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HYPERBOLIC LEBESGUE CONSTANTS IN DIMENSION TWO

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Abstract
We present a very detailed proof of the growth of the Lebesgue constants of hyperbolic Bochner-Riesz means for double Fourier series.

Keyword Fourier transform · Hyperbolic cross · Lebesgue constant · Multiplier

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Introduction

Since the appearance of Babenko's paper [1], interest has continued in various questions of Approximation Theory and Fourier analysis in $\mathbb{R}^n$ connected with the study of linear means with harmonics in "hyperbolic crosses":

$$\Gamma(N,\gamma)=\{k\in\mathbb{Z}^n\,:\,h(N,k,\gamma)=\prod_{j=1}^{n}\left(\frac{|k_j|}{N}\right)^{\gamma_j}\le 1,\quad \gamma_j>0, j=1,...,n\}.$$

We are interested in the hyperbolic means of Bochner-Riesz type of order $\alpha\ge 0$

$$L^{\alpha}_{\Gamma(N,\gamma)}\,:f(x)\mapsto\sum_{k\in\Gamma(N,\gamma)}(1-h(N,k,\gamma))^{\alpha}\hat{f}(k)e^{ikx},$$

where for $x=(x_1,...,x_n)\in\mathbb{T}^n$ and $k=(k_1,...,k_n)\in\mathbb{Z}^n$, the inner product $kx=k_1x_1+...+k_nx_n$. Hyperbolic Bochner-Riesz means (for the two-dimensional Fourier integrals with $\gamma_1=\gamma_2=2$) appeared for the first time in the paper of El-Kohen [10] in connection with the study of their L^p-norms. That result not being sharp shortly after was strengthened by Carbery [7]. Of course, taking $\gamma_1=\gamma_2=2$ (or any other integer parameters) simplifies the calculations. Such calculations would be similarly simplified if the parameters γ_j were large enough. This is not the problem for partial sums. Taking N in the appropriate power allows one to have these parameters as large as needed. However, for the Bochner-Riesz type means, we have to adjust the means themselves. By this, the dimension and order α are taken into account.

Dedicated to the memory of N.K. Karapetyants, a remarkable person and excellent mathematician.

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The case $\alpha = 0$, the hyperbolic partial sums $L_{\Gamma(N,\gamma)} = L^0_{\Gamma(N,\gamma)}$, has been investigated separately earlier. The exact degree of growth for them $\|L_{\Gamma(N,\gamma)}\| \asymp N^{\frac{n-1}{2}}$ was established in the two-dimensional case independently by Belinsky [5] and by A. and V. Yudins [18], and afterwards was generalized to the case of arbitrary dimension in [13]. Recently, these results were applied to problems of uniform convergence in [9].

For $\alpha \geq 0$, the estimates are announced in [16, Ch.7]. The strategy for the proof is only outlined there. It is inductive and strongly based on the two-dimensional version of the result. The proof of the latter is also only outlined there. Here we present an extremely detailed proof for the double case. The corresponding result reads as follows.

Theorem 1.1 *For $\gamma_1, \gamma_2 \geq 1$, the following assertions hold.*

1. *For $0 \leq \alpha < \frac{1}{2}$, we have*

$$\|L^\alpha_{\Gamma(N,\gamma)}\| \asymp N^{\frac{1}{2}-\alpha}.$$

2. *For $\alpha = \frac{1}{2}$, we have*

$$\|L^{\frac{1}{2}}_{\Gamma(N,\gamma)}\| = \omega_\gamma \ln^2 N + O(\ln N).$$

3. *For $\alpha > \frac{1}{2}$, we have*

$$\|L^\alpha_{\Gamma(N,\gamma)}\| = \omega_{\gamma,\alpha} \ln N + O(1).$$

Here and below ω with subscripts denotes, generally speaking, different constants depending only on the indicated indices. Also, $A \asymp B$ denotes $A \lesssim B \lesssim A$, where we use here and below the notation "$\lesssim$" as abbreviations for "$\leq C$" with C being a positive constant, may be different in different occurrences (and of course different in $A \lesssim B \lesssim A$). Such constants never depend on varying essential parameters. What is important in this work is that in estimates with α the constants may depend on α. Being fixed in every particular case, such constants may grow when α approaches $\frac{1}{2}$. Also, there will appear a parameter β in the sequel on which the constants in the estimates may depend.

Observe that the critical order $\frac{1}{2}$ is the same as in the spherical case (in dimension two; for dimension n, it is $\frac{n-1}{2}$). But if for the values lower than the critical one, the orders of growth of the Lebesgue constants are such as in the spherical case, the difference between the uniform boundedness in the spherical case and **2)** in Theorem 1.1 is striking, moreover, for the orders greater than $\frac{1}{2}$: in the latter case the Lebesgue constants of the usual spherical Bochner-Riesz means are bounded. In order to establish Theorem 1.1, especially **2)** and **3)**, we need the following result. We present only the two-dimensional version; besides that it is worth noting that in fact the estimates obtained in [14] are valid for a wider range of γ.

Theorem 1.2 ([14, 17]) *Let γ and α be the same as in Theorem 1.1. For the norms of operators*

$$\bar{L}^\alpha_{\Gamma(N,\gamma)}: \qquad f(x) \mapsto \sum_{|k_j|\leq N, j=1,2} (1 - h(N,k,\gamma))^\alpha \widehat{f}(k) e^{ikx},$$

the following asymptotic equality holds true

$$\|\bar{L}^\alpha_{\Gamma(N,\gamma)}\| = \omega_{\gamma,\alpha} \ln N + O(1).$$

This is a strengthening of Kivinukk's result [12], where bilateral ordinal inequalities were obtained; by this it was shown for the first time the influence of smoothness at the corner points on the order drop of a logarithmic growth, as compared with the Lebesgue constants of the cubic partial sums.

It should be mentioned that our results are proved by step by step passage from sums to corresponding integrals. The Fourier transform of the function generating the method of summability under consideration is thus to be estimated. This will give the upper estimates for $0 \leq \alpha < \frac{1}{2}$, while the lower ones of the same order are given in [15], the leading term of the asymptotics for $\alpha = \frac{1}{2}$, and remainder terms for $\alpha > \frac{1}{2}$. The leading term for $\alpha > \frac{1}{2}$ comes from Theorem 1.2. Some ideas from [5] are used here.

Passage to the Fourier transform: first steps

Since the norms of the operators

$$f \to \int_{\mathbb{T}} f(x_1, x_2) dx_1 \quad \text{and} \quad f \to \int_{\mathbb{T}} f(x_1, x_2) dx_2,$$

taking $C(\mathbb{T}^2)$ into $C(\mathbb{T}^2)$, are bounded, we can restrict our estimates to those for the norm of the operator

$$f \to \sum_{1 \le |m_1|^{\gamma_1} |m_2|^{\gamma_2} \le N^{\gamma_1+\gamma_2}} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} \hat{f}(m) e^{imx}.$$

This norm is equal to

$$\begin{aligned} &\frac{1}{4\pi^2} \int_{\mathbb{T}^2} \left| \sum_{1 \le |m_1|^{\gamma_1} |m_2|^{\gamma_2} \le N^{\gamma_1+\gamma_2}} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{imx} \right| dx \\ &= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \left| \sum_{1 \le |m_1| \le N} e^{im_1x_1} \sum_{1 \le |m_2|^{\gamma_2} \le \frac{N^{\gamma_1+\gamma_2}}{|m_1|^{\gamma_1}}} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{im_2x_2} \right. \\ &+ \sum_{1 \le |m_2| \le N} e^{im_2x_2} \sum_{1 \le |m_1|^{\gamma_1} \le \frac{N^{\gamma_1+\gamma_2}}{|m_2|^{\gamma_2}}} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{im_1x_1} \\ &\left. - \sum_{1 \le |m_1|, |m_2| \le N} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{imx} \right| dx \end{aligned} \tag{2.1}$$

