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ON TYPE-2 FUZZY SETS AND TYPE-2 FUZZY SYSTEMS

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Abstract. One of the advantages of systems based on fuzzy logic (fuzzy systems) is the possibility of a soft switch from one set of values of input parameters of the system to another, when different conclusions are drawn for different sets of these values. A fuzzy set of type 2 is a direct generalization of an ordinary fuzzy set. In this paper, we review some branches of the theory of type-2 fuzzy sets and the theory of type-2 fuzzy systems. We discuss operations on type-2 fuzzy sets, type-2 fuzzy relations, and the centroids of type-2 fuzzy sets and describe type-2 functional fuzzy systems and type-2 relational fuzzy systems.

Keywords and phrases: fuzzy set of type 2, fuzzy relation of type 2, functional fuzzy system, relational fuzzy system.

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Introduction. In classical set theory, for any subset of a certain set (called the universal set), the membership degree of an element of the universal set in the subset considered can be either 0 (the element does not belong to the subset) or 1 (the element belongs to the subset). In the theory of fuzzy set, the membership degree of an element of the universal set in a fuzzy subset (fuzzy set) can be arbitrary real number from $[0;1]$. In the theory of fuzzy sets of type 2, the membership degree itself of an element of the universal set is not an unambiguously number: it is a fuzzy number whose support belongs to the segment $[0,1]$.

The theory of fuzzy sets is of great theoretical and practical importance in mathematical modeling of uncertainty; there is an extensive literature on this scientific direction (see, e.g., [12, 19] and the references therein). This theory allows one to study a different kind of uncertainty than probability theory. In probability theory, a certain probability is assigned to each group of values of an unknown quantity. In the theory of fuzzy sets, the values themselves are determined ambiguously. The theory of type-2 fuzzy sets and type-2 fuzzy systems is a scientific direction that is in a state of intensive development and has great practical significance (see, e.g., [1, 15]).

Type-2 fuzzy sets were introduced in [28]. The theory of these sets was further developed in [16, 17]. Reviews of the theory and its applications in various applied fields are given, for example, in [2, 7, 9, 13, 23, 25]. Various results related to the theory of type-2 fuzzy systems are presented, in particular, in [8, 10, 14, 26, 30], as well as in many other works.

By fuzzy systems, fuzzy control systems are often understood. Such systems are considered, for example, in [5, 6, 11, 24, 27, 29]. (A classic example of a fuzzy control system is as follows. If an obstacle is close and the speed is high, then one must brake sharply; if an obstacle is far or the speed is low, then one must slow down smoothly. Two fuzzy rules and six fuzzy sets are used here, namely, the following three "bunches" of fuzzy sets: close obstacle—distant obstacle, low speed—high speed, and smooth braking—sharp braking. For example, the membership degree of the speed 40 km/h to a fuzzy set of high speeds, depending on the task, can be taken equal to 0.3, but it can also be taken equal to 0.8 in another problem. The membership degree of the speed 60 km/h to the fuzzy set of high speeds for various problems can be, respectively, 0.5 and 0.9. Thus, the ambiguity of the concept

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(value) “high speed” can be taken into account. Such bunches are called linguistic variables. Of course, a bunch can include more than two fuzzy sets.) However, the term “fuzzy system” is still broader than “fuzzy control system.” For example, a software package designed for study the stock market and developing trading strategies is presented in [3, 4]; this package is based on fuzzy systems, but the controls are not included in the model. Similar econometric fuzzy systems were also considered by other authors. Fuzzy systems are also called systems based on fuzzy logic. One of the advantages of this approach is the ability to gently switch. So, in the example considered, a soft switch is made from a situation with a close obstacle to a situation with a distant obstacle.

If fuzzy rules are based on type-2 fuzzy sets, then we speak of fuzzy systems of type-2. This modification allows one to achieve better results in a number of problems. Type-2 fuzzy systems, like ordinary fuzzy systems, are divided into two classes:functional and relational. There are few reviews on the theory of type-2 fuzzy systems in Russian; for example, we indicate [20, Chap. 5] and [18]. However, none of these two papers discuss, for example, functional type-2 fuzzy systems. This review partially fills the gap.

