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ABELIAN GROUPS ISOMORPHIC TO A PROPER FULLY INVARIANT SUBGROUP

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Abstract. The present paper is a survey of the authors' results related to studying groups containing a proper fully invariant subgroup isomorphic to the group.

Introduction

In the theory of Abelian groups, the study of groups containing a proper subgroup isomorphic to the group is one of research lines.

R. A. Beaumont studied such groups in [1]. He called them I-groups. In [1], it was proved that each primary Abelian group that can be decomposed into an infinite sum of cocyclic groups is an I-group. I-modules were studied in [2] by R. A. Beaumont and R. S. Pierce. In particular, in [2] it was proved that a torsion-free R-module M that is not divisible is an I-module, and a torsion module M of finite rank is not an I-module. In [3], in addition to I-groups, they considered IP-groups (groups isomorphic to a proper pure subgroup) and ID-groups (groups isomorphic to a proper direct summand).

In [5], P. Crawley constructs an example of an infinite primary Abelian group that does not contain elements of infinite height and is not isomorphic to a proper subgroup.

In [19], P. Hill and Ch. Megibben present a more general and simple construction of primary groups without proper isomorphic subgroups as compared to that by P. Crawley. In their work, they also show that an infinite reduced primary group is a group without proper subgroups isomorphic to the group only if this group is unbounded, uncountable, and has finite Ulm–Kaplansky invariants.

In [23], G. S. Monk studies Abelian p-groups not containing proper pure dense subgroups, isomorphic to the group.

In recent time, the interest in the groups containing a proper subgroup isomorphic to the group does not wane. In particular, in [8] B. Goldsmith, S. Óhógáin, and S. Wallutis study quasi-minimal groups (groups isomorphic to each of their subgroups of the same cardinality as the groups), pure quasi-minimal groups (groups isomorphic to each of their pure subgroups of the same cardinality as the groups), and direct quasi-minimal groups (groups isomorphic to each of their direct summand of the same cardinality as the groups).

The present article is a review. It presents our results concerning to the research of groups containing a fully invariant subgroup isomorphic to the group.

All groups in the article are Abelian.

1. Basic Definitions and Known Results

In this section, basic definitions and known results used below are given.

Let A be a torsion group. By A_p we define the subgroup of the group A, containing all the elements $a \in A$, whose order is a prime power p. The subgroup A_p is called the *p-component* of the group A.

Theorem 1.1 ([6, Sec. 8]). *A torsion group A is a direct sum of the p-group A_p belonging to different primes p. The groups A_p are uniquely defined by the group A.*

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If the group A is not a torsion group, then by the *p-component* of this group the p-component of its periodic part $T(A)$ is meant.

A group D is called *divisible* if $nD = D$ for each natural number n. We recall some properties of divisible groups.

Theorem 1.2 ([6, Sec. 20]).

(1) *If D_i $(i \in I)$ are divisible subgroups of the group A, then also $\sum D_i$ is a divisible subgroup of the group A.*

(2) *A direct sum and a direct product are divisible groups if and only if each component is a divisible group.*

Theorem 1.3 ([6, Sec. 23]). *Any divisible group D is a direct sum of quasi-cyclic groups and groups isomorphic to the full rational group. The cardinality of the component sets $\mathbb{Z}(p^\infty)$ (for each p) and $\mathbb{Q}$ form a complete and independent system of invariants of the group D.*

A group C is called *reduced* if it has no nonzero divisible subgroups.

Theorem 1.4 ([6, Sec. 21]). *Any group A is a direct sum of a divisible group D and a reduced group C, $A = D \oplus C$. The subgroup D of A is defined here uniquely and the subgroup C is defined uniquely up to an isomorphism.*

Consider direct sums of cyclic groups. The following result of L. Ya. Kulikov is a criterion allowing one to establish when a given p-group is decomposable into a direct sum of cyclic p-groups.

Theorem 1.5 ([22]). *The p-group A is a direct sum of cyclic groups if and only if A is a union of the ascending sequence of the subgroups*

$$A_1 \subset A_2 \subset \dots \subset A_n \subset \dots, \quad \bigcup_{n=1}^{\infty} A_n = A,$$

where heights of nonzero elements entering into A_n are less then a fixed number k_n.

Two important results follow from this theorem.

Theorem 1.6 ([6, Sec. 17]). *A bounded group is a direct sum of cyclic groups.*

Theorem 1.7 ([6, Sec. 17]). *A countable p-group is a direct sum of cyclic groups if and only if this group has no nonzero elements of infinite height.*

If the group A is decomposable into a direct sum of cyclic groups, then this group can have many decompositions. However, when considering only the components' orders, the uniqueness holds.

Theorem 1.8 ([6, Sec. 17]). *Any two decompositions of a group into a direct sum of cyclic groups of infinite orders and of orders that are equal to a prime degree are isomorphic.*

A subgroup B of the group A that is mapped into itself under every endomorphism of A is said to be a *fully invariant subgroup* of A. We consider fully invariant subgroups of a direct sum of groups.

Theorem 1.9. *If*

$$A = \bigoplus_{i \in I} A_i$$

and S is a fully invariant subgroup of the group A, then

$$S = \bigoplus_{i \in I} (S \cap A_i),$$

where $S \cap A_i$ is a fully invariant subgroup of the group A_i for each $i \in I$.

This theorem can be proved by generalizing the reasonings presented in [6] in the proof of Lemma 9.3.

The following result describes fully invariant subgroups of direct sums of cyclic p-groups.

Theorem 1.10 ([4])**.** *Let*

$$B = \bigoplus_{k \in \mathbb{N}} B_k,$$

where

$$B_k = \bigoplus \mathbb{Z}(p^k).$$

L is a fully invariant subgroup of the group B if and only if

$$L = \bigoplus_{k \in \mathbb{N}} p^{n_k} B_k,$$

where

(1) $n_k \le k$ *for all* $k \in \mathbb{N}$;
(2) $n_k \le n_{k+r} \le n_k + r$ *for all* $k \in \mathbb{N}$, $r \in \mathbb{N}$.

Basic subgroups introduced by L. Ya. Kulikov play an important role when studying p-groups. A subgroup B of a p-group A is called *basic* if this group satisfies the following conditions:

(1) B is a direct sum of cyclic p-groups;
(2) B is a pure subgroup of the group A;
(3) A/B is a divisible group.

The following theorem is valid.

Theorem 1.11 ([6, Secs. 32 and 35])**.**

(1) *Any p-group contains basic subgroups.*
(2) *Any two basic subgroups of this p-group are isomorphic.*

Sometimes it is convenient to speak about a basic subgroup of a torsion-free group A. By this we mean a direct sum $\bigoplus_p B_p$ of subgroups B_p that are basic in p-components of the group A.

A fully invariant subgroup L of a p-group A is called a *large subgroup* if $L + B = A$ for each basic subgroup B of the group A [24]. The following statements are true for large subgroups:

(1) 0 is a large subgroup of the group A if and only if A is a bounded group;
(2) any fully invariant subgroup of a bounded group is a large subgroup;
(3) if L is a large subgroup of the group A, then $p^n L$ for each n is also a large subgroup.

The following interesting result is connected with the factor group of the group by its subgroup.

