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APPLICATIONS OF COVERING MAPPINGS IN THE THEORY OF IMPLICIT DIFFERENTIAL EQUATIONS

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Abstract. This paper is a brief review of results in the theory of covering mappings of metric spaces and vector metric spaces and its applications to implicit differential equations. For the Cauchy problem and boundary-value problems, we obtain existence conditions, estimates of solutions, and conditions of the continuous dependence of solutions on the parameters of the equation and initial and boundary conditions.

***Keywords and phrases*:** covering mapping, point of coincidence of mappings, metric space, partially ordered space, implicit differential equation, Cauchy problem, boundary-value problem.

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1. Introduction. One of the main tools for proving solvability conditions for explicit ordinary differential equations (i.e., equations resolved with respect to the derivative of the desired function) are fixed-point theorems. These statements are also used to obtain estimates of solutions, stability studies, approximate integration of equations, etc. A similar role in the study of implicit differential equations can be played by theorems on coincidence points of two mappings and theorems on Lipschitzian perturbations of covering mappings. In this paper, we present a brief review of modern results and propose new results on the theory of covering mappings of metric and vector metric spaces and its applications to implicit differential equations.

2. Covering mappings of metric spaces and their applications to the study of the Cauchy problem.

2.1. Operator equations in metric spaces. Let $X = (X, \rho_X)$ and $Y = (Y, \rho_Y)$ be two metric spaces. We denote by $B_X(x_0, r)$ the closed ball $\{x \in X : \rho_X(x, x_0) \le r\}$ of radius $r \ge 0$ centered at $x_0 \in X$.

Definition 2.1. Let a number $\alpha > 0$ be given. A mapping $\Psi : X \to Y$ is called an *α-covering* if

$$B_Y(\Psi(x_0), \alpha r) \subset \Psi(B_X(x_0, r))$$

for all $x_0 \in X$ and all $r \ge 0$.

Definition 2.2. Let a number $\kappa \ge 0$ be given. A mapping $\Psi : X \to Y$ is called (*metrically*) *κ-regular* if for all $x_0 \in X$ and $y \in Y$, there exists $x \in X$ such that $\Psi(x) = y$ and

$$\rho_X(x, x_0) \le \kappa \rho_Y(y, \Psi(x_0)).$$

These two properties are equivalent: a mapping Ψ is an α-covering if and only if it is κ-regular with $\kappa = \alpha^{-1}$. Here we use the term "covering mapping," but in the next section, a vector analog of the regularity property will be more convenient for mappings of vector metric spaces. The property of being a covering and the metric regularity of mappings were investigated in detail in works of

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E. R. Avakov, A. V. Arutyunov, B. D. Gelman, L. M. Graves, A. V. Dmitruk, A. D. Ioffe, A. A. Milyutin, B. S. Mordukhovich, N. P. Osmolovsky, A. Uderzo, and others. Here we state results that have applications to differential equations.

Let two mappings $\Psi, \Phi : X \to Y$ be given. Their *coincidence point* is an element $x \in X$ such that $\Psi(x) = \Phi(x)$. A. V. Arutyunov proved the following theorem on coincidence points.

Theorem 2.1 (see [1]). *Assume that a space X is complete, a mapping $\Psi : X \to Y$ is a continuous α-covering, and a mapping $\Phi : X \to Y$ satisfies the Lipschitz condition with a constant $\beta < \alpha$. Then for arbitrary $x_0 \in X$, there is a coincidence point $\xi = \xi(x_0) \in X$ of the mappings Ψ and Φ for which*

$$\rho_X(\xi, x_0) \le (\beta - \alpha)^{-1} \rho_Y\big(\Psi(x_0), \Phi(x_0)\big).$$

Proof. Introduce a sequence $\{x_i\} \subset X$ such that

$$\Psi(x_i) = \Phi(x_{i-1}), \quad \rho_X(x_{i+1}, x_i) \le \frac{\beta}{\alpha} \rho_X(x_i, x_{i-1}), \quad i = 1, 2, \dots . \tag{1}$$

This sequence is fundamental and converges in the complete space X to the coincidence point ξ of the given mappings. □

The iterations (1) can be used for approximate calculation of coincidence points. Application of this method requires the study of the influence of errors that arise when finding elements x_i on the limit of the iterative sequence. This problem was examined in [2, 9].

The result was refined by A. V. Arutyunov and other authors, various modifications of the covering condition were used (the most general statements were obtained in [3], and a comparative analysis of various definitions of coverings was performed in [14]). We introduce the notion that allows one to combine various definitions of the covering property.

Definition 2.3. The *α-covering set* of a mapping $\Psi : X \to Y$ is the set of all pairs $(u, y) \in X \times Y$ for which there exists $x \in X$ such that

$$\Psi(x) = y, \quad \rho_X(x, u) \le \alpha^{-1} \rho_Y(y, \Psi(u)).$$

Let us denote the α-covering set of the mapping $\Psi : X \to Y$ by $\mathfrak{B}_\alpha(\Psi)$.

In [4, 7], the operator equation (with respect to the unknown $x \in X$)

$$F(x) = y \tag{2}$$

was examined by using the covering property; here $F : X \to Y$ is a given mapping, $y \in Y$. Conditions of the solvability, correct solvability, and estimates of solutions of Eq. (2) obtained in the above-mentioned papers; these results were applied to the study of the Cauchy problem for an implicit differential equation. We present statements about Eq. (2) similar to the corresponding theorems from [4, 7] under somewhat less restrictive assumptions using Definition 2.3.

