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BRANCHING RANDOM WALKS WITH ALTERNATING SIGN INTENSITIES OF BRANCHING SOURCES

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Abstract. We consider a continuous-time symmetric branching random walk on a multidimensional lattice with a finite set of particle generation centers, i.e., branching sources. The existence of a positive eigenvalue of the evolutionary operator means the exponential growth of the first moment of the total number of particles both at an arbitrary point and on the entire lattice. Branching random walks with positive or negative intensities of sources that have a simplex configuration are presented in the paper. It is established that the amount of positive eigenvalues of the evolutionary operator, counting their multiplicity, does not exceed the amount of the branching sources with positive intensity, while the maximal eigenvalue is simple. For branching random walk with different positive intensities of sources and arbitrary configuration for both finite and infinite variance of jumps, the critical values of sources' intensities are found, which allows us to prove the existence of positive eigenvalues of the evolutionary operator.

1. Introduction

We consider a branching random walk with continuous time on the multidimensional lattice $\mathbb{Z}^d$, $d \geq 1$, with a finite number of branching sources on it. Walk is assumed to be homogeneous in time and space, symmetric and irreducible. Branching random walks are widely used to describe the population dynamics of objects with non-overlapping generations, for example, for describing the spread of viral infections [1,5], simulating the epidemic and vaccination [7], various physical, biological, and genetic systems [10].

Branching random walks on $\mathbb{Z}^d$ with continuous time in recent decades have been considered in a number of publications, see, for example, [4,11,13,14,16]. Evolution equations for transition probabilities and moments of particle numbers are often conveniently represented as linear differential equations in Banach spaces. By virtue of their linearity, the investigation of the asymptotic behavior of the solutions as $t \to \infty$ leads to the study of the spectrum of the corresponding operators. In particular, the presence of a positive eigenvalue in the spectrum of the evolutionary operator ensures an exponential growth of the particle numbers both at each point and the entire lattice. Branching random walks with exponential growth of particle numbers are called supercritical.

Analysis of the evolutionary operator of branching random walk with several sources in general form was performed in [14], where it was noted, in particular, that the presence of branching sources can lead to the appearance of positive eigenvalues of the operator. In [2] it was proved that for the case of equal source intensities and finite variance of jumps, the number of eigenvalues (counting multiplicity) does not exceed the number of sources N and the multiplicity of each eigenvalue of the operator does not exceed $N-1$. In [15], for sources with equal intensities in the case of an infinite variance of jumps, it is shown that the appearance of several lower eigenvalues in the spectrum of the evolution operator can be caused by a simplicial configuration of sources.

The structure of the work is as follows. In Sec. 2, a formal description of the model of branching with several branching sources is presented, basic definitions and equations for the average particle numbers are introduced, it is shown that the asymptotic behavior of the particle numbers is related to the spectra of the corresponding operators. In Sec. 3, models with sources that can have both positive and

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negative intensities forming simplicial configurations are considered. In Sec. 4, critical values are found for intensities of the sources in an arbitrary configuration.

2. Model Description

We consider branching random walk on the multidimensional lattice $\mathbb{Z}^d$, $d \geq 1$, in which branching (birth or death) occurs in the sources $x_1, x_2, \ldots, x_N$. We assume that random walk is given by a matrix of transient intensities $A = \big(a(x,y)\big)_{x,y\in\mathbb{Z}^d}$, with the properties $a(x,y) = a(y,x) = a(0, y-x) = a(y-x)$, for all x and y. Thus, a random walk is symmetric and spatially homogeneous. Moreover, we assume that the regularity properties $\sum\limits_{z\in\mathbb{Z}^d} a(z) = 0$ and irreducibility are fulfilled, i.e., for all $z \in \mathbb{Z}^d$ there exists a set of vectors $z_1, z_2, \ldots, z_k \in \mathbb{Z}^d$ such that $z = \sum\limits_{i=1}^{k} z_i$ and $a(z_i) \neq 0$ for $i = 1, 2, \ldots, k$.

The transition probability $p(t, \cdot, y)$ is conveniently considered as a function $p(t)$ in $l^2(\mathbb{Z}^d)$, depending on the time t and parameter y. For $h \to 0$, the following equalities hold:

$$\begin{aligned} &p(h,x,y) = a(x,y)h + o(h) \text{ for } y \neq x, \\ &p(h,x,x) = 1 + a(x,x)h + o(h). \end{aligned} \tag{1}$$

As is known from [6], the transition probabilities satisfy the system of inverse Kolmogorov equations:

$$\frac{\partial p(t,x,y)}{\partial t} = \sum_{x'} a(x,x')p(t,x',y), \quad p(0,x,y) = \delta(x-y),$$

where δ is the discrete δ-function of Kronecker on $\mathbb{Z}^d$.