The estimate for the last sum is given in Theorem 1.2. The first two sums are similar, so we will handle only one of them. We are going to replace summation in m_2 by the corresponding integration, that is, passing from the trigonometric sum to the Fourier transform. We have

$$\begin{aligned} &\int_{\mathbb{T}^2} \left| \sum_{1 \le |m_1| \le N} e^{im_1x_1} \left\{ \sum_{1 \le |m_2|^{\gamma_2} \le \frac{N^{\gamma_1+\gamma_2}}{|m_1|^{\gamma_1}}} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{im_2x_2} \right. \right. \\ &\left. \left. - \int_{|y_2|^{\gamma_2} \le \frac{N^{\gamma_1+\gamma_2}}{|m_1|^{\gamma_1}}} \left(1 - \frac{|m_1|^{\gamma_1} |y_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{iy_2x_2} dy_2 \right\} \right| dx \\ &= \int_{\mathbb{T}^2} \left| \sum_{1 \le |m_1| \le N} e^{im_1x_1} \left\{ \sum_{1 \le |m_2|^{\gamma_2} \le \frac{N^{\gamma_1+\gamma_2}}{|m_1|^{\gamma_1}}} \left(1 - \frac{|m_1|^{\gamma_1} |m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^{\alpha} e^{im_2x_2} \right. \right. \\ &\left. \left. - \frac{N^{\frac{\gamma_1}{\gamma_2}+1}}{|m_1|^{\frac{\gamma_1}{\gamma_2}}} \int_{-1}^{1} (1 - |z|^{\gamma_2})^{\alpha} e^{ix_2 z \frac{N^{\gamma_1/\gamma_2+1}}{|m_1|^{\gamma_1/\gamma_2}}} dz \right\} \right| dx, \end{aligned} \tag{2.2}$$

where z substitutes for $\frac{y_2 |m_1|^{\frac{\gamma_1}{\gamma_2}}}{N^{\frac{\gamma_1}{\gamma_2}+1}}$. The right-hand side may be rewritten as:

$$\int_{\mathbb{T}^2} \Big| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Big\{ \sum_{1\le|m_2|\le M} \Big(1-\Big(\frac{|m_2|}{M}\Big)^{\gamma_2}\Big)^{\alpha} e^{im_2x_2}$$
$$- M\int_{-1}^{1}(1-|z|^{\gamma_2})^{\alpha} e^{ix_2zM}\,dz \Big\}\Big|\,dx,$$

with $M=\frac{N^{\frac{\gamma_1}{\gamma_2}+1}}{|m_1|^{\frac{\gamma_1}{\gamma_2}}}$. We now represent this as

$$\int_{\mathbb{T}^2} \Big| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Big\{ \sum_{1\le|m_2|\le M} \Big(1-\Big(\frac{|m_2|}{M}\Big)^{\gamma_2}\Big)^{\alpha} e^{im_2x_2}$$
$$- M\Lambda(x_2M) \Big\}\Big|\,dx,$$

where Λ is the one-dimensional inverse Fourier transform, times 2π, of the function $\lambda_{\gamma_2,\alpha}(z)=(1-|z|^{\gamma_2})_+^{\alpha}$, that is,

$$\Lambda(u)=2\pi\check{\lambda}_{\gamma_2,\alpha}(u)=2\pi\int_{-1}^{1}(1-|z|^{\gamma_2})^{\alpha}e^{iuz}\,dz.$$

The m_2-th Fourier coefficient of the function $M\Lambda(\cdot M)$ (understood as extended periodically) is

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} M\Lambda(x_2M)e^{-im_2x_2}\,dx_2 =&\frac{1}{2\pi}\int_{\mathbb{T}} 2\pi M\check{\lambda}_{\gamma_2,\alpha}(x_2M)e^{-im_2x_2}\,dx_2\\ &=\Big(\int_{\mathbb{R}}-\int_{|t|>M\pi}\Big)\check{\lambda}_{\gamma_2,\alpha}(t)e^{-im_2\frac{t}{M}}\,dt \\ =\Big(1-\frac{|m_1|^{\gamma_2}|m_2|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\Big)_+^{\alpha}&-\int_{|t|>M\pi}\check{\lambda}_{\gamma_2,\alpha}(t)e^{-im_2\frac{t}{M}}\,dt.\end{aligned} \tag{2.3}$$

Omitting the case $m_2=0$ so far, we will return to it later. Combining () and (2.3), applying successively the Cauchy-Schwarz inequality and Parseval's identity to

$$\int_{\mathbb{T}^2}\Big|\sum_{1\le|m_1|\le N}e^{im_1x_1}\sum_{m_2\ne0}e^{im_2x_2}\int_{|t|>M\pi}\Lambda(t)e^{-im_2\frac{t}{M}}\,dt\Big|\,dx,$$

we arrive at the following value to be estimated:

$$\Big\{\sum_{1\le|m_1|\le N}\sum_{m_2\ne0}\Big|\int_{|t|>\pi M}\Lambda(t)e^{-im_2\frac{t}{M}}\,dt\Big|^2\Big\}^{1/2}.$$

Without loss of generality, we can deal with

$$\Big\{\sum_{m=1}^{N}\sum_{k=1}^{\infty}\Big|\int_{|t|>\pi M}\Lambda(t)e^{-ik\frac{t}{M}}\,dt\Big|^2\Big\}^{\frac12},$$

where now

$$M := M(m) := \frac{N^{1+\gamma}}{m^{\gamma}}, \quad \text{with} \quad \gamma = \frac{\gamma_1}{\gamma_2} > 0, \tag{2.4}$$

and

$$\Lambda(t) = 2\pi \int_{-1}^{1} (1-|z|^{\delta})^{\alpha} e^{itz}\, dz = 4\pi \int_{0}^{1} (1-z^{\delta})^{\alpha} \cos tz\, dz, \tag{2.5}$$

with δ replacing more tedious γ_2. Denoting $\Phi(t) = \frac{\Lambda(t)}{4\pi}$, we have to estimate the error of the passage from the trigonometric sum to the Fourier transform in one variable by means of

$$\left\{ \sum_{m=1}^{N} \sum_{k=1}^{\infty} \Big| \int_{|t|>\pi M} \Phi(t) e^{-ik\frac{t}{M}}\, dt \Big|^2 \right\}^{\frac{1}{2}}. \tag{2.6}$$

For subsequent estimates, we need to know certain asymptotic properties of $\Phi(t)$ for large t. We mention that the following assertions are of interest if $0 < \alpha < 1$. For $\alpha \ge 1$, we will just apply the estimates for the remainder terms in the asymptotic relations. Also, certain problems might appear if $\delta < \alpha$. However, in our case, δ is either γ_1 or γ_2 and it does not play any role in the estimates. The stationary phase method yields (see, e.g., [8])

$$\Phi(t) = a\frac{e^{it}}{t^{1+\alpha}} + b\frac{e^{-it}}{t^{1+\alpha}} + O\Big(\frac{1}{t^2}\Big). \tag{2.7}$$

Here a and b are some constants, independent of t, that have no influence on the decay in t in the estimates. Since

$$\Phi'(t) = -\int_{0}^{1} z(1-z^{\delta})^{\alpha} \sin zt\, dz, \tag{2.8}$$

the stationary phase method gives the same formula (2.7), maybe with different a and b.

Double hyperbolic partial sums

Though the estimates for partial sums are better studied, as mentioned above, we present our proof. It differs from the earlier ones. Not that it is really simpler but goes along more general lines, better adjusted to the case of any dimension.

Passage to the Fourier transform continued

We observe that for $\alpha = 0$, the case of the double hyperbolic partial sums,

$$\Phi(t) = \frac{\sin t}{t},$$

and

$$\Phi'(t) = \frac{\cos t}{t} - \frac{\sin t}{t^2},$$

which fits the above asymptotic relations. Let us, for the sake of completeness, fulfil all the calculations for the double hyperbolic partial sums separately. Also, this proof will differ from that in [18] and, in certain respects, from that in [5]. Here (2.6) is (recall that M is given in (2.4))

$$\left\{ \sum_{m=1}^{N} \sum_{k=1}^{\infty} \left| \int\limits_{|t|>\pi M} \frac{\sin t}{t} e^{-ik\frac{t}{M}}\, dt \right|^2 \right\}^{\frac{1}{2}}. \tag{3.1}$$

Before treating it, we complete certain simpler cases. The omitted case $k = 0$ reduces to

$$\left\{ \sum_{m=1}^{N} \left| \int\limits_{|t|>\pi M} \frac{\sin t}{t}\, dt \right|^2 \right\}^{\frac{1}{2}}.$$

This is

$$\left\{ \sum_{m=1}^{N} \frac{1}{M^2} \right\}^{\frac{1}{2}} = O(N^{-\frac{1}{2}}),$$

more than enough.

Also, we wish to treat the case $M - 2 < k < M + 2$ immediately. It follows from the formula (see (5) in [6, Ch.I, §4]; it is mentioned in Remark 12 in the cited literature of [6] that the formula goes back to Fourier)

$$\int_{-\infty}^{\infty} \frac{\sin at}{t} \cos yt\, dx = \begin{cases} \pi, & y < a; \\ \frac{\pi}{2}, & y = a; \\ 0, & y > a. \end{cases}$$

In the same book, this integral is called the Dirichlet discontinuous factor. The same integral, with $\sin yt$ in place of $\cos yt$, vanishes if it exists because of oddness of the integrated function. Both integrals exist if understood in the principal value sense. Therefore, the integral in (3.1) is bounded, uniformly in M, and the estimate for this case reduces to

$$\left\{ \sum_{m=1}^{N} C \right\}^{\frac{1}{2}} = O(N^{\frac{1}{2}}),$$

which is the claimed bound.

