DOI 10.1007/s10958-021-05639-1
Journal of Mathematical Sciences, Vol. 259, No. 4, December, 2021

DEFINABILITY OF COMPLETELY DECOMPOSABLE TORSION-FREE ABELIAN GROUPS BY SEMIGROUPS OF ENDOMORPHISMS AND GROUPS OF HOMOMORPHISMS

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UDC 512.541

Abstract. Let C be an Abelian group. A class X of Abelian groups is called a ${}_CE^\bullet H$-class if for any groups $A, B \in X$, it follows from the existence of isomorphisms $E^\bullet(A) \cong E^\bullet(B)$ and $\mathrm{Hom}(C, A) \cong \mathrm{Hom}(C, B)$ that there is an isomorphism $A \cong B$. In this paper, conditions are studied under which the class $\Im^{\mathrm{ad}}_{\mathrm{cd}}$ of completely decomposable almost divisible Abelian groups and class $\Im^*_{\mathrm{cd}}$ of completely decomposable torsion-free Abelian groups A where $\Omega(A)$ contains only incomparable types are ${}_CE^\bullet H$-classes, where C is a completely decomposable torsion-free Abelian group.

The well-known result of R. Baer [2] and I. Kaplansky [4] about determinability of periodic Abelian groups by their endomorphisms ring in the class of periodic groups initiated multiple researches in this direction. A class X of Abelian groups is called an E-class if for any groups $A, B \in X$ it follows from the existence of an isomorphism $E(A) \cong E(B)$ that there is an isomorphism $A \cong B$. There is the same question both for rings of endomorphisms $E(A)$ of the group A and for its multiplicative semigroup $E^\bullet(A)$, which is called an endomorphism semigroup of the group A. P. Puusemp [6] and A. M. Sebeldin [11] researched the problem of determinability of Abelian groups by their multiplicative semigroups. In connection with the above, it is logical to search the problems of determinability of Abelian groups by their endomorphism semigroups together with the supplementary condition of isomorphism of homomorphism groups.

Let C be an Abelian group. A class X of Abelian groups is called an ${}_CE^\bullet H$-class if for any groups $A, B \in X$, it follows from the existence of isomorphisms $E^\bullet(A) \cong E^\bullet(B)$ and $\mathrm{Hom}(C, A) \cong \mathrm{Hom}(C, B)$ that there is an isomorphism $A \cong B$. In this paper, we describe necessary and sufficient conditions on a completely decomposable torsion-free Abelian group C for a given specified class of torsion-free Abelian groups to be a ${}_CE^\bullet H$-class.

Introduce the following notation: Ω is the set of distinct types of torsion-free Abelian groups of rank 1; $\tau(A)$ is the type of a torsion-free Abelian group A of rank 1; $\Omega(A)$ is the set of distinct types of the direct summands of rank 1 of a torsion-free Abelian group; Ω_0 is the set of all types in Ω , whose characteristics do not contain the symbol ∞; $\Omega_0(A)$ is the set of all types in $\Omega(A)$, whose characteristics do not contain the symbol ∞; $\aleph_0$ is the least infinite cardinal; $|M|$ is the cardinality of the set M.

The set Ω can be split like that:

$$\Omega = \bar{\Omega} \cup \Omega^*, \quad \bar{\Omega} \cap \Omega^* = \varnothing,$$

where Ω^* is the set of all types of almost divisible torsion-free groups of rank 1. We similarly have that any completely decomposable torsion-free Abelian group A can be represented in the form $A = \bar{A} \oplus A^*$, where $\bar{A}$ does not contain almost divisible groups of rank 1.

Theorem 1. *Let C be a completely decomposable torsion-free Abelian group. The class $\Im^{\mathrm{ad}}_{\mathrm{cd}}$ of completely decomposable almost divisible torsion-free Abelian groups is a ${}_CE^\bullet H$-class if and only if the group C satisfies one of the following conditions*:

(1) $\Omega_0(C) \neq \varnothing$;

Translated from Fundamentalnaya i Prikladnaya Matematika, Vol. 22, No. 5, pp. 145–152, 2019.