We assume that the branching occurs in the sources x_i and is determined by the infinitesimal generating functions

$$f_i(u) = \sum_{n=0}^{\infty} b_{i,n} u^n, \quad 0 \leq u \leq 1,$$

where $\sum\limits_n b_n(x_i) = 0$, $b_n(x_i) \geq 0$ for $n \neq 1$ and $b_1(x_i) < 0$, $f_i^{(r)}(1) < \infty$ for all $r \in \mathbb{N}$.

Definition 1. The intensity of the source x_i is the quantity

$$\beta_i = f'(1, x_i) = \sum_n n b_n(x_i) = -\big(b_1(x_i)\big)\left(\sum_{n\neq 1} n \frac{b_n(x_i)}{-\big(b_1(x_i)\big)} - 1\right),$$

characterizing the average number of descendants that are born in it.

Let $\mu_t(y)$ denote the number of particles at time t at the point y and let $m_1(t,x,y) := \mathbb{E}_x \mu_t(y)$ denote the expected value of the number of particles at the point y at time t under the condition that at the initial instant of time $t = 0$ there was one particle in the system located at the point x. Then, according to [6],

$$\frac{\partial m_1(t,x,y)}{\partial t} = \sum_{x'} a(x,x')m_1(t,x',y) + \sum_{i=1}^{N} \beta_i \delta(x-x_i) m_1(t,x,y), \quad m_1(0,x,y) = \delta(x-y). \tag{2}$$

On the set of functions $u(x)$, $x \in \mathbb{Z}^d$, we consider the operator

$$(\mathcal{A}u)(x) = \sum_{x'\in\mathbb{Z}^d} a(x-x')u(x')$$

and for each of the sources $x_i \in \mathbb{Z}^d$, the operators

$$(\Delta_{x_i} u)(x) = \delta(x-x_i)u(x),$$

where $\delta(\cdot)$ is the discrete δ-function of Kronecker on $\mathbb{Z}^d$. The operator $\mathcal{A}$, as an operator in the Hilbert space $l^2(\mathbb{Z}^d)$, is self-adjoint, the operators Δx_i act in each of the function spaces $l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$ (see [14]).

The behavior of the mean number of particles both at an arbitrary point and on the entire lattice can be described in terms of an evolutionary operator of a special type

$$\mathcal{H}_{\beta_1,\ldots,\beta_N} = \mathcal{A} + \sum_{i=1}^{N} \beta_i \Delta_{x_i}, \quad x_i \in \mathbb{Z}^d, \tag{3}$$

which is a perturbation of the generator $\mathcal{A}$ of the symmetric random walk. This operator can be treated as a linear bounded operator acting in each of the function spaces $l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$ (see [14]).

According to [14] the evolution equations for the transition probabilities (1) and the moments of particle numbers (2) can be represented as the following differential equation in the space $l^2(\mathbb{Z}^d)$ and $l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$, respectively:

$$\frac{dp(t,x,y)}{dt} = \big(\mathcal{A}p(t,\cdot,y)\big)(x), \quad p(0,x,y) = \delta(x-y),$$
$$\frac{dm_1(t,x,y)}{dt} = \big(\mathcal{H}_{\beta_\infty,\ldots,\beta_N} m_1(t,\cdot,y)\big)(x), \quad m_1(0,x,y) = \delta(x-y).$$

The Green function of the operator $\mathcal{A}$ can be represented as the Laplace transform of the transition probability $p(t, x, y)$:

$$G_\lambda(x,y) := \int_0^\infty e^{-\lambda t} p(t,x,y)\, dt = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{e^{i(\theta, y-x)}}{\lambda - \phi(\theta)}\, d\theta, \quad \lambda \geq 0, \tag{4}$$

where

$$\phi(\theta) = \sum_{z \in \mathbb{Z}^d} a(z) e^{i(\theta,z)}$$

for $\theta \in [-\pi, \pi]$. For further research, the value $G_0 := G_0(0,0)$ plays an important role.

If the inequality

$$\sum_{z \in \mathbb{Z}^d} |z|^2 a(z) < \infty, \tag{5}$$

where $|z|$ is the Euclidean norm of the vector z, is fulfilled, then the variance of jumps is finite and $G_0 = \infty$ for $d = 1$ and $d = 2$ and $G_0 < \infty$ for $d \geq 3$ [16].