Theorem 1.12 ([7, Sec. 67])**.** *If L is a large subgroup of the p-group A, then A/L is a direct sum of cyclic groups.*

Let us present a result showing the interconnection between divisible and reduced subgroups of an Abelian group and its fully invariant subgroup. Let A be a p-group. As usual [6, Sec. 1], we denote by $A[p^k]$, where k is an integer non-negative number, next subgroup of the group A: $\{a \in A \mid p^k a = 0\}$; if $k = \infty$, then set $A[p^\infty] = A$. If a is an element of the order p^k of the group A, then let its exponent be denoted as $e(a)$, i.e., $e(a) = k$.

Theorem 1.13 ([10])**.** *Let A be a group,*

$$A = R \oplus D_0 \oplus \left(\bigoplus_p D_p \right),$$

where R is a reduced group, D_0 is a divisible torsion-free group, and D_p are divisible p-groups. A subgroup S of the group A is a fully invariant subgroup in A if and only if it has one of the following forms:

(1) $S = R' \oplus \Big(\bigoplus_p D_p[p^{k_p}]\Big)$, *where* $R' = \bigoplus_p R'_p$ *is a torsion fully invariant subgroup of the group* R (R'_p *is a* p*-component of the group* R') *and* $k_p \geq \sup\{e(r) \mid r \in R'_p\}$ (k_p *is an integer non-negative number or a symbol* ∞);

(2) $S = R' \oplus D_0 \oplus \Big(\bigoplus_p D_p\Big)$, *where* R' *is a fully invariant subgroup of the group* R.

Given a in A; the greatest non-negative integer number r for which the equation $p^r x = a$ is solvable in A is called the *p-height* $h_p(a)$ of a. If $p^r x = a$ is solvable whatever r is, then a is an element of *infinite p-height*, $h_p(a) = \infty$. The zero is of infinite height at every prime. If it is completely clear from the context which prime p is meant, we call $h_p(a)$ simply the *height of* a and write $h(a)$.

A sequence of the non-negative integer number and symbols of ∞ is called a *characteristic*. We denote by $\mathfrak{X}$ the set of this sequences. If $\chi_1 = (k_1, \dots, k_n, \dots)$ and $\chi_2 = (l_1, \dots, l_n, \dots)$, then we suppose that $\chi_1 \leq \chi_2$ if and only if $k_n \leq l_n$ for all $n \in \mathbb{N}$.

Let A be a torsion-free group and $a \in A$. The sequence of p-heights

$$\chi(a) = (h_{p_1}(a), \dots, h_{p_n}(a), \dots),$$

where $p_1, \dots, p_n, \dots$ is the sequence of all prime numbers ordered by increasing magnitude, is said to be the *characteristic* or the *height-sequence* of a. Since this depends on A, we sometimes write $\chi_A(a)$ to emphasize the role of A.

If $\chi_1 = (k_1, \dots, k_n, \dots)$ and $\chi_2 = (l_1, \dots, l_n, \dots)$ are characteristics, then their *sum* is defined as a characteristic

$$\chi_1 + \chi_2 = (k_1 + l_1, \dots, k_n + l_n, \dots),$$

where, naturally, the sum of ∞ and anything is ∞. A *difference* $\chi_1 - \chi_2$ of two characteristics $\chi_1 \geq \chi_2$ is defined as a characteristic

$$\chi_1 - \chi_2 = (k_1 - l_1, \dots, k_n - l_n, \dots),$$

here, we set $\infty - k = \infty$ for each k. A characteristic χ is called *idempotent* if $\chi + \chi = \chi$. Note, for these operations for characteristics in [7] is used the multiplicative notation, but for our research the additive notation is the most conveniently.

Two characteristics $(k_1, \dots, k_n, \dots)$ and $(l_1, \dots, l_n, \dots)$ are be considered as *equivalent* if $k_n \neq l_n$ holds only for a finite number of n if and only if k_n and l_n are finite. An equivalence class of characteristics is called a *type*. If $\chi(a)$ belongs to the type $\mathbf{t}$, then we say that an element a is of type $\mathbf{t}$ and write $\mathbf{t}(a) = \mathbf{t}$ or $\mathbf{t}_A(a) = \mathbf{t}$ if it is necessary to indicate that the type of a is computed in A.

We shall represent a type by a characteristic in this class. In other words, we write

$$\mathbf{t} = (k_1, \dots, k_n, \dots)$$

and keep in mind that the characteristic $(k_1, \dots, k_n, \dots)$ can be replaced by an equivalent one. For two types $\mathbf{t}_1$ and $\mathbf{t}_2$ it is set $\mathbf{t}_1 \leq \mathbf{t}_2$ if there exist two characteristics χ_1 and χ_2 belonging to the types $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively, such that $\chi_1 \leq \chi_2$.

Since the addition of characteristics is matched with the equivalence relation in the set of characteristics, one can naturally introduce a sum and difference of types in the type set, as well as the concept of the idempotent type $\mathbf{t}$ ($\mathbf{t} = \mathbf{t} + \mathbf{t}$). We note that the inequality $\mathbf{t}(a) \leq \mathbf{t}\big(\varphi(a)\big)$ holds for every homomorphism $\varphi\colon A \to B$ and the element $a \in A$.

A torsion-free group A in which all nonzero elements have the same type $\mathbf{t}$ is called a *homogeneous* group (of the type $\mathbf{t}$). If a homogeneous group has a type $\mathbf{t}$, one writes $\mathbf{t}(A) = \mathbf{t}$. It is clear that any torsion-free group of rank 1 is a homogeneous group.

A type $\mathbf{t}$ is called p_k*-divisible* ($p_k \in \Pi$, where Π is the set of all prime numbers enumerated in the order of increasing) if for each characteristic $v = (v^{(1)}, v^{(2)}, \dots, v^{(n)}, \dots)$ belonging to the type $\mathbf{t}$, we have $v^{(k)} = \infty$. If A is a homogeneous group of the type $\mathbf{t}$ and the type $\mathbf{t}$ is a p_k-divisible, then $p_k A = A$.

Let A be an arbitrary group. A finite system $\{a_1, a_2, \ldots, a_k\}$ of nonzero elements of a group A is called *linearly independent* or briefly *independent* if

$$n_1a_1 + n_2a_2 + \cdots + n_ka_k = 0 \quad (n_i \in \mathbb{Z})$$

implies

$$n_1a_1 = n_2a_2 = \cdots = n_ka_k = 0.$$

This means $n_i = 0$ if the order of the element a_i is infinite; if the order of the element a_i is finite, then n_i is divisible by the order of the element a_i.

A system of elements is *dependent* if it is not independent.

An infinite system $L = \{a_i\}_{i\in I}$ of elements of A is called *independent* if every finite subsystem of L is independent.

An independent system M of A is called *maximal* if there is no independent system in A containing M properly. By Zorn's lemma, every independent system in A can be extended to a maximal one.

The *rank* $r(A)$ of a group A is the cardinal number of a maximal independent system containing only elements of infinite and prime power orders. If we restrict ourselves to elements of infinite order in A, i.e., we select an independent system which contains elements of infinite order only and which is maximal with respect to this property, then the cardinality of this system is called the *torsion-free rank* $r_0(A)$ of A. If we select an independent system that contains elements whose orders are powers of a fixed prime p only and is maximal with respect to this property, then the cardinality of this system is called the *p-rank* $r_p(A)$ of A. It is clear from these definitions that the following equality holds true for every group A:

$$r(A) = r_0(A) + \sum_p r_p(A).$$

The following theorem holds.

