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CONSTRUCTION OF APPROXIMATION FUNCTIONALS FOR MINIMAL SPLINES

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This paper presents formulas for constructing quadratic minimal splines, which explicitly depend on the components of a generating vector function. Formulas for various approximation functionals for minimal splines used as coefficients in local approximation methods are obtained. Examples of special cases of approximation schemes, known as quasi-interpolation, are provided. Results of numerical experiments on approximating a circular arc by minimal splines are considered. Bibliography: 37 titles.

1. Introduction

One of the most important tasks of spline approximation is determining the coefficients at the basis functions. In solving an interpolation problem with polynomial splines of odd order, the set of interpolation points and the grid used in constructing the splines usually coincide, whereas for splines of even order they are different. In general, the solution of an interpolation problem is a *plain* polynomial spline of maximal smoothness. In order to compute such splines, it is necessary to solve a system of linear algebraic equations; its order coincides with the number of interpolation points. There are two classical approaches to solving an interpolation problem with parabolic splines, which are due to Subbotin and Marsden. Subbotin suggested to select the nodes of a spline grid as the midpoints of the intervals between neighboring interpolation points, whereas Marsden considered a spline grid to be known and selected the interpolation points as the midpoints of the spline grid intervals. On a uniform grid, these two approaches yield the same construction. However, on a nonuniform grid, we have two different approximation constructions, possessing different properties, which can be used in solving specific problems (see [1, 2] for more detail).

In the last few decades, local approximation methods have actively been studied. Their main feature is that the coefficients at the basis functions are determined as values of the approximation functionals, which are, for example, linear combinations of values of the function and those of its derivatives at some points (see [3–9] for more detail). Local methods allowing one to obtain the maximal order of accuracy are called *quasi-interpolation methods*, and the functionals used in constructing them are called *quasi-interpolation functionals* or *quasi-interpolants*. These functionals have been constructed for different spline functions, see [10–17]. Approximation approaches based on quasi-interpolation are widely used in solving boundary-value problems of mathematical physics [18–21], in numerical differentiation and integration methods [22], in computer-aided geometric design systems, etc. In particular, circles and circular arcs are widely used in such systems; various methods for approximating these curves were studied in a number of publications (see, e.g., [23–27] and the references therein).

The goal of this paper is to construct the approximation functionals (quasi-interpolants) for minimal splines. Splines obtained from the so-called approximation relations using a complete chain of vectors and a generating vector function and having a minimal support are called *minimal splines* (see, e.g., [28–30]). A special approach to choosing the above-mentioned chain of vectors allows one to consider the minimal splines of maximal smoothness and to

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establish the uniqueness of the space of these splines [31]. Minimal splines provide a nice tool for approximation because they are obtained from the approximation relations.

It turns out that the approaches used in quasi-interpolation with quadratic polynomial splines are similar to Subbotin's and Marsden's methods. Grebennikov [10], similarly to Subbotin, considered averaging local methods and selected spline grid points in between known function values. Later, Sablonniere [12], similarly to Marsden, selected the points for quasi-interpolation in between the nodes of a given spline grid. These approaches can be regarded as special cases of the methods developed in the present paper, where a polynomial generating vector function is used. Moreover, the ideas of these two approaches can be combined, which leads to n-point quasi-interpolation. In this case, given a spline grid, the quasi-interpolation points are selected either at the grid nodes or at some auxiliary points, located in between the spline nodes. This approach was studied, for example, in [13]. In the present paper, we generalize this approach to the case of minimal splines using the three-point method.

This paper continues the research initiated in [32,33], where a system of biorthogonal functionals for minimal splines of small orders was constructed. An approximation approach based on these functionals was successfully used in a number of practical applications, such as approximation of transcendental curves [34], circular arcs [27], and also some functions with large gradients in a boundary layer, which frequently occur in solving singularly perturbed boundary-value problems [35, 36]. The paper presents results of numerical experiments on approximation of a circular arc by minimal splines generated by various vector functions, the coefficients at them being computed as values of the approximation functionals. Also we provide formulas for de Boor–Fix type functionals, which are used in the numerical experiments.

2. The space of minimal quadratic splines

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_+ := \{j \mid j \geq 0,\ j \in \mathbb{Z}\}$, $\mathbb{R}^1$ be the set of reals. The linear (vector) space of three-dimensional column vectors is denoted by $\mathbb{R}^3$; vectors of this space are identified with one-column matrices, and the usual matrix operations are applied to these vectors. In particular, given two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the expression $\mathbf{a}^T\mathbf{b}$ denotes the Euclidean inner product of the vectors. A square matrix with columns $\mathbf{a}_0$, $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$ (in the order indicated) is denoted by $(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$, and $\det(\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2)$ is its determinant. The vector components are denoted by square brackets and indexed by integers; for instance, $\mathbf{a} = ([\mathbf{a}]_0, [\mathbf{a}]_1, [\mathbf{a}]_2)^T$. For an arbitrary number $S \in \mathbb{Z}_+$ we denote $C^S[a,b] := \{u \mid u^{(i)} \in C[a,b],$ $i = 0, 1, \dots, S\}$, setting $C^0[a,b] := C[a,b]$. We write $\mathbf{u} \in \mathbf{C}^S[a,b]$ if the components of a vector function $\mathbf{u} \in \mathbb{R}^3$ are S times continuously differentiable on an interval $[a,b]$.

On $[a,b] \subset \mathbb{R}^1$, we consider a grid X,

$$a = x_{-2} = x_{-1} = x_0 < x_1 < \cdots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = b. \tag{1}$$

Denote $J_{i,k} := \{i, i+1, \dots, k\}$, $i,\ k \in \mathbb{Z}$, $i < k$. An ordered set $\mathbf{A} := \{\mathbf{a}_j\}_{j \in J_{-2,n-1}}$ of vectors $\mathbf{a}_j \in \mathbb{R}^3$ is called a *vector chain.* A chain $\mathbf{A}$ is said to be *complete* if $\det(\mathbf{a}_{j-2}, \mathbf{a}_{j-1}, \mathbf{a}_j) \neq 0$ for all $j \in J_{0,n-1}$.

The union of elementary grid intervals is denoted by $M := \cup_{j \in J_{0,n-1}}(x_j, x_{j+1})$. By $\mathbb{X}(M)$ we denote the linear space of real-valued functions defined on M. We set $S_j := [x_j, x_{j+3}]$, $j \in J_{-2,n-1}$.