In the other case, if for sufficiently large (of the norm) $z \in \mathbb{Z}^d$, the asymptotic relation

$$a(z) \sim \frac{H(z/|z|)}{|z|^{d+\alpha}}, \quad \alpha \in (0,2), \tag{6}$$

holds, where $H(\cdot)$ is a continuous positive function symmetric on the sphere $S^{d-1} = \{z \in R^d\colon |Z| = 1\}$, then $G_0 = \infty$ for $d = 1$, $\alpha \in [1, 2)$ and $G_0 < \infty$ for $d = 1$, $\alpha \in (0, 1)$ or for $d \geq 2$, $\alpha \in (0, 2)$ [12]. Condition (6) leads to the divergence of the series $\sum_{z \in \mathbb{Z}^d} |z|^2 a(z)$ and to the infinity of the variance of jumps.

3. Simplicial Configuration of Sources

3.1. Three Branching Sources with Different Intensities. We consider branching random walk on $\mathbb{Z}^d$, $d \geq 3$, with three branching sources with arbitrary intensities β_1, β_2, and β_3, located at the vertices of some simplex, $|x_1 - x_2| = |x_1 - x_3| = |x_2 - x_3|$. Simplexes of this kind on $\mathbb{Z}^d$ exist, for example, as their vertices one can select points $(0, 0, t, 0, 0, \ldots)$, $(0, t, 0, 0, 0, \ldots)$, $(t, 0, 0, 0, 0, \ldots)$, $t \in \mathbb{Z}$. Denote

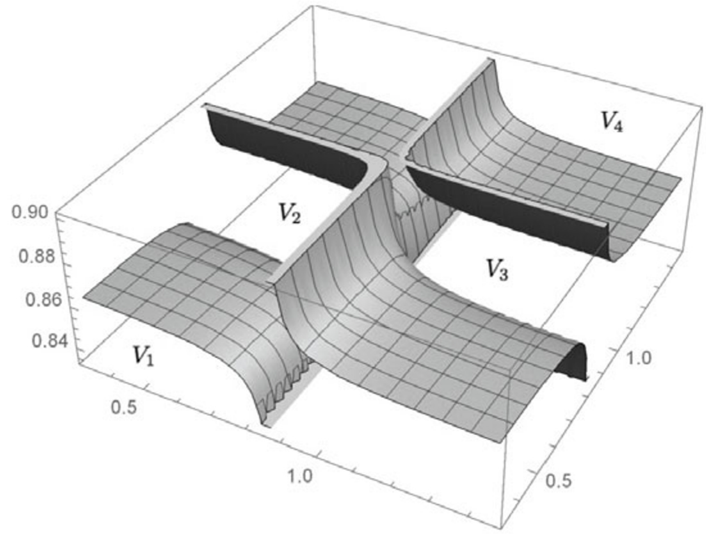

Fig. 1. Ranges of the parameters values β_1, β_2, and β_3.

$\tilde{G}_\lambda := G_\lambda(x_i, x_j)$ for $i \neq j$, it does not depend on i, j because of the simplicial configuration of the sources. In this case, the operator (3) takes the form

$$\mathcal{H}_{\beta_1,\beta_2,\beta_3} = \mathcal{A} + \beta_1\Delta_{x_1} + \beta_2\Delta_{x_2} + \beta_3\Delta_{x_3}.$$

Note that λ is an eigenvalue of the operator $\mathcal{H}_{\beta_1,\beta_2,\beta_3}$ if and only if

$$\beta_1\beta_2\beta_3(3\tilde{G}_0^2G_0 - 2\tilde{G}_0^3 - G_0^3) + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)(G_0^2 - \tilde{G}_0^2) - (\beta_1 + \beta_2 + \beta_3)G_0 + 1 = 0 \tag{7}$$

(see [14, Theorem 6]).

Figure 1 shows the ranges of the parameters β_1, β_2, and β_3 by the number of eigenvalues of the operator $\mathcal{H}_{\beta_1,\beta_2,\beta_3}$ in space $l^2(\mathbb{Z}^3)$. On this figure, the coordinates of the sources are: $x_1 = (0,0,1)$, $x_2 = (0,1,0)$, and $x_3 = (1,0,0)$, operator $\mathcal{A}$ is the standard Laplacian, i.e., $a(0) = -1$, $a(z) = 1/6$ for $z = \{z_1, z_2, z_3\}$ such that $|z_1| + |z_2| + |z_3| = 1$ and $a(z) = 0$ otherwise. As a result of modeling in the Wolfram® Alpha values $G_0 = 1.1564$ and $\tilde{G}_0 = 0.0414$ were obtained.

The area V_1 represents the range of the parameters $\{\beta_1, \beta_2, \beta_3\}$, for which the operator $\mathcal{H}_{\beta_1,\beta_2,\beta_3}$ has no positive eigenvalues. The area V_2 corresponds to a single, V_3 to two, and V_4 to three eigenvalues counting multiplicity.