The widespread use of type-1 fuzzy systems in various applied fields is largely due to the fact that such systems are universal approximators (see, e.g., [22]). The dependence between input and output variables of the system may have a nonlinear character, not known in advance. Fuzzy systems allow one to give correct approximations for a wide class of such dependencies. Of course, type-2 fuzzy systems are also universal approximators since they are form a wider class of systems than type-1 fuzzy systems. The development of the theory of type-2 fuzzy systems and applications of this theory is a promising scientific field.

This paper is organized as follows. Section 1 provides necessary definitions of the theory of type-2 fuzzy sets. In Sec. 2, the centroid of a type-2 fuzzy set is described and functional type-2 fuzzy systems are considered. Section 3 contains definitions related to type-2 fuzzy relations, extended t-norms, and the t-conorm; also relational type-2 fuzzy systems are considered. Definitions and historical references related to ordinary fuzzy sets are not discussed in this paper; they can be found, for example, in the review [21].

1. Type-2 fuzzy sets.

Let X be a certain universal set. For an ordinary fuzzy set A, the membership degree of an element $x \in X$ in this fuzzy set is a real $\mu_A(x)$ from the segment $[0, 1]$.

However, in many applications it remains unclear how to determine the membership degree of each element x in a fuzzy set A. Therefore, the idea arose that the membership degree is a fuzzy set whose support belongs to the segment $[0, 1]$. It turns out that this approach allows one to get the best results in a number of applied problems.

Consider the function $\mu : X \times [0, 1] \to [0, 1]$.

Definition 1. The graph of the function μ, i.e., the set of pairs

$$\big((x, w),\ \mu(x, w)\big), \quad x \in X,\ w \in [0, 1],$$

is called a *type-2 fuzzy set* (FS2).

Below, ordinary fuzzy set will be called type-1 fuzzy sets (FS1).

Remark. Type-2 fuzzy sets can be considered as type-1 fuzzy sets with the universal set $X \times [0, 1]$. However, this approach is not convenient, for example, when introducing the operations of union, intersection, and complement for FS2. In addition, in the sense of the considered applied problems, the universal set is X, not $X \times [0, 1]$. We denote by μ_A the function μ for an FS2 A.

Let A be an FS2; for any $x \in X$, we consider the set

$$J_x = \big\{w \in [0, 1] :\ \mu_A(x, w) > 0\big\}.$$

In Definitions 2–7 below, A is an FS2.

Definition 2. The *primary membership function* for A is a set-valued function, which assigns the set J_x ti each element $x \in X$.

Definition 3. The *lower membership function* for A is the function $\underline{\mu}_A : X \to [0,1]$ defined by the relation

$$\underline{\mu}_A(x) = \inf J_x, \quad x \in X.$$

Definition 4. The *upper membership function* for A is the function $\overline{\mu}_A : X \to [0,1]$ defined by the relation

$$\overline{\mu}_A(x) = \sup J_x, \quad x \in X.$$

Definition 5. If for any $x \in X$ there exists a unique element $w_x \in [0,1]$ for which $\mu_A(x, w_x) = 1$, then the function, which assigns the element w_x to an element x, is called the *principal membership function* for A.

Definition 6. For fixed $x \in X$, the function $\mu_A(x, w)$ considered as a function of the argument w is called the *secondary membership function* for A.

Definition 7. The *figure of uncertainty* of A is the set

$$FOU(A) = \big\{(x, w) : \ x \in X, \ w \in J_x\big\}.$$

Definition 8. A type-2 fuzzy set A is called an *interval type*-2 *fuzzy set* (IFS2) if $\mu_A(x, w) = 1$ for all $x \in X$, $w \in J_x$.

Definition 9. A type-1 fuzzy set C with an universal set X is said to be *imbedded* in an FS2 A if

$$\underline{\mu}_A(x) \le \mu_C(x) \le \overline{\mu}_A(x)$$

for all $x \in X$.

Remark. We can also consider type-3 fuzzy sets, when for any $x \in X$ and $w \in [0,1]$, the value $\mu(x, w)$ is not a real number, but FS1 with a support belonging to the segment $[0,1]$, as well as fuzzy sets of higher types. However, the practical value of such mathematical constructions is much less than the practical value of fuzzy sets of types 1 and 2.