Integrating in (3.1) by parts, we get

$$\int\limits_{|t|>\pi M} \frac{\sin t}{t} e^{-ik\frac{t}{M}}\, dt = \frac{M}{ik} \frac{\sin \pi M}{\pi M} e^{-i\pi k} + \frac{M}{ik} \int\limits_{|t|>\pi M} \left[\frac{\cos t}{t} - \frac{\sin t}{t^2} \right] e^{-ik\frac{t}{M}}\, dt.$$

For the integrated term, we obtain the bound

$$\left\{ \sum_{m=1}^{N} \sum_{k=1}^{\infty} \frac{M^2}{k^2} \frac{1}{M^2} \right\}^{\frac{1}{2}} = O(N^{\frac{1}{2}}).$$

The term $\frac{\sin t}{t^2}$ leads to the same estimate after routine calculations. The remaining value to be estimated is convenient to represent as

$$\left\{ \sum_{m=1}^{N} \sum_{\substack{1 \le k \le M-2, \\ k \ge M+2}} \frac{M^2}{k^2} \left| \int\limits_{|t|>\pi M} \frac{e^{it}}{t} e^{-ik\frac{t}{M}}\, dt \right|^2 \right\}^{\frac{1}{2}}.$$

We mention that the case where we have e^{-it} and $e^{-ik\frac{t}{M}}$ runs along the same lines, with simpler estimates. Integrating by parts, without loss of generality, in the integral

$$\int_M^\infty \frac{e^{it(1-\frac{k}{M})}}{t}\,dt,$$

we obtain

$$\int_M^\infty \frac{1}{t}e^{it(1-\frac{k}{M})}\,dt = -\frac{M}{i(M-k)M}e^{i(M-k)} + \frac{M}{i(M-k)}\int_M^\infty \frac{1}{t^2}e^{it(1-\frac{k}{M})}\,dt,$$

which gives the value

$$\frac{1}{|M-k|}$$

to be estimated. This implies

$$\left\{\sum_{m=1}^{N}\sum_{\substack{1\le k\le M-2,\\ k\ge M+2}}\frac{M^2}{k^2}\frac{1}{|M-k|^2}\right\}^{\frac{1}{2}}.$$

Passing from sums to integrals, we can easily calculate that

$$\sum_{1\le k\le M-2}\frac{M^2}{k^2}\frac{1}{|M-k|^2} = O(1),$$

which yields the same $O(N^{\frac{1}{2}})$. More or less similarly,

$$\left\{\sum_{m=1}^{N}\sum_{k\ge M+2}\frac{M^2}{k^2}\frac{1}{|M-k|^2}\right\}^{\frac{1}{2}} \lesssim \left\{\sum_{m=1}^{N}\sum_{k\ge M+2}\frac{1}{|M-k|^2}\right\}^{\frac{1}{2}} \lesssim \left\{\sum_{m=1}^{N}\right\}^{\frac{1}{2}} = O(N^{\frac{1}{2}}).$$

Now, we have to pass from the discrete parameter m_1 to the corresponding continuous in

$$\int_{\mathbb{T}^2}\left|\sum_{1\le|m_1|\le N} e^{im_1x_1}\,M\int_{|z|\le 1} e^{ix_2zM}\,dz\right|dx.$$

Using the relation

$$e^{im_1x_1} = \frac{x_1}{2\sin(x_1/2)}\int_{m_1-1/2}^{m_1+1/2} e^{ix_1u}\,du, \tag{3.2}$$

we have to estimate the difference

$$\int_{\mathbb{T}^2}\Bigg|\sum_{1\le|m_1|\le N}\int_{m_1-1/2}^{m_1+1/2} e^{ix_1u}\,du\Bigg\{\int_{|t|\le N^{\gamma+1}|m_1|^{-\gamma}} e^{ix_2t}\,dt - \int_{|t|\le N^{\gamma+1}|u|^{-\gamma}} e^{ix_2t}dt\Bigg\}\Bigg|\,dx.$$

Substituting $u \to m + u$ and applying simple inequalities, and then the Cauchy-Schwarz inequality, Parseval's identity, and mean-value theorem, we estimate this difference via

$$\int_{-1/2}^{1/2}\int_{\mathbb{T}}\Bigg\{\sum_{1\le|m_1|\le N}\Bigg|\int_{|t|\le N^{\gamma+1}|m_1|^{-\gamma}} e^{ix_2t}\,dt - \int_{|t|\le N^{\gamma+1}|u|^{-\gamma}} e^{ix_2t}dt\Bigg|^2\Bigg\}^{\frac{1}{2}} dx_2\,du$$
$$\le \int_{-1/2}^{1/2}\int_{\mathbb{T}}\Bigg\{\sum_{1\le|m_1|\le N}\frac{1}{x_2^2}\Big|e^{ix_2N^{\gamma+1}|m_1|^{-\gamma}} - e^{ix_2N^{\gamma+1}|m_1+u|^{-\gamma}}\Big|^2\Bigg\}^{\frac{1}{2}} dx_2\,du.$$

It is obvious that for $|x_2| \ge 1$, the bound is $O(N^{\frac{1}{2}})$. Considering the integral in x_2 to be over $|x_2| < 1$, we split the sum in m_1 into two parts. First,

$$\int_{|x_2|<1}\Bigg\{\sum_{1\le|m_1|\le N|x_2|^{\beta}}\frac{1}{x_2^2}\Big|e^{ix_2N^{\gamma+1}|m_1|^{-\gamma}} - e^{ix_2N^{\gamma+1}|m_1+u|^{-\gamma}}\Big|^2\Bigg\}^{\frac{1}{2}} dx_2$$
$$\lesssim \int_{|x_2|<1}\Bigg\{\sum_{1\le|m_1|\le N|x_2|^{\beta}}\frac{1}{x_2^2}\Bigg\}^{\frac{1}{2}} dx_2 \lesssim N^{\frac{1}{2}}\int_{|x_2|<1}|x_2|^{\frac{\beta}{2}-1}\,dx_2,$$

which is $O(N^{\frac{1}{2}})$ for any $\beta > 0$. On the other hand,

$$\int_{|x_2|<1}\Bigg\{\sum_{|m_1|>N|x_2|^{\beta}}\frac{1}{|x|_2^2}\Big|e^{ix_2N^{\gamma+1}|m_1|^{-\gamma}} - e^{ix_2N^{\gamma+1}|m_1+u|^{-\gamma}}\Big|^2\Bigg\}^{\frac{1}{2}} dx_2$$
$$\lesssim \int_{|x_2|<1}\Bigg\{\sum_{|m_1|>N|x_2|^{\beta}}\frac{N^{2(\gamma+1)}}{|m_1|^{2(1+\gamma)}}\Bigg\}^{\frac{1}{2}} dx_2$$
$$\lesssim N^{\frac{1}{2}}\int_{|x_2|<1}|x_2|^{-\beta(1+\gamma)+\frac{\beta}{2}}\,dx_2,$$

which is $O(N^{\frac{1}{2}})$ for any $\beta > 0$ such that $-\beta\gamma + \frac{\beta}{2} > -1$. This is true for any $\beta < \frac{1}{\frac{1}{2}+\gamma}$, and such a β always exists.
Thus, we have replaced the summation by integration with the error $O(N^{\frac{1}{2}})$.