In the case of the equal intensities $\beta := \beta_1 = \beta_2 = \beta_3$, we denote by β_c and β_{c_1} critical values for the intensity β such that for $\beta \leq \beta_c$, the spectrum of the operator $\mathcal{H}_\beta$ does not contain eigenvalues, for $\beta \in (\beta_c, \beta_{c_1})$, the operator has single and for $\beta \geq \beta_{c_1}$, more than one eigenvalue. The equation (7) in this case has the form

$$\beta^3(3\tilde{G}_0^2G_0 - 2\tilde{G}_0^3 - G_0^3) + 3\beta^2(G_0^2 - \tilde{G}_0^2) - 3\beta G_0 + 1 = 0,$$

consequently,

$$\beta_c = \frac{1}{G_0 + 2\tilde{G}_0}, \quad \beta_{c_1} = \frac{1}{G_0 - \tilde{G}_0}.$$

The calculations performed in the system Wolfram® Alpha lead to the following results: $\beta_c = 0.8070$, $\beta_{c_1} = 0.8969$, the solution is illustrated on Fig. 2.

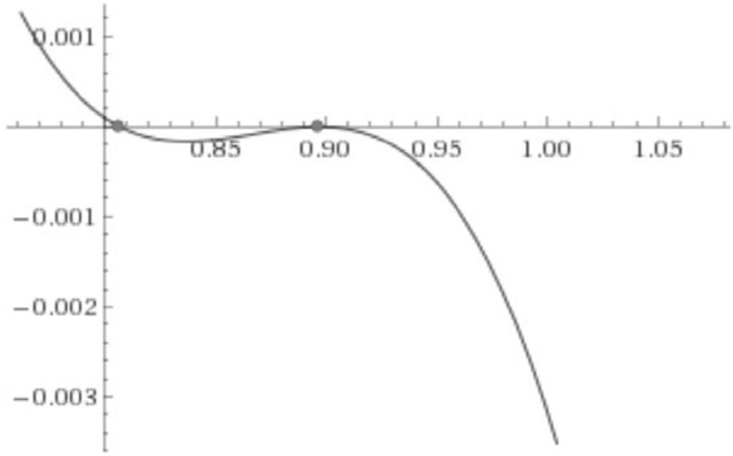

Fig. 2. Values β_c and β_{c_1}.

3.2. Alternating Sign Intensities of the Sources. We consider the branching random walk with p positive intensity sources $\beta > 0$ and n negative intensity sources $(-\beta) < 0$, located at the vertices of the simplex, $|x_i - x_j| = \text{const}$ for $i \neq j$. Denote $\tilde{G}_\lambda := G_\lambda(x_i, x_j) = G_\lambda(0, |x_i - x_j|)$ for $i \neq j$. Sources with positive intensities indicate points where the birth rate prevails over the degree of death, and in sources with negative intensity it is the other way around.

Theorem 1. *The number of eigenvalues $\lambda > 0$ counting their multiplicity of the evolution operator $\mathcal{H}_{\beta_1,\dots,\beta_{p+n}}$, where $\beta_1 = \dots = \beta_p = \beta$ and $\beta_{p+1} = \dots = \beta_{p+n} = -\beta$, does not exceed the number of branching sources with positive intensity, the maximum of these eigenvalues is simple.*

Proof. According to (3) the operator $\mathcal{H}_{\beta_1,\dots,\beta_{p+n}}$ has the form

$$\mathcal{H}_{\beta_1,\dots,\beta_{p+n}} = \mathcal{A} + \beta\Delta_{x_1} + \beta\Delta_{x_2} + \dots + \beta\Delta_{x_p} - \beta\Delta_{x_{p+1}} - \beta\Delta_{x_{p+2}} - \dots - \beta\Delta_{x_{p+n}}.$$

Note that λ is an eigenvalue of the operator $\mathcal{H}_{\beta_1,\dots,\beta_{p+n}}$ if and only if the system of linear equations with respect to variables $X_1, \dots, V_{p+n}$

$$\begin{cases} -X_1 + \beta G_\lambda X_1 + \beta\tilde{G}_\lambda X_2 + \dots + \beta\tilde{G}_\lambda X_{p+n} = 0, \\ -X_2 + \beta\tilde{G}_\lambda X_1 + \beta G_\lambda X_2 + \dots + \beta\tilde{G}_\lambda X_{p+n} = 0, \\ \quad \dots \\ X_{p+1} + \beta\tilde{G}_\lambda X_1 + \beta\tilde{G}_\lambda X_2 + \dots + \beta\tilde{G}_\lambda X_{p+n} = 0, \\ \quad \dots \\ X_{p+n} + \beta\tilde{G}_\lambda X_1 + \beta\tilde{G}_\lambda X_2 + \dots + \beta G_\lambda X_{p+n} = 0 \end{cases} \tag{8}$$