Theorem 2.2. *Let a metric space X be complete, $x_0 \in X$, $\alpha > \beta \ge 0$, $r = (\alpha - \beta)^{-1} \rho_Y\,(F(x_0), y)$, and let the mapping $F : X \to Y$ be representable in the form*

$$F(x) = \Upsilon(x, x)$$

for all $x \in X$, where the mapping $\Upsilon : X^2 \to Y$ satisfies the following conditions:

$$\forall x \in B_X(x_0, r) : \quad (x, y) \in \mathfrak{B}_\alpha\left(\Upsilon(\cdot, x)\right); \tag{3}$$

$$\forall x, \hat{x} \in B_X(x_0, r) : \quad \Upsilon(\hat{x}, x) = y \quad \Rightarrow \quad \rho_Y\left(\Upsilon(\hat{x}, \hat{x}), y\right) \le \beta \rho_X(\hat{x}, x); \tag{4}$$

$$\forall \{x_i\} \subset B_X(x_0, r), \quad \forall \xi \in X \ x_i \to \xi, \ \forall i = 1, 2, \dots : \quad \Upsilon(x_i, x_{i-1}) = y \quad \Rightarrow \quad \Upsilon(\xi, \xi) = y. \tag{5}$$

Then there exists a solution of Eq. (2) in the ball $B_X(x_0, r)$.

Proof. As in the proof of Theorem 2.1, we use the iterative sequence constructed as follows. By virtue of (3), the following statement is valid:

$$\exists x_1 \in X: \quad \Upsilon(x_1, x_0) = y, \quad \rho_X(x_1, x_0) \le \frac{\rho_Y\left(y, \Upsilon(x_0, x_0)\right)}{\alpha}.$$

Obviously, $x_1 \in B_X(x_0, r)$. According to (3), the following inequality holds:

$$\rho_Y\left(\Upsilon(x_1, x_1), y\right) \le \beta \rho_X(x_1, x_0),$$

and according to (4),

$$\exists x_2 \in X: \quad \Upsilon(x_2, x_1) = y, \quad \rho_X(x_2, x_1) \le \frac{\rho_Y\left(y, \Upsilon(x_1, x_1)\right)}{\alpha}.$$

Thus,

$$\rho_X(x_2, x_1) \le \frac{\beta}{\alpha} \rho_X(x_1, x_0),$$
$$\rho_X(x_2, x_0) \le \left(1 + \frac{\beta}{\alpha}\right) \rho_X(x_1, x_0) \le \left(1 + \frac{\beta}{\alpha}\right) \frac{\rho_Y\left(y, \Upsilon(x_0, x_0)\right)}{\alpha} \le r.$$

Similarly, for any $i = 3, 4, \ldots$, we can prove the existence of x_i such that

$$\Upsilon(x_i, x_{i-1}) = y, \quad \rho_X(x_i, x_{i-1}) \le \frac{\beta}{\alpha} \rho_X(x_{i-1}, x_{i-2}),$$
$$\rho_X(x_i, x_0) \le \left(1 + \frac{\beta}{\alpha} + \ldots + \frac{\beta^{i-1}}{\alpha^{i-1}}\right) \frac{\rho_Y\left(y, \Upsilon(x_0, x_0)\right)}{\alpha} \le r.$$

Due to the assumption $\beta < \alpha$, the sequence constructed is a Cauchy sequence; this follows from the inequality

$$\rho_X(x_i, x_{i+j}) \le \left(1 + \frac{\beta}{\alpha} + \ldots + \frac{\beta^{j-1}}{\alpha^{j-1}}\right) \rho_X(x_i, x_{i+1}) \le \left(1 + \frac{\beta}{\alpha} + \ldots + \frac{\beta^{j-1}}{\alpha^{j-1}}\right) \frac{\beta^i}{\alpha^i} \rho_X(x_0, x_1).$$

According to (5), the sequence $\{x_i\} \subset B_X(x_0, r)$ converges to a solution of Eq. (2). □

From Theorem 2.2 we derive the well-posedness conditions for Eq. (2). Let a sequence of mappings $\Upsilon_i : X^2 \to Y$, $i = 1, 2, \ldots$, and elements $x_0 \in X$, $y \in Y$ be given. Consider the sequence of equations

$$F_i(x) \stackrel{\text{def}}{=} \Upsilon_i(x, x) = y, \quad i = 1, 2, \ldots. \tag{6}$$

We are interested in conditions that ensure the convergence of the sequence of solutions of Eqs. (6) to $x_0 \in X$ if the sequence $F_i(x_0)$ converges to y.

Corollary 2.1. *Assume that the metric space X is complete, the mappings $\Upsilon_i : X^2 \to Y$, $i = 1, 2, \ldots$, satisfy the conditions of Theorem 2.2, i.e., the numbers $\alpha_i > \beta_i \ge 0$ and $r_i = (\alpha_i - \beta_i)^{-1} \rho_Y\left(F(x_0), y\right)$ are given and the relations (3), (4), and (5) are fulfilled for them. If $r_i \to 0$ as $i \to \infty$, then for any i, there exists a solution $x = \xi_i$ of Eq. (6) such that $\xi_i \to x_0$.*

Proof. According to Theorem 2.2, for any i, there exists a solution ξ_i of Eq. (6) belonging to the ball $B_X(x_0, r_i)$; therefore, $\xi_i \to x_0$. □

Corollary 2.1 is a more general statement than [4, Theorem 2].

2.2. Covering properties of the Nemytsky operator. In order to apply the properties of covering mappings proved in Sec. 2.1 to various functional (including differential) equations, it is necessary to verify the conditions of these statements for specific operators in various functional spaces. Covering properties of the Nemytsky operator are of the greatest interest.

Let $f : [0,\tau] \times \mathbb{R}^m \to \mathbb{R}^l$ be a function satisfying the Carathéodory conditions (that is, measurable with respect to the first argument and continuous with respect to each component of the second argument, which is a vector of $\mathbb{R}^m$). The Nemytsky operator assigns the measurable function

$$[0,\tau] \ni t \longmapsto (N_f x)(t) \stackrel{\text{def}}{=} f(t, x(t)) \in \mathbb{R}^l$$

to each measurable function $x : [0,\tau] \to \mathbb{R}^m$. For Nemytsky operators acting in spaces of essentially bounded functions, the following fact was proved in [4, 7]: if a function $f(t,\cdot)$ is an α-covering for almost all $t \in [0,\tau]$, then the Nemytsky operator N_f is also an α-covering. For an operator N_f acting in spaces of summable functions, conditions of being a covering were obtained in [12].