Consider a three-component (column) vector function $\boldsymbol{\varphi} : [a,b] \to \mathbb{R}^3$ with components in the space $\mathbf{C}^2[a,b]$ and nonzero Wronskian determinant $W(t)$,

$$W(t) := \det(\boldsymbol{\varphi}(t), \boldsymbol{\varphi}'(t), \boldsymbol{\varphi}''(t)) \neq 0, \quad t \in [a,b]. \tag{2}$$

Let $\mathbf{A}$ be a complete vector chain. Assume that functions $\omega_j \in \mathbb{X}(M)$, $j \in J_{-2,n-1}$, satisfy the relations

$$\sum_{j'=k-2}^{k} \mathbf{a}_{j'}\,\omega_{j'}(t) \equiv \boldsymbol{\varphi}(t), \quad t \in (x_k, x_{k+1}), \quad k \in J_{0,n-1},$$
$$\omega_j(t) \equiv 0, \quad t \in M\backslash S_j, \quad j \in J_{-2,n-1}. \tag{3}$$

For every fixed $t \in (x_k, x_{k+1})$ and all $k \in J_{0,n-1}$, relations (3) can be regarded as a system of linear algebraic equations in the unknowns $\omega_j(t)$. Since, by assumption, the vector chain $\mathbf{A}$ is complete, system (3) has a unique solution. Using Cramer's rule, we find

$$\omega_j(t) = \frac{\det\Big(\{\mathbf{a}_{j'}\}_{j' \in J_{k-2,k}, j' \neq j} \ \| \ {}'^{j}\, \boldsymbol{\varphi}(t)\Big)}{\det\Big(\mathbf{a}_{k-2}, \mathbf{a}_{k-1}, \mathbf{a}_k\Big)}, \quad t \in (x_k, x_{k+1}), \quad j \in J_{k-2,k},$$

where the symbolic notation $\| \ {}'^{j}$ means that the determinant in the numerator is obtained from that in the denominator by replacing the column $\mathbf{a}_j$ with the column $\boldsymbol{\varphi}(t)$ (the column order being preserved). It follows that $\operatorname{supp}\omega_j \subset S_j$.

The linear span of the functions $\omega_j(t)$ is called *the space of quadratic minimal coordinate* $(\mathbf{A}, \boldsymbol{\varphi})$*-splines*. We denote it by

$$\mathbb{S}(X, \mathbf{A}, \boldsymbol{\varphi}) := \left\{ u(t) \ : \ u(t) = \sum_{j=-2}^{n-1} c_j\,\omega_j(t), \ c_j \in \mathbb{R}^1, \ t \in [a,b] \right\}.$$

Identities (3) are called *the approximation relations*. The vector function $\boldsymbol{\varphi}$ is called the *generating* vector function.

Given a vector-valued function $\boldsymbol{\varphi} \in \mathbf{C}^1[a,b]$ and its derivatives, we set

$$\boldsymbol{\varphi}_j := \boldsymbol{\varphi}(x_j), \quad \boldsymbol{\varphi}'_j := \boldsymbol{\varphi}'(x_j), \quad j \in J_{-2,n+2},$$

and consider the vectors $\mathbf{d}_j \in \mathbb{R}^3$ determined by the identity

$$\mathbf{d}_j^T \mathbf{x} \equiv \det(\boldsymbol{\varphi}_j, \boldsymbol{\varphi}'_j, \boldsymbol{\varphi}''_j, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{4}$$

Define the vector chain $\mathbf{A} = \mathbf{A}^N := \{\mathbf{a}_j^N\}_{j \in J_{-2,n-1}}$ by the formula

$$\mathbf{a}_j^N := \boldsymbol{\varphi}_{j+1} - \alpha_{j+1}\boldsymbol{\varphi}'_{j+1}, \tag{5}$$

where $\alpha_{j+1} := \dfrac{\mathbf{d}_{j+2}^T\,\boldsymbol{\varphi}_{j+1}}{\mathbf{d}_{j+2}^T\,\boldsymbol{\varphi}'_{j+1}}$.

As is known [31], if the condition

$$|W(t)| \geq c = \text{const} > 0, \quad t \in [a,b],$$

is fulfilled, then, for a sufficiently small $h_X := \sup\limits_{j \in J_{0,n-1}} (x_{j+1} - x_j)$, the chain of vectors $\{\mathbf{a}_j^N\}$, $j \in J_{0,n-1}$, is complete, and $\omega_j \in C^1[a,b]$ for all $j \in J_{-2,n-1}$. Moreover, if $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1$ and the vector function $\boldsymbol{\varphi}_1$ is such that $[\boldsymbol{\varphi}_1(t)]_0 \equiv 1$, then *the partition of unity property* is valid:

$$\sum_{j=-2}^{n-1} \omega_j(t) = 1, \quad t \in [a,b].$$

In this case, the functions $\omega_j(t)$ are called *the normalized quadratic minimal coordinate* $B_{\boldsymbol{\varphi}}$*-splines*, and the corresponding space is denoted by $\mathbb{S}(X) := \mathbb{S}(X, \mathbf{A}^N, \boldsymbol{\varphi}_1)$. The following theorem is known [32].

Theorem 1 ([32]). *The function $\omega_j \in C^1[a,b]$ and its derivative are determined by the following formulas* ($i = 0, 1$):

$$\omega_j^{(i)}(t) = \frac{\mathbf{d}_j^T \boldsymbol{\varphi}^{(i)}(t)}{\mathbf{d}_j^T \mathbf{a}_j^N}, \quad t \in [x_j, x_{j+1}), \tag{6}$$

$$\omega_j^{(i)}(t) = \frac{\mathbf{d}_j^T \boldsymbol{\varphi}^{(i)}(t)}{\mathbf{d}_j^T \mathbf{a}_j^N} - \frac{\mathbf{d}_j^T \mathbf{a}_{j+1}^N}{\mathbf{d}_j^T \mathbf{a}_j^N} \frac{\mathbf{d}_{j+1}^T \boldsymbol{\varphi}^{(i)}(t)}{\mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^N}, \quad t \in [x_{j+1}, x_{j+2}), \tag{7}$$

$$\omega_j^{(i)}(t) = \frac{\mathbf{d}_{j+3}^T \boldsymbol{\varphi}^{(i)}(t)}{\mathbf{d}_{j+3}^T \mathbf{a}_j^N}, \quad t \in [x_{j+2}, x_{j+3}). \tag{8}$$

Remark 1. For $\boldsymbol{\varphi}(t) := (1, t, t^2)^T$, the functions $\omega_j(t)$ coincide with the following known quadratic polynomial B-splines $\omega_j^B(t)$ of the third order:

$$\omega_j^B(t) = \begin{cases} \dfrac{(t-x_j)^2}{(x_{j+1}-x_j)(x_{j+2}-x_j)}, & t \in [x_j, x_{j+1}), \\ \dfrac{1}{x_{j+1}-x_j}\left(\dfrac{(t-x_j)^2}{x_{j+2}-x_j} - \dfrac{(t-x_{j+1})^2(x_{j+3}-x_j)}{(x_{j+2}-x_{j+1})(x_{j+3}-x_{j+1})}\right), & t \in [x_{j+1}, x_{j+2}), \\ \dfrac{(t-x_{j+3})^2}{(x_{j+3}-x_{j+1})\,(x_{j+3}-x_{j+2})}, & t \in [x_{j+2}, x_{j+3}). \end{cases} \tag{9}$$