Estimates for the Fourier transform

It remains to estimate

$$\int_{\mathbb{T}^2}\Bigg|\int_{1/2\le|u|\le[N]+1/2} e^{ix_1u}\int_{|t|\le N^{\gamma+1}|u|^{-\gamma}} e^{ix_2t}\,dt\,du\Bigg|\,dx.$$

Of course, we can simplify the calculations by considering, without loss of generality,

$$\int_{\mathbb{T}_+^2} \left| \int_{1/2\le u\le [N]+1/2} e^{ix_1 u} \int_{|t|\le N^{\gamma+1}u^{-\gamma}} e^{ix_2 t}\,dt\,du \right| dx,$$

since the calculations for the remaining integrals go along the same lines. The latter integral is

$$\int_{\mathbb{T}_+^2} \frac{1}{x_2} \left| \int_{1/2\le u\le [N]+1/2} e^{ix_1 u} \sin(x_2 N^{\gamma+1}u^{-\gamma})\,du \right| dx.$$

Again, and only for the sake of simplicity, we can take the upper limit to be N, since the bound for the difference of the two values is

$$\int_N^{[N]+1/2} \int_{\mathbb{T}_+} \frac{1}{x_2} |\sin(x_2 N^{\gamma+1}u^{-\gamma})|\,dx_2\,du,$$

which is equivalent to $\ln N$.
By this, what remains to be estimated is

$$\int_{\mathbb{T}_+^2} \frac{1}{x_2} \left| \int_{1/2}^{N} e^{ix_1 u} \sin(x_2 N^{\gamma+1}u^{-\gamma})\,du \right| dx.$$

The next two steps correspond to two changes of variables: first

$$t = x_1^{\frac{1}{1+\gamma}} x_2^{-\frac{1}{1+\gamma}} N^{-1} u,$$

and then $v_1 = Nx_1$ and $v_2 = Nx_2$. The estimated integral becomes

$$\begin{aligned} &\int_{N\mathbb{T}_+^2} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \left| \int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} e^{iv_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t} \right. \\ &\left. \times \sin(v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma})\,dt \right| dv. \end{aligned} \tag{3.3}$$

Further estimates will be undertaken for two different cases. Denoting $V = v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}}$, we consider separately $V \le 1$ and $V > 1$.
Estimates for the case $V \le 1$ We do not have the same approach to all $\gamma = \frac{\gamma_1}{\gamma_2}$. Let first $\gamma \ge \frac{1}{2}$.
(a1) The simplest part is $0 \le v_1 \le \frac{N^{1/2}}{\ln N}$, $\frac{1}{N^\gamma} \le v_2 \le N$. Since

$$\frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \left(\frac{v_1}{v_2}\right)^{\frac{\gamma_2}{\gamma_1+\gamma_2}} = \frac{1}{v_2},$$

we obtain

$$\int_0^{\frac{N^{1/2}}{\ln N}} \int_{\frac{1}{N^\gamma}}^{N} \frac{dv_2}{v_2}\,dv_1 \lesssim N^{1/2}.$$

(a2) Furthermore,

$$\int_0^N \int_0^{\frac{1}{N^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \left|\sin(v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma})\right| dt\, dv_2\, dv_1$$

$$\leq \int_0^N \int_0^{\frac{1}{N^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} \int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} t^{-\gamma}\, dt\, dv_2\, dv_1$$

$$=|1-\gamma| \int_0^N \int_0^{\frac{1}{N^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} \left(\frac{v_1}{v_2}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}} |1-(2N)^{\gamma-1}|\, dv_2\, dv_1$$

$$=|1-\gamma| \int_0^N \int_0^{\frac{1}{N^\gamma}} |1-(2N)^{\gamma-1}|\, dv_2\, dv_1 = |1-\gamma| N^{1-\gamma} + O(1),$$

which is $O(N^{\frac{1}{2}})$ provided $\gamma \geq \frac{1}{2}$. Of course, the above calculations are true for $\gamma \neq 1$. However, for $\gamma = 1$, the corresponding calculations are quite similar and even easier (see [5]).
(a3) Finally, observing that $V \leq 1$ is equivalent to $v_2 \leq \frac{1}{v_1^\gamma}$ and fulfilling exactly the same estimates as in (a2), we get, for $\gamma \leq 1$,

$$\int_{\frac{N^{1/2}}{\ln N}}^N \int_{\frac{1}{N^\gamma}}^{\frac{1}{v_1^\gamma}} |1-(2N)^{\gamma-1}|\, dv_2\, dv_1 \lesssim \int_{\frac{N^{1/2}}{\ln N}}^N \frac{dv_1}{v_1^\gamma} = O(N^{1-\gamma}),$$

which is $O(N^{\frac{1}{2}})$ provided $\gamma \geq \frac{1}{2}$. Needless to say that the above remark 0n the case $\gamma = 1$ is true here as well. If $\gamma > 1$, we provide instead of (a2) the simpler estimate

$$\int_{\frac{N^{1/2}}{\ln N}}^N \int_{\frac{1}{N^\gamma}}^{\frac{1}{v_1^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} dt\, dv_2\, dv_1$$

$$\leq \int_{\frac{N^{1/2}}{\ln N}}^N \int_{\frac{1}{N^\gamma}}^{\frac{1}{v_1^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \left(\frac{v_1}{v_2}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}} dv_2\, dv_1 \lesssim \int_{\frac{N^{1/2}}{\ln N}}^N \frac{dv_1}{v_1^2} = O(1).$$

Let now $\gamma < \frac{1}{2}$. The first two steps are very similar to (a1) and (a2). We only replace $\frac{1}{N^\gamma}$ by $\frac{1}{N^{1/2}}$. This shows a sort of saturation for the considered γ. We also note that estimates on these domains can be provided for all γ. The first step is such in any case. On the second step, this can be achieved by replacing

$$\left|\sin(v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma})\right| \leq v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma}$$

with

$$\left|\sin(v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma})\right| \leq (v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma})^\beta$$

and choosing an appropriate β.
Let finally $\frac{N^{1/2}}{\ln N} \le v_1 \le N$ and $\frac{1}{N^{1/2}} \le v_2 \le \frac{1}{v_1^{\gamma}}$. We now treat the integral

$$\int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} e^{iVt} e^{\frac{iV}{t^{\gamma}}}\, dt$$

in a different way. Integrating by parts, we get for the next estimates the bounds $\frac{1}{V}$ and

$$\int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} t^{-1-\gamma}\, dt.$$

The latter, in turn, reduces to

$$(\frac{v_1}{v_2})^{-\frac{\gamma_1}{\gamma_1+\gamma_2}}(2N)^{\gamma}.$$

The first value times $\frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}}\frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}}$ gives $\frac{1}{v_1 v_2}$ and $\ln^2 N$ in general. The second one, times the same value, leaves the integral

$$N^{\gamma}\int_{\frac{N^{1/2}}{\ln N}}^{N} \frac{1}{v_1}\int_{\frac{1}{N^{1/2}}}^{\frac{1}{v_1^{\gamma}}} dv_2\, dv_1$$

to be estimated. It is controlled by

$$N^{\gamma}\int_{\frac{N^{1/2}}{\ln N}}^{N} \frac{dv_1}{v_1^{1+\gamma}},$$

which is small enough even for $\gamma < 1$.
In conclusion, for $V \le 1$, the Fourier integral that replaced the initial sum is of the desired $O(N^{\frac{1}{2}})$ growth.
Estimates for the case $V > 1$ Here, the Stationary Phase Method will be applied to the inner integral in

$$\int_{N\mathbb{T}_+^2} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}}\frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}}\left|\int_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} e^{iv_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(t\pm t^{-\gamma})}\, dt\right| dv. \tag{3.4}$$

We mention that this integral is in the form equivalent to (3.3) used for $V \le 1$; it will be more convenient to deal with it just in this form. The version of the Stationary Phase Method we are going to apply reads, for any dimension n, as follows (it can be found in [11]).

Theorem 3.1 *For the integer $k \ge 1$, the following asymptotic formula is valid*

$$Q_n(R) = \int_{\mathbb{R}_+^{n-1}} \varphi(v)e^{iRS(v)}dv$$
$$= (2\pi)^{\frac{n-1}{2}} R^{\frac{1-n}{2}} e^{i(RS(v_0)+\theta(v_0))} \times |\det S''(v_0)|^{-\frac{1}{2}} \left(\varphi(v_0) + O(R^{-1})\right) \tag{3.5}$$
$$+ R^{\frac{1-n}{2}} e^{iRS(v_0)} \sum_{j=1}^{k-1} a_j R^{-j} + O(R^{\frac{1-n}{2}-k}),$$

where $v_0 = (v_1^0, v_2^0, \dots, v_{n-1}^0)$ *is a stationary point of* S; S'' *is the Hessian matrix of the second derivatives of* S *such that* $S''(v_0) \neq 0; \theta(v_0)$ *is a real number depending on* $\det S''(v_0)$; *and* a_j *are some (complex) numbers.*

In our case, it is rather simple. Calculating the first and the second derivatives of the function $t \pm \frac{1}{t^\gamma}$, we mainly see that the second one does not vanish. Therefore, the worst possible estimate is of the order $V^{-\frac{1}{2}}$. We thus have to estimate

$$\int\limits_{N\mathbb{T}_+^2, V>1} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}+\frac{\gamma_1}{2(\gamma_1+\gamma_2)}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}+\frac{\gamma_2}{2(\gamma_1+\gamma_2)}}} \, dv_2 \, dv_1.$$

We observe that both v_1 and v_2 in the denominators are in the power less than 1. Being integrated, they give

$$v_1^{1-\frac{\gamma_2}{\gamma_1+\gamma_2}-\frac{\gamma_1}{2(\gamma_1+\gamma_2)}}$$

and

$$v_2^{1-\frac{\gamma_1}{\gamma_1+\gamma_2}-\frac{\gamma_2}{2(\gamma_1+\gamma_2)}},$$

both now in positive powers. Obviously, the maximum of their product is attained at $v_1 = v_2 = N$. This gives

$$N^{1-\frac{\gamma_2}{\gamma_1+\gamma_2}-\frac{\gamma_1}{2(\gamma_1+\gamma_2)}} N^{1-\frac{\gamma_1}{\gamma_1+\gamma_2}-\frac{\gamma_2}{2(\gamma_1+\gamma_2)}} = N^{\frac{1}{2}},$$

which completes the proof for the above estimate of the Lebesgue constants of double hyperbolic partial sums.