Theorem 1.14 ([6, Sec. 16])**.** *The ranks $r(A)$, $r_0(A)$, and $r_p(A)$ of a group A are invariants of this group.*

A group A is called *separable* if any finite subset of its elements can be embedded into a direct summand of the group A which is a direct sum of groups of rank 1. By virtue of their structure, all divisible groups are separable, and it is easy to establish that a group is separable if and only if its reduced part is separable. The following result is true.

Theorem 1.15 ([7, Sec. 65])**.** *A reduced p-group is separable if and only if it contains no nonzero elements of infinite height.*

Let A be a reduced p-group and σ be an ordinal number. By $p^\sigma A$ we denote the subgroup of the group A defined by induction as follows: $p^0A = A$, $p^{\sigma+1}A = p(p^\sigma A)$, and $p^\sigma A = \bigcap_{\rho<\sigma} p^\rho A$ if σ is a limit ordinal number. The least ordinal number τ for which $p^\tau A = 0$ is called the *length* $\lambda(A)$ *of the group* A; by the *σth Ulm–Kaplansky invariant* $f_A(\sigma)$ *of the group* A we denote the cardinal number equal to the rank of the factor group $(p^\sigma A)[p]/(p^{\sigma+1}A)[p]$ [6, Sec. 37].

2. α-Copies of Separable p-Groups

In this section, we use I. Kaplansky's description of fully invariant subgroups of a class of p-groups in the language of sequences of ordinal numbers and symbols ∞ and investigate connections between some properties of an increasing sequence α of non-negative integers and properties of a reduced separable p-group A when the group A is isomorphic to its fully invariant subgroup S defined by the sequence α.

Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots)$ be an increasing sequence of non-negative integers and symbols ∞ (for any pair of indices (i, j), where $i < j$, $\alpha_i < \alpha_j$ if $\alpha_i \neq \infty$, and $\alpha_i = \alpha_j$ if $\alpha_i = \infty$). If $\alpha_i + 1 < \alpha_{i+1}$, then it is said that the *sequence* α *has a jump in* α_{i+1}. Denote by $\mathbb{N}_0$ the set of all non-negative integers. By the *length* $\lambda(\alpha)$ *of the sequence* α we mean the least number $i \in \mathbb{N}_0$ such that $\alpha_i = \infty$; and also we suppose $\lambda(\alpha) = \infty$ if and only if $\alpha_i < \infty$ for all $i \in \mathbb{N}_0$.

An increasing sequence $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ of non-negative integers and symbols ∞ is called a *U-sequence* for a reduced separable p-group A if for every $\alpha_i \neq \infty$ we have $\alpha_i < \lambda(A)$ and anytime when there exist a jump in α_n, the α_{n-1}th Ulm–Kaplansky invariant of the group A is different from zero [20]. Denote by $A(\alpha)$ the following subgroup of the group A:

$$A(\alpha) = \{a \in A \mid h(p^n a) \geq \alpha_n \text{ for each } n \in \mathbb{N}_0\}.$$

It is clear that $A(\alpha)$ is a fully invariant subgroup of the group A. It follows from I. Kaplansky's results [20, pp. 56–66] that every fully invariant subgroup S of the reduced separable p-group A has the form $A(\alpha)$ for some U-sequence α; the group S is represented in this form uniquely. We need the following result.

Theorem 2.1 ([28]). *Let $S = A(\alpha)$ be an unbounded fully invariant subgroup of a reduced separable p-group A, where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ is a U-sequence for the group A. Then for all $i \in \mathbb{N}_0$,*

$$f_S(i) = \sum_{j=0}^{k_i} f_A(\alpha_i + j), \quad k_i = \alpha_{i+1} - 1 - \alpha_i. \tag{1}$$

Definition 2.2. Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ be an increasing sequence of non-negative integers and symbols ∞. We say that a reduced separable p-group A *has an α-copy* if α is a U-sequence for the group A and $A \cong A(\alpha)$.

Consider α-copies of unbounded reduced separable p-groups. Note that if A is an unbounded reduced separable p-group and A has an α-copy, then $\lambda(\alpha) = \infty$, i.e., the sequence α has no symbols ∞. Let W denote the set of all increasing sequences of non-negative integers.

Theorem 2.3. *Let $\alpha \in W$ and A be an unbounded reduced separable p-group having an α-copy. Then*

(1) *if the sequence α for some $i \in \mathbb{N}_0$ has a jump in α_{i+1}, then $f_A(i) \neq 0$ and $f_A(i) \geq f_A(\alpha_i)$;*
(2) *if it has no jump in α_{i+1}, then $f_A(i) = f_A(\alpha_i)$.*

Proof. Since the group A has an α-copy, we have $A \cong A(\alpha)$ and hence $f_A(i) = f_{A(\alpha)}(i)$ for any $i \in \mathbb{N}_0$. Using Theorem 2.1, we obtain that for each $i \in \mathbb{N}_0$,

$$f_A(i) = \sum_{j=0}^{k_i} f_A(\alpha_i + j), \quad \text{where} \quad k_i = \alpha_{i+1} - 1 - \alpha_i. \tag{2}$$

We consider each case separately.

(1) If the sequence α for some $i \in \mathbb{N}_0$ has a jump in α_{i+1}, then $\alpha_i + 1 < \alpha_{i+1}$, and, by definition of the U-sequence, we have $f_A(\alpha_i) \neq 0$. Hence, $f_A(i) \neq 0$ and the right-hand part of equality (2) contains not less than two summands (the first summand is $f_A(\alpha_i)$, the last summand is $f_A(\alpha_{i+1} - 1)$). Therefore, $f_A(i) \geq f_A(\alpha_i)$.

(2) If the sequence α has no jump in α_{i+1}, then $\alpha_i + 1 = \alpha_{i+1}$ and from (2) we obtain

$$f_A(i) = f_A(\alpha_i).$$

□

Let W_0 denote the set of all increasing sequences consisting of non-negative integers and beginning from zero.

Theorem 2.4. *Let $\alpha \in W_0$ and A be an unbounded reduced separable p-group whose all Ulm–Kaplansky invariants are finite. Then if α has at least one jump, then the group A has no α-copy.*

Proof. Suppose that the group A has an α-copy. Let α have the first jump in α_{t+1}. Then

$$\alpha = (0, 1, \dots, t, t + 2 + n_1, t + 3 + n_2, \dots, t + m + 1 + n_m, \dots),$$

where $0 \leq n_1 \leq n_2 \leq \dots$.

By Theorem 2.1, we obtain that for each $i \in \mathbb{N}_0$

$$f_A(i) = \sum_{j=0}^{k_i} f_A(\alpha_i + j), \quad \text{where} \quad k_i = \alpha_{i+1} - 1 - \alpha_i. \tag{3}$$

Using (3), we find $f_A(t)$. We have $\alpha_t = t$, $\alpha_{t+1} = t+2+n_1$, $k_t = \alpha_{t+1} - 1 - \alpha_t = t+2+n_1-1-t = n_1+1$. Thus,

$$f_A(t) = f_A(t) + f_A(t+1) + \cdots + f_A(t+1+n_1). \tag{4}$$

Similarly, for each $j \in \mathbb{N}$, we obtain

$$f_A(t+j) = f_A(t+j+1+n_j) + f_A(t+j+2+n_j) + \cdots + f_A(t+j+1+n_{j+1}). \tag{5}$$

From (4), with allowance for the finiteness of Ulm–Kaplansky invariants of the group A, we have

$$f_A(t+1) + \cdots + f_A(t+1+n_1) = 0,$$

and, hence,

$$f_A(t+1) = 0.$$

Let $m \in \mathbb{N}$ and $m \geq 2$. Suppose that for all $i < m$ ($i \in \mathbb{N}$) $f_A(t+i) = 0$. Let us prove that $f_A(t+m) = 0$. Considering equalities (5) for all j varying from 1 to $m-1$, we obtain the following system of equations:

$$\left.\begin{aligned} &0 = f_A(t+2+n_1) + \cdots + f_A(t+2+n_2), \\ &\ldots \\ &0 = f_A(t+m+n_{m-1}) + \cdots + f_A(t+m+n_m). \end{aligned}\right\} \tag{6}$$

$f_A(t+m)$ enters into the right-hand side of a certain equality in system (6) having a zero left-hand side; therefore, $f_A(t+m) = 0$. Thus, $f_A(t+i) = 0$ for each $i \in \mathbb{N}$; this contradicts the fact that the group A is unbounded. □

3. Some Properties of IF-Groups

We study groups containing proper fully invariant subgroups isomorphic to the group.