Theorem 2. *Let $g \in \mathbb{S}(X)$, i.e., assume that the spline $g(t)$ on an interval $[a,b]$ can be represented as $g(t) = \sum_{j=-2}^{n-1} c_j\, \omega_j(t)$, $c_j \in \mathbb{R}^1$. Then the following assertions are valid:*

(1) *if $t \in [x_k, x_{k+1})$ for a certain k, $0 \le k \le n-1$, then $g(t) = \sum_{j=k-2}^{k} c_j\, \omega_j(t)$;*

(2) *$g(a) = c_{-2}$ and $g(b) = c_{n-1}$.*

Proof. In order to prove the first assertion, we consider the location of the supports $\operatorname{supp} \omega_j = [x_j, x_{j+3}]$ of the spline functions for different values of j.

Prove the second assertion. As is known (see [37] for more detail), if $x^* := x_j = x_{j+1} = x_{j+2} < x_{j+3}$, then $\omega_j(x^*) = 1$ and $\omega_{j'}(x^*) = 0$ for $j' \ne j$. Thus, in accordance with the first assertion of the theorem, the equalities $g(a) = g(x_0) = c_{-2}\, \omega_{-2}(x_0) + c_{-1}\, \omega_{-1}(x_0) + c_0\, \omega_0(x_0) = c_{-2}$ are valid. The equalities $g(b) = g(x_n) = c_{n-1}$ are proved similarly. ☐

Let $\boldsymbol{\varphi}(t) := (1, \rho(t), \sigma(t))^T$, where $\rho, \sigma \in C^2[a,b]$. By using the notation (see [33])

$$\Delta_j(\rho, \sigma) := \begin{vmatrix} \rho_j & \rho_j' \\ \sigma_j & \sigma_j' \end{vmatrix},$$

where $\rho_j := \rho(x_j)$ and $\sigma_j := \sigma(x_j)$, from (4) we obtain

$$\mathbf{d}_j = (\Delta_j(\rho, \sigma), -\sigma_j', \rho_j')^T. \tag{10}$$

Denote

$$S_j(\rho, \sigma, \tau) := -\frac{\begin{vmatrix} \Delta_j(\rho, \sigma) & \Delta_{j+1}(\rho, \sigma) \\ \tau_j & \tau_{j+1} \end{vmatrix}}{\begin{vmatrix} \rho_j & \rho_{j+1} \\ \sigma_j & \sigma_{j+1} \end{vmatrix}},$$

where $\tau \in C^1[a,b]$, $\tau_j := \tau(x_j)$.

Then, by using (5), for the vector $\mathbf{a}_j^N$ we obtain

$$\mathbf{a}_j^N = \big(1,\, S_{j+1}(\rho,\, \sigma,\, \rho'),\, S_{j+1}(\rho,\, \sigma,\, \sigma')\big)^T. \tag{11}$$

In accordance with (10) and (11), we have

$$\mathbf{d}_j^T \boldsymbol{\varphi}(t) = \begin{vmatrix} \rho'_j & \rho(t) - \rho_j \\ \sigma'_j & \sigma(t) - \sigma_j \end{vmatrix}, \tag{12}$$

$$\mathbf{d}_j^T \mathbf{a}_k^N = \begin{vmatrix} \rho'_j & S_{k+1}(\rho,\, \sigma,\, \rho') - \rho_j \\ \sigma'_j & S_{k+1}(\rho,\, \sigma,\, \sigma') - \sigma_j \end{vmatrix}. \tag{13}$$

Now, by using (12), (13), and (6)–(8), we obtain the following closed-form expressions for the minimal quadratic normalized $B_{\boldsymbol{\varphi}}$-splines:

if $t \in [x_j,\, x_{j+1})$, then

$$\omega_j(t) = \frac{\begin{vmatrix} \rho'_j & \rho(t) - \rho_j \\ \sigma'_j & \sigma(t) - \sigma_j \end{vmatrix}}{\begin{vmatrix} \rho'_j & S_{j+1}(\rho, \sigma, \rho') - \rho_j \\ \sigma'_j & S_{j+1}(\rho, \sigma, \sigma') - \sigma_j \end{vmatrix}}; \tag{14}$$

if $t \in [x_{j+1},\, x_{j+2})$, then the following string of equalities is valid:

$$\omega_j(t) = \frac{\mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^N\, \mathbf{d}_j^T \boldsymbol{\varphi}(t) - \mathbf{d}_j^T \mathbf{a}_{j+1}^N\, \mathbf{d}_{j+1}^T \boldsymbol{\varphi}(t)}{\mathbf{d}_j^T \mathbf{a}_j^N\, \mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^N} = \frac{\begin{vmatrix} \mathbf{d}_j^T \boldsymbol{\varphi}(t) & \mathbf{d}_j^T \mathbf{a}_{j+1}^N \\ \mathbf{d}_{j+1}^T \boldsymbol{\varphi}(t) & \mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^N \end{vmatrix}}{\mathbf{d}_j^T \mathbf{a}_j^N\, \mathbf{d}_{j+1}^T \mathbf{a}_{j+1}^N}$$

$$= \frac{\begin{vmatrix} \rho'_j & S_{j+2}(\rho, \sigma, \rho') - \rho_j & \rho(t) - \rho_j & 0 \\ \sigma'_j & S_{j+2}(\rho, \sigma, \sigma') - \sigma_j & \sigma(t) - \sigma_j & 0 \\ 0 & S_{j+2}(\rho, \sigma, \rho') - \rho_{j+1} & \rho(t) - \rho_{j+1} & \rho'_{j+1} \\ 0 & S_{j+2}(\rho, \sigma, \sigma') - \sigma_{j+1} & \sigma(t) - \sigma_{j+1} & \sigma'_{j+1} \end{vmatrix}}{\begin{vmatrix} \rho'_j & S_{j+1}(\rho, \sigma, \rho') - \rho_j \\ \sigma'_j & S_{j+1}(\rho, \sigma, \sigma') - \sigma_j \end{vmatrix} \begin{vmatrix} \rho'_{j+1} & S_{j+2}(\rho, \sigma, \rho') - \rho_{j+1} \\ \sigma'_{j+1} & S_{j+2}(\rho, \sigma, \sigma') - \sigma_{j+1} \end{vmatrix}}; \tag{15}$$

if $t \in [x_{j+2},\, x_{j+3})$, then

$$\omega_j(t) = \frac{\begin{vmatrix} \rho'_{j+3} & \rho(t) - \rho_{j+3} \\ \sigma'_{j+3} & \sigma(t) - \sigma_{j+3} \end{vmatrix}}{\begin{vmatrix} \rho'_{j+3} & S_{j+1}(\rho, \sigma, \rho') - \rho_{j+3} \\ \sigma'_{j+3} & S_{j+1}(\rho, \sigma, \sigma') - \sigma_{j+3} \end{vmatrix}}. \tag{16}$$