has a nontrivial solution (see [14, Theorem 6]). Let $S_{p,n}$ denote the matrix of the system (8):

$$S_{p,n} = \begin{pmatrix} \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \beta\tilde{G}_\lambda \\ \beta\tilde{G}_\lambda & \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \beta\tilde{G}_\lambda \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \dots & \dots & \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \dots & \beta\tilde{G}_\lambda \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \beta\tilde{G}_\lambda & \beta G_\lambda + 1 & \dots & \beta\tilde{G}_\lambda \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \beta G_\lambda + 1 \end{pmatrix}.$$

We prove in this case by induction the following equality:

$$|S_{p,n}| = (\beta G_\lambda - \beta\tilde{G}_\lambda - 1)^{p-1}(\beta G_\lambda - \beta\tilde{G}_\lambda + 1)^{n-1} \times \big((\beta G_\lambda)^2 + (p+n-2)\beta^2 G_\lambda\tilde{G}_\lambda - (p+n-1)(\beta\tilde{G}_\lambda)^2 + (p-n)\beta\tilde{G}_\lambda - 1\big). \quad (9)$$

A direct calculation shows that

$$|S_{p,1}| = \begin{vmatrix} \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta G_\lambda - 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \beta G_\lambda - 1 & \beta\tilde{G}_\lambda \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \beta G_\lambda + 1 \end{vmatrix}$$

$$= \begin{vmatrix} \beta G_\lambda - 1 - \beta\tilde{G}_\lambda & 0 & \dots & \beta\tilde{G}_\lambda - \beta G_\lambda + 1 & 0 \\ 0 & \beta G_\lambda - 1 - \beta\tilde{G}_\lambda & \dots & \beta\tilde{G}_\lambda - \beta G_\lambda + 1 & 0 \\ 0 & 0 & \dots & \beta\tilde{G}_\lambda - \beta G_\lambda + 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \beta G_\lambda - 1 & \beta\tilde{G}_\lambda \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \beta G_\lambda + 1 \end{vmatrix}$$

$$= \begin{vmatrix} \beta G_\lambda - 1 - \beta\tilde{G}_\lambda & \dots & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \dots & \dots & \dots & \beta G_\lambda - 1 + (p-1)\beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda \\ \beta\tilde{G}_\lambda & \dots & \dots & \dots & \dots & \beta G_\lambda + 1 \end{vmatrix}$$

$$= (\beta G_\lambda - \beta\tilde{G}_\lambda - 1)^{p-1} \cdot \big((\beta G_\lambda)^2 + (p-1)\beta^2 G_\lambda\tilde{G}_\lambda - p(\beta\tilde{G}_\lambda)^2 + (p-1)\beta\tilde{G}_\lambda - 1\big).$$

We proceed by induction:

$$|S_{p,n+1}| = \begin{vmatrix} \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta G_\lambda - 1 & \beta\tilde{G}_\lambda & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta G_\lambda - 1 & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta G_\lambda - 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \beta\tilde{G}_\lambda & \dots & \dots & \beta G_\lambda + 1 \end{vmatrix}$$

$$= \begin{vmatrix} \beta G_\lambda - 1 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \dots & \beta G_\lambda - 1 & \dots & \dots & \dots \\ \beta\tilde{G}_\lambda & \dots & \dots & \beta G_\lambda + 1 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \beta G_\lambda + 1 & \beta\tilde{G}_\lambda \\ 0 & \dots & \dots & & \dots & \beta G_\lambda - \beta\tilde{G}_\lambda + 1 \end{vmatrix}$$

$$= (\beta G_\lambda - \beta\tilde{G}_\lambda + 1) \cdot |S_{p,n}| + (\beta G_\lambda - \beta\tilde{G}_\lambda + 1) \cdot (\beta G_\lambda - \beta\tilde{G}_\lambda - 1)^p \times (\beta G_\lambda - \beta\tilde{G}_\lambda + 1)^{n-1} \cdot \beta\tilde{G}_\lambda = (\beta G_\lambda - \beta\tilde{G}_\lambda - 1)^{p-1}(\beta G_\lambda - \beta\tilde{G}_\lambda + 1)^n \times \big((\beta G_\lambda)^2 + (p+n-1)\beta^2 G_\lambda\tilde{G}_\lambda - (p+n)(\beta\tilde{G}_\lambda)^2 + (p-n-1)\beta\tilde{G}_\lambda - 1\big).$$

The inductive transition is carried out, therefore, the required equality (9) is proved.