Definition 3.1. A group is called an *IF-group* if it isomorphic to some proper fully invariant subgroup.

At first, we consider a case of bounded group.

Theorem 3.2. *Every bounded p-group is not an IF-group.*

Proof. Let B be a bounded p-group and p^m be the largest of element orders of the group B. Then $B = \bigoplus_{k=1}^{m} B_k$, where $B_k = \bigoplus \mathbb{Z}(p^k)$ [6]. Let L be a fully invariant subgroup of the group B. Then by Theorem 1.10

$$L = p^{n_1} B_1 \oplus p^{n_2} B_2 \oplus \cdots \oplus p^{n_m} B_m,$$

where n_k satisfies inequalities (1) and (2) of this theorem. If $n_m = 0$, then by $n_1 \leq n_2 \leq \cdots \leq n_m = 0$ we obtain that $L = B_1 \oplus B_2 \oplus \cdots \oplus B_m = B$. Therefore, L is not a proper subgroup of the group B. Hence, $n_m \geq 1$. We have $p^{n_m} B_m = \bigoplus \mathbb{Z}(p^{m-n_m})$, whence the group L has no cyclic direct summand of order p^m and, therefore, L is not isomorphic to B. □

The following Lemma is more useful to us for the study of IF-group that are direct sums.

Lemma 3.3. *Let* $A = \bigoplus_{i \in I} A_i$ *and* $C = \bigoplus_{i \in I} C_i = \bigoplus_{i \in I} C_i'$, *where* C_i *and* C_i' *are subgroups of the group* A_i *for each* $i \in I$. *Then* $C_i = C_i'$ *for each* $i \in I$.

Proof. Let us show that $C_i \subset C'_i$. Let $c_i \in C_i$. Then $c_i \in C$ and, hence, $c_i \in \bigoplus_{i\in I} C'_i$. We have $c_i = c'_{i_1} + c'_{i_2} + \cdots + c'_{i_k}$, where $c'_{i_j} \in C'_{i_j}$, $i_j \in I$, $j = \overline{1,k}$. Since C_i and C'_i are subgroups of the group A_i for each $i \in I$, we have that $c_i \in A_i$, $c'_{i_j} \in A_{i_j}$. Taking into account that A is a direct sum of groups A_i ($i \in I$), we obtain that for some j ($j = \overline{1,k}$) $i_j = i$ and $c'_{i_j} = c_i$, and for all other j $c'_{i_j} = 0$. Hence, $c_i \in C'_i$. Similarly, $C'_i \subset C_i$ for each $i \in I$. Therefore, $C_i = C'_i$ for each $i \in I$. □

Theorem 3.4. *Let $B = \bigoplus_{i\in I} B_i$, where B_i is a fully invariant subgroup of the group B for each $i \in I$. B is an IF-group if and only if there exists at least one index $i \in I$ for which the group B_i is an IF-group.*

Proof. Necessity. Let S be a proper fully invariant subgroup of the group B such that $B \cong S$. We have $S = \bigoplus_{i\in I} S_i$, where $S_i = S \cap B_i$ is a fully invariant subgroup of the group B_i for each $i \in I$. Let φ is an isomorphic mapping of the group B on S. φ can be considered as an endomorphism of the group B. Let φ_i ($i \in I$) denote a restriction of the endomorphism φ on the group B_i. Since B_i is a fully invariant subgroup of the group B, then φ_i is an endomorphism of the group B_i for each $i \in I$. Let $b \in B$, $b = b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_j} \in B_{i_j}$, $i_j \in I$, $j = \overline{1,k}$. We have

$$\varphi b = \varphi(b_{i_1} + b_{i_2} + \cdots + b_{i_k}) = \varphi b_{i_1} + \varphi b_{i_2} + \cdots + \varphi b_{i_k} = \varphi_{i_1} b_{i_1} + \varphi_{i_2} b_{i_2} + \cdots + \varphi_{i_k} b_{i_k}.$$

Therefore, $\varphi B = \sum_{i\in I} \varphi_i B_i$. Since $\varphi_i B_i \subset B_i$,

$$\sum_{i\in I} \varphi_i B_i = \bigoplus_{i\in I} \varphi_i B_i$$

(see [6]). Thus, $S = \varphi B = \bigoplus_{i\in I} \varphi_i B_i$ and $S = \bigoplus_{i\in I} S_i$, where $\varphi_i B_i$ and S_i are subgroups of the group B_i for each $i \in I$. By Lemma 3.3, $\varphi_i B_i = S_i$. Since φ is an isomorphism, then $\operatorname{Ker}\varphi = 0$, and, therefore, $\operatorname{Ker}\varphi_i = 0$ for each $i \in I$. Thus, φ_i is an isomorphic mapping of B_i on S_i. Taking into account that $S \neq B$, we obtain that there exists at least one index $i_0 \in I$ such that $B_{i_0} \cong S_{i_0}$ and $B_{i_0} \neq S_{i_0}$, i.e., the group B_{i_0} is an IF-group.

Sufficiency. Let $B = \bigoplus_{i\in I} B_i$, where B_i is a fully invariant subgroup of the group B for each $i \in I$. Let for some $i_0 \in I$ B_{i_0} be an IF-group. Let us prove that B is an IF-group. Since B_{i_0} is an IF-group, there exists a proper fully invariant subgroup S_{i_0} of the group B_{i_0} such that $S_{i_0} \cong B_{i_0}$. Let

$$S = S_{i_0} + \Bigg(\bigoplus_{\substack{j\in I\\ j\neq i_0}} B_j\Bigg).$$

By properties of direct sums [6], we obtain

$$S = S_{i_0} \oplus \Bigg(\bigoplus_{\substack{j\in I\\ j\neq i_0}} B_j\Bigg).$$

Since $S_{i_0} \neq B_{i_0}$, S is a proper subgroup of the group B. By virtue of the fact that $S_{i_0} \cong B_{i_0}$, we obtain

$$S = S_{i_0} \oplus \Bigg(\bigoplus_{\substack{j\in I\\ j\neq i_0}} B_j\Bigg) \cong B_{i_0} \oplus \Bigg(\bigoplus_{\substack{j\in I\\ j\neq i_0}} sB_j\Bigg) = \bigoplus_{i\in I} B_i = B,$$

i.e., $S \cong B$. Let η be an arbitrary endomorphism of the group B and $s \in S$. Then $s = s_{i_0} + b_{i_1} + b_{i_2} + \cdots + b_{i_k}$, where $b_{i_j} \in B_{i_j}$, $i_j \in I$, $j = \overline{1,k}$, $s_{i_0} \in S_{i_0}$. We have

$$\eta s = \eta(s_{i_0} + b_{i_1} + b_{i_2} + \cdots + b_{i_k}) = \eta s_{i_0} + \eta b_{i_1} + \eta b_{i_2} + \cdots + \eta b_{i_k}.$$