3. On the approximation approach

We consider the approach to approximating a given function $f : [a, b] \to \mathbb{R}^1$ by splines from the space $\mathbb{S}(X)$ described by the following algorithm:

(1) Choose a subinterval $I = [x_\mu,\, x_\nu] \subset [a, b]$ such that $I \cap (x_j, x_{j+3}) \neq \oslash$. By f^I we denote the restriction of the function f to the interval I, i.e., $f^I := f|_{[x_\mu,\, x_\nu]}$.

(2) Using a local approximation method P^I, we determine an approximation g^I of f^I in the form

$$g^I = P^I f^I = \sum_{i=\mu-2}^{\nu-1} b_i\, \omega_i, \quad b_i \in \mathbb{R}^1. \tag{17}$$

(3) Denote the approximation of the function f on $[a,b]$ by

$$Pf = \sum_{j=-2}^{n-1} c_j\,\omega_j, \quad c_j \in \mathbb{R}^1. \tag{18}$$

The value of b_j obtained at the previous step is taken as the coefficient c_j of the global approximation, i.e., we set $c_j = b_j$.

Let $\mathbb{S}(X^I)$ denote the restriction of $\mathbb{S}(X)$ to the interval I. Then the suggested approximation approach possesses the following property.

Theorem 3. *If a local approximation approach P^I reproduces all functions in the space $\mathbb{S}(X^I)$, then the approximation Pf reproduces all functions in $\mathbb{S}(X)$.*

Proof. Consider $f \in \mathbb{S}(X)$. Then, in accordance with (18), $f = \sum\limits_{j=-2}^{n-1} c_j\,\omega_j$, whence the restriction f^I can be written in the form $f^I = \sum\limits_{j=\mu-2}^{\nu-1} c_j\,\omega_j$.

If the method P^I reproduces a function f^I, then, in view of (17), we have

$$f^I = P^I f^I = \sum_{i=\mu-2}^{\nu-1} b_i\,\omega_i.$$

Thus, $\sum\limits_{i=\mu-2}^{\nu-1} c_i\,\omega_i = \sum\limits_{i=\mu-2}^{\nu-1} b_i\,\omega_i$. From the linear independence of the functions $\{\omega_i\}$ on the interval I it follows that $c_i = b_i$ for all $i \in J_{\mu-2,\nu-1}$. Since this property holds for all values of $j \in J_{-2,n-1}$, we conclude that $Pf = f$, which proves that the approximation considered reproduces the space $\mathbb{S}(X)$. □

4. Three-point approximation functionals

Consider an interval $I = [x_{j+1}, x_{j+2}] \subset [a,b]$. We need to construct a local approximation method P^I reproducing functions $f \in \{[\boldsymbol{\varphi}]_i \,|\, i = 0,1,2\}$. For example, as P^I we can use interpolation at arbitrary three distinct points x_{j+1}, $x_{j+3/2} := x_{j+1} + \theta(x_{j+2} - x_{j+1})$, where $\theta \in (0,1)$, and x_{j+2} of the interval I.

Represent (17) in the form

$$P^I f(t) := \alpha\,\omega_{j-1}(t) + \beta\,\omega_j(t) + \gamma\,\omega_{j+1}(t), \quad t \in I, \tag{19}$$

where $\alpha, \beta, \gamma \in \mathbb{R}^1$.

Then, in order to determine the coefficients in (19), we must solve the following three simultaneous linear equations in three unknowns, which express the coincidence of $P^I f$ and f at the points x_{j+1}, $x_{j+3/2}$, and x_{j+2}:

$$\begin{cases} \alpha\,\omega_{j-1}(x_{j+1}) + \beta\,\omega_j(x_{j+1}) + \gamma\,\omega_{j+1}(x_{j+1}) = f(x_{j+1}), \\ \alpha\,\omega_{j-1}(x_{j+3/2}) + \beta\,\omega_j(x_{j+3/2}) + \gamma\,\omega_{j+1}(x_{j+3/2}) = f(x_{j+3/2}), \\ \alpha\,\omega_{j-1}(x_{j+2}) + \beta\,\omega_j(x_{j+2}) + \gamma\,\omega_{j+1}(x_{j+2}) = f(x_{j+2}). \end{cases} \tag{20}$$

In accordance with the approach (18) described above, the value of the approximation functional c_j can be obtained as the value of β in representation (19).

From the location of the support of the function $\omega_j(t)$ it follows that

$$\omega_{j-1}(x_{j+2}) = \omega_{j+1}(x_{j+1}) = 0.$$

Then, by expressing the coefficients α and γ from the first and third equations in (20), respectively, and substituting them into the second one, we obtain

$$\beta = \frac{f(x_{j+3/2}) - f(x_{j+1})\,\frac{\omega_{j-1}(x_{j+3/2})}{\omega_{j-1}(x_{j+1})} - f(x_{j+2})\,\frac{\omega_{j+1}(x_{j+3/2})}{\omega_{j+1}(x_{j+2})}}{\omega_j(x_{j+3/2}) - \omega_j(x_{j+1})\,\frac{\omega_{j-1}(x_{j+3/2})}{\omega_{j-1}(x_{j+1})} - \omega_j(x_{j+2})\,\frac{\omega_{j+1}(x_{j+3/2})}{\omega_{j+1}(x_{j+2})}}. \tag{21}$$

Note that on every grid interval (14)–(16), the function $\omega_j(t)$ is represented as a fraction whose denominator is independent of t. Therefore, the denominators of the representations of the function $\omega_{j-1}(t)$ at the points x_{j+1} and $x_{j+3/2}$, as well as those of the function $\omega_{j+1}(t)$ at the points $x_{j+3/2}$ and x_{j+2}, coincide. The corresponding numerators are denoted (depending on an interval) by $\omega_j^l(t)$, $\omega_j^c(t)$, and $\omega_j^r(t)$ for formulas (14), (15), and (16), respectively. Replacing the ratios on the right-hand side of (21) with the ratios of their numerators and multiplying the numerator and denominator of the right-hand side of (21) by $\omega_{j-1}^r(x_{j+1})\,\omega_{j+1}^l(x_{j+2})$, we obtain

$$\beta = \Delta^{-1}\left(\lambda^{\langle 1\rangle} f(x_{j+1}) - \lambda^{\langle 3/2\rangle} f(x_{j+3/2}) + \lambda^{\langle 2\rangle} f(x_{j+2})\right).$$