Double Bochner-Riesz type means

We continue our calculations for the general Bochner-Riesz type means.

Passage to the Fourier transform continued

We first estimate the case $m_2 = 0$. Of course, it may cause no serious problem. We have to estimate

$$\int\limits_{\mathbb{T}^2} \Bigg| \sum_{1\le|m_1|\le N} e^{im_1x_1} \Bigg(\int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma/\gamma_2}} \Lambda(t)\, dt - 1 \Bigg) \Bigg| \, dx$$

$$= \omega_2 \log N + O\Bigg(\sum_{1\le|m_1|\le N} \Bigg| \int\limits_{|t|>\pi N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}} \Lambda(t)\, dt \Bigg| \Bigg).$$

It suffices to use a simple bound $\Lambda(t) = O(t^{-1-\alpha})$. This yields the estimate $O(N^{-\alpha})$ for the remainder term on the right-hand side. Actually, this is the case for any individual m_2, or a finite number of m_2-s. So, we should have the same good estimate for

$$\left\{ \sum_{m=1}^{N} \sum_{|k-M|\le 1} \left| \int_{|t|>\pi M} \Phi(t) e^{-ik\frac{t}{M}}\,dt \right|^2 \right\}^{\frac{1}{2}},$$

where for the simplifying notation M, Φ, and other, see (2.4) and further remarks. Integrating by parts (for simplicity and without loss of generality, only over $[\pi M, \infty)$), we have

$$\int_{\pi M}^{\infty} \Phi(t) e^{-ik\frac{t}{M}}\,dt = \frac{M}{ik}\Phi(\pi M) + \frac{M}{ik}\int_{\pi M}^{\infty} \Phi'(t) e^{-ik\frac{t}{M}}\,dt. \tag{4.1}$$

For the last integral in (4.1), we use (2.7). This leads to estimating, with $\gamma = \frac{\gamma_1}{\gamma_2}$,

$$\left\{ \sum_{m=1}^{N} M^{-2\alpha} \right\}^{\frac{1}{2}} = N^{-\alpha(1+\gamma)} \left\{ \sum_{m=1}^{N} m^{2\alpha\gamma} \right\}^{\frac{1}{2}} \lesssim N^{\frac{1}{2}-\alpha}. \tag{4.2}$$

For the integrated term on the right-hand side of (4.1), we similarly obtain, using (2.7),

$$\left\{ \sum_{m=1}^{N} M^{-2(1+\alpha)} \right\}^{\frac{1}{2}} \lesssim N^{-\frac{1}{2}-\alpha}.$$

This is "too good" to be taken into account.
Denoting $\Psi(z) = z(1-z^2)_+^{\alpha}$, we have to estimate

$$\begin{aligned} &\left\{ \sum_{m=1}^{N} \sum_{\substack{1\le k\le M-2,\\ k\ge M+2}} \frac{M^2}{k^2} \left| \int_{\pi M}^{\infty} \Phi'(t) e^{-ik\frac{t}{M}}\,dt \right|^2 \right\}^{\frac{1}{2}} \\ \le &\left\{ \sum_{m=1}^{N} \sum_{\substack{1\le k\le M-2,\\ k\ge M+2}} \frac{M^2}{k^2} \left| \int_{\pi M}^{\infty} \left[\Phi'(t) - \left(\frac{\delta}{2}\right)^{\alpha} \Psi(t)\right] e^{-ik\frac{t}{M}}\,dt \right|^2 \right\}^{\frac{1}{2}} \\ +\left(\frac{\delta}{2}\right)^{\alpha} &\left\{ \sum_{m=1}^{N} \sum_{\substack{1\le k\le M-2,\\ k\ge M+2}} \frac{M^2}{k^2} \left| \int_{\pi M}^{\infty} \Psi(t) e^{-ik\frac{t}{M}}\,dt \right|^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

We continue with the first summand on the right-hand side of (4.3). Integrating by parts, we obtain

$$\begin{aligned} \Phi'(t) - \left(\frac{\delta}{2}\right)^{\alpha}\Psi(t) &= \int_0^1 \left[z(1-z^{\delta})^{\alpha} - \left(\frac{\delta}{2}\right)^{\alpha} z(1-z^2)^{\alpha}\right] \sin tz\,dz \\ &= \frac{1}{t}\int_0^1 \left[(1-z^{\delta})^{\alpha} - \left(\frac{\delta}{2}\right)^{\alpha}(1-z^2)^{\alpha}\right] \cos zt\,dz \\ &+ \frac{1}{t}\int_0^1 \alpha\delta\left[\left(\frac{\delta}{2}\right)^{\alpha-1} z^2(1-z^2)^{\alpha-1} - z^{\delta}(1-z^{\delta})^{\alpha-1}\right] \cos zt\,dz. \end{aligned}$$

For the first summand on the right-hand side of the last relation, the estimate $O(\frac{1}{t^{2+\alpha}})$ follows immediately, since both functions are Lipα. For the second one, the same assertion needs a somewhat more delicate analysis. We rewrite it as

$$\alpha\delta z^{\delta}(1-z^{\delta})^{\alpha-1}\left[\left(\frac{\delta}{2}\right)^{\alpha-1}z^{2-\delta}\left(\frac{1-z^2}{1-z^{\delta}}\right)^{\alpha-1}-1\right].$$

Since, by l'Hôpital's rule, the limit of the value in brackets as $z\to 1$ is 0, the singularity $\alpha-1$ is neutralized and we again live in Lipα. Therefore, we have $\Phi'(t)-(\frac{\delta}{2})^{\alpha}\Psi(t)=O(t^{-2-\alpha})$, which gives after integration $O(M^{-1-\alpha})$. Finally,

$$\left\{\sum_{m=1}^{N}\sum_{k=1}^{\infty}\frac{M^2}{k^2}M^{-2-2\alpha}\right\}^{\frac12}=\left\{\sum_{m=1}^{N}N^{-2\alpha-2\alpha\gamma}m^{2\alpha\gamma}\right\}^{\frac12}=O(N^{\frac12-\alpha}),$$

the desired bound.
Since (see [4, Ch. II, §2.3, (9)])

$$\int_0^1 z(1-z^2)^{\alpha}\sin tz\,dz=2^{\alpha-\frac12}\sqrt{\pi}\Gamma(\alpha+1)t^{-\alpha-\frac12}J_{\alpha+\frac32}(t),$$

what remains to estimate in (4.3) is

$$\left\{\sum_{m=1}^{N}\sum_{\substack{1\le k\le M-2,\\ k\ge M+2}}\frac{M^2}{k^2}\left|\int_{\pi M}^{\infty}t^{-\alpha-\frac12}J_{\alpha+\frac32}(t)e^{-ik\frac{t}{M}}\,dt\right|^2\right\}^{\frac12}.$$

