider an operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$ where Σ is the above defined $\mathbb{G}$−periodic hypersurface in $\mathbb{R}^n$, We assume that the real-valued potentials A,Φ have the form: $A = A^0 + A'$, $\Phi = \Phi^0 + \Phi'$, where A^0 is a $\mathbb{G}$−periodic magnetic potential, Φ^0 is a $\mathbb{G}$−periodic electrostatic potential, $m\in\mathbb{R}$ is the mass of the particle, and Γ is a Hermitian $\mathbb{G}$−periodic matrix on Σ such that the local Lopatinsky-Shapiro condition is satisfied at every point $x\in\Sigma$. We assume that the perturbations A' and Φ' are slowly oscillating at infinity, such that their partial derivatives tend to zero at infinity. In this case the limit operators $\mathcal{D}_{A^h,\Phi^h,m,\mathfrak{B}_\Sigma}$ are such that $A^h = A^0 + A'_h$, $\Phi^h = \Phi^0 + \Phi'_h$ where $A'_h\in\mathbb{R}^n$, $\Phi'_h\in\mathbb{R}$. Then

$$sp\mathcal{D}_{A^h,\Phi^h,m,\mathfrak{B}_\Sigma} = sp(\mathcal{D}_{A^0,\Phi^0,m,\mathfrak{B}_\Sigma} + \Phi'_h I) = \sum_{j\in\mathbb{Z}} \left[a_j + \Phi'_h, b_j + \Phi'_h\right].$$

Applying formula (46), we obtain that

$$sp\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma} = \sum_{j\in\mathbb{Z}} \left[a_j + \mathfrak{m}(\Phi'), b_j + \mathfrak{M}(\Phi')\right],$$

where $\mathfrak{m}(\Phi') = \liminf_{x\to\infty} \Phi'(x)$, $\mathfrak{M}(\Phi') = \limsup_{x\to\infty} \Phi'(x)$.

Hence, if

$$a_{j+1} - b_j < \mathfrak{M}(\Phi') - \mathfrak{m}(\Phi')$$

the gap (b_j, a_{j+1}) in the spectrum of operator $\mathcal{D}_{A^0,\Phi^0,m,\mathfrak{B}_\Sigma}$disappears in the spectrum of perturbed operator $\mathcal{D}_{A,\Phi,m,\mathfrak{B}_\Sigma}$.

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