Here, the following notation is used:

$$\begin{aligned} \lambda^{\langle 1\rangle} &:= \omega_{j-1}^r(x_{j+3/2})\,\omega_{j+1}^l(x_{j+2}),\\ \lambda^{\langle 3/2\rangle} &:= \omega_{j-1}^r(x_{j+1})\,\omega_{j+1}^l(x_{j+2}),\\ \lambda^{\langle 2\rangle} &:= \omega_{j-1}^r(x_{j+1})\,\omega_{j+1}^l(x_{j+3/2}),\\ \Delta &:= \lambda^{\langle 1\rangle}\,\omega_j(x_{j+1}) - \lambda^{\langle 3/2\rangle}\,\omega_j(x_{j+3/2}) + \lambda^{\langle 2\rangle}\,\omega_j(x_{j+2}). \end{aligned}$$

Now it remains to find closed-form expressions for $\lambda^{\langle 1\rangle}$, $\lambda^{\langle 3/2\rangle}$, $\lambda^{\langle 2\rangle}$, and Δ. By using (14)–(16), we obtain

$$\lambda^{\langle 1\rangle} = -\begin{vmatrix} \rho'_{j+1} & \rho_{j+2}-\rho_{j+1} \\ \sigma'_{j+1} & \sigma_{j+2}-\sigma_{j+1} \end{vmatrix} \times \begin{vmatrix} \rho'_{j+2} & \rho_{j+2}-\rho_{j+3/2} \\ \sigma'_{j+2} & \sigma_{j+2}-\sigma_{j+3/2} \end{vmatrix},$$

$$\lambda^{\langle 3/2\rangle} = -\begin{vmatrix} \rho'_{j+1} & \rho_{j+2}-\rho_{j+1} \\ \sigma'_{j+1} & \sigma_{j+2}-\sigma_{j+1} \end{vmatrix} \times \begin{vmatrix} \rho'_{j+2} & \rho_{j+2}-\rho_{j+1} \\ \sigma'_{j+2} & \sigma_{j+2}-\sigma_{j+1} \end{vmatrix},$$

$$\lambda^{\langle 2\rangle} = -\begin{vmatrix} \rho'_{j+1} & \rho_{j+3/2}-\rho_{j+1} \\ \sigma'_{j+1} & \sigma_{j+3/2}-\sigma_{j+1} \end{vmatrix} \times \begin{vmatrix} \rho'_{j+2} & \rho_{j+2}-\rho_{j+1} \\ \sigma'_{j+2} & \sigma_{j+2}-\sigma_{j+1} \end{vmatrix},$$

$$\Delta = \lambda^{\langle 1\rangle}\,\frac{\begin{vmatrix} \rho'_j & \rho_{j+1}-\rho_j \\ \sigma'_j & \sigma_{j+1}-\sigma_j \end{vmatrix}}{\begin{vmatrix} \rho'_j & S_{j+1}(\rho,\sigma,\rho')-\rho_j \\ \sigma'_j & S_{j+1}(\rho,\sigma,\sigma')-\sigma_j \end{vmatrix}} + \lambda^{\langle 2\rangle}\,\frac{\begin{vmatrix} \rho'_{j+3} & \rho_{j+2}-\rho_{j+3} \\ \sigma'_{j+3} & \sigma_{j+2}-\sigma_{j+3} \end{vmatrix}}{\begin{vmatrix} \rho'_{j+3} & S_{j+1}(\rho,\sigma,\rho')-\rho_{j+3} \\ \sigma'_{j+3} & S_{j+1}(\rho,\sigma,\sigma')-\sigma_{j+3} \end{vmatrix}}$$

$$-\lambda^{\langle 3/2\rangle}\,\frac{\begin{vmatrix} \rho'_j & S_{j+2}(\rho,\sigma,\rho')-\rho_j & \rho_{j+3/2}-\rho_j & 0 \\ \sigma'_j & S_{j+2}(\rho,\sigma,\sigma')-\sigma_j & \sigma_{j+3/2}-\sigma_j & 0 \\ 0 & S_{j+2}(\rho,\sigma,\rho')-\rho_{j+1} & \rho_{j+3/2}-\rho_{j+1} & \rho'_{j+1} \\ 0 & S_{j+2}(\rho,\sigma,\sigma')-\sigma_{j+1} & \sigma_{j+3/2}-\sigma_{j+1} & \sigma'_{j+1} \end{vmatrix}}{\begin{vmatrix} \rho'_j & S_{j+1}(\rho,\sigma,\rho')-\rho_j \\ \sigma'_j & S_{j+1}(\rho,\sigma,\sigma')-\sigma_j \end{vmatrix}\,\begin{vmatrix} \rho'_{j+1} & S_{j+2}(\rho,\sigma,\rho')-\rho_{j+1} \\ \sigma'_{j+1} & S_{j+2}(\rho,\sigma,\sigma')-\sigma_{j+1} \end{vmatrix}}.$$

Note that our conclusions are correct for the grid points $x_{j+1} < x_{j+2}$, $j \in J_{-1,n-2}$. For the multiple points of the grid (1) the coefficients c_j for $j = -2$ and $j = n-1$ in expansion (18) are determined, in accordance with Theorem 2, from the equalities $g(x_0) = c_{-2}$ and $g(x_n) = c_{n-1}$ for an arbitrary spline $g \in \mathbb{S}(X)$. Therefore, we set $c_{-2} = f(x_0)$ and $c_{n-1} = f(x_n)$.

Thus, the approximation (18) of the function f can be written as

$$Pf = \sum_{j=-2}^{n-1} \lambda_j(f)\,\omega_j, \tag{22}$$

where the approximation functional $\lambda_j(f)$ is defined as follows:

$$\lambda_j(f) := \begin{cases} f(x_0), & j = -2, \\ \Delta^{-1}\left(\lambda^{\langle 1\rangle} f(x_{j+1}) - \lambda^{\langle 3/2\rangle} f(x_{j+3/2}) + \lambda^{\langle 2\rangle} f(x_{j+2})\right), & j = -1, \dots, n-2, \\ f(x_n), & j = n-1. \end{cases} \tag{23}$$

Remark 2. Our approach to constructing an approximation allows one to select the interval I in a different way, say, one may set $I = [x_j, x_{j+1}]$. Then the local approximation (17) is constructed in the form

$$P^I f(t) := \alpha'\,\omega_{j-2}(t) + \beta'\,\omega_{j-1}(t) + \gamma'\,\omega_j(t), \qquad t \in I,$$

where $\alpha', \beta', \gamma' \in \mathbb{R}^1$, and the value of the functional is determined by the coefficient γ' from the equations that express the coincidence of $P^I f$ and f at the points x_j, $x_{j+1/2}$, and x_{j+1}. However, the most concise representation of the functional is obtained if the interval $I = [x_{j+1}, x_{j+2}]$ is considered.