(2) $\Omega_0(C) = \varnothing$ *and for any almost divisible type* τ_0 *there is a type* $\tau' \in \Omega(C)$ *such that* $\tau' \le \tau_0$.

Proof. Sufficiency. Let

$$A = \bigoplus_{\tau \in \Omega(A)} \bigoplus_{i \in I(\tau)} A_i = A_1 \oplus A_2,$$
$$B = \bigoplus_{\tau \in \Omega(B)} \bigoplus_{j \in J(\tau)} B_j = B_1 \oplus B_2,$$

where

$$A_1 = \bigoplus_{\tau \in \Omega_1(A)} A^{(\tau)}, \quad A_2 = \bigoplus_{\tau \in \Omega_2(A)} A^{(\tau)}, \quad A^{(\tau)} = \bigoplus_{i \in I(\tau)} A_i,$$
$$B_1 = \bigoplus_{\tau \in \Omega_1(B)} B^{(\tau)}, \quad B_2 = \bigoplus_{\tau \in \Omega_2(B)} B^{(\tau)}, \quad B^{(\tau)} = \bigoplus_{j \in J(\tau)} B_j,$$

and

$$\Omega_1(A) = \{\tau \in \Omega(A) \colon |I(\tau)| = 1\}, \quad \Omega_2(A) = \{\tau \in \Omega(A \colon |I(\tau)| > 1\},$$
$$\Omega_1(B) = \{\tau \in \Omega(B) \colon |J(\tau)| = 1\}, \quad \Omega_2(B) = \{\tau \in \Omega(B) \colon |J(\tau)| > 1\}.$$

As far as $\mathrm{Hom}(C, A) \cong \mathrm{Hom}(C, B)$, since [3, Theorem 43.1], [9], we obtain

$$\prod_{k \in K} \bigoplus_{i \in I} \mathrm{Hom}(C_k, A_i) \cong \prod_{k \in K} \bigoplus_{j \in J} \mathrm{Hom}(C_k, B_j).$$

Show that $\Omega(A) = \Omega(B)$. Let us consider a random $\tau(A_i) \in \Omega(A)$.

(1) Let the group C satisfy the first condition of the theorem, i.e., there is a group C_k in the decomposition of the group C such that $\tau(C_k) \in \Omega_0(C)$. Then, since A_i is almost divisible, we obtain

$$\tau\bigl(\mathrm{Hom}(C_t, A_i)\bigr) = \tau(A_i) - \tau(C_t) = \tau(A_i)$$

and then $\Omega(A) \subset \Omega\bigl(\mathrm{Hom}(C, A)\bigr)$.

(2) Let the group C satisfy the second condition of the theorem, i.e., there is a group C_t in the decomposition of the group C such that $\tau(C_t) \le \tau(A_i)$. Then, since A_i is almost divisible, we obtain

$$\tau\bigl(\mathrm{Hom}(C_k, A_i)\bigr) = \tau(A_i) - \tau(C_k) = \tau(A_i)$$

and then $\Omega(A) \subset \Omega\bigl(\mathrm{Hom}(C, A)\bigr)$. On the other hand, due to the fact that A_i is almost divisible, we have $\mathrm{Hom}(C_k, A_i) \cong A_i$ or $\mathrm{Hom}(C_k, A_i) = 0$. This means that $\Omega\bigl(\mathrm{Hom}(C, A)\bigr) \subset \Omega(A)$. Therefore, $\Omega\bigl(\mathrm{Hom}(C, A)\bigr) = \Omega(A)$. Similarly, $\Omega\bigl(\mathrm{Hom}(C, B)\bigr) = \Omega(B)$. Then it follows form the existence of the isomorphism $\mathrm{Hom}(C, A) \cong \mathrm{Hom}(C, B)$, that there is $\Omega(A) = \Omega(B)$. Then $A_1 \cong B_1$.