We need the following properties of the Bessel function J_ν (see, e.g., [3, §7.2.8(50),(51); §7.13.1(3); §7.12(8)]):

$$\frac{d}{dt}\left[t^{\pm\nu}J_\nu(t)\right]=\pm t^{\pm\nu}\,J_{\nu\mp1}(t);\tag{4.4}$$

$$\begin{aligned}J_\nu(t)=&\sqrt{\frac{2}{\pi t}}\cos\left(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\\&+\sqrt{\frac{2}{\pi}}\,\frac{1-4\nu^2}{8}\,t^{-\frac32}\sin\left(t-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)\\&+O\left(t^{-\frac52}\right)\qquad\text{as}\quad t\to\infty;\end{aligned}\tag{4.5}$$

$$J_\nu=O(t^\nu)\qquad\text{for small}\quad t.\tag{4.6}$$

Since

$$\begin{aligned}&\left\{\sum_{m=1}^{N}\sum_{\substack{1\le k\le M-2,\\ k\ge M+2}}\frac{M^2}{k^2}\left|\int_{\pi M}^{\infty}t^{-\alpha-2}\,dt\right|^2\right\}^{\frac12}\\&\lesssim\left\{\sum_{m=1}^{N}\sum_{\substack{1\le k\le M-2,\\ k\ge M+2}}\frac{M^2}{k^2}M^{-2\alpha-2}\right\}^{\frac12}=O(N^{\frac12-\alpha}),\end{aligned}$$

it remains to estimate

$$\left\{\sum_{m=1}^{N}\sum_{\substack{1\le k\le M-2,\\ k\ge M+2}}\frac{M^2}{k^2}\left|\int_{\pi M}^{\infty}t^{-\alpha-1}e^{it(1-\frac{k}{M})}\,dt\right|^2\right\}^{\frac12}.$$

Needless to say that, as above, the case where we have e^{-it} and $e^{-ik\frac{t}{M}}$ runs along the same lines, with simpler estimates. Integrating by parts, without loss of generality, in the integral

$$\int_M^\infty \frac{e^{it(1-\frac{k}{M})}}{t^{1+\alpha}}\,dt,$$

we obtain

$$\int_M^\infty \frac{1}{t^{\alpha+1}} e^{it(1-\frac{k}{M})}\,dt = -\frac{M}{i(M-k)M^{1+\alpha}} e^{-i\pi(M-k)} + \frac{(1+\alpha)M}{i(M-k)} \int_M^\infty \frac{1}{t^{2+\alpha}} e^{it(1-\frac{k}{M})}\,dt,$$

which gives the value

$$\frac{1}{|M-k|M^\alpha}$$

to be estimated. This implies

$$\left\{ \sum_{m=1}^N \sum_{\substack{1\le k\le M-2,\\ k\ge M+2}} \frac{M^{2-2\alpha}}{k^2} \frac{1}{|M-k|^2} \right\}^{\frac{1}{2}}.$$

Passing from sums to integrals, we can easily calculate that

$$\sum_{1\le k\le M-2} \frac{M^2}{k^2} \frac{1}{|M-k|^2} = O(1),$$

which yields (4.2). Similarly,

$$\left\{ \sum_{m=1}^N \sum_{k\ge M+2} \frac{M^{2-2\alpha}}{k^2} \frac{1}{|M-k|^2} \right\}^{\frac{1}{2}} \lesssim \left\{ \sum_{m=1}^N M^{-2\alpha} \sum_{k\ge M+2} \frac{1}{|M-k|^2} \right\}^{\frac{1}{2}} = \left\{ \sum_{m=1}^N M^{-2\alpha} \right\}^{\frac{1}{2}} = O(N^{\frac{1}{2}-\alpha}).$$

Now, we have to pass from the discrete parameter m_1 to the corresponding continuous in

$$\int_{\mathbb{T}^2} \left| \sum_{1\le |m_1|\le N} e^{im_1x_1} N^{\gamma_1/\gamma_2+1} |m_1|^{-\gamma_1/\gamma_2} \Lambda(x_2 N^{\gamma_1/\gamma_2+1} |m_1|^{-\gamma_1/\gamma_2}) \right| dx.$$

Using (3.2), we have to estimate the difference

$$\int_{\mathbb{T}^2}\Big|\sum_{1\le|m_1|\le N}\int_{m_1-1/2}^{m_1+1/2} e^{ix_1u}\,du$$
$$\Big\{\int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}}(1-|m_1|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}\,dt$$
$$-\int_{|t|\le N^{\gamma_1/\gamma_2+1}|u|^{-\gamma_1/\gamma_2}}(1-|u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}dt\Big\}\Big|\,dx.$$

Substituting $u\to m_1+u$ and applying simple inequalities, and then the Cauchy-Schwarz inequality, Parseval's identity, and mean-value theorem, we estimate this difference via

$$\int_{\mathbb{T}} dx_2\int_{-1/2}^{1/2} du\Big|\sum_{1\le m_1|\le N} e^{ix_1(m_1+u)}$$
$$\Big\{\int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1|^{-\gamma_1/\gamma_2}}(1-|m_1|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}\,dt$$
$$-\int_{|t|\le N^{\gamma_1/\gamma_2+1}|m_1+u|^{-\gamma_1/\gamma_2}}(1-|m_1+u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}dt\Big\}\Big|$$
$$\le\int_{-1/2}^{1/2}\int_{\mathbb{T}}\Big\{\sum_{1\le|m_1|\le N}\Big|\int_{|t|\le N^{\gamma+1}|m_1|^{-\gamma}}(1-|m_1|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}\,dt$$
$$-\int_{|t|\le N^{\gamma+1}|m_1+u|^{-\gamma}}(1-|m_1+u|^{\gamma_1}|t|^{\gamma_2}N^{-\gamma_1-\gamma_2})^\alpha e^{ix_2t}\,dt\Big|^2\Big\}^{1/2}dx_2\,du.$$

Substituting $t\frac{|m_1|^\gamma}{N^{1+\gamma}}\to t$ in the first integral under the sign of absolute value and $t\frac{|m_1+u|^\gamma}{N^{1+\gamma}}\to t$ in the second one, we then have to deal with

$$\int_{\mathbb{T}}\Big\{\sum_{1\le|m_1|\le N}\Big|\frac{N^{1+\gamma}}{|m_1|^\gamma}\int_{|t|\le1}(1-|t|^{\gamma_2})^\alpha e^{itx_2\frac{N^{1+\gamma}}{|m_1|^\gamma}}\,dt$$
$$-\frac{N^{1+\gamma}}{|m_1+u|^\gamma}\int_{|t|\le1}(1-|t|^{\gamma_2})^\alpha e^{itx_2\frac{N^{1+\gamma}}{|m_1+u|^\gamma}}\,dt\Big|^2\Big\}^{1/2}dx_2,$$

for any $u\in[-\frac12,\frac12]$. The latter value is equal to

$$\sqrt{2}\int_{\mathbb{T}}\Big\{\sum_{1\le|m_1|\le N}\Big|\frac{N^{1+\gamma}}{|m_1|^\gamma}\int_0^1(1-t^{\gamma_2})^\alpha\cos tx_2\frac{N^{1+\gamma}}{|m_1|^\gamma}\,dt$$
$$-\frac{N^{1+\gamma}}{|m_1+u|^\gamma}\int_0^1(1-t^{\gamma_2})^\alpha\cos tx_2\frac{N^{1+\gamma}}{|m_1+u|^\gamma}\,dt\Big|^2\Big\}^{1/2}dx_2.$$

Integrating by parts in both inner integrals, we get (omitting the constants unnecessary for the estimates)

$$\int_{\mathbb{T}} \Bigg\{ \sum_{1\le|m_1|\le N} x_2^{-2} \Bigg| \int_0^1 t^{\gamma_2-1}(1-t^{\gamma_2})^{\alpha-1} \sin tx_2 \frac{N^{1+\gamma}}{|m_1|^\gamma}\, dt$$
$$- \int_0^1 t^{\gamma_2-1}(1-t^{\gamma_2})^{\alpha-1} \sin tx_2 \frac{N^{1+\gamma}}{|m_1+u|^\gamma}\, dt \Bigg|^2 \Bigg\}^{1/2} dx_2.$$

We shall use the following well-known relation:

$$\int_0^1 t^{\gamma_2-1}(1-t^{\gamma_2})^{\alpha-1} \sin yt\, dt \sim |y|^{-\alpha}. \tag{4.7}$$

As above, the case $|x_2| \ge 1$ is very simple. We just have the bound

$$\int_{|x_2|\ge 1} |x_2|^{-1-\alpha} \Bigg\{ \sum_{1\le|m_1|\le N} \frac{|m_1|^{2\alpha\gamma}}{N^{2\alpha(1+\gamma)}} \Bigg\}^{1/2} dx_2 = O(N^{\frac{1}{2}-\alpha}).$$