Remark 3. For $\boldsymbol{\varphi}(t) = (1,\, t,\, t^2)^T$ and $\theta = 1/2$, the functional (23) has the form

$$\lambda_j(f) = -\frac{1}{2}\left(f(x_{j+1}) - 4f(x_{j+3/2}) + f(x_{j+2})\right), \quad j = -1, \dots, n-2,$$

and coincides with the well-known quasi-interpolation functional for the quadratic B-splines (see, e.g., [13]).

5. Averaging approximation functionals

In this section, for convenience, all the necessary objects defined on the grid X are supplied with the superscript X, i.e., we write ω_j^X, $S_j^X(\rho, \sigma, \tau)$, etc. Then, in accordance with representation (11), in componentwise form the approximation relations (3) are as follows:

$$\begin{cases} \sum\limits_{j=-2}^{n-1} \omega_j^X(t) = 1, \\ \sum\limits_{j=-2}^{n-1} S_{j+1}^X(\rho,\, \sigma,\, \rho')\,\omega_j^X(t) = \rho(t), \\ \sum\limits_{j=-2}^{n-1} S_{j+1}^X(\rho,\, \sigma,\, \sigma')\,\omega_j^X(t) = \sigma(t). \end{cases} \tag{24}$$

Consider another grid Y with the nodes

$$y_j := \begin{cases} x_0, & j = -2, \\ x_{j+1} + \theta(x_{j+2} - x_{j+1}), \quad \theta \in [0, 1], & j = -1, \dots, n-2, \\ x_n, & j = n-1. \end{cases} \tag{25}$$

We construct the approximation Pf of a function f in the form

$$Pf = \sum_{j=-2}^{n-1} \mu_j(f)\,\omega_j^X,$$

where the approximation functional $\mu_j(f)$ is defined as follows:

$$\mu_j(f) := \begin{cases} f(y_{-2}), & j = -2, \\ a_j f(y_{j-1}) + b_j f(y_j) + c_j f(y_{j+1}), & j = -1, \dots, n-2, \\ f(y_{n-1}), & j = n-1. \end{cases} \tag{26}$$

As above, the approximation Pf must reproduce all functions $f \in \{[\boldsymbol{\varphi}]_i \mid i = 0, 1, 2\}$. This time, this property is ensured by an appropriate choice of the coefficients a_j, b_j, c_j of the functional (26). From the reproduction conditions we obtain the following system of equations:

$$\begin{cases} \sum\limits_{j=-2}^{n-1} (a_j + b_j + c_j)\, \omega_j^X(t) = 1, \\ \sum\limits_{j=-2}^{n-1} (a_j \rho(y_{j-1}) + b_j \rho(y_j) + c_j \rho(y_{j+1}))\, \omega_j^X(t) = \rho(t), \\ \sum\limits_{j=-2}^{n-1} (a_j \sigma(y_{j-1}) + b_j \sigma(y_j) + c_j \sigma(y_{j+1}))\, \omega_j^X(t) = \sigma(t). \end{cases} \tag{27}$$

Upon equating the left-hand sides of the respective equations in (24) and (27), we obtain the following system of equations for determining the coefficients a_j, b_j, c_j:

$$\begin{cases} a_j + b_j + c_j = 1, \\ a_j \rho(y_{j-1}) + b_j \rho(y_j) + c_j \rho(y_{j+1}) = S_{j+1}^X(\rho,\, \sigma,\, \rho'), \\ a_j \sigma(y_{j-1}) + b_j \sigma(y_j) + c_j \sigma(y_{j+1}) = S_{j+1}^X(\rho,\, \sigma,\, \sigma'). \end{cases}$$

Solving this system, we find

$$\begin{aligned} a_j &= 1 - b_j - c_j. \\ b_j &= \frac{(\rho(y_{j+1}) - \rho(y_{j-1}))(S_{j+1}^X(\rho,\, \sigma,\, \sigma') - \sigma(y_{j-1}))}{(\rho(y_{j+1}) - \rho(y_{j-1}))(\sigma(y_j) - \sigma(y_{j-1}))} \\ &\qquad - \frac{(S_{j+1}^X(\rho,\, \sigma,\, \rho')\rho(y_{j-1}))(\sigma(y_{j+1}) - \sigma(y_{j-1}))}{(\rho(y_j)\rho(y_{j-1}))(\sigma(y_{j+1}) - \sigma(y_{j-1}))}, \\ c_j &= \frac{(S_{j+1}^X(\rho,\, \sigma,\, \rho') - \rho(y_{j-1}))(\sigma(y_j) - \sigma(y_{j-1}))}{(\rho(y_{j+1}) - \rho(y_{j-1}))(\sigma(y_j) - \sigma(y_{j-1}))} \\ &\qquad - \frac{(\rho(y_j) - \rho(y_{j-1}))(S_{j+1}^X(\rho,\, \sigma,\, \sigma') - \sigma(y_{j-1}))}{(\rho(y_j) - \rho(y_{j-1}))(\sigma(y_{j+1}) - \sigma(y_{j-1}))}. \end{aligned}$$

Remark 4. In order to compute the value of $\mu_{-2}(f)$, we must determine the additional node y_{-3}. However, this can be avoided because, by Theorem 2, every spline in the space $\mathbb{S}(X)$ reproduces the first and last coefficients of the expansion at the ends of the interval $[a, b]$. Thus, we may set $\mu_{-2}(f) := f(y_{-2})$. Similarly, $\mu_{n-1}(f) := f(y_{n-1})$.

Remark 5. For $\boldsymbol{\varphi}(t) := (1, t, t^2)^T$ and $\theta = 1/2$, the functional (26) on the uniform grid has the form

$$\mu_j(f) = -\frac{1}{8}\,(f(y_{j-1}) - 10 f(y_j) + f(y_{j+1}))$$

and coincides with the well-known quasi-interpolation functional for the quadratic B-splines (see, e.g., [12]).

Consider a grid Z with the nodes

$$z_j := \begin{cases} x_0, & j = -1, 0, 1, \\ x_{j-2} + \theta(x_{j-1} - x_{j-2}), & \theta \in [0,1], \quad j = 2, \ldots, n+1, \\ x_n, & j = n+2, n+3, n+4. \end{cases} \tag{28}$$

We construct the spline functions on the grid (28) using (14)–(16). All objects considered on the grid Z are supplied with the superscript Z, e.g., ω_j^Z, $S_j^Z(\rho, \sigma, \tau)$, etc.