We give a statement about the Nemytsky operator, which allows one to verify the validity of the relation (3) for various functional equations.

Lemma 2.1. *Let*

$$u : [0,\tau] \to \mathbb{R}^m, \quad y : [0,\tau] \to \mathbb{R}^l, \quad \alpha : [0,\tau] \to \mathbb{R}_+$$

be measurable functions. Assume that for almost every $t \in [0,\tau]$, there exists a point $x \in \mathbb{R}^m$ such that

$$f(t,x) = y(t), \quad |x - u(t)| \le \frac{1}{\alpha(t)} |y(t) - f(t,u(t))| . \tag{7}$$

Then there exists a measurable function $\hat{x} : [0,\tau] \to \mathbb{R}^m$ satisfying the relations

$$(N_f\hat{x})(t) = y(t), \quad |\hat{x}(t) - u(t)| \le \frac{1}{\alpha(t)} |y(t) - (N_f u)(t)| \quad \text{a.e. on } [0,\tau]. \tag{8}$$

Proof. We set

$$R(t) = \frac{1}{\alpha(t)} |f(t,u(t)) - y(t)| , \quad U(t) = B_{\mathbb{R}^m}(u(t), R(t)) , \quad t \in [0,\tau].$$

The function $R : [0,\tau] \to \mathbb{R}$ is measurable; therefore, the multi-valued mapping $U : [0,\tau] \rightrightarrows \mathbb{R}^m$ is measurable. By the assumption (7), the inclusion $y(t) \in f\big(t, U(t)\big)$ holds. According to Filippov's lemma (see, e.g., [8, Sec. 1.5.2]), there exists a measurable section $\hat{x} : [0,\tau] \to \mathbb{R}^m$ of the mapping U such that $y(t) = f(t, \hat{x}(t))$, $t \in [0,\tau]$. Thus, $y = N_f\hat{x}$. The second relation in (8) directly follows from the inclusion $\hat{x}(t) \in U(t)$. □

As usual, we denote by $L_p^m = L_p([0,\tau], \mathbb{R}^m)$ the space of functions $v : [0,\tau] \to \mathbb{R}^m$ that are summable with power $p \in [0,\infty)$ with the norm

$$\|v\|_{L_p^m} = \left(\int_0^\tau |v(t)|^p dt \right)^{1/p};$$

also, we denote by $L_\infty^m = L_\infty([0,\tau], \mathbb{R}^m)$ the space of essentially bounded functions $v : [0,\tau] \to \mathbb{R}^m$ with the norm

$$\|v\|_{L_\infty^m} = \operatorname*{vrai\,sup}_{t \in [0,\tau]} |v(t)|.$$

Lemma 2.1 allows one to formulate conditions of being a covering (with respect to the corresponding metrics) for Nemytsky operators acting in specific spaces of measurable functions. Thus, important results related to Nemytsky operators in Lebesgue spaces can be obtained (see [4, 7, 12]). For example, if

$$(y - N_f u) \in L_p^l, \quad \alpha_0 = \operatorname*{vrai\,inf}_{t \in [0,\tau]} \alpha(t) > 0,$$

then by virtue of (8), we have $(\hat{x}-u) \in L_p^m$ and

$$\|\hat{x}-u\|_{L_p^m} \le \frac{1}{\alpha_0}\|y - N_f\, u\|_{L_p^l}.$$

This means that for an essentially bounded function α, the inclusion

$$(u(t), y(t)) \in \mathfrak{B}_{\alpha(t)}\left(f(t,\cdot)\right) \subset \mathbb{R}^m \times \mathbb{R}^l$$

implies the inclusion

$$(u, y) \in \mathfrak{B}_{\alpha_0}(N_f) \subset L_p^m \times L_p^l.$$

Similarly, if $(y - N_f\, u) \in L_p^l$ and

$$\alpha_0 = \left(\int\limits_0^\tau \frac{1}{\alpha(t)^q} dt\right)^{-1/q} < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{9}$$

then $(\hat{x}-u) \in L_1^m$ and

$$\|\hat{x}-u\|_{L_1^m} \le \frac{1}{\alpha_0}\|y - N_f\, u\|_{L_p^l}.$$

Thus, if the condition (9) is fulfilled, then the inclusion

$$(u(t), y(t)) \in \mathfrak{B}_{\alpha(t)}\left(f(t,\cdot)\right) \subset \mathbb{R}^m \times \mathbb{R}^l$$

implies the inclusion

$$(u, y) \in \mathfrak{B}_{\alpha_0}(N_f) \subset L_1^m \times L_p^l.$$

2.3. Cauchy problem for an implicit differential equation. Introduce some function spaces used below. Let $y : [0,\tau] \to \mathbb{R}^l$ be a measurable function. We define the space $W_\tau^l(y) = W(y,[0,\tau],\mathbb{R}^l)$ of all measurable functions $v : [0,\tau] \to \mathbb{R}^l$ for which the difference vy is essentially bounded, and introduce the function

$$\rho_{W_\tau^l(y)}(v,\hat{v}) = \operatorname*{vrai\,sup}_{t\in[0,\tau]} |v(t) - \hat{v}(t)|.$$

Obviously, the space $W_\tau^l(y)$ is complete. Note that if a function y is essentially bounded, then the space $W_\tau^l(y)$ coincides with the “ordinary” space L_∞^l of essentially bounded functions.

Let $y : \mathbb{R}_+ \to \mathbb{R}^l$ be a measurable function and $f : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^l$ be a function satisfying the Carathéodory conditions (i.e.,, measurable with respect to the first argument and continuous with respect to each component of the second and third arguments). Let us consider the differential equation

$$f(t, x, \dot{x}) = y(t), \quad t \ge 0, \tag{10}$$

with the initial condition

$$x(0) = \gamma, \quad \gamma \in \mathbb{R}^m. \tag{11}$$

Let $\tau > 0$. An absolutely continuous function on the interval on $[0,\tau]$ satisfying the equation for almost all $t \in [0,\tau]$ is called a *solution* of Eq. (10) on $[0,\tau]$.