$E^\bullet(A_1) \cong E^\bullet(B_1)$ and $E^\bullet(A_2) \cong E^\bullet(B_2)$ [10] follow from $E^\bullet(A) \cong E^\bullet(B)$. Hence, we obtain $A_2 \cong B_2$ according to [5, 7]. So, $A \cong B$.

Necessity. Arguing by contradiction, assume that $\Omega_0(C) = \varnothing$ and there is $\tau^* \in \Omega^*$ such that $\tau^* \not\geq \tau$ for all $\tau \in \Omega(C)$. Consider two distinct almost divisible type τ_1 and τ_2, for which $|P_\infty(\tau_1)| = |P_\infty(\tau_2)| = |P_\infty(\tau^*)| - 1$ and $P_\infty(\tau_1) \subset P_\infty(\tau^*)$, $P_\infty(\tau_2) \subset P_\infty(\tau^*)$. It is clear that for any $\tau \in \Omega(C) \tau_1 \not\geq \tau$ and $\tau_2 \not\geq \tau$. Then there are two nonisomorphic groups of rank 1 A and B from $\Im^{\mathrm{ad}}_{\mathrm{cd}}$, whose types are equal to τ_1 and τ_2, respectively. On the other hand, $E^\bullet(A) \cong E^\bullet(B)$ and $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, B) = 0$. Therefore, $\Im^{\mathrm{ad}}_{\mathrm{cd}}$ is not ${}_C E^\bullet H$-class. A contradiction. The theorem is proved. □

Theorem 2. *Let C be a completely decomposable torsion-free Abelian group. The class $\Im^*_{\mathrm{cd}}$ of completely decomposable torsion-free Abelian groups A, where $\Omega(A)$ contains only noncomparable types, is ${}_C E^\bullet H$-class if and only if the group C satisfies the condition*:

> *for any type $\tau_0 \in \bar{\Omega}$ in the decomposition of the group $\bar{C}$ there is a non-empty finite set of groups of rank 1 of idempotent types that is smaller than type τ_0.*

Proof. Sufficiency. Let

$$A = \bigoplus_{\tau\in\Omega(A)} \bigoplus_{i\in I(\tau)} A_i = A_1 \oplus A_2,$$
$$B = \bigoplus_{\tau\in\Omega(B)} \bigoplus_{j\in J(\tau)} B_j = B_1 \oplus B_2,$$

where

$$A_1 = \bigoplus_{\tau\in\Omega_1(A)} A^{(\tau)}, \quad A_2 = \bigoplus_{\tau\in\Omega_2(A)} A^{(\tau)}, \quad A^{(\tau)} = \bigoplus_{i\in I(\tau)} A_i,$$
$$B_1 = \bigoplus_{\tau\in\Omega_1(B)} B^{(\tau)}, \quad B_2 = \bigoplus_{\tau\in\Omega_2(B)} B^{(\tau)}, \quad B^{(\tau)} = \bigoplus_{j\in J(\tau)} B_j,$$

and

$$\Omega_1(A) = \{\tau \in \Omega(A) \colon |I(\tau)| = 1\}, \quad \Omega_2(A) = \{\tau \in \Omega(A) \colon |I(\tau)| > 1\},$$
$$\Omega_1(B) = \{\tau \in \Omega(B) \colon |J(\tau)| = 1\}, \quad \Omega_2(B) = \{\tau \in \Omega(B) \colon |J(\tau)| > 1\}.$$

As far as $\mathrm{Hom}(C, A) \cong \mathrm{Hom}(C, B)$, by [3, Theorem 43.1] and [9], we obtain

$$\prod_{k\in K} \bigoplus_{i\in I} \mathrm{Hom}(C_k, A_i) \cong \prod_{k\in K} \bigoplus_{j\in J} \mathrm{Hom}(C_k, B_j). \tag{$*$}$$