In the study of FS2, we use the following notation customary for FS1:

$$A = \int\limits_{x \in X} \int\limits_{w \in [0,1]} \frac{\mu_A(x, w)}{(x, w)};$$

moreover, the sign $\int$ is often used instead of the sign $\sum$ even in the cases where each of the sets X and J_x contains only a finite number of elements. Sometimes, it is more convenient to write the last formula in the form

$$A = \int\limits_{x \in X} \Bigg(\int\limits_{w \in [0,1]} \frac{\mu_A(x, w)}{w} \Bigg) \Big/ x,$$

emphasizing that with each x, a certain FS1 is associated whose support belongs to the segment $[0,1]$.

For $0 < \eta \le 1$, we consider the η-shears of the FS2 A:

$$A_\eta = \big\{(x, w) : \ \mu_A(x, w) \ge \eta\big\}.$$

We have

$$\mu_A(x, w) = \sup_{\eta \in (0,1]} \eta \cdot I_{A_\eta}(x, w)$$

for all $x \in X$, $w \in [0,1]$, where I_{A_η} is the indicator of the set A_η. The proof of this formula is the same as the proof of a similar formula for FS1.

Let A and B be two FS2 with the same universal set X.

Definition 10. The *union* of A and B is the FS2

$$A \cup B = \int\limits_{x \in X} \left(\int\limits_{w \in [0,1]} \frac{\sup\limits_{u \in [0,1],\ v \in [0,1]:\ u \vee v = w} \mu_A(x,u) \wedge \mu_B(x,v)}{w} \right) \bigg/ x,$$

where $\wedge$ and $\vee$ mean the minimum and the maximum, respectively.

Definition 11. The *intersection* of A and B is the FS2

$$A \cap B = \int\limits_{x \in X} \left(\int\limits_{w \in [0,1]} \frac{\sup\limits_{u \in [0,1],\ v \in [0,1]:\ u \wedge v = w} \mu_A(x,u) \wedge \mu_B(x,v)}{w} \right) \bigg/ x.$$

Definition 12. The *complement* of A is the FS2

$$\overline{A} = \int\limits_{x \in X} \left(\int\limits_{w \in [0,1]} \frac{\mu_A(x,w)}{1-w} \right) \bigg/ x.$$

2. Centroid of a type-2 fuzzy set and functional fuzzy systems. Let the universal set X be a finite set whose elements are real numbers, $X = \{x_1, \ldots, x_m\}$.

Recall that if A is an FS1, then the centroid of A is the real number

$$\left(\sum_{i=1}^{m} x_i \mu_A(x_i) \right) \bigg/ \left(\sum_{i=1}^{m} \mu_A(x_i) \right).$$

Let A be an FS2 and let each of the sets J_{x_i}, $i = 1, \ldots, m$, contain a finite number of elements. Construct an FS1, which can serve as the centroid of A.

Assume that the support of the desired FS1 is the set of all real numbers of the form

$$\left(\sum_{i=1}^{m} x_i w_i \right) \bigg/ \left(\sum_{i=1}^{m} w_i \right), \tag{1}$$

where $w_1 \in J_{x_1}, \ldots, w_m \in J_{x_m}$. This is the union of the centroids of all imbedded FS1. If the set J_{x_i} contains n_i elements, then the total number of the numbers of the form (1) may be equal to $\prod\limits_{i=1}^{m} n_i$. However, some of these numbers can coincide.

For a real number of the form (1), its membership degree in the centroid A is assumed to be

$$\mu_A(x_1, w_1) \wedge \cdots \wedge \mu_A(x_m, w_m). \tag{2}$$

Note (see [10]) that in this case, we cannot use the product instead the minimum. If an element of the support admits several distinct representations of the form (1), then, in accordance with the expansion principle, one need to take the maximum of the numbers (2) as its membership degree in FS1. The centroid of the FS2 A constructed by this way is denoted by C_A; as was already mentioned, it is a FS1.

If A is an IFS2, then each of the numbers (2) is equal to 1.

Now we pass to the case where each of sets J_{x_i}, $i = 1, \ldots, m$, is a segment of a real line (belonging to the segment $[0,1]$); A is an IFS2. We construct an FS1, which can serve as the centroid of A.

Assume that the support of the desired FS1 is the set of real numbers of the form (1), where $w_1 \in J_{x_1}, \ldots, w_m \in J_{x_m}$. It is easy to see that the support constructed by this way is a segment of the real number; we denote it by $[y_l, y_r]$.