Since B_{i_j} are fully invariant subgroups of the group B for each $j = \overline{1,k}$, $\eta b_{i_j} \in B_{i_j}$. S_{i_0} is a fully invariant subgroup of the group B_{i_0}, and B_{i_0} is a fully invariant subgroup of the group B. Thus, S_{i_0} is a fully invariant subgroup of the group B and, therefore, $\eta s_{i_0} \in S_{i_0}$. We obtain that $\eta s_{i_0} \in S_{i_0}$ for an arbitrary element $s \in S$, and hence S is a fully invariant subgroup of the group B. Thus, B contains a proper fully invariant subgroup S such that $S \cong B$, i.e., B is an IF-group. □

Corollary 3.5. *A torsion group is an IF-group if and only if some of its p-components is an IF-group.*

Proof. Indeed, let A be a torsion group. Then by Theorem 1.1 $A = \bigoplus\limits_p A_p$, where A_p are p-components of the group A. A_p are fully invariant subgroups of the group A. By Theorem 3.4, we obtain our corollary. □

Theorem 3.6. *Any bounded group is not an IF-group.*

Proof. Let B be a bounded group. Then B is a torsion group. It is clear that any p-component of the group B is also a bounded p-group. Since bounded p-groups are not IF-groups by Theorem 3.2, the group B is not an IF-group by Corollary 3.5. □

4. Nonreduced Torsion-Free IF-Groups

In this section, we find the conditions equivalent to the fact that a nonreduced torsion-free group is an IF-group. It was proved that a divisible torsion-free group is not an IF-group.

Theorem 4.1. *A nonreduced p-group A is an IF-group if and only if its reduced part is an IF-group.*

Proof. Necessity. Let A be a nonreduced p-group. Then it has the form $A = R \oplus D_p$, where D_p is a divisible p-group and R is a reduced p-group. Let A be an IF-group. Then there exists a fully invariant subgroup S of the group A such that $S \cong A$ and $S \neq A$. By Theorem 1.13, S has one of the following forms forms:

(1) $S = R' \oplus D_p[p^{k_p}]$, where $k_p \geq \sup\{e(r) \mid r \in R'\}$, R' is a fully invariant subgroup of the group R;
(2) $S = R' \oplus D_p$, where R' is a fully invariant subgroup of the group R.

Consider the first case. Let $k_p \neq \infty$, then

$$D_p[p^{k_p}] = \{d \in D_p \mid p^{k_p} d = 0\}$$

is a bounded group; therefore, it has no divisible subgroups. Hence S is a reduced group. Since A is a nonreduced group and $S \cong A$, we have a contradiction.

If $k_p = \infty$, then the first case coincides with the second case.

Consider the second case. Let $S = R' \oplus D_p$, where R' is a fully invariant subgroup of the group R. Since $A = R \oplus D_p$ and $S \cong A$, we obtain that $R' \cong R$ and R' is a proper subgroup of the group R. Hence R is an IF-group.

Sufficiency. Let A be a nonreduced p-group of the form $A = R \oplus D_p$. Let R be an IF-group. Then there exists a fully invariant subgroup R' of the group R such that $R' \cong R$ and $R' \neq R$. Consider the group $S = R' \oplus D_p$. S is a proper fully invariant subgroup of the group A and $S \cong A$. Hence, A is an IF-group. □

Theorem 4.2. *A divisible p-group is not an IF-group.*

Proof. Let A be a divisible p-group, then $A = D_p$. Assume the contrary, let A be an IF-group. Then there exists a fully invariant subgroup S of the group A such that $S \cong A$ and $S \neq A$. Then by Theorem 1.13 the following cases hold:

(1) $S = D_p[p^{k_p}]$, where k_p is an arbitrary non-negative integer or ∞;
(2) $S = D_p$.

Consider the first case. If $k_p \neq \infty$, then

$$S = D_p[p^{k_p}] = \{d \in D_p \mid p^{k_p} d = 0\}$$

is a bounded group; therefore, it is not a divisible group. Since A is a divisible p-group and $S \cong A$, we have a contradiction.

If $k_p = \infty$, then $S = D_p$, and the first case coincides with the second case.

Consider the second case. Let $S = D_p$. Since $A = D_p$, S is not a proper subgroup of the group A. Hence A is not an IF-group. □

Theorem 4.3. *A divisible torsion-free group is not an IF-group.*

Proof. Let A be a divisible torsion-free group. Then $A = \bigoplus_p A_p$, where A_p are divisible p-groups. By Theorem 4.2 and Corollary 3.5, we obtain that A is not an IF-group. □

Theorem 4.4. *For a nonreduced torsion-free group A, the following conditions are equivalent*:

(1) *A is an IF-group*;
(2) *some p-component of the group A is not a divisible group and has a reduced part which is an IF-group*;
(3) *a reduced part of the group A is an IF-group.*

Proof. (1) $\Longrightarrow$ (2). Let A be a nonreduced torsion-free group, that is, an IF-group. Then by Corollary 3.5 some p-component of this group is an IF-group; by Theorem 4.2, this p-component is not a divisible group. Applying Theorem 4.1, we obtain that the reduced part of this p-component is an IF-group.

(2) $\Longrightarrow$ (3). All of the p-component A_p of the group A can be written in the form $A_p = R_p \oplus D_p$, where R_p is a reduced p-group, and D_p is a divisible p-group. Then the reduced part R of the group A can be written in the form $R = \bigoplus_p R_p$. Since at least one of the groups R_p by condition (2) is an IF-group, the group R is an IF-group by Corollary 3.5.

(3) $\Longrightarrow$ (1). Let R_p and D_p be the reduced and the divisible parts of the p-component A_p of the group A, respectively, i.e., $A_p = R_p \oplus D_p$. Then we have

$$A = \left(\bigoplus_p R_p\right) \oplus \left(\bigoplus_p D_p\right).$$

It is clear that $\bigoplus_p R_p$ is the reduced part of the group A. By Corollary 3.5 the group R_p is an IF-group for some prime number p. By Theorem 4.1 we obtain that the group A_p is an IF-group for this prime number p; by Corollary 3.5, the group A is an IF-group. □

Theorems 4.3 and 4.4 reduce the research of the torsion-free IF-groups to studying reduced primary IF-groups.

5. Primary IF-Groups

Now we turn to the research of the primary separable IF-groups. At first, consider the direct sum of cyclic p-group. Since a bounded group is not an IF-group, we have to consider unbounded groups. We introduce the following definition.

Definition 5.1. Let A be a separable p-group. A strictly increasing sequence of non-negative integers $i_0, i_1, \dots, i_n, \dots$ is said to be *admissible* for the group A if the system of equalities

$$f_A(k) = \sum_{i=i_k}^{i_{k+1}-1} f_A(i), \quad k \in \mathbb{N}_0. \tag{7}$$

is valid for the Ulm–Kaplansky invariants of this group.