Show that $\Omega(A) = \Omega(B)$. It follows from the conditions of the theorem that, for any group A_i from the decomposition of the group A in the decomposition of the group $\bar{C}$, there is a group C_k that has idempotent type $\tau(C_k) \leq \tau(A_i)$. Then $\mathrm{Hom}(C_k, A_i) \cong A_i$. Hence by [1] and $(*)$, we obtain that for any group A_i from A there will be a group $\mathrm{Hom}(C_t, B_j)$ from $\mathrm{Hom}(C, B)$ such that $A_i \cong \mathrm{Hom}(C_t, B_j)$. Hence, by [8, Corollary 1], $\tau(B_j) \geq \tau(A_i)$. Similarly, for a group B_j there will be a group $\mathrm{Hom}(C_k, A_l)$ from $\mathrm{Hom}(C, A)$ such that $B_j \cong \mathrm{Hom}(C_k, A_l)$. Then $\tau(B_j) \leq \tau(A_l)$. Next, $\tau(A_i) \leq \tau(B_j) \leq \tau(A_l)$. Since all types from $\Omega(A)$ are noncomparable, we have that $A_i \cong B_j$. Hence $\Omega(A) \subset \Omega(B)$.

Similarly, for any group B_j from B there is a group A_i from A such that $B_j \cong A_i$. Therefore, $\Omega(A) = \Omega(B)$. Then $A_1 \cong B_1$.

Let us represent A_2 and B_2 in the form $A_2 = \bar{A}_2 \oplus A_2^*$, $B_2 = \bar{B}_2 \oplus B_2^*$. $E^\bullet(A) \cong E^\bullet(B)$ implies $E^\bullet(A_2^*) \cong E^\bullet(B_2^*) and E^\bullet(\bar{A}_2) \cong E^\bullet(\bar{B}_2)$ [10]. Hence, according to [5, 7], we obtain $A_2 \cong B_2$ and $E(\bar{A}_2) \cong E(\bar{B}_2)$. Suppose that $\bar{A}_2 \ncong \bar{B}_2$, i.e., there is a type $\tau_1 \in \Omega(A) = \Omega(B)$ such that $|I(\tau_1)| \neq |J(\tau_1)|$. According to the condition of the theorem in the decomposition group $\bar{C}$ there are only finite set of cardinality of m groups C_t of rank 1 of idempotent types such that $\tau(C_t) \leq \tau_1$. Then in $\mathrm{Hom}(C, A)$, the cardinality of the set of groups of rank 1 of the type τ_1 is $m|I(\tau_1)|$, and in $\mathrm{Hom}(C, B)$ is $m|J(\tau_1)|$. Since $|I(\tau_1)| \neq |J(\tau_1)|$, we have that $m|I(\tau_1)| \neq m|J(\tau_1)|$, i.e., cardinalities of sets of direct summands of rank 1 of the type τ_1 in $\mathrm{Hom}(C, A)$ and $\mathrm{Hom}(C, B)$ are different. This means that $\mathrm{Hom}(C, \bar{A}) \ncong \mathrm{Hom}(C, \bar{B})$. The contradiction. Therefore, $\bar{A}_2 \cong \bar{B}_2$. Hence $A \cong B$.

Necessity. Let the group C does not satisfy the condition of the theorem. Then there are two cases:

(1) there is not almost divisible type τ such that in the decomposition of the group $\bar{C}$ there is no summand of the idempotent type smaller than the type τ;
(2) there is a type $\bar{\tau} \in \bar{\Omega}$ such that in the decomposition of the group $\bar{C}$ there is an infinite set of groups of rank 1, whose types are idempotent and smaller than the type $\bar{\tau}$.