Let $x_0 : \mathbb{R}_+ \to \mathbb{R}^m$ be a function, which is absolutely continuous on each finite segment $[0,\tau]$, $\tau > 0$, and satisfies the condition $x_0(0) = \gamma$. We set $y_0(t) = f\left(t, x_0(t), \dot{x}_0(t)\right)$, $t \in \mathbb{R}_+$. This function is measurable; let $R(t) = \operatorname*{vrai\,sup}_{s\in[0,t]}|y(s) - y_0(s)| < \infty$ for any $t > 0$. Introduce the function

$$r(t) = \int\limits_0^t R(s)ds.$$

Let $\alpha, \varepsilon > 0$ be given. We introduce multi-valued mappings

$$U_{\alpha\varepsilon} : \mathbb{R}_+ \rightrightarrows \mathbb{R}^m, \quad U_{\alpha\varepsilon}(t) = B_{\mathbb{R}^m}\left(x_0(t), \alpha^{-1} r(t) + \varepsilon t\right),$$
$$V_{\alpha\varepsilon} : \mathbb{R}_+ \rightrightarrows \mathbb{R}^m, \quad V_{\alpha\varepsilon}(t) = B_{\mathbb{R}^m}\left(\dot{x}_0(t), \alpha^{-1} R(t) + \varepsilon\right).$$

The mapping $U_{\alpha\varepsilon}$ is continuous and the mapping $V_{\alpha\varepsilon}$ is measurable.

Theorem 2.3. *Assume that for some positive numbers α and ε and functions $\mu, \mathcal{M} : \mathbb{R}_+ \to \mathbb{R}_+$ that are essentially bounded on each finite segment, following relations hold for almost all $t \in \mathbb{R}_+$ and arbitrary $x, \hat{x} \in U_{\alpha\varepsilon}(t)$ and $v \in V_{\alpha\varepsilon}(t)$:*

$$(v, y(t)) \in \mathfrak{B}_\alpha\left(f(t, x, \cdot)\right) \subset \mathbb{R}^m \times \mathbb{R}^l, \tag{12}$$
$$|f(t, x, v) - f(t, \hat{x}, v)| \le \mu(t)|x - \hat{x}|, \tag{13}$$
$$|f(t, x_0(t), v) - f(t, x_0(t), \dot{x}_0(t))| \le \mathcal{M}(t). \tag{14}$$

Then there exists $\tau > 0$ such that there exists a solution of the problem (10), (11) *on* $[0, \tau]$.

Proof. We set

$$\hat{R} = \operatorname*{vrai\,sup}_{t \in [0,1]} R(t), \quad \hat{\mu} = \operatorname*{vrai\,sup}_{t \in [0,1]} \mu(t), \quad \tau = \min\left\{1,\ 2^{-1}\hat{R}^{-1}\hat{\mu}^{-1}\alpha^2\varepsilon\right\} \tag{15}$$

and denote by $\mathcal{V}_\tau$ the set of measurable sections of multi-valued mapping $V_{\alpha\varepsilon}(\cdot)$ defined on $[0, \tau]$. We have $\mathcal{V}_\tau \subset W^m_\tau(\dot{x}_0)$; therefore, we assume that $\mathcal{V}_\tau$ is a metric space with a reduced metric. Introduce the mapping

$$\Upsilon_\tau(z_1, z_2)(t) = f\left(t, \gamma + \int_0^t z_2(s)ds, z_1(t)\right), \quad t \in [0, \tau].$$

For $t \in [0, \tau]$, the problem (10), (11) can be represented as the equation

$$\Upsilon_\tau(z, z) = f \tag{16}$$

for the unknown function $z \in \mathcal{V}_\tau$, which is the derivative of the solution of the problem (10), (11). □

Due to the assumptions (13) and (14), the mapping Υ_τ acts from $\mathcal{V}_\tau \times \mathcal{V}_\tau$ to $W^l_\tau(y)$. The relation (13) implies that for any $z_1 \in \mathcal{V}_\tau$, the mapping $\Upsilon_\tau(z_1, \cdot) : \mathcal{V}_\tau \to W^l_\tau(y)$ is Lipschitzian with coefficient $\beta = \hat{\mu}\tau \le 2^{-1}\hat{R}^{-1}\alpha^2\varepsilon$; without loss of generality, we may assume that ε is sufficiently small so that the inequality $\beta \le 2^{-1}\alpha$ holds. Since the function f satisfies the Carathéodory conditions, the mapping $\Upsilon_\tau(\cdot, z_2) : \mathcal{V}_\tau \to W^l_\tau(y)$ is closed for any $z_2 \in \mathcal{V}_\tau$. According to Lemma 2.1, the assumption (12) implies that for any $z \in \mathcal{V}_\tau$, the following inclusion holds:

$$(z, y) \in \mathfrak{B}_\alpha\left(\Upsilon_\tau(\cdot, z)\right) \subset \mathcal{V}_\tau \times W^l_\tau(y).$$

It remains to note that

$$V_{\alpha\varepsilon}(t) \subset B_{\mathbb{R}^m}\left(\dot{x}_0(t), (\alpha - \beta)^{-1} R(t)\right);$$

thus, the conditions of Theorem 2.2 are satisfied. Therefore, Eq. (16) has a solution $\hat{z} \in \mathcal{V}_\tau$, and then the absolutely continuous function

$$\hat{x}(\cdot) = \gamma + \int_0^{(\cdot)} \hat{z}(s)ds$$

is a solution of the problem (10), (11) defined on $[0, \tau]$.

Remark 2.1. The relation (15) determines the length of the existence interval of the solution of the problem (10), (11). The solution x defined on this interval satisfies the inequality

$$\operatorname*{vrai\,sup}_{t\in[0,1]} \left|\dot{x}(t) - \dot{x}_0(t)\right| \le \frac{2}{\alpha} R(t).$$

Note that the Cauchy problem for a special case of Eq. (10) of the form $g(t, \dot{x}) = \varphi(t, x)$ was considered in [5], where the theorem of coincidence points obtained in [1] was used.