Theorem 5.2. *Let B be an unbounded p-group and a direct sum of cyclic groups. The group B is an IF-group if and only if exists admissible sequence, different from the sequence of all non-negative integers ordered by increasing magnitude.*

Proof. Necessity. Let the group B be an IF-group. Note that the the sequence of all non-negative integers ordered by increasing magnitude is admissible for each separable p-group since the system of equalities (7) which defines the admissible sequence has a trivial form in this case: $f_A(k) = f_A(k)$, $k \in \mathbb{N}_0$. Assume that the sequence of all non-negative integers ordered by increasing magnitude is a unique admissible sequence for the group B. If L is a fully invariant subgroup of the group B, then, by Theorem 1.10, it has the following form:

$$L = \bigoplus p^{n_k} B_k,$$

where n_k satisfies inequalities (1) and (2) of Theorem 1.10. We have

$$\begin{aligned} f_L(n) &= r\Bigg(\bigoplus_{k\in\mathbb{N}} p^{n_k} B_k \mid p^{n_k} B_k = \bigoplus \mathbb{Z}(p^{n+1})\Bigg) = r\Bigg(\bigoplus_{k\in\mathbb{N}} p^{n_k} B_k \mid k - n_k = n+1\Bigg) \\ &= \sum_{k\in\mathbb{N}} (r(p^{n_k} B_k) \mid k - n_k = n+1) = \sum_{k\in\mathbb{N}} (r(B_k) \mid k - n_k - 1 = n) \\ &= \sum_{k\in\mathbb{N}} (f_B(k-1) \mid k - n_k - 1 = n). \end{aligned}$$

Thus,

$$f_L(n) = \sum_{k\in\mathbb{N}} (f_B(k-1) \mid k - n_k - 1 = n). \tag{8}$$

Theorem 1.10 implies the following relations:

$$(k+1) - n_{k+1} - 1 \geq (k+1) - (n_k + 1) - 1 = k - n_k - 1, \tag{9}$$

$$(k+1) - n_{k+1} - 1 \leq (k+1) - n_k - 1 = (k - n_k - 1) + 1. \tag{10}$$

Let

$$i_n = \min_{k\in\mathbb{N}} \{k - 1 \mid k - n_k - 1 = n\}.$$

Then we obtain from (8)–(10)

$$f_L(n) = \sum_{i=i_n}^{i_{n+1}-1} f_B(i). \tag{11}$$

Some sums in the right-hand side of equalities (11) can be degenerate, i.e., consisting of one summand (this holds when $i_{n+1} = i_n + 1$). Let $L \cong B$. Then, by equality (11), for each non-negative integer n,

$$f_B(n) = f_L(n) = \sum_{i=i_n}^{i_{n+1}-1} f_B(i).$$

The sequence $i_0, i_1, \ldots, i_n, \ldots$ is admissible for the group B; therefore, under the condition of the theorem, it follows that $i_n = n$ for any n. Taking into account that

$$i_n = \min_{k\in\mathbb{N}} \{k - 1 \mid k - n_k - 1 = n\},$$

we obtain $n_k = 0$ for any k, i.e., $L = B$. This contradicts the fact that B is an IF-group.

Sufficiency. Let us write the group B in the form $B = \bigoplus_{k\in\mathbb{N}} B_k$, where $B_k = \bigoplus \mathbb{Z}(p^k)$. Suppose that there is an admissible sequence $r_0, r_1, r_2, \ldots$ for the group B different from the admissible sequence

$0, 1, 2, \dots$. Then for each $m \in \mathbb{N}_0$ we have

$$f_B(m) = \sum_{r=r_m}^{r_{m+1}-1} f_B(r). \tag{12}$$

Two cases are possible:

(1) $r_0 \neq 0$;
(2) $r_0 = 0$.

We consider all the cases.

(1) Let $r_0 \neq 0$. We construct a subgroup L of the group B in the following way:

$$\begin{aligned} L = pB_1 \oplus p^2 B_2 \oplus \dots \oplus p^{r_0} B_{r_0} \oplus p^{r_0} B_{r_0+1} \oplus p^{r_0+1} B_{r_0+2} \oplus \dots \\ \oplus p^{r_1-1} B_{r_1} \oplus p^{r_1-1} B_{r_1+1} \oplus p^{r_1} B_{r_1+2} \oplus p^{r_1+1} B_{r_1+3} \oplus \dots \\ \oplus p^{r_2-2} B_{r_2} \oplus p^{r_2-2} B_{r_2+1} \oplus p^{r_2-1} B_{r_2+2} \oplus p^{r_2} B_{r_2+3} \oplus \dots \oplus p^{r_3-3} B_{r_3} \oplus \dots, \end{aligned}$$

i.e.,

$$L = \bigoplus p^{n_k} B_k,$$

where $n_{r_j} = n_{r_{j+1}} = r_j - j$ $(j \in \mathbb{N}_0)$; $n_{r_j+k} = r_j - j + k - 1$ $(1 < k < r_{j+1} - r_j + 1)$. L is a proper subgroup of the group B. Using Theorem 1.10, we obtain that L is a fully invariant subgroup of B. Moreover, $L \cong B$ due to the fact that the corresponding Ulm–Kaplansky invariants are equal. Indeed, as follows from the construction of the group L and equalities (7), we obtain for each $m \in \mathbb{N}_0$ that

$$f_L(m) = f_B(r_m) + f_B(r_m + 1) + \dots + f_B(r_{m+1} - 1) = f_B(m).$$

This means that $L \cong B$ but $L \neq B$. Hence, B is an IF-group.

(2) Let $r_0 = 0$. Denote by $k + 1$ $(k \in \mathbb{N}_0)$ the smallest natural number for which $r_{k+1} > k + 1$. Then $r_0 = 0$, $r_1 = 1$, $\dots$, $r_k = k$, and the admissible sequence has the form $0, 1, \dots, k, r_{k+1}, r_{k+2}, \dots$. Equalities (12) for this sequence are written as follows:

$$\begin{aligned} &f_B(0) = f_B(0), \\ &f_B(1) = f_B(1), \\ &\quad \dots \\ &f_B(k-1) = f_B(k-1), \\ &f_B(k) = f_B(k) + f_B(k+1) + \dots + f_B(r_{k+1} - 1), \\ &f_B(q) = f_B(r_q) + \dots + f_B(r_{q+1} - 1) \text{ for each } q > k \quad (q \in \mathbb{N}_0). \end{aligned} \tag{13}$$

The sum in the right-hand side of the $(k + 1)$th equality in (13) is the first non-degenerate sum, i.e., it contains more than one summand.

Consider the following subgroup L of the group B:

$$\begin{aligned} L = B_1 \oplus B_2 \oplus \dots \oplus B_k \oplus B_{k+1} \oplus pB_{k+2} \oplus \dots \\ \oplus p^{r_{k+1}-k-1} B_{r_{k+1}} \oplus p^{r_{k+1}-k-1} B_{r_{k+1}+1} \oplus p^{r_{k+1}-k} B_{r_{k+1}+2} \oplus \dots \\ \oplus p^{r_{k+2}-k-2} B_{r_{k+2}} \oplus p^{r_{k+2}-k-2} B_{r_{k+2}+1} \oplus p^{r_{k+2}-k-1} B_{r_{k+2}+2} \oplus \dots \\ \oplus p^{r_{k+3}-k-3} B_{r_{k+3}} \oplus \dots. \end{aligned}$$

Using Theorem 1.10, we obtain that L is a fully invariant subgroup of B. As follows from the construction of the group B, we have

$$\begin{aligned}
&f_L(0) = f_B(0),\\
&f_L(1) = f_B(1),\\
&\quad \dots\\
&f_L(k-1) = f_B(k-1),\\
&f_L(k) = f_B(k) + f_B(k+1) + \dots + f_B(r_{k+1}-1),\\
&f_L(q) = f_B(r_q) + \dots + f_B(r_{q+1}-1) \text{ for each } q > k \quad (q \in \mathbb{N}_0).
\end{aligned} \tag{14}$$

Comparing (13) and (14), we obtain that $L \cong B$. Since $L \neq B$, B is an IF-group. □

Let us consider arbitrary separable p-groups.