Let us consider the first case. It is obvious that $\tau(Z) \notin \Omega(C)$. Let us assume that

$$\Omega_{(01)} = \{\tau \in \Omega_0 \colon \tau \ni (\ldots, h_p^\tau, \ldots),\ (0,0,0,\ldots) < (\ldots, h_p^\tau, \ldots) < (1,1,1,\ldots)\},$$
$$\Omega_{(01)}(C) = \Omega_0(C) \cap \Omega_{(01)}.$$

Note that $\Omega_{(01)}(C) \neq \varnothing$. Indeed, let $\Omega_{(01)}(C) = \varnothing$. Take any two incomparable types $\tau_1, \tau_2 \in \Omega_{(01)}$. Let us consider the torsion-free Abelian groups A and B of rank 1 of the type $\tau(A) = \tau_1$, $\tau(B) = \tau_2$. Then $E(A) \cong E(B)$, $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, B) = 0$, but $A \not\cong B$. Then $\Omega_{(01)}(C) \neq \varnothing$.

Then there are two subcases.

(a) $\Omega_{(01)}(C)$ contains minimal types.

Let τ_0 be a minimal type in $\Omega_{(01)}(C)$. Take two noncomparable types τ_1 and τ_2 such that $\tau_1, \tau_2 < \tau_0$. Let us consider torsion-free groups A and B of rank 1 whose types are $\tau(A) = \tau_1$ and $\tau(B) = \tau_2$. It is obvious that $E^\bullet(A) \cong E^\bullet(B)$ and $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, B) = 0$, but $A \not\cong B$. The contradiction.

(b) $\Omega_{(01)}(C)$ does not contain minimal types, i.e., for any type $\tau \in \Omega_{(01)}$ there is a type $\tau' \in \Omega_{(01)}(C)$ such that $\tau' \neq \tau(\mathbf{Z})$ and $\tau' < \tau$. Let us take a random type $\tau_0 \in \Omega_{(01)}(C)$. Let us consider the set $\Omega_{(01)}(\tau_0)$ of all noncomparable types that contains the type τ_0. Let $\tau^* \in \Omega_{(01)}(\tau_0)$, $\tau^* \neq \tau_0$. Let us consider the group

$$A = \bigoplus_{\tau \in \Omega_{(01)}(\tau_0)} \bigoplus_{r(C)} \mathbf{Q}(\tau), \quad B = \bigoplus_{\tau \in \Omega_{(01)}(\tau_0),\ \tau \neq \tau^*} \bigoplus_{r(C)} \mathbf{Q}(\tau),$$

where $\mathbf{Q}(\tau)$ is a rational group of the type τ. It is obvious that $E(A) \cong E(B)$. Show that

$$\Omega\Big(\mathrm{Hom}\big(C, \mathbf{Q}(\tau^*)\big)\Big) \subset \Omega\big(\mathrm{Hom}(C, B)\big).$$

Let us take

$$\tau' \in \Omega\Big(\mathrm{Hom}\big(C, \mathbf{Q}(\tau^*)\big)\Big).$$

Then $\tau' = \tau^* - \tau(C_0)$, where $\tau(C_0) \in \Omega_{(01)}(C)$. Let $\tau^* \ni (\ldots, \alpha_p^*, \ldots)$, $\tau(C_0) \ni (\ldots, \gamma_p, \ldots)$, and $\tau' \ni (\ldots, \alpha_p^* - \gamma_p, \ldots)$. Let us assume for the type τ containing characteristic $(\ldots, h_p, \ldots)$ that

$$P_0(\tau) = \{p \in P \colon h_p = 0\}, \quad P_1(\tau) = \{p \in P \colon h_p = 1\}.$$

Let us consider a type τ_1 such that $P_1(\tau_1) = P_0(\tau^*)$ and $P_0(\tau_1) = P_1(\tau^*)$. According to the assumption, in $\Omega_{(01)}(C)$ there is a type τ_1^C such that $\tau_1^C \neq \tau(\mathbf{Z})$ and $\tau_1^C < \tau_1$. Then $P_1(\tau_1^C) \subset P_1(\tau_1)$ and $|P_1(\tau_1) \setminus P_1(\tau_1^C)| = \aleph_0$ and $|P_1(\tau_1^C)| = \aleph_0$. Let us consider the type $\tau_2 \in \Omega_{(01)}$ such that $P_1(\tau_2) = P_1(\tau_1^C) \cup P_1(\tau')$. Note that τ_2 is not compared with τ^*, so $\tau_2 \in \Omega(B)$. Then $\tau' \in \Omega\big(\mathrm{Hom}(C, B)\big)$, since $\tau_2 - \tau_1^C = \tau'$. Thus,

$$\Omega\Big(\mathrm{Hom}\big(C, \mathbf{Q}(\tau^*)\big)\Big) \subset \Omega\big(\mathrm{Hom}(C, B)\big).$$