Indeed, if

$$y' = \left(\sum_{i=1}^{m} x_i w_i'\right) \Big/ \left(\sum_{i=1}^{m} w_i'\right), \quad y'' = \left(\sum_{i=1}^{m} x_i w_i''\right) \Big/ \left(\sum_{i=1}^{m} w_i''\right),$$

then for any $y \in [y', y'']$, the segment connecting the points $(w_1', \dots, w_m')$ and $(w_1'', \dots, w_m'')$ contains a point $(w_1, \dots, w_m)$ for which the value of the fraction (1) is equal to y since the function (1) considered as a function of the argument $(w_1, \dots, w_m)$ is continuous on this segment. The boundedness of the function (1) considered as a function of the argument $(w_1, \dots, w_m)$ for fixed $x_1, \dots, x_m$ is obvious. The closedness of the set of all values of the function (1) for $(w_1, \dots, w_m) \in J_{x_1} \times \dots \times J_{x_m}$ is also obvious. The case where $0 \in J_{x_1}, \dots, 0 \in J_{x_m}$ must be considered separately. In this case, if each of the segments $J_{x_1}, \dots, J_{x_m}$ has a nonzero length, then $[y_l, y_r] = \left[\min\limits_{1 \le i \le m} x_i, \ \max\limits_{1 \le i \le m} x_i\right]$. At the point $(0, \dots, 0)$, the function (1) is undefined.

To complete the construction of the centroid C_A, we set $\mu_{C_A}(y) = 1$ for any $y \in [y_l, y_r]$.

Functional fuzzy systems are also called Takagi–Sugeno fuzzy systems. Consider a system with real numbers $x_1^0, \dots, x_n^0$ at the input and a real number y^0 at the output. Let $x_1^0 \in X_1, \dots, x_n^0 \in X_n$, and $y^0 \in Y$, where $X_1, \dots, X_n$ and Y are certain subsets of the set of real numbers $\mathbb{R}$.

The base consists of k fuzzy rules of the following form:

$$\text{IF } (x_1 = A_{j1}) \text{ AND } \dots \text{ AND } (x_n = A_{jn}) \text{ THEN } y = f_j(x_1, \dots, x_n),$$

where $j = 1, \dots, k$. We assume that A_{jp} is an IFS2 with the universal set X_p, $p = 1, \dots, n$, and $f_j : X_1 \times \dots \times X_n \to \mathbb{R}$, $j = 1, \dots, k$, are some functions for each j.

The essence of the fuzzy system is that for various domains of input parameter values $(x_1^0, \dots, x_n^0)$, it is necessary to draw different conclusions (use different forms of dependence of the output parameters on the input parameters) and ensure soft switching. If the activation degree of the fuzzy rule j for a given set of input parameters is large, then the contribution of f_j to the final general dependence is also large, and vice versa, for a small activation degree of the fuzzy rule, the contribution is small.

A description of Takagi–Sugeno fuzzy systems for the case where A_{jp}, $p = 1, \dots, n$, are FS1, can be found, e.g., in [19].

Remark. The most important case is the case where functions $f_1, \dots, f_k$ are linear:

$$f_j(x_1, \dots, x_n) = a_{j0} + a_{j1}x_1 + \dots + a_{jn}x_n,$$

where $a_{j0}, a_{j1}, \dots, a_{jn}$ are real numbers. Theoretically, we can also consider models for which a_{j0}, a_{j1}, $\dots, a_{jn}$ are FS1, but in practice such models are apparently not yet used.

Let $\underline{\mu}_{A_{j1}}, \dots, \underline{\mu}_{A_{jn}}$ and $\overline{\mu}_{A_{j1}}, \dots, \overline{\mu}_{A_{jn}}$ are the lower and upper membership functions of IFS2 $A_{j1}, \dots, A_{jn}$, $j = 1, \dots, k$, respectively. For each j, $1 \le j \le k$, consider the segments $\left[\underline{\tau}_j, \overline{\tau}_j\right]$, where

$$\underline{\tau}_j = \min\left(\underline{\mu}_{A_{j1}}(x_1^0), \dots, \underline{\mu}_{A_{jn}}(x_n^0)\right), \quad \overline{\tau}_j = \min\left(\overline{\mu}_{A_{j1}}(x_1^0), \dots, \overline{\mu}_{A_{jn}}(x_n^0)\right),$$

or

$$\underline{\tau}_j = \underline{\mu}_{A_{j1}}(x_1^0) \cdots \underline{\mu}_{A_{jn}}(x_n^0), \quad \overline{\tau}_j = \overline{\mu}_{A_{j1}}(x_1^0) \cdots \overline{\mu}_{A_{jn}}(x_n^0).$$

Except for the minimum and the product, other t-norms can be used. (For the definitions of t-norms and t-conorms, see, e.g., [20].) An FS1 with the support $\left[\underline{\tau}_j, \overline{\tau}_j\right]$ is called the *activation degree* of the jth fuzzy rule.