We examine the problem on the continuous dependence of solutions of the problem (10), (11) on the parameters (i.e., on the function f and the initial value γ).

Assume that for each natural i, the following data are defined:

(i) a function $f_i : \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^l$ satisfying the Carathéodory conditions,
(ii) a measurable function $y_i : \mathbb{R}_+ \to \mathbb{R}^l$,
(iii) and a vector $\gamma_i \in \mathbb{R}^m$.

Consider the sequence of Cauchy problems

$$f_i(t, x, \dot{x}) = y_i(t),\ t \ge 0, \quad x(0) = \gamma_i,\ i = 1, 2, \dots. \tag{17}$$

Let $x_0 : \mathbb{R}_+ \to \mathbb{R}^m$, $x_0(0) = \gamma$ be a function absolutely continuous on each finite segment. We set

$$x_{0i}(t) = \gamma_i + \int_0^t \dot{x}_0(s)ds, \quad y_{0i}(t) = f_i\left(t, x_{0i}(t), \dot{x}_0(t)\right), \quad t \in \mathbb{R}_+.$$

We formulate the solvability conditions for the problem (17) for each i and the conditions of convergence of the sequence of solutions to the function $x_0 : \mathbb{R}_+ \to \mathbb{R}^m$ (in the space of absolutely continuous functions on each finite segment) provided that γ_i converges to γ and y_{0i} uniformly converges to y.

We note the following. In Theorem 2.3, which establishes the solvability conditions for the Cauchy problem, there are no restrictions on the covering coefficient α of the function f with respect to the second argument and the Lipschitz coefficient $\mu(t)$ of this function with respect to the third argument. The fact is that if we consider the corresponding operators in spaces of functions defined on $[0, \tau]$, the inequality $\beta < \alpha$ required in Theorem 2.2 holds for sufficiently small $\tau > 0$. In the study of the sequence of Cauchy problems, we must assume the existence of the coefficients α and $\mu(t)$ that are common for all functions f_i; otherwise, for each i we can obtain a solution defined on $[0, \tau_i]$ with $\tau_i \to 0$, or on any segment $[0, \tau]$ the sequence of solutions may turn out to be divergent.

Example 2.1. Consider the Cauchy problem for the simplest linear equation:

$$\dot{x} - ix = 1,\ t \ge 0, \quad x(0) = i^{-1},\ i = 1, 2, \dots. \tag{18}$$

We set $f_i(t, x, v) = -ix + v$, $y(t) \equiv 1$, $\gamma_i = i^{-1}$, $x_{0i}(t) \equiv i^{-1}$. The function f_i is 1-covering with respect to the third argument and i-Lipschitzian with respect to the second argument. We have $f_i\left(t, x_{0i}(t), \dot{x}_{0i}(t)\right) \equiv 1$, $\gamma_i \to 0$, and $\dot{x}_{0i}(t) \equiv 0$ as $i \to \infty$. The solution of the problem (18) is $x_i(t) = i^{-1}\exp(it) - i^{-1}$, and for any $t > 0$, we have $x_i(t) \to \infty$.

For $\varepsilon > 0$, $i = 1, 2, \dots$, we set

$$U_{\varepsilon i}, V_{\varepsilon i} : \mathbb{R}_+ \rightrightarrows \mathbb{R}^m, \quad U_{\varepsilon i}(t) = B_{\mathbb{R}^m}\left(x_{0i}(t), \varepsilon t\right), \quad V_{\varepsilon i}(t) = B_{\mathbb{R}^m}\left(\dot{x}_0(t), \varepsilon\right).$$

Theorem 2.4. *Assume that for some positive numbers α and ε and a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, which is essentially bounded on each finite segment, for any $i = 1, 2, \dots$, almost all $t \in \mathbb{R}_+$, and arbitrary*

$x, \hat{x} \in U_{\varepsilon i}(t)$ and $v \in V_{\varepsilon i}(t)$, the following relations hold:

$$(v, y(t)) \in \mathfrak{B}_\alpha \left(f_i(t, x, \cdot)\right) \subset \mathbb{R}^m \times \mathbb{R}^l, \tag{19}$$

$$|f_i(t, x, v) - f_i(t, \hat{x}, v)| \leq \mu(t)|x - \hat{x}|, \tag{20}$$

$$\forall \tau > 0 \ \ \operatorname*{vrai\,sup}_{s \in [0,\tau]} |f_i(s, x_{0i}(s), v) - f_i(s, x_{0i}(s), \dot{x}_{0i}(s))| \leq \infty. \tag{21}$$

If

$$R_i = \operatorname*{vrai\,sup}_{s \in [0,T]} |y_i(s) - y_{0i}(s)| \to 0$$

as $i \to \infty$ for some $T > 0$, then there exists a number i_0 such that for all $i \geq i_0$, there exists a solution x_i of the problem (17) defined on $[0, T]$, and the sequence of these solutions satisfies the following condition:

$$\operatorname*{vrai\,sup}_{s \in [0,T]} |\dot{x}_i(s) - \dot{x}_0(s)| \to 0.$$

Proof. We set

$$\hat{\mu} = \operatorname*{vrai\,sup}_{t \in [0,T]} \mu(t), \quad \tau = \min\left\{T, 2^{-1}\hat{\mu}^{-1}\alpha\right\} \tag{22}$$

and take i_0 so that $\alpha^{-1}2R_i \leq \varepsilon$ for any $i > i_0$. We denote by $\mathcal{V}_{\tau i}$ the subspace in $W_\tau^m(\dot{x}_0)$ containing measurable functions $v : [0, \tau] \to \mathbb{R}^m$ such that $|v(t) - \dot{x}_{0i}(t)| \leq \varepsilon$ for almost all $t \in [0, \tau]$. Introduce the mapping

$$\Upsilon_{\tau i}(z_1, z_2)(t) = f_i\left(t, \gamma_i + \int_0^t z_2(s)ds, z_1(t)\right), \quad t \in [0, \tau].$$

For $t \in [0, \tau]$, the problem (17) can be rewritten as the equation

$$\Upsilon_{\tau i}(z, z) = f_i, \quad i = 1, 2, \ldots, \tag{23}$$

for the unknown function $z \in \mathcal{V}_{\tau i}$, which is the derivative of the solution to the problem (17).