Let us return to the question of the existence of eigenvalues of the operator, which is equivalent to the existence of a nontrivial solution for the system of linear equations (8) and the equality of the determinant of its matrix to 0. We get

$$(\beta G_\lambda - \beta\tilde{G}_\lambda - 1)^{p-1}(\beta G_\lambda - \beta\tilde{G}_\lambda + 1)^{n-1} \times \big((\beta G_\lambda)^2 + (p+n-2)\beta^2 G_\lambda\tilde{G}_\lambda - (p+n-1)(\beta\tilde{G}_\lambda)^2 + (p-n)\beta\tilde{G}_\lambda - 1\big) = 0.$$

Here the first factor has at most $p-1$ roots counting their multiplicity, the second factor has no roots due to $G_\lambda > \tilde{G}_\lambda$, the third factor has no more than one root. Thus, the operator $\mathcal{H}_{\beta_1,\dots,\beta_{p+n}}$ has no more than p eigenvalues. To simplify the subsequent calculations, we denote

$$D_\lambda := (G_\lambda - \tilde{G}_\lambda)\big(G_\lambda + \ +\tilde{G}_\lambda(n+p-1)\big).$$

The eigenvalues of the operator $\mathcal{H}_{\beta_1,\dots,\beta_{p+n}}$ are found from the equations

$$\frac{1}{\beta} = G_\lambda - \tilde{G}_\lambda,$$

$$\frac{1}{\beta} = \frac{2D_\lambda}{(n-p)\tilde{G}_\lambda + \sqrt{(n-p)^2(\tilde{G}_\lambda)^2 + 4D_\lambda}}. \tag{10}$$

Note that $G_\lambda - \tilde{G}_\lambda$ monotonically decreases with λ [15], while

$$\frac{2D_\lambda}{(n-p)\tilde{G}_\lambda + \sqrt{(n-p)^2(\tilde{G}_\lambda)^2 + 4D_\lambda}} > G_\lambda - \tilde{G}_\lambda.$$

Consequently, the leading eigenvalue λ_0 is found from the equation (10) and has a unit multiplicity.

Recall that we denoted by $\beta_c := \beta_c(n,p)$, $p \geq 2$, the minimal positive value of the intensity of the sources that for $\beta > \beta_c$ the spectrum of the operator $\mathcal{H}_{\beta_1,\dots,\beta_{p+n}}$ has positive eigenvalues, and $\beta_{c_1} > \beta_c$ such that for $\beta \in (\beta_c, \beta_{c_1})$ it has single eigenvalue $\lambda_0(\beta)$. Then

$$\beta_c = \frac{(n-p)\tilde{G}_0 + \sqrt{(n-p)^2(\tilde{G}_0)^2 + 4D_0}}{2D_0}, \quad \beta_{c_1} = \frac{1}{G_0 - \tilde{G}_0}.$$

The theorem is proved. □

4. Arbitrary Configuration of Branching Sources with Positive Intensities

We consider branching random walk with N sources with arbitrary positive intensities $\beta_1, \dots, \beta_N$, which are in an arbitrary configuration. We denote $\beta_{\min} := \min\limits_i\{\beta_i\}$ and $\beta_{\max} := \max\limits_i\{\beta_i\}$ and assume that there exist values $\beta_{c_{\min}}$ and $\beta_{c_{\max}}$ such that for $\beta_{\min} > \beta_{c_{\min}}$, the operator $\mathcal{H}_{\beta_1,\dots,\beta_N}$ contains positive eigenvalues, one or more, and for $\beta_{\max} < \beta_{c_{\max}}$, the operator $\mathcal{H}_{\beta_1,\dots,\beta_N}$ does not contain positive eigenvalues.

Theorem 2. *Let branching random walk satisfy the conditions* (5) *or* (6). *If* $G_0 = \infty$, *then* $\beta_{c_{\min}} = 0$ *for* $N \geq 1$. *If* $G_0 < \infty$, *then* $\beta_{c_{\min}} = \beta_{c_{\max}} = G_0^{-1}$ *for* $N = 1$, $0 < \beta_{c_{\min}}$ *and* $0 < \beta_{c_{\max}} < G_0^{-1}$ *for* $N > 1$.