To determine the output of a fuzzy system, the activation degree of each fuzzy rule must be linked with the output value for this fuzzy rule. Consider the segment $[y_l, y_r]$ consisting of all points of the form

$$\left(\sum_{j=1}^{k} \tau_j f_j\left(x_1^0, \ldots, x_n^0\right)\right) \Big/ \left(\sum_{j=1}^{k} \tau_j\right), \tag{3}$$

where $\tau_j \in \left[\underline{\tau}_j, \overline{\tau}_j\right]$, $j = 1, \ldots, k$. The proof of the fact that this set of points is a segment is similar to the above. Thus,

$$y_l = \min_{\substack{\tau_j \in \left[\underline{\tau}_j, \overline{\tau}_j\right], \\ j=1,\ldots,k}} \frac{\sum_{j=1}^{k} \tau_j f_j\left(x_1^0, \ldots, x_n^0\right)}{\sum_{j=1}^{k} \tau_j}, \quad y_r = \max_{\substack{\tau_j \in \left[\underline{\tau}_j, \overline{\tau}_j\right], \\ j=1,\ldots,k}} \frac{\sum_{j=1}^{k} \tau_j f_j\left(x_1^0, \ldots, x_n^0\right)}{\sum_{j=1}^{k} \tau_j}.$$

Further, $y^0 = \left(y_l + y_r\right)/2$.

The method of finding y^0 presented above is one of the most common. However, one can use a simpler method in which y^0 is defined by the formula (3), in which $\tau_j = \left(\underline{\tau}_j + \overline{\tau}_j\right)/2$.

3. Type-2 fuzzy relations and relational fuzzy system.

Let X and Y be two sets used as universal sets below.

Definition 13. A *type-2 fuzzy relation* between the sets X and Y is an FS2 with the universal set $X \times Y$.

Let $D_1, \ldots, D_n$ be FS1 with the segment $[0,1]$ as the universal sets and $\mu_{D_1}, \ldots, \mu_{D_n}$ are the membership functions of these fuzzy sets.

Let Δ be a certain t-norm and ∇ be a certain t-conorm. As is known, for any numbers $a_1, \ldots, a_n$ from the segment $[0,1]$, $a_1 \Delta \ldots \Delta a_n$ and $a_1 \nabla \ldots \nabla a_n$ are also numbers from $[0,1]$. Common examples of t-norms are the product and minimum operations; an example of a t-conorm is the maximum operation.

In the theory of type-2 fuzzy systems, the operations of extended t-norm $\overset{\Delta}{*}$ and extended t-conorm $\overset{\nabla}{*}$ are used. The results of application of these operations to the membership functions $\mu_{D_1}, \ldots, \mu_{D_n}$, are also membership functions whose supports lie in $[0,1]$. Let $\overline{\Delta}$ be a t-norm, which can coincide or not with the t-norm Δ.

Definition 14. The result of application of the *extended t-norm* $\overset{\Delta}{*}$ to the membership functions $\mu_{D_1}, \ldots, \mu_{D_n}$ is the membership function

$$\left(\mu_{D_1} \overset{\Delta}{*} \ldots \overset{\Delta}{*} \mu_{D_n}\right)(w) = \sup_{\substack{u_1 \in [0,1], \ldots, u_n \in [0,1]: \\ u_1 \Delta \ldots \Delta u_n = w}} \mu_{D_1}(u_1) \overline{\Delta} \ldots \overline{\Delta} \mu_{D_n}(u_n),$$

where $w \in [0,1]$.

Definition 15. The result of application of the *extended t-conorm* $\overset{\nabla}{*}$ to the membership functions $\mu_{D_1}, \ldots, \mu_{D_n}$ is the membership function

$$\left(\mu_{D_1} \overset{\nabla}{*} \ldots \overset{\nabla}{*} \mu_{D_n}\right)(w) = \sup_{\substack{u_1 \in [0,1], \ldots, u_n \in [0,1]: \\ u_1 \nabla \ldots \nabla u_n = w}} \mu_{D_1}(u_1) \overline{\Delta} \ldots \overline{\Delta} \mu_{D_n}(u_n),$$

where $w \in [0,1]$.