For $|x_2| \le 1$, we split the sum into two. For $1 \le |m_1| \le N|x_2|^\beta$, we obtain

$$\int_{|x_2|\le 1} \Bigg\{ \sum_{1\le|m_1|\le N|x_2|^\beta} x_2^{-2} \frac{|m_1|^{2\alpha\gamma}}{N^{2\alpha(1+\gamma)}} \Bigg\}^{1/2} dx_2$$
$$\lesssim \int_{|x_2|\le 1} \Big\{ N^{2\gamma\alpha+1} N^{-2\alpha(\gamma+1)} |x_2|^{-2+\beta(2\alpha\gamma+1)} \Big\}^{1/2} dx_2$$
$$\lesssim N^{\frac{1}{2}-\alpha} \int_{|x_2|\le 1} |x_2|^{-1+\beta(\alpha\gamma+\frac{1}{2})}\, dx_2,$$

which is $O(N^{\frac{1}{2}-\alpha})$ for any fixed $\beta > 0$.
For $|m_1| > N|x_2|^\beta$, we have to estimate

$$\int_{|x_2|\le 1} \Bigg\{ \sum_{|m_1|>N|x_2|^\beta} x_2^{-2} \Bigg| \int_0^1 t^{\gamma_2-1}(1-t^{\gamma_2})^{\alpha-1} \Bigg[\sin tx_2 \frac{N^{1+\gamma}}{|m_1|^\gamma}$$
$$- \sin tx_2 \frac{N^{1+\gamma}}{|m_1+u|^\gamma} \Bigg] dt \Bigg|^2 \Bigg\}^{1/2} dx_2.$$

The value in the brackets can be expressed as

$$t \int_{x_2 \frac{N^{1+\gamma}}{|m_1+u|^\gamma}}^{x_2 \frac{N^{1+\gamma}}{|m_1|^\gamma}} \cos ts\, ds.$$

Changing the order of integration under the sign of absolute value, we have to estimate

$$\int\limits_{x_2\frac{N^{1+\gamma}}{|m_1+u|^\gamma}}^{x_2\frac{N^{1+\gamma}}{|m_1|^\gamma}} \int\limits_0^1 t^{\gamma_2}(1-t^{\gamma_2})^{\alpha-1}\cos ts\,dt\,ds.$$

It follows from (4.7) that it is dominated by

$$|x_2|^{1-\alpha}N^{(1-\alpha)(1+\gamma)}\left|\frac{1}{|m_1+u|^{(1-\alpha)\gamma}}-\frac{1}{|m_1+u|^{(1-\alpha)\gamma}}\right|.$$

Representing the difference as

$$\int\limits_{\frac{1}{|m_1+u|^\gamma}}^{\frac{1}{|m_1|^\gamma}} s^{-\alpha}\,ds$$

and fulfilling standard calculations, we get the bound

$$|x_2|^{1-\alpha}N^{(1-\alpha)(1+\gamma)}|m_1|^{\alpha\gamma-(1+\gamma)}.$$

The integral in question is now dominated by

$$\int\limits_{|x_2|\le 1}\left\{\sum_{|m_1|>N|x_2|^\beta}|x_2|^{-2\alpha}N^{2(1-\alpha)(1+\gamma)}|m_1|^{2\alpha\gamma-2(1+\gamma)}\right\}^{1/2}dx_2$$
$$\lesssim N^{\frac12-\alpha}\int\limits_{|x_2|\le 1}|x_2|^{-\alpha-\beta+\beta\gamma+\beta\alpha\gamma}\,dx_2,$$

which is $O(N^{\frac12-\alpha})$ provided $-\alpha-\beta+\beta\gamma+\beta\alpha\gamma>-1$. The latter is true if β is chosen small enough.
Finally, all this means that the passage from the trigonometric sum to the Fourier transform is properly estimated.

Estimates for the Fourier transform

What has to be estimated now is the following integral

$$\int\limits_{\mathbb{T}^2}\Bigg|\int\limits_{\frac12\le|u|\le N+\frac12} e^{ix_1u}\int\limits_{|t|\le\frac{N^{\gamma_1/\gamma_2+1}}{|u|^{\gamma_1/\gamma_2}}}\left(1-\frac{|u|^{\gamma_1}|t|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^\alpha e^{ix_2t}\,dt\,du$$
$$+\int\limits_{\frac12\le|t|\le N+\frac12} e^{ix_2t}\int\limits_{|u|\le\frac{N^{\gamma_2/\gamma_1+1}}{|t|^{\gamma_2/\gamma_1}}}\left(1-\frac{|u|^{\gamma_1}|t|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^\alpha e^{ix_1u}\,du\,dt\Bigg|\,dx.$$

The two inner integrals are similar; it suffices to estimate one of them. Let it be, with the above notation in hand,

$$\int\limits_{\mathbb{T}^2}\Bigg|\int\limits_{\frac12\le|u|\le N+\frac12} e^{ix_1u}\int\limits_{|t|\le\frac{N^{\gamma+1}}{|u|^{\gamma}}}\left(1-\frac{|u|^{\gamma_1}|t|^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^\alpha e^{ix_2t}\,dt\,du\Bigg|\,dx.$$

Without loss of generality, we can estimate the simpler integral

$$\int_{\mathbb{T}^2_+}\left|\int_{\frac{1}{2}}^{N+\frac{1}{2}}\cos x_1u\int_0^{\frac{N^{\gamma+1}}{u^\gamma}}\left(1-\frac{u^{\gamma_1}t^{\gamma_2}}{N^{\gamma_1+\gamma_2}}\right)^\alpha\cos x_2t\,dt\,du\right|dx.$$

Substituting $\frac{tu^\gamma}{N^{\gamma+1}}\to t$, we get

$$\int_{\mathbb{T}^2_+}\left|\int_{\frac{1}{2}}^{N+\frac{1}{2}}\frac{N^{\gamma+1}}{u^\gamma}\cos x_1u\int_0^1\left(1-t^{\gamma_2}\right)^\alpha\cos x_2t\frac{N^{\gamma+1}}{u^\gamma}\,dt\,du\right|dx.$$

As in the case $\alpha=0$, we can use a more convenient upper limit N in place of $N+\frac{1}{2}$. Indeed, for $0\le x_2<\frac{1}{N}$, we obtain the bound

$$\int_0^{\frac{1}{N}}\int_N^{N+\frac{1}{2}}\frac{N^{\gamma+1}}{u^\gamma}\,du\,dx_2=O(1).$$

For greater x_2, we have the bound

$$\int_{\frac{1}{N}}^{\pi}x_2^{-\alpha-1}\int_N^{N+\frac{1}{2}}\frac{N^{\gamma+1}}{u^\gamma}\frac{u^{\gamma(\alpha+1)}}{N^{(\alpha+1)(\gamma+1)}}\,du\,dx_2=O(1).$$

Thus, we proceed to

$$\int_{\mathbb{T}^2_+}\left|\int_{\frac{1}{2}}^{N}\frac{N^{\gamma+1}}{u^\gamma}\cos x_1u\int_0^1\left(1-t^{\gamma_2}\right)^\alpha\cos x_2t\frac{N^{\gamma+1}}{u^\gamma}\,dt\,du\right|dx.$$

Integrating by parts, we obtain

$$\alpha\gamma_2\int_{\mathbb{T}^2_+}\frac{1}{x_2}\left|\int_{\frac{1}{2}}^{N}\cos x_1u\int_0^1 t^{\gamma_2-1}\left(1-t^{\gamma_2}\right)^{\alpha-1}\sin x_2t\frac{N^{\gamma+1}}{u^\gamma}\,dt\,du\right|dx.$$

Hence, we have to estimate

$$\int_{\mathbb{T}^2_+}\frac{1}{x_2}\left|\int_0^1 t^{\gamma_2-1}\left(1-t^{\gamma_2}\right)^{\alpha-1}\int_{\frac{1}{2}}^{N}\cos x_1u\sin x_2t\frac{N^{\gamma+1}}{u^\gamma}\,du\,dt\right|dx.$$

The next two steps correspond to two changes of variables: first

$$s=x_1^{\frac{1}{1+\gamma}}x_2^{-\frac{1}{1+\gamma}}N^{-1}u,$$

and then $v_1=Nx_1$ and $v_2=Nx_2$. The estimated integral becomes

$$\int\limits_{N\mathbb{T}_+^2} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \Bigg| \int\limits_0^1 t^{\gamma_2-1}\left(1-t^{\gamma_2}\right)^{\alpha-1} \int\limits_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \cos v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} s \sin(t v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} s^{-\gamma})\, ds\, dt \Bigg| \, dv.$$

Further estimates will be undertaken for two different cases. Denoting again $V = v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}}$, we consider separately $V \le 1$ and $V > 1$. Estimates for the case $V \le 1$ We do not have the same approach to all $\gamma = \frac{\gamma_1}{\gamma_2}$. Let first $\gamma \ge \frac{1}{2} + \alpha$. The estimates are very similar to those for partial sums; the main difference is that we replace $N^{\frac{1}{2}}$ in the integrals by

$$A_N = \begin{cases} N^{\frac{1}{2}-\alpha}, & \text{if } \alpha < \frac{1}{2}, \\ 1, & \text{if } \alpha \ge \frac{1}{2}. \end{cases}$$

(b1) The simplest part is $0 \le v_1 \le \frac{A_N}{\ln N}$, $\frac{1}{N^\gamma} \le v_2 \le N$. Since

$$\frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \left(\frac{v_1}{v_2}\right)^{\frac{\gamma_2}{\gamma_1+\gamma_2}} = \frac{1}{v_2},$$

we obtain

$$\int\limits_0^{\frac{A_N}{\ln N}} \int\limits_{\frac{1}{N^\gamma}}^{N} \frac{dv_2}{v_2}\, dv_1 \lesssim A_N,$$

as requested.