Let us take $\tau \in \Omega\big(\mathrm{Hom}(C, A)\big)$. Let $|T^A(\tau)|$ and $|T^B(\tau)|$ be cardinalities of sets of direct summands of rank 1 of the type τ in $\mathrm{Hom}(C, A)$ and $\mathrm{Hom}(C, B)$, respectively. Then

$$|T^A(\tau)| = r(C) m^B(\tau), \quad |T^B(\tau)| = r(C)\big(m^B(\tau) + m(\tau^*)\big),$$

where $m^B(\tau)$ is the cardinality of the set of direct summands of C_k of rank 1 of the group C, for which in $\Omega(\tau_0)$ there are types $\tau_k \neq \tau^*$ such that $\tau_k - \tau(C_k) = \tau$; $m(\tau^*)$ is the cardinality of the set of direct summands C_r of rank 1 of the group C so that $\tau^* - \tau(C_r) = \tau$. Since $\Omega_{(01)}(C)$ does not contain minimal types, we have that $r(C) \geq \aleph_0$, which means that $|T^A(\tau)| = |T^B(\tau)|$. Therefore, $\mathrm{Hom}(C, A) \cong \mathrm{Hom}(C, B)$, $E^\bullet(A) \cong E^\bullet(B)$ but $A \not\cong B$. The contradiction.

Let us consider the second case. There is a type $\bar{\tau} \in \bar{\Omega}$ such that in the decomposition of the group $\bar{C}$ there is an infinite set of cardinality α of groups of rank 1, whose types are idempotent relative to the type $\bar{\tau}$.

Let us consider the set

$$\tilde{\Omega}(\bar{C}) = \{\tau \in \Omega(\bar{C}) \colon P_\infty(\tau) \subset P_\infty(\bar{\tau}) \text{ and } \tau \text{ is not idempotent relative to the type } \bar{\tau}\}.$$

There are possible subcases.

(1) $\tilde{\Omega}(\bar{C}) = \varnothing$. In Ω there are two noncomparable types $\tau^{(1)}$ and $\tau^{(2)}$ such that $P_\infty(\tau^{(1)}) = P_\infty(\tau^{(2)}) = P_\infty(\bar{\tau})$. Let us assume $A = A_1 \oplus A_1 \oplus A_2$, $B = A_1 \oplus A_2 \oplus A_2$, where $\tau(A_1) = \tau^{(1)}$, $\tau(A_2) = \tau^{(2)}$. Let us consider $\mathrm{Hom}(C, A)$ and $\mathrm{Hom}(C, B)$. We have

$$\mathrm{Hom}(C,A) = \mathrm{Hom}(C,A_1) \oplus \mathrm{Hom}(C,A_1) \oplus \mathrm{Hom}(C,A_2) \cong \prod_\alpha A_1 \oplus \prod_\alpha A_1 \oplus \prod_\alpha A_2 \cong \prod_\alpha A_1 \oplus \prod_\alpha A_2.$$

Similarly,

$$\mathrm{Hom}(C,B) \cong \prod_\alpha A_1 \oplus \prod_\alpha A_2.$$

Thus, $A \not\cong B$ but $E(A) \cong E(B)$, $\mathrm{Hom}(C,A) \cong \mathrm{Hom}(C,B)$. A contradiction.