Proof. According to (3) the operator $\mathcal{H}_{\beta_1,\dots,\beta_N}$ has the form

$$\mathcal{H}_{\beta_1,\dots,\beta_N} = \mathcal{A} + \sum_{i=1}^{N} \beta_i \Delta_{x_i}.$$

Note that $\lambda > 0$ is an eigenvalue of the operator $\mathcal{H}_{\beta_1,\dots,\beta_N}$ if and only if the system of linear equations

$$V_i - \sum_{j=1}^{N} \beta_j G_\lambda(x - I, x_j) V_j = 0, \quad i = 1, \dots, N,$$

has a nontrivial solution with respect to the variables V_i (see [14]). Thus, the determinant of corresponding matrix $\Gamma_{\beta_1,\dots,\beta_N}(\lambda) - I$ equals 0, where

$$\Gamma_{\beta_1,\dots,\beta_N}(\lambda) = [\beta_i G_\lambda(x_i, x_j)].$$

This matrix is not symmetric due to β_i are different in different rows of the matrix $\Gamma_{\beta_1,\dots,\beta_N}(\lambda)$, but if each ij element is decomposed into $\sqrt{\beta_i}$ and multiplied by $\sqrt{\beta_j}$, then we obtain a symmetric matrix $\Gamma_\beta(\lambda) - I$, where

$$\Gamma_\beta(\lambda) = [\sqrt{\beta_i \beta_j} G_\lambda(x_i, x_j)].$$

Note that

$$\det(\Gamma_{\beta_1,\dots,\beta_N}(\lambda) - I) = 0$$

if and only if

$$\det(\Gamma_\beta(\lambda) - I) = 0,$$

thus 1 is an eigenvalue of the matrix $\Gamma_\beta(\lambda)$. Consider the matrix $D(\lambda_1, \lambda_2) := \Gamma_\beta(\lambda_1) - \Gamma_\beta(\lambda_2)$. From the representation of the Green function (4) it follows that the elements of the matrix $D(\lambda_1, \lambda_2)$ have the form

$$D_{ij}(\lambda_1, \lambda_2) = \frac{\sqrt{\beta_i \beta_j}(\lambda_2 - \lambda_1)}{(2\pi)^d} \int\limits_{[-\pi,\pi]^d} \frac{e^{i(\theta, x_i - x_j)}}{\big(\lambda_1 - \phi(\theta)\big)\big(\lambda_2 - \phi(\theta)\big)}\, d\theta.$$

Let $s = \max\limits_{\theta \in [-\pi,\pi]^d} \{-\phi(\theta)\}$. Then

$$\frac{1}{\big(\lambda_1 - \phi(\theta)\big)\big(\lambda_2 - \phi(\theta)\big)} \ge \frac{1}{(\lambda_1 + s)(\lambda_2 + s)} > 0.$$

Denote $\tilde{D}(\lambda_1, \lambda_2) = [\tilde{D}_{ij}(\lambda_1, \lambda_2)]$, where

$$\tilde{D}_{ij}(\lambda_1, \lambda_2) = \int\limits_{[-\pi,\pi]^d} \frac{e^{i(\theta, x_i - x_j)}}{\big(\lambda_1 - \phi(\theta)\big)\big(\lambda_2 - \phi(\theta)\big)}\, d\theta.$$

Since the function $\theta(\phi)$ is even, the matrix $\tilde{D}(\lambda_1, \lambda_2)$ is real and symmetric. Let us prove its positive definiteness:

$$(\tilde{D}(\lambda_1, \lambda_2) z_i, z_j) = \sum_{i,j=1}^{N} \int\limits_{[-\pi,\pi]^d} \frac{e^{i(\theta, x_i - x_j)}}{\big(\lambda_1 - \phi(\theta)\big)\big(\lambda_2 - \phi(\theta)\big)} z_i z_j \, d\theta$$

$$= \int\limits_{[-\pi,\pi]^d} \frac{1}{\big(\lambda_1 - \phi(\theta)\big)\big(\lambda_2 - \phi(\theta)\big)} \sum_{i=1}^{N} \sum_{j=1}^{N} \big(e^{i(\theta, x_i)} z_i\big)\big(e^{-i(\theta, x_j)} z_j\big)$$

$$= \int\limits_{[-\pi,\pi]^d} \frac{|e^{i(\theta, x_1)} z_1 + \dots + e^{i(\theta, x_N)} z_N|^2}{\big(\lambda_1 - \phi(\theta)\big)\big(\lambda_2 - \phi(\theta)\big)} \ge \int\limits_{[-\pi,\pi]^d} \frac{|e^{i(\theta, x_1)} z_1 + \dots + e^{i(\theta, x_N)} z_N|^2}{(\lambda_1 + s)(\lambda_2 + s)} > 0.$$

By the Sylvester criterion [3] all the angular minors of the matrix $\tilde{D}(\lambda_1, \lambda_2)$ are positive, thus the angular minors of the matrix $D(\lambda_1, \lambda_2)$,

$$\Delta_{D(\lambda_1, \lambda_2), i} = \frac{(\lambda_2 - \lambda_1) \prod\limits_{j=1}^{i} \beta_j \times \Delta_{\tilde{D}(\lambda_1, \lambda_2), i}}{(2\pi)^d}, \quad i = 1, \dots, N,$$

are positive and the matrix $D(\lambda_1, \lambda_2)$ is positive-definite.