(b2) Furthermore,

$$\begin{aligned}
&\int\limits_0^N \int\limits_0^{\frac{1}{N^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \int\limits_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \left| \sin(v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} t^{-\gamma}) \right| dt\, dv_2\, dv_1 \\
\le &\int\limits_0^N \int\limits_0^{\frac{1}{N^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} \int\limits_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} t^{-\gamma}\, dt\, dv_2\, dv_1 \\
= &|1-\gamma| \int\limits_0^N \int\limits_0^{\frac{1}{N^\gamma}} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} v_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}} \left(\frac{v_1}{v_2}\right)^{\frac{\gamma_2-\gamma_1}{\gamma_1+\gamma_2}} |1-(2N)^{\gamma-1}|\, dv_2\, dv_1 \\
= &|1-\gamma| \int\limits_0^N \int\limits_0^{\frac{1}{N^\gamma}} |1-(2N)^{\gamma-1}|\, dv_2\, dv_1 = |1-\gamma| N^{1-\gamma} + O(1),
\end{aligned}$$

which is $O(N^{\frac{1}{2}-\alpha})$ provided $\gamma \ge \frac{1}{2} + \alpha$. Of course, the above calculations are true for $\gamma \ne 1$. However, for $\gamma = 1$, the corresponding calculations are quite similar and even easier, as well as in what follows.

The last step is exactly the same as (a3) for the partial sums, just with A_N in place of $N^{\frac{1}{2}}$ and resulting estimate $O(N^{\frac{1}{2}-\alpha})$ provided $\gamma \ge \frac{1}{2} + \alpha$.

Let now $\gamma < \frac{1}{2} + \alpha$. The first step is exactly (a1), with $\frac{1}{N^\gamma}$ replaced by $\frac{1}{N^{\frac{1}{2}+\alpha}}$. What remains is the domain $\frac{A_N}{\ln N} \le v_1 \le N$ and $\frac{1}{N^{\frac{1}{2}+\alpha}} \le v_2 \le \frac{1}{v_1^\gamma}$. We manage it in the same way like for the case of the partial sums ($\alpha = 0$). The logarithmic estimate there is satisfactory. Arriving at

$$N^\gamma \int\limits_{\frac{A_N}{\ln N}}^{N} \frac{1}{v_1} \int\limits_{\frac{1}{N^{1/2+\alpha}}}^{\frac{1}{v_1^\gamma}} dv_2\, dv_1,$$

we see that it is just bounded.

Estimates for the case $V > 1$ Here, the Stationary Phase Method (Theorem 3.1) will be used for the inner integral in

$$\int\limits_{N\mathbb{T}_+^2} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}}} \Bigg| \int\limits_0^1 t^{\gamma_2-1}\left(1-t^{\gamma_2}\right)^{\alpha-1} \int\limits_{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(2N)^{-1}}^{(v_1/v_2)^{\frac{\gamma_2}{\gamma_1+\gamma_2}}} e^{iv_1^{\frac{\gamma_1}{\gamma_1+\gamma_2}} v_2^{\frac{\gamma_2}{\gamma_1+\gamma_2}}(s\pm ts^{-\gamma})}\, ds\, dt \Bigg|\, dv. \tag{4.8}$$

We mention that this integral is in the form equivalent to that for $V \le 1$; it will be more convenient to deal with it just in this form. On this stage, it is rather simple and similar to the case of partial sums. The difference is that we need a precise asymptotic formula to be applied to the integral in t rather than just the estimate $O(V^{-\frac{1}{2}})$. Calculating the first and the second derivatives of the function $S(s,t) = s + \frac{t}{s^\gamma}$ (the sign is chosen for the most problematic case; $s - \frac{t}{s^\gamma}$ is such when the rest of $\mathbb{T}^2$ is considered), we mainly see that the second one does not vanish. The first derivative vanishes at $s_0 = (t\gamma)^{\frac{1}{1+\gamma}}$. Correspondingly, we get

$$S(s_0, t) = (t\gamma)^{\frac{1}{1+\gamma}} + \frac{t}{(t\gamma)^{\frac{\gamma}{1+\gamma}}} = \omega t^{\frac{1}{1+\gamma}},$$

with $\omega = \gamma^{\frac{1}{1+\gamma}} + \gamma^{-\frac{\gamma}{1+\gamma}}$. The second derivative is $\gamma(\gamma+1)ts^{-\gamma-2}$; at s_0 it takes the value

$$\gamma(\gamma+1)\frac{t}{(t\gamma)^{\frac{\gamma+2}{1+\gamma}}} = \gamma^{-\frac{1}{1+\gamma}}(\gamma+1)t^{-\frac{1}{1+\gamma}}.$$

Using the main term of the asymptotic relation, we now have to estimate

$$\int\limits_0^1 t^{\gamma_2-1}\left(1-t^{\gamma_2}\right)^{\alpha-1} t^{\frac{1}{2(1+\gamma)}} e^{i\omega V t^{\frac{1}{1+\gamma}}}\, dt. \tag{4.9}$$

The estimates we will fulfil further also show that for the remainder term in the Stationary Phase Method the bounds are better. Now, for $\alpha < \frac{1}{2}$, the worst possible estimate - along with the obtained $V^{-\frac{1}{2}}$ - is of the order $V^{-\frac{1}{2}-\alpha}$. We thus have to estimate

$$\int\limits_{N\mathbb{T}_+^2, V>1} \frac{1}{v_1^{\frac{\gamma_2}{\gamma_1+\gamma_2}+(\frac{1}{2}+\alpha)\frac{\gamma_1}{\gamma_1+\gamma_2}}} \frac{1}{v_2^{\frac{\gamma_1}{\gamma_1+\gamma_2}+(\frac{1}{2}+\alpha)\frac{\gamma_2}{\gamma_1+\gamma_2}}}\, dv_2\, dv_1.$$

We observe that both v_1 and v_2 in the denominators are in the power less than 1. Being integrated, they give

$$v_1^{1-\frac{\gamma_2}{\gamma_1+\gamma_2}-(\frac{1}{2}+\alpha)\frac{\gamma_1}{\gamma_1+\gamma_2}}$$

and

$$v_2^{1-\frac{\gamma_1}{\gamma_1+\gamma_2}-(\frac{1}{2}+\alpha)\frac{\gamma_2}{\gamma_1+\gamma_2}},$$

both now in positive powers. Obviously, the maximum of their product is attained at $v_1 = v_2 = N$. This gives

$$N^{1-\frac{\gamma_2}{\gamma_1+\gamma_2}-(\frac{1}{2}+\alpha)\frac{\gamma_1}{\gamma_1+\gamma_2}} N^{1-\frac{\gamma_1}{\gamma_1+\gamma_2}-(\frac{1}{2}+\alpha)\frac{\gamma_2}{\gamma_1+\gamma_2}} = N^{\frac{1}{2}-\alpha},$$

which completes the proof for the above estimate of the Lebesgue constants provided $\alpha < \frac{1}{2}$.
If $\alpha > \frac{1}{2}$, then both v_1 and v_2 will be in the powers strictly less than -1, which yields a constant bound. Here, the main term of the asymptotics comes from Theorem 1.2, which completes the case **3)** in Theorem 1.1. For $\alpha = \frac{1}{2}$, we get v_1 and v_2 as $\frac{1}{v_1 v_2}$. Since we need asymptotic estimate, this upper estimate is not enough. However, treating (4.9) in an asymptotic manner, we see, on the one hand, that the remainder term leads to the final $O(1)$ bound, and, on the other hand, that the leading term will contain, in addition to $v_1 v_2$ in the denominator, trigonometric functions of v_1 and v_2 in the nominator. This leads to the classical asymptotics with $\ln^2 N$ times a constant as the leading term, as desired in the case **2)** in Theorem 1.1.

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