(2) $\tilde{\Omega}(\bar{C}) \neq \varnothing$. Let us bring the type τ^0 containing characteristic $(\dots, \alpha_p^0, \dots)$ in compliance with each type $\tau \in \tilde{\Omega}$ containing characteristic $(\dots, \alpha_p, \dots)$, where $\alpha_p^0 = 0$ if $p \in P_\infty(\bar{\tau}) and \alpha_p^0 = \alpha_p$ in other cases. The set of all such types τ^0 forms a $\tilde{\Omega}_0$. There are two variants.

(a) $\tilde{\Omega}_0$ contains the minimal types. Let τ_0 be one of the minimal types in $\tilde{\Omega}_0$. Let us take two noncomparable types $\tau_0^{(1)}, \tau_0^{(2)} < \tau_0$, where $\tau_0^{(1)} \ni (\dots, h_p^{(1)}, \dots)$ and $\tau_0^{(2)} \ni (\dots, h_p^{(2)}, \dots)$. Let us consider the types τ_1 and τ_2 containing the characteristics $(\dots, \alpha_p^{(1)}, \dots)$ and $(\dots, \alpha_p^{(2)}, \dots)$, respectively, where $\alpha_p^{(1)} = \alpha_p^{(2)} = \infty$ if $p \in P_\infty(\bar{\tau})$ and $\alpha_p^{(1)} = h_p^{(1)}$, $\alpha_p^{(2)} = h_p^{(2)}$ in other cases. Let us assume

$$A = A_1 \oplus A_1 \oplus A_2, \quad B = A_1 \oplus A_1 \oplus A_2,$$

where $\tau(A_1) = \tau_1$ and $\tau(A_2) = \tau_2$. Then as in the case (1) we will obtain a contradiction.

(b) $\tilde{\Omega}_0$ does not contain the minimal types. Let us take any type τ^* such that $P_\infty(\tau^*) = P_\infty(\bar{\tau})$, $P_1(\tau^*) \cup P_0(\tau^*) = P \setminus P_\infty(\tau^*)$, and $|P_1(\tau^*)| = |P_0(\tau^*)| = \aleph_0$. Let us assume that

$$\Omega(\tau^*) = \{\tau \in \Omega \colon P_\infty(\tau) = P_\infty(\tau^*),\ \tau \text{ is not compared with } \tau^*, \\ \tau \ni (\dots, \beta_p, \dots), \text{ where } \beta_p = 0 \text{ or } \beta_p = 1 \text{ for all } p \notin P_\infty(\tau)\}.$$

Let us consider the group

$$A = \bigoplus_{\tau \in \Omega(\tau^*)} \mathbf{Q}(\tau), \quad B = A \oplus \mathbf{Q}(\tau^*).$$

The proof that $\Omega(\mathrm{Hom}(C,A)) = \Omega(\mathrm{Hom}(C,B))$ is similar to the case (1)(b). Let us prove that for any τ from $\Omega(\mathrm{Hom}(C,A))$, there is an equality $|T^A(\tau)| = |T^B(\tau)|$. Let $\tau \in \Omega(\tau^*)$. Then $\tau \in \Omega(\mathrm{Hom}(C,A))$ and $|T^A(\tau)| = |T^B(\tau)| = \alpha$. If $\tau \in \Omega(\mathrm{Hom}(C,A)) \setminus \Omega(\tau^*)$, i.e., $\tau = \tau_0 - \tau(C_k)$, where $\tau_0 \in \Omega(\tau^*)$, $\tau(C_k) \in \Omega(C)$. As was shown above (the case (1)(b)), in $\Omega(\tau^*)$, there is an infinite set of τ_i, for which in $\tilde{\Omega}_0$ there are types τ_k such that $\tau_i - \tau_k = \tau$. Thus, $|T^A(\tau)| = |T^B(\tau)| \geq \aleph_0$. This means that $\mathrm{Hom}(C,A) \cong \mathrm{Hom}(C,B)$, $E(A) \cong E(B)$, but $A \not\cong B$. A contradiction. The theorem is proved. □

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