Theorem 5.3. *A separable p-group is not an IF-group if its basic subgroup is not an IF-group.*

Proof. Let A be a separable p-group whose basic subgroup B is not an IF-group. Without loss of generality, one can suppose that A is a reduced p-group. If A is a bounded group, then A is not an IF-group by Theorem 3.2 (note that in this case the basic subgroup of the group A coincides with A). Let A be a nonbounded group. Suppose that A is an IF-group. Then there exists a proper fully invariant subgroup S of the group A such that $S \cong A$. Since A is a reduced separable p-group, it does not contain elements of infinite height [7, Sec. 65]. S is a nonbounded fully invariant subgroup of the group A and, therefore, S is a large subgroup of the group A [4, p. 423]. Thus, $S \cap B$ is a basic subgroup of the group S [4, p. 422].

If $S \cap B = 0$, then, taking into account the fact that the factor group of any p-group with respect to its basic subgroup is a divisible group, we obtain that S is a divisible group. This is impossible because A is a reduced group.

If $S \cap B = B$, then S contains a basic subgroup B of the group A. We have $S + B = A$ as S is a large subgroup of the group A; and the inclusion $B \subset S$ implies $S + B = S$ what is impossible because S is a proper subgroup of the group A.

Thus, $S \cap B$ is a proper non-zero subgroup of the group B. Since $S \cong A$, basic subgroups of the groups S and A are also isomorphic, i.e., $S \cap B \cong B$. Since S is a large subgroup of the group A, $S \cap B$ is a large subgroup of the group B [24, Corollary 2.8]. Thus, we obtain that the basic subgroup B of the group A has a proper fully invariant subgroup $S \cap B$ isomorphic to B. Contradiction. □

Theorem 5.4. *If an unbounded separable p-group is an IF-group, then there exist an admissible sequence for this group different from the sequence of all non-negative integers ordered by increasing magnitude.*

Proof. Let A be an unbounded separable p-group being an IF-group, and let B be a basic subgroup of the group A. Then B is an IF-group by Theorem 5.3. Applying Theorem 5.2, we obtain that there exists an admissible sequence for the group B different from the sequence of all non-negative integers ordered by increasing magnitude. Since $f_A(k) = f_B(k)$ for each $k \in \mathbb{N}_0$ [24, p. 186], then the same sequence is admissible for the group A. □

A *torsion complete p-group* is the torsion part $T(\bar{B})$ of the p-adic completion $\bar{B}$ of a direct sum B of cyclic p-groups [7, Sec. 68]. These groups were first studied by L. Ya. Kulikov; he called them closed groups [21]. Such groups play a very important part in studying p-groups.

Theorem 5.5. *For a torsion complete p-group A, the following conditions are equivalent*:

(1) *A is an IF-group*;
(2) *a basic subgroup of the group A is an IF-group*;
(3) *A is an unbounded group for which there exists an admissible sequence different from the sequence of all non-negative integers ordered by increasing magnitude.*

Proof. (1) $\sim$ (2). By virtue of Theorem 5.3, one should prove only (2) $\Longrightarrow$ (1). Let A be a torsion complete p-group and B be its basic subgroup being an IF-group. By Theorem 3.2 B is an unbounded group, so A is also an unbounded group. Since B is an IF-group, then there exists a proper fully invariant subgroup S of the group B such that $B \cong S$. It is clear that S is a proper large subgroup of the group B. There exists a proper large subgroup S^* of the group A such that $S^* \cap B = S$ [4, Theorem 2.9], and S is a basic subgroup of the group S^* [4, p. 422]. The group S^* is a torsion complete group as a large subgroup of a torsion complete group [9]. Therefore, we obtain that, in the group A, there exists a proper fully invariant subgroup S^* such that the basic subgroup B of the group A isomorphic to the basic subgroup S of the group S^*. Since A and S^* are torsion complete groups, $A \cong S^*$, i.e., A is an IF-group.

(2) $\Longrightarrow$ (3). Let B be a basic subgroup of the group A, and B be an IF-group. If A is a bounded group, then $A = B$. Hence, B is a bounded IF-group, which contradicts to Theorem 3.6. If A is an unbounded group, then B is unbounded. Taking into account Theorem 5.2 and the fact that for each $k \in \mathbb{N}_0$ $f_A(k) = f_B(k)$, we obtain that for the group A there exists an admissible sequence different from the sequence of all non-negative integers ordered by increasing magnitude.

(3) $\Longrightarrow$ (1). Let A be an unbounded group for which there exists an admissible sequence different from the sequence of all non-negative integers ordered by increasing magnitude; then the basic subgroup B of the group A has the same property. Then by Theorem 5.2 B is an IF-group, and, by equivalence (2) $\sim$ (1), the group A is also an IF-group. □

We say that the sequence of the Ulm–Kaplansky invariants of an unbounded separable p-group A is *periodic* if there exists $k \in \mathbb{N}$ such that the equality $f_A(n) = f_A(n+k)$ holds for all $n \in \mathbb{N}_0$.

Corollary 5.6. *Let A be a torsion complete p-group. If the sequence of the Ulm–Kaplansky invariants of the group A is periodic, then A is an IF-group.*

Proof. Let A be a torsion complete p-group and let there exist $k \in \mathbb{N}$ such that the equality $f_A(n) = f_A(n+k)$ holds for all $n \in \mathbb{N}_0$. Then for this group the sequence $k, k+1, k+2, \dots$ is admissible and then by Theorem 5.5 A is an IF-group. □

Corollary 5.7. *If for a torsion complete p-group A there exists a cardinal number γ such that $f_A(n) = \gamma$ for each $n \in \mathbb{N}_0$, then A is an IF-group.*

Proof. Let A be a torsion complete p-group and let $f_A(n) = \gamma$ for each $n \in \mathbb{N}_0$, where γ is some cardinal number. Then such a sequence of the Ulm–Kaplansky invariants is periodic because $f_A(n) = f_A(n+1)$ for each $n \in \mathbb{N}_0$. Applying Corollary 5.6, we obtain that A is an IF-group. □

6. Nonreduced and Divisible Torsion-Free Groups Containing Fully Invariant Subgroups Isomorphic to the Group

Let us consider torsion-free groups that contain proper fully invariant subgroups isomorphic to the group.

Theorem 6.1. *The torsion-free group A contains the proper fully invariant subgroup isomorphic to the group if and only if A is a non-divisible group.*

Proof. Necessity. Let A be a torsion-free group that contains a proper fully invariant subgroup S isomorphic to the group A. Let us suppose that A is a divisible group. By Theorem 1.13 we obtain that $S = A$, which contradicts to the fact that S is a proper subgroup of the group A.