Let R be a type-2 fuzzy relation between sets X and Y, S be a type-2 fuzzy relation between sets Y and Z, and A be an FS2 with the universal set X. Let $\overset{\Delta}{*}$ be a certain extended t-norm and $\overset{\nabla}{*}$ be an extended t-conorm.

Let Y be a finite set.

Definition 16. The *combination* of the type-2 fuzzy relations R and S is the type-2 fuzzy relation $R \circ S$ between the sets X and Z whose secondary membership functions has the form

$$\mu_{R\circ S}(x,z) = \overset{\triangledown}{\underset{y\in Y}{\ast}} \Big(\mu_R(x,y) \overset{\vartriangle}{\ast} \mu_S(y,z)\Big), \quad x \in X, \quad z \in Z.$$

Let X be a finite set and Y be an arbitrary set.

Definition 17. The *combination* of an FS2 A and a type-2 fuzzy relation R is the FS2 $A \circ R$ with the universal set Y whose secondary membership functions have the form

$$\mu_{A\circ R}(y) = \overset{\triangledown}{\underset{x\in X}{\ast}} \Big(\mu_A(x) \overset{\vartriangle}{\ast} \mu_R(x,y)\Big), \quad y \in Y.$$

We emphasize that in Definitions 16 and 17, the argument of all membership functions is w.

Let A_p be FS2 with the universal sets X_p, $p = 1,\dots,n$. We denote the secondary membership functions of the FS2 A_j by $\mu_{A_j}(x_j)$, where $x_j \in X_j$. Each of the functions $\mu_{A_j}(x_j)$ is a function of the argument w, $w \in [0,1]$.

Definition 18. The *Cartesian product* of the FS2 $A_1,\dots,A_n$ is the FS2 $A_1 \times \dots \times A_n$ whose universal set is $X_1 \times \dots \times X_n$ and the secondary membership functions have the form

$$\mu_{A_1\times\dots\times A_n}\big(x_1,\dots,x_n\big) = \mu_{A_1}(x_1) \overset{\vartriangle}{\ast} \dots \overset{\vartriangle}{\ast} \mu_{A_n}(x_n),$$

where $\overset{\vartriangle}{\ast}$ is a certain extended t-norm.

Relational fuzzy systems are also called Mamdani fuzzy systems. Consider a system with real numbers $x_1^0,\dots,x_m^0$ at the input and a real number y^0 at the output. Let $x_1^0 \in X_1, \dots, x_n^0 \in X_n$, $y^0 \in Y$, and let $X_1,\dots,X_n$, Y be certain subsets of the set of real numbers $\mathbb{R}$.

The base consists of k fuzzy rules of the following form:

$$\text{IF } (x_1 = A_{j1}) \text{ AND } \dots \text{ AND } (x_n = A_{jn}) \text{ THEN } (y = B_j),$$

where $j = 1,\dots,k$. We assume that A_{jp} are FS2 with the universal sets X_p, $p = 1,\dots,n$, and B_j are FS2 with the universal set Y for each j.

A description of Mamdani fuzzy systems for the case where $A_{j1},\dots,A_{jn}$, and B_j are FS1 can be found, e.g., in [19].

Type-2 fuzzy relations R_j between the sets $X = X_1 \times \dots \times X_n$ and Y, $j = 1,\dots,k$, are assumed to be known.

At the fuzzification stage, we construct an FS2 A' with the universal set X by the input data $x_1^0 \in X_1, \dots, x_n^0 \in X_n$. At the decision-making stage, we first find the FS2 $B'_j = A' \circ R_j$, $j = 1,\dots,k$, and then, using some extended t-conorm, we determine the FS2 B' with the secondary membership functions

$$\mu_{B'}(y) = \mu_{B'_1}(y) \overset{\triangledown}{\ast} \dots \overset{\triangledown}{\ast} \mu_{B'_n}(y), \quad y \in Y.$$

At the final stage, we first reduce the order of the FS2 B' (for example, by constructing the centroid $C_{B'}$) and the we perform the defuzzification.

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