By repeating the same arguments as in the proof of Theorem 2.3, we conclude that the conditions of Theorem 2.2 for the mapping $\Upsilon_{\tau i} : \mathcal{V}_{\tau i} \times \mathcal{V}_{\tau i} \to W_\tau^l(y_i)$ are satisfied; therefore, Eq. (23) for $i \geq i_0$ has a solution $\hat{z}_i \in \mathcal{V}_{\tau i}$ satisfying the inequality

$$\operatorname*{vrai\,sup}_{t \in [0,\tau]} \left|\hat{z}_i(t) - \dot{x}_0(t)\right| \leq \frac{2}{\alpha} R_i \leq \varepsilon.$$

Therefore, the function

$$x_i(\cdot) = \gamma_i + \int_0^{(\cdot)} \hat{z}_i(s)ds$$

is a solution to the problem (17) defined on $[0, \tau]$. For this solution, the inclusion $x_i(t) \in U_{\varepsilon i}(t)$ is satisfied.

Similarly, we can prove that for any $i > i_0$, there exists a solution of the differential equation on the interval $[\tau, 2\tau]$ with the initial condition $x(\tau) = x_i(\tau)$; this solution x_i satisfies the conditions

$$x_i(t) \in U_{\varepsilon i}(t), \quad \dot{x}_i(t) \in V_{\varepsilon i}(t), \quad |\dot{x}_i(t) - \dot{x}_0(t)| \leq \alpha^{-1}2R_i,$$

and so on. Thus, for any $i > i_0$, for a finite number $\tau^{-1}T$ of "steps," we obtain a solution x_i of the problem (17) defined on $[0, T]$ and satisfying the conditions

$$\dot{x}_i(t) \in V_{\varepsilon i}(t), \quad |\dot{x}_i(t) - \dot{x}_0(t)| \leq \alpha^{-1}2R_i.$$

Now the theorem obviously follows from the convergence $R_i \to 0$. □

By virtue of the assumptions (20) and (21), the mapping $\Upsilon_{\tau i}$ acts from $\mathcal{V}_{\tau i} \times \mathcal{V}_{\tau i}$ into $W^l_\tau(y_i)$. The relation (20) implies that for any $z_1 \in \mathcal{V}_{\tau i}$, the mapping $\Upsilon_\tau(z_1, \cdot) : \mathcal{V}_\tau \to W^l_\tau(y)$ is Lipschitzian with the coefficient $\beta = \hat{\mu}\tau \le 2^{-1}\alpha$. Since the function f satisfies the Carathéodory conditions, the mapping $\Upsilon_\tau(\cdot, z_2) : \mathcal{V}_\tau \to W^l_\tau(y)$ is closed for any $z_2 \in \mathcal{V}_\tau$. According to Lemma 2.1, the assumption (12) implies that for any $z \in \mathcal{V}_\tau$, the following inclusion holds:

$$(z, y) \in \mathfrak{B}_\alpha\left(\Upsilon_\tau(\cdot, z)\right) \subset \mathcal{V}_\tau \times W^l_\tau(y).$$

Thus, the conditions of Theorem 2.2 are fulfilled, and Eq. (16) has a solution $\hat{z} \in \mathcal{V}_\tau$, and then the absolutely continuous function

$$\hat{x}(\cdot) = \gamma + \int\limits_0^{(\cdot)} \hat{z}(s)ds$$

is a solution to the problem (10), (11) defined on $[0, \tau]$.

From Theorems 2.3 and 2.4 we can deduce theorems on the existence and the continuous dependence on parameters of solutions of the Cauchy problem for the implicit differential equation, which were earlier obtained in [4, 5, 7].

3. Regular mappings of products of metric spaces and their applications to the study of boundary-value problems.

3.1. Systems of operator equations in metric spaces. Let X_k, $k = \overline{1, m}$, and Y_j, $j = \overline{1, l}$, be metric spaces. We introduce the spaces

$$\overline{X} = \prod_{k=1}^m X_k, \quad \overline{Y} = \prod_{j=1}^l Y_j.$$

It is natural to introduce the distance between elements of the products $\overline{X}$ and $\overline{Y}$ as the vectors $\overline{\rho}_{\overline{X}} = (\rho_{X_1}, \dots, \rho_{X_m})$ and $\overline{\rho}_{\overline{Y}} = (\rho_{Y_1}, \dots, \rho_{Y_l})$. A ball in $\overline{X}$ of radius $d = (d_1, \dots, d_m) \in \mathbb{R}^n_+$ centered at $u = (u_1, \dots, u_m) \in \overline{X}$ is the set

$$\overline{B}_{\overline{X}}(u, d) = \prod_{k=1}^n B_{X_k}(u_k, d_k).$$

We say that a sequence $\{x_i\}$ converges to x_0 in $\overline{X}$ if $\overline{\rho}_{\overline{X}}(x_i, x_0) \to 0$ in the space $\mathbb{R}^m$. Cauchy sequences and the notion of completeness are defined in a standard way. The space $(\overline{X}, \overline{\rho}_{\overline{X}})$ is complete if and only if the spaces X_k, $k = \overline{1, m}$, are complete.