Sufficiency. Let A be a torsion-free group that is not a divisible group. There exist a natural number n, different from 1, such that $nA \neq A$. Consider $S = nA$. Then S is a fully invariant subgroup of the group A. Since A is a torsion-free group, then $S \cong A$. This means that A contains a fully invariant subgroup isomorphic to the group. □

Thus, a torsion-free group A that is not divisible always contains a proper fully invariant subgroup of the form nA isomorphic to the group. Below, we consider torsion-free groups that have a proper fully invariant subgroup different from nA and isomorphic to the group A.

Definition 6.2. A torsion-free group A is called *IF-group* if it contains a proper fully invariant subgroup different from nA that is isomorphic to the group.

The result below follows from Theorem 6.1.

Theorem 6.3. *A divisible torsion-free group is not an IF-group.*

Consider nonreduced torsion-free groups.

Theorem 6.4. *A nonreduced torsion-free group is an IF-group if and only if its reduced part is an IF-group.*

Proof. Necessity. Let A be a nonreduced torsion-free group. Then it has the form $A = R \oplus D_0$, where D_0 is a divisible torsion-free group and R is a reduced torsion-free group. Let A be an IF-group. Then, there exist a fully invariant subgroup S of the group A such that $S \cong A$, $S \neq A$, and $S \neq nA$. By Theorem 1.13, S has the following form: $S = R' \oplus D_0$, where R' is a fully invariant subgroup of the group R. Since $A = R \oplus D_0$ and $S \cong A$, we obtain that $R' \cong R$ and R' is a proper subgroup of the group R. $S \neq nA$; therefore, $R' \oplus D_0 \neq n(R \oplus D_0) = nR \oplus D_0$. We obtain that $R' \neq nR$. Hence R is an IF-group.

Sufficiency. Let A be a nonreduced torsion-free group. $A = R \oplus D_0$, where R is a reduced torsion-free group, D_0 is a divisible torsion-free group. Let R be an IF-group. Then there exists a fully invariant subgroup R' of the group R such that $R' \cong R$, $R' \neq R$, and $R' \neq nR$ for each $n \in \mathbb{N}$. Consider the group $S = R' \oplus D_0$. S is a proper fully invariant subgroup of the group A, $S \cong A$, and $S \neq nA$. Hence, A is an IF-group. □

7. Homogeneous χ-Groups

Below, by virtue of Theorems 6.3 and 6.4, we consider only reduced groups.

Let $\mathbf{t}$ be a type. Consider a characteristics v that satisfies the following conditions:

(a) $v = (v^{(1)}, v^{(2)}, \ldots, v^{(n)}, \ldots) \leq w$ for some characteristic $w \in \mathbf{t}$;

(b) $v^{(k)} = \infty$ if the type $\mathbf{t}$ is p_k-divisible.

Let us denote by $\mathfrak{F}(\mathbf{t})$ a set consisting of all characteristics satisfying the properties and characteristics whose elements are only symbols ∞.

Let A be a torsion-free group. If $v \in \mathfrak{X}$, we denote by $A(v)$ the following subgroup of the group A:

$$A(v) = \{a \in A \mid \chi(a) \geq v\}.$$

$A(v)$ is a fully invariant subgroup of the group A. Note that if A is a reduced group and the characteristic v consists only of symbols ∞, then $A(v) = 0$.

A reduced torsion-free group A is called a *χ-group* if each fully invariant subgroup S of the group A is given by $S = A(v)$, where v is a characteristic [10]. The reduced torsion-free group A is called *fully transitive* if for any two elements a and b of its group such that $\chi(a) \leq \chi(b)$, there exists an endomorphism φ of its group such that $\varphi(a) = b$ [12].

Let A be a homogeneous χ-group of the type $\mathbf{t}$. It is proved in [11] that each fully invariant subgroup S of the group A is uniquely presented in the form $S = A(v)$, where v is a characteristic belonging to $\mathfrak{F}(\mathbf{t})$. Note that if $v \in \mathfrak{F}(\mathbf{t})$, where $v = \big(v^{(1)}, v^{(2)}, \ldots, v^{(n)}, \ldots\big)$, and $v \leq w$, where $w = \big(w^{(1)}, w^{(2)}, \ldots, w^{(n)}, \ldots\big) \in \mathbf{t}$, then the type of the group $A(v)$ is defined by the characteristic

$$w - v = \big(w^{(1)} - v^{(1)}, w^{(2)} - v^{(2)}, \ldots, w^{(n)} - v^{(n)}, \ldots\big).$$

Let us prove main theorem of this section.

Theorem 7.1. *Homogeneous χ-groups are not IF-groups.*

Proof. Let A be a homogeneous χ-group of the type $\mathbf{t}$. Let us suppose that A is an IF-group. Then there exists a fully invariant subgroup S of the group A such that $S \cong A$ and $S \neq nA$. We have $S = A(v)$, where $v \in \mathfrak{F}(\mathbf{t})$. $A(v)$ is a homogeneous group. As $S \cong A$, $A(v) \cong A$; hence the type of the group $A(v)$ is the same as the type of the group A.

Taking into account that $v \in \mathfrak{F}(\mathbf{t})$, we prove the existence of a characteristic

$$w = \left(w^{(1)}, w^{(2)}, \ldots, w^{(n)}, \ldots\right)$$

belonging to the type $\mathbf{t}$ such that $v \leq w$. Let

$$I(w) = \left\{i \in \mathbb{N} \mid w^{(i)} \neq 0 \text{ and } w^{(i)} \neq \infty\right\}.$$

At first, we consider the case when the type $\mathbf{t}$ is not idempotent. Then $I(w)$ is an infinite set. Since $v \in \mathfrak{F}(\mathbf{t})$, $v^{(i)} = w^{(i)}$ when $i \in \mathbb{N} \setminus I(w)$. Let I' be a subset of the set $I(w)$, consisting of all natural numbers i, for which $v^{(i)} \neq 0$. Since $A(v)$ is a proper subgroup of the group A, then $I' \neq \varnothing$. Then $w^{(i)} - v^{(i)} \neq w^{(i)}$ for any $i \in I'$. Taking into account that the type of the group $A(v)$ is defined by the characteristic $w - v$ and $t\big(A(v)\big) = t(A)$, we obtain that I' is a finite set. Let $n = \prod\limits_{i \in I'} p_i^{v^{(i)}}$. Then $S = A(v) = nA$. Contradiction.

Let the type $\mathbf{t}$ be idempotent. Then the set $I(w)$ is finite. Since $v \in \mathfrak{F}(\mathbf{t})$ and $v \leq w$, $v^{(i)} = w^{(i)}$ when $i \in \mathbb{N} \setminus I(w)$. Let

$$I' = \left\{i \in \mathbb{N} \mid v^{(i)} \neq 0 \text{ and } i \in I(w)\right\}.$$

I' is an non-empty finite subset of the set $I(w)$. Then $S = A(v) = nA$, where $n = \prod\limits_{i \in I'} p_i^{v^{(i)}}$. Contradiction. $\square$

Using the fact that any homogeneous fully transitive group is a χ-group and that any homogeneous reduced separable group is a fully transitive group [12], we obtain the following results.

Corollary 7.2. *A proper fully invariant subgroup S of a homogeneous fully transitive group A is isomorphic to the group A if and only if $S = nA$ for some natural number n different from unity.*

Corollary 7.3. *A proper fully invariant subgroup S of a homogeneous reduced separable group A is isomorphic to the group A if and only if $S = nA$ for some natural number n different from unity.*

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