Now we construct the approximation Pf of a function f in the form

$$Pf = \sum_{j=-1}^{n+1} \nu_j(f)\, \omega_j^Z,$$

where the functional $\nu_j(f)$ is defined as follows:

$$\nu_j(f) := \begin{cases} f(x_0), & j = -1, \\ a'_j f(x_{j-1}) + b'_j f(x_j) + c'_j f(x_{j+1}), & j = 0, \ldots, n, \\ f(x_n), & j = n+1. \end{cases} \tag{29}$$

Since the approximation Pf must reproduce all functions $f \in \{[\varphi]_i \,|\, i = 0, 1, 2\}$, we obtain the system of linear equations

$$\begin{cases} \sum\limits_{j=-1}^{n+1} (a'_j + b'_j + c'_j)\, \omega_j^Z(t) = 1, \\ \sum\limits_{j=-1}^{n+1} (a'_j \rho(x_{j-1}) + b'_j \rho(x_j) + c'_j \rho(x_{j+1}))\, \omega_j^Z(t) = \rho(t), \\ \sum\limits_{j=-1}^{n+1} (a'_j \sigma(x_{j-1}) + b'_j \sigma(x_j) + c'_j \sigma(x_{j+1}))\, \omega_j^Z(t) = \sigma(t). \end{cases} \tag{30}$$

Then, on the grid Z, we can write the following equations, similar to (24):

$$\begin{cases} \sum\limits_{j=-1}^{n+1} \omega_j^Z(t) = 1, \\ \sum\limits_{j=-1}^{n+1} S_{j+1}^Z(\rho,\, \sigma,\, \rho')\, \omega_j^Z(t) = \rho(t), \\ \sum\limits_{j=-1}^{n+1} S_{j+1}^Z(\rho,\, \sigma,\, \sigma')\, \omega_j^Z(t) = \sigma(t). \end{cases} \tag{31}$$

On equating the coefficients at the spline functions in the left-hand sides of the respective equations in (31) and (30), we again obtain a system of linear equations for determining the coefficients a'_j, b'_j, c'_j. Solving this system, we find the following closed–form expressions for the coefficients of the desired approximation functional:

$$a'_j = 1 - b'_j - c'_j,$$

$$b'_j = \frac{(\rho(x_{j+1}) - \rho(x_{j-1}))(S^Z_{j+1}(\rho,\, \sigma,\, \sigma') - \sigma(x_{j-1}))}{(\rho(x_{j+1}) - \rho(x_{j-1}))(\sigma(x_j) - \sigma(x_{j-1}))} - \frac{(S^Z_{j+1}(\rho,\, \sigma,\, \rho') - \rho(x_{j-1}))(\sigma(x_{j+1}) - \sigma(x_{j-1}))}{(\rho(x_j) - \rho(x_{j-1}))(\sigma(x_{j+1}) - \sigma(x_{j-1}))},$$

$$c'_j = \frac{(S^Z_{j+1}(\rho,\, \sigma,\, \rho') - \rho(x_{j-1}))(\sigma(x_j) - \sigma(x_{j-1}))}{(\rho(x_{j+1}) - \rho(x_{j-1}))(\sigma(x_j) - \sigma(x_{j-1}))} - \frac{(\rho(x_j) - \rho(x_{j-1}))(S^Z_{j+1}(\rho,\, \sigma,\, \sigma') - \sigma(x_{j-1}))}{(\rho(x_j) - \rho(x_{j-1}))(\sigma(x_{j+1}) - \sigma(x_{j-1}))}.$$

Remark 6. In order to compute $\nu_{-1}(f)$, we can use the following string of equalities:

$$\nu_{-1}(f) = a'_{-1} f(x_{-2}) + b'_{-1} f(x_{-1}) + c'_{-1} f(x_0) = (a'_{-1} + b'_{-1} + c'_{-1}) f(x_0) = f(x_0).$$

Similarly, $\nu_{n+1}(f) = f(x_n)$.

Remark 7. For $\boldsymbol{\varphi}(t) := (1, t, t^2)^T$ and $\theta = 1/2$, the functional (29) on the uniform grid has the form

$$\nu_j(f) = -\frac{1}{8}\,(f(x_{j-1}) - 10 f(x_j) + f(x_{j+1}))$$

and coincides with the well-known quasi-interpolation averaging functional for the quadratic B-splines (see [10]).

6. Biorthogonal functionals of de Boor–Fix type

Consider yet another type of linear functionals $\xi_j^{\langle r\rangle}$, $j = -2, \dots, n-1$, $r = 0, 1, 2$, which are defined by the following formulas:

$$\begin{aligned}
\xi_j^{\langle 0\rangle}(f) := f(x_j) &+ \Big((\rho_{j+1}\sigma'_{j+1} - \rho'_{j+1}\sigma_{j+1})(\rho''_j\sigma'_{j+2} - \rho'_{j+2}\sigma''_j) \\
&+ (\rho'_{j+1}\sigma'_{j+2} - \rho'_{j+2}\sigma'_{j+1})(\rho''_j\sigma_j - \rho_j\sigma''_j) + (\rho_{j+2}\sigma'_{j+2} - \rho'_{j+2}\sigma_{j+2})(\rho'_{j+1}\sigma''_j - \rho''_j\sigma'_{j+1})\Big) \\
&\times \frac{f'(x_j)}{(\rho'_{j+1}\sigma'_{j+2} - \rho'_{j+2}\sigma'_{j+1})(\rho'_j\sigma''_j - \rho''_j\sigma'_j)} + \Big((\rho_{j+1}\sigma'_{j+1} - \rho'_{j+1}\sigma_{j+1})(\rho'_{j+2}\sigma'_j - \rho'_j\sigma'_{j+2}) \\
&+ (\rho'_{j+1}\sigma'_{j+2} - \rho'_{j+2}\sigma'_{j+1})(\rho_j\sigma'_j - \rho'_j\sigma_j) + (\rho_{j+2}\sigma'_{j+2} - \rho'_{j+2}\sigma_{j+2})(\rho'_j\sigma'_{j+1} - \rho'_{j+1}\sigma'_j)\Big) \\
&\times \frac{f''(x_j)}{(\rho'_{j+1}\sigma'_{j+2} - \rho'_{j+2}\sigma'_{j+1})(\rho'_j\sigma''_j - \rho''_j\sigma'_j)}, \quad f \in C^2[a,b],
\end{aligned}$$

$$\xi_j^{\langle 1\rangle}(f) := f(x_{j+1}) + \frac{(\sigma_{j+2} - \sigma_{j+1})\rho'_{j+2} - (\rho_{j+2} - \rho_{j+1})\sigma'_{j+2}}{\rho'_{j+2}\sigma'_{j+1} - \rho'_{j+1}\sigma'_{j+2}}\, f'(x_{j+1}), \quad f \in C^1[a,b] \tag{32}$$

$$\xi_j^{\langle 2\rangle}(f) := f(x_{j+2}) + \frac{(\sigma_{j+2} - \sigma_{j+1})\rho'_{j+1} - (\rho_{j+2} - \rho_{j+1})\sigma'_{j+1}}{\rho'_{j+2}\sigma'_{j+1} - \rho'_{j+1}\sigma'_{j+2}}\, f'(x_{j+2}), \quad f \in C^1[a,b]. \tag{33}$$