Definition 3.1. Let K be an $(m \times l)$-matrix with nonnegative elements κ_{kj}, $k = \overline{1, m}$, $j = \overline{1, l}$. The set of all pairs $(u, y) \in \overline{X} \times \overline{Y}$, for each of which there exists an element $x \in \overline{X}$ such that $\Psi(x) = y$ and $\overline{\rho}_{\overline{X}}(x, u) \le K\overline{\rho}_{\overline{Y}}(y, \Psi(u))$ is called the *set* (*of vector metric*) *K-regularity* of the mapping $\Psi : \overline{X} \to \overline{Y}$. The set of K-regularity of the mapping $\Psi : \overline{X} \to \overline{Y}$ is denoted by $\mathfrak{B}_K(\Psi)$.

Obviously, Definitions 3.1 and 2.3 are equivalent in the case $m = l = 1$.

The regularity property (also called the covering property) of mappings in products of metric spaces and in more general spaces with vector metrics was introduces and studied in [6, 10–12]. Using the regularity set of the corresponding mappings, we obtain here a solvability condition for systems of equations, which is similar to the results of the papers mentioned above, under somewhat less restrictive assumptions.

Consider a vector $y \in \overline{Y}$, a vector mapping $F = (F_1, \dots, F_l) : \overline{X} \to \overline{Y}$, and the system of equations $F_j(x_1, \dots, x_m) = y_j$, $j = \overline{1, l}$; we write this system in the form

$$F(x) = y. \tag{24}$$

Theorem 3.1. *Let X_k, $k = \overline{1,m}$, be complete metric spaces and $x_0 \in \overline{X}$ be a given vector. Assume that the matrices $K_{m\times l}$ and $B_{l\times m}$ with nonnegative elements are such that the spectral radius of their product satisfies the condition* $\mathrm{sr}(BK) < 1$. *Let*

$$r = K(I_l - BK)^{-1}\overline{\rho}_{\overline{Y}}\,(F(x_0), y)\,,$$

where I_l is the identity $(l \times l)$-matrix. Assume that mapping $F : \overline{X} \to \overline{Y}$ can be represented in the form

$$F(x) = \Upsilon(x, x) \quad \forall x \in \overline{X},$$

where the mapping $\Upsilon : \overline{X}^2 \to \overline{Y}$ satisfies the following conditions:

$$\forall x \in \overline{B}_{\overline{X}}(x_0, r) \quad (x, y) \in \mathfrak{B}_K\,(\Upsilon(\cdot, x))\,; \tag{25}$$

$$\forall x, \hat{x} \in \overline{B}_{\overline{X}}(x_0, r) \quad \Upsilon(\hat{x}, x) = y \Rightarrow \overline{\rho}_{\overline{Y}}\,(\Upsilon(\hat{x}, \hat{x}), y) \le \beta\overline{\rho}_{\overline{X}}(\hat{x}, x); \tag{26}$$

$$\forall \{x_i\} \subset \overline{B}_{\overline{X}}(x_0, r) \quad \forall \xi \in X \quad x_i \to \xi, \quad \forall i = 1, 2, \ldots \quad \Upsilon x_i, x_{i-1}) = y \Rightarrow \Upsilon(\xi, \xi) = y. \tag{27}$$

Then there exists a solution of Eq. (24) *in the ball $\overline{B}_{\overline{X}}(x_0, r)$.*

Proof. Due to the assumption $\mathrm{sr}(BK) < 1$, the series $\sum\limits_{i=0}^{\infty}(BK)^i$ converges to the matrix $(I_l - BK)^{-1}$. Since elements of the matrices B and K are nonnegative, the following inequality holds for any l:

$$\sum_{i=0}^{l}(BK)^i \le (I_l - BK)^{-1}.$$

Further arguments are similar to those used in the proof of Theorem 2.2. The assumptions (25) and (26) allow one to define a sequence $\{x_i\} \subset \overline{X}$ such that

$$\Upsilon(x_i, x_{i-1}) = y, \quad \overline{\rho}_{\overline{X}}(x_i, x_{i-1}) \le BK\overline{\rho}_{\overline{X}}(x_{i-1}, x_{i-2}),$$

$$\overline{\rho}_{\overline{X}}(x_i, x_0) \le K\sum_{n=0}^{i-1}(BK)^n\overline{\rho}_{\overline{Y}}\,(y, \Upsilon(x_0, x_0)) \le r.$$

Since $\mathrm{sr}(BK) < 1$, the sequence $\{x_i\} \subset \overline{B}_{\overline{X}}(x_0, r)$ in the complete space $\overline{X}$ is a Cauchy sequence. According to the condition (27), this sequence converges to a solution of Eq. (24). ☐

From Theorem 3.1, we can derive the well-posedness conditions for the vector equation (24). Let $\Upsilon_i : \overline{X}^2 \to \overline{Y}$, $i = 1, 2, \ldots$, be a sequence of mappings, $x_0 \in \overline{X}$, and $y \in \overline{Y}$. We consider the sequence of equations

$$F_i(x) \stackrel{\mathrm{def}}{=} \Upsilon_i(x, x) = y, \quad i = 1, 2, \ldots. \tag{28}$$

Corollary 3.1. *Let X_k, $k = \overline{1,m}$, be complete metric spaces and mappings $\Upsilon_i : \overline{X}^2 \to \overline{Y}$, $i = 1, 2, \ldots$, satisfy the conditions of Theorem* 3.1, *i.e., $(m \times l)$- and $(l \times m)$-matrices K_i and B_i with nonnegative components be given,* $\mathrm{sr}(B_iK_i) < 1$, *and the relations* (25), (26), *and* (27) *be satisfied, where*

$$r_i = K(I_l - BK)^{-1}\overline{\rho}_{\overline{Y}}\,(F(x_0), y)\,.$$

If $r_i \to 0$ as $i \to \infty$, then for any i, there exists a solution $x = \xi_i$ of Eq. (28) *such that $\xi_i \to x_0$.*

3.2. Regularity of the Nemytsky operator in products of spaces of measurable functions. We formulate a vector analog of the results obtained in Sec. 2.2.

Let $x = (x_1, \dots, x_m)$ be a vector in $\mathbb{R}^m$; we introduce the notation $\mathcal{H}_m x = (|x_1|, \dots, |x_m|) \in \mathbb{R}^m_+$.