We denote by

$$\zeta_0(\lambda) \ge \zeta_1(\lambda) \ge \dots \ge \zeta_{N-1}(\lambda)$$

the eigenvalues of the real Hermitian positive-definite matrix $\Gamma_\beta(\lambda)$ and by

$$\gamma_0(\lambda) \geq \gamma_1(\lambda) \geq \cdots \geq \gamma_{N-1}(\lambda)$$

the eigenvalues of the real Hermitian positive definite matrix $\Gamma(\lambda) = [G_\lambda(x_i, x_j)]$. The eigenvalues $\zeta_i(\lambda)$, $i = 0, \ldots, N-1$, by Weyl's theorem [9, Theorem 4.3.1] satisfy the inequalities $\zeta_i(\lambda_1) > \zeta_i(\lambda_2)$ for $\lambda_2 > \lambda_1$. Therefore, each of the equations $\zeta_i(\lambda) = 1$ has no more than one solution λ and the operator $\mathcal{H}_{\beta_1,\ldots,\beta_N}$ has no more than N eigenvalues.

If $G_0 = \infty$ and $G_\lambda \to \infty$ for $\lambda \to 0$, then $\|\Gamma_\beta(\lambda)\| \to \infty$ and $\zeta_0(\lambda) \to \infty$ for $\lambda \to 0$, then the equation $\zeta_0(\lambda) = 1$ has a solution for any $\beta_i > 0$, $i = 1, \ldots, N$. Thus, $\beta_{c_{\min}} = 0$.

For the case $G_0 < \infty$ and $N = 1$, the statement of the theorem $\beta_{c_{\min}} = \beta_{c_{\max}} = G_0^{-1}$ is the corollary of Theorem 3 from [16].

We now turn to the case $G_0 < \infty$ for $N > 1$, $G_0(x, y) < \infty$ for all x and y. In this case,

$$\beta_{\min}\|\Gamma(\lambda)\| \leq \|\Gamma_\beta(\lambda)\| \leq \beta_{\max}\|\Gamma(\lambda)\| < \infty$$

and

$$\Gamma_\beta(\lambda) \to \Gamma_\beta(0), \quad \Gamma(\lambda) \to \Gamma(0) \quad \text{for} \quad \lambda \to 0,$$

there exists $\gamma_* < \infty$ such that $\gamma_0(\lambda) \leq \gamma_* < \infty$ for all λ. Then the equation $\beta_{\max}\gamma_0(\lambda) = 1$ has no solutions for $\beta_{\max} \to 0$. Therefore, $\beta_{c_{\max}} > 0$.

Consider the situation where $\beta' - \delta < \beta_i < \beta' + \delta$, $\beta' \geq G_0^{-1}$, $i = 1, \ldots, N$. Then

$$\forall \epsilon > 0 \; \exists \; \delta > 0 \colon \max_{i,j} |\Gamma_{\beta_1,\ldots,\beta_N,ij}(\lambda) - \beta' G_\lambda(x_i, x_j)| < \epsilon,$$

$$\Gamma_{\beta_1,\ldots,\beta_N}(\lambda) \to [\beta' G_\lambda(x_i, x_j)].$$

According to [16, Theorem 3] for $\beta_i = \beta' \geq G_0^{-1}$, $i = 1, \ldots, N$, the operator

$$\mathcal{H}_{\beta'} = \mathcal{A} + \sum_{i=1}^{N} \beta' \Delta_{x_i}$$

has a positive eigenvalue. The eigenvalues of the difference of matrices $\Gamma_{\beta_1,\ldots,\beta_N}(\lambda)$ and $[\beta' G_\lambda(x_i, x_j)]$ do not exceed

$$\|\Gamma_{\beta_1,\ldots,\beta_N}(\lambda) - [\beta' G_\lambda(x_i, x_j)]\| \to 0,$$

and by Weyl's theorem [9, Theorem 4.3.1], the eigenvalues of the operator $\mathcal{H}_{\beta_1,\ldots,\beta_N}$ converge to the eigenvalues of the operator $\mathcal{H}_{\beta'}$. Thus, $\mathcal{H}_{\beta_1,\ldots,\beta_N}$ has positive eigenvalues and $\beta_{c_{\max}} < G_0^{-1}$. The theorem is proved. □

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