As is known [33], the approximation

$$Pf = \sum_{j=-2}^{n-1} \xi_j^{\langle r\rangle}(f)\,\omega_j$$

of a function f reproduces all functions $f \in \{[\boldsymbol{\varphi}]_i \mid i = 0,1,2\}$, i.e., for these functions we have $Pf \equiv f$. Moreover, for every fixed $r = 0,1,2$, the functionals $\xi_j^{\langle r\rangle}$ are biorthogonal to the functions $\omega_{j'}$, i.e., $\xi_j^{\langle r\rangle}(\omega_{j'}) = \delta_{j,j'}$, where $\delta_{j,j'}$ is the Kronecker symbol.

These functionals are known to reproduce all splines $g \in \mathbb{S}(X)$, and (in accordance with the biorthogonality property) they yield the value of the coefficient c_j at the function ω_j in the related expansion (18):

$$\xi_j^{\langle r\rangle}(g) = \xi_j^{\langle r\rangle}\left(\sum_{j'=-2}^{n-1} c_{j'}\,\omega_{j'}\right) = c_j.$$

Remark 8. For the polynomial generating vector function $\boldsymbol{\varphi}(t) = (1, t, t^2)^T$, the functionals $\xi_j^{\langle r\rangle}$ coincide with the well-known functionals of de Boor and Fix [32]:

$$\xi_j^{\langle 0\rangle}(f) = f(x_j) + \left(\frac{x_{j+1}+x_{j+2}}{2} - x_j\right) f'(x_j) + \frac{1}{2}(x_{j+1}-x_j)(x_{j+2}-x_j)f''(x_j),$$

$$\xi_j^{\langle 1\rangle}(f) = f(x_{j+1}) + \frac{1}{2}(x_{j+2}-x_{j+1})f'(x_{j+1}),$$

$$\xi_j^{\langle 2\rangle}(f) = f(x_{j+2}) - \frac{1}{2}(x_{j+2}-x_{j+1})f'(x_{j+2}).$$

7. Numerical experiments

In this section, we study the error of approximating a circular arc by the B-splines and minimal splines with various generating vector functions and approximation functionals.

We use the circular arc $u(t) = \sqrt{1-t^2}$ as a test function, which is approximated on a uniform grid on the interval $[-0.5, 0.5]$, whereas the error is estimated using another uniform grid that is ten times finer than the original one. The error is estimated by the absolute value of the largest deviation of the constructed approximation u^h from the value of the function u at the nodes of the finer grid, i.e.,

$$E = \max_{t\in[-0.5,\,0.5]} |u^h(t) - u(t)|.$$

In Table 1, we present the results of numerical approximation of the considered circular arc by the B-splines. The approximations were constructed for different numbers of grid nodes, depending on the choice of the approximation functionals. The three-point functional (23) is denoted by λ_j; the averaging functionals (26) and (29) are denoted by μ_j and ν_j, respectively (in both cases, the midpoints of the grid intervals were used as the nodes of the auxiliary grid). The de Boor–Fix type functional (32) is denoted by $\xi_j^{\langle 1\rangle}$.

Table 1. The approximation error for the B-splines as a function of the node number n.

Functional	$n = 10$	$n = 20$	$n = 30$
λ_j	2.8×10^{-5}	3.4×10^{-6}	1.0×10^{-6}
μ_j	3.6×10^{-5}	5.3×10^{-6}	1.7×10^{-6}
$\xi_j^{\langle 1\rangle}$	1.2×10^{-4}	1.6×10^{-5}	5.0×10^{-6}

Now consider approximation by the minimal splines. Tables 2, 3, and 4 provide the approximation errors for different generating vector functions and for the averaging functionals (26), (29) and de Boor–Fix type functional (32), respectively. Here, we select the nodes of the auxiliary grid as the midpoints of the intervals of the original grid. The results of our numerical experiments show that the suggested approximation approach based on minimal splines allows one to construct more accurate approximations of circular arcs using the approximation functionals considered.

Table 2. The approximation error for the averaging functional (26) as a function of the node number n.

$\varphi(t)$	$n = 10$	$n = 20$	$n = 30$
$(1,\ t,\ t^2)^T$	3.6×10^{-5}	5.3×10^{-6}	1.7×10^{-6}
$(1, \sinh t, \cosh t)^T$	2.8×10^{-5}	4.2×10^{-6}	1.3×10^{-6}
$(1, \sqrt{1-t}, \sqrt{1+t})^T$	7.5×10^{-6}	1.1×10^{-6}	3.3×10^{-7}

Table 3. The approximation error for the averaging functional (29) as a function of the node number n.

$\varphi(t)$	$n = 10$	$n = 20$	$n = 30$
$(1,\ t,\ t^2)^T$	2.7×10^{-5}	3.4×10^{-6}	1.1×10^{-6}
$(1, \sinh t, \cosh t)^T$	9.4×10^{-6}	1.5×10^{-6}	1.3×10^{-7}
$(1, \sqrt{1-t}, \sqrt{1+t})^T$	5.1×10^{-6}	6.8×10^{-7}	2.2×10^{-7}

Table 4. The approximation error for the de Boor–Fix type functional (32) as a function of the node number n.

$\varphi(t)$	$n = 10$	$n = 20$	$n = 30$
$(1,\ t,\ t^2)^T$	1.2×10^{-4}	1.6×10^{-5}	5.0×10^{-6}
$(1, \sinh t, \cosh, t)^T$	9.2×10^{-5}	1.3×10^{-5}	4.0×10^{-6}
$(1, \sqrt{1-t}, \sqrt{1+t})^T$	2.3×10^{-5}	3.1×10^{-6}	9.6×10^{-7}

Remark 9. The quadratic minimal splines generated by the vector function

$$\varphi(t) = (1, \sinh t, \cosh t)^T$$

are called the *hyperbolic splines*. For their properties and application to approximation of catenary and other transcendental curves, see [34].

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