Let $f : [0,\tau] \times \mathbb{R}^m \to \mathbb{R}^l$ be a function satisfying the Carathéodory conditions. Introduce the Nemytsky operator, which assigns the measurable function

$$[0,\tau] \ni t \mapsto (N_f x)(t) \stackrel{\text{def}}{=} f(t, x(t)) \in \mathbb{R}^l$$

to each measurable function $x : [0,\tau] \to \mathbb{R}^m$.

The following assertion allows one to verify the validity of the relation (25) for functional equations.

Lemma 3.1. *Let*

$$u : [0,\tau] \to \mathbb{R}^m, \quad y : [0,\tau] \to \mathbb{R}^l, \quad K : [0,\tau] \to \mathbb{R}^{m\times l}_+$$

be measurable functions. Assume that for almost all $t \in [0,\tau]$, there exists a vector $x \in \mathbb{R}^m$ such that

$$f(t,x) = y(t), \quad \mathcal{H}_m (x - u(t)) \le K(t)\mathcal{H}_l (y(t) - f(t, u(t))).$$

Then there exists a measurable function $\hat{x} : [0,\tau] \to \mathbb{R}^m$ satisfying the relations

$$(N_f \hat{x})(t) = y(t), \quad \mathcal{H}_m (\hat{x}(t) - u(t)) \le K(t)\mathcal{H}_m (y(t) - (N_f u)(t))$$

almost everywhere on $[0,\tau]$.

3.3. Boundary-value problem for an implicit differential equation. Consider the boundary-value problem for the implicit differential equation (10), i.e., the problem of finding a solution to this equation on a certain interval $[0,T]$ satisfying the boundary condition. For definiteness, we consider the two-point problem with the condition

$$Ax(0) + Dx(T) = \gamma, \tag{29}$$

where $A, D \in \mathbb{R}^{n\times m}$ and $\gamma \in \mathbb{R}^n$). We write the boundary-value problem (10), (29) in the form of the system of equations

$$N_f \left(\dot{x}, x(0) + \int_0^{(\cdot)} \dot{x}(s)ds \right) = \gamma, \quad (A + D)x(0) + D \int_0^T \dot{x}(s)ds = \gamma,$$

for the unknown $(\dot{x}, x(0))$. The results on regular mappings acting in products of metric spaces obtained above are applicable to this system.

Let $x_0 : [0,T] \to \mathbb{R}^m$ be an absolutely continuous function. We introduce the function

$$y_0 : [0,T] \to \mathbb{R}^l, \quad y_0(t) = f(t, x_0(t)\dot{x}_0(t)),$$

which is obviously measurable. Let

$$R_j = \operatorname*{vrai\,sup}_{t\in[0,T]} |y_0(t) - y(t)| < \infty, \quad j = \overline{1,l}.$$

Introduce the vectors $R = (R_j)_{j=\overline{1,l}}$, $\gamma_0 = Ax_0(0) + Dx_0(T)$, $d = \mathcal{H}_n(\gamma_0 - \gamma)$.

Assume that the rank of the matrix $A + D$ is equal to n. In this case, there exists the right inverse matrix $(A + D)^{-1}_+ \in \mathbb{R}^{m\times n}$. We set

$$K^b = \mathcal{H}_{m\times n} \left((A + D)^{-1}_+ \right).$$

For the mapping $A + D : \mathbb{R}^m \to \mathbb{R}^n$, the K^b-regularity set coincides with the whole space $\mathbb{R}^m \times \mathbb{R}^n$. We introduce the notation $|D| = \mathcal{H}_{n\times m} D$.

Let K^f and M be $(m \times l)$- and $(l \times m)$-matrices with nonnegative components. We define the $2m \times (l+n)$- and $(l+n) \times 2m$-matrices K and B and their product by the formulas

$$K = \begin{pmatrix} K^f & 0 \\ 0 & K^b \end{pmatrix}, \quad B = \begin{pmatrix} TM & M \\ T|D| & 0 \end{pmatrix}, \quad BK = \begin{pmatrix} TMK^f & MK^b \\ T|D|K^f & 0 \end{pmatrix}.$$

Assume that $\mathrm{sr}(BK) < 1$. Then there exists the matrix $G = K(I_{l+n} - BK)^{-1}$. We write this matrix in the form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

and define its block structure as follows: G_{11} and G_{21} are $(m \times l)$-matrices and G_{12} and G_{22} are $(m \times n)$-matrices. Consider the sets

$$V(t) = \overline{B}_{\overline{\mathbb{R}^m}}\left(\dot{x}_0(t), G_{11}R + G_{12}d\right),$$
$$U(t) = \overline{B}_{\overline{\mathbb{R}^m}}\left(x_0(t), G_{21}R + G_{22}d + tG_{11}R + tG_{12}d\right).$$

Theorem 3.2. *Assume that for some essentially bounded function $\mathcal{M} : [0,T] \to \mathbb{R}_+$, the relations*

$$(v, y(t) \in \mathfrak{B}_{K^f}\left(f(t,x,\cdot)\right) \subset \mathbb{R}^m \times \mathbb{R}^l,$$
$$\mathcal{H}_l\left(f(t,x,v) - f(t,\hat{x},v)\right) \le M\mathcal{H}_m(x - \hat{x}),$$
$$|f(t,x_0(t),v) - f(t,x_0(t),\dot{x}_0(t))| \le \mathcal{M}(t)$$

hold for almost all $t \in \mathbb{R}_+$ and arbitrary $x, \hat{x} \in U(t)$ and $v \in V(t)$. Then there exists a solution x of the boundary-value problem (10)*,* (29) *for which $\dot{x}(t) \in V(t)$ and $x(0) \in \overline{B}_{\overline{\mathbb{R}^m}}(\gamma_0, G_{21}R + G_{22}d)$.*

In conclusion, we note that methods based on covering (regular) mappings are also applicable to the study of implicit functional-differential equations. For example, an equation with a deviating argument was examined in [13] by using assertions on vector covering mappings.

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