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FUNDAMENTAL FREQUENCY SOLUTIONS WITH PRESCRIBED ACTION VALUE TO NONLINEAR SCHRÖDINGER EQUATIONS

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We apply the nonlinear generalized Rayleigh quotients method to develop new tools that can be used to study ground states of nonlinear Schrödinger equations. We introduce a new type variational functional, the global minimizer of which corresponds to the so-called fundamental frequency solutions with a prescribed action value. We find the ground state of the problem and uniquely determine the corresponding values of the mass, frequency, and action level. Based on this approach, we obtain new results on the existence and absence of nonnegative solutions to the zero mass problem. Bibliography: 15 *titles.*

1 Introduction

We consider the nonlinear Schrödinger equation

$$i\psi_t = \Delta\psi + f_\mu(|\psi|\psi), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.1}$$

with combined power type nonlinearity $f_\mu(|\psi|\psi) = \mu|\psi|^{p-2}\psi - |\psi|^{q-2}\psi$, where ψ is a complex-valued function of (t,x), $p,q \in (2, 2^* := 2N/(N-2))$, $N \geqslant 3$, and $\mu \in \mathbb{R}$. For physical background of such equations we refer the reader to [1] and the references therein. As known (cf., for example, [2, 3]), the Cauchy problem for Equation (1.1) with the initial data $\psi_0 \in H^1 := H^1(\mathbb{R}^N)$ is locally well posed and has a unique local solution $\psi \in C([0,T(\psi_0)), H^1) \cap C^1([0,T(\psi_0)), H^{-1})$ for some $T(\psi_0) > 0$ that satisfies the energy conservation law

$$E \equiv H_\mu(\psi(t)) := \int \Big(\frac{1}{2}|\nabla\psi|^2 - F_\mu(|\psi|\psi)\Big)dx,$$

where

$$F_\mu(|\psi|\psi) = \frac{\mu}{p}|\psi|^p - \frac{1}{q}|\psi|^q,$$

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and the mass (charge, particle numbers) conservation law

$$m \equiv Q(\psi(t)) := \frac{1}{2} \int |\psi|^2 dx.$$

In the present paper, we study the existence of the standing waves $\psi_\lambda = e^{i\lambda t} u$ of Equation (1.1), where the amplitude function u satisfies the equation

$$-\Delta u - \lambda u - \mu |u|^{p-2} u + |u|^{q-2} u = 0, \quad x \in \mathbb{R}^N. \tag{1.2}$$

Here, $\lambda \in \mathbb{R}$ is the standing wave frequency. We note that the corresponding action functional

$$S_{\lambda,\mu}(u) := H_\mu(u) - \lambda Q(u)$$

is a conserved quantity. For given $\lambda \in \mathbb{R}$ a solution $\overline{u}$ to Equation (1.2) is called a *ground state* if $S_{\lambda,\mu}(\overline{u}) \leqslant S_{\lambda,\mu}(w)$ for any $w \in H^1 \setminus 0$ such that $DS_{\lambda,\mu}(w) = 0$.

We also deal with the so-called *prescribed action solution* to Equation (1.2), i.e., a function $u^S \in H^1$ that for a given action $S \in \mathbb{R}^+$ satisfies the conditions

$$S_{\lambda,\mu}(u^S) = S, \quad DS_{\lambda,\mu}(u^S) = 0 \tag{1.3}$$

with some $\lambda \leqslant 0$. The standard approach to studying Equation (1.2) considers solutions with prescribed frequency λ, whereas the action S and mass m are unknown (cf., for example, [3, 4]). An alternative approach is to find a solution u to Equation (1.2) with prescribed mass $m = Q(u)$ and unknown λ and S (cf., for example, [5, 6]).

We note that the frequency λ can be also considered as the value of the following conserved quantity:

$$\lambda = \Lambda^S_\mu(\psi(t)) := \frac{H_\mu(\psi) - S}{Q(\psi)}. \tag{1.4}$$

In what follows, the frequency λ, action S, and mass m are referred to as the *main parameters* of the problem.

We note that the approach with prescribed action can be applied to the study of inverse problems and the so-called spectral and scattering control problems (cf., for example, [7, 8] and the references therein).

There is an extensive literature on problems with prescribed frequency and mass, but little is known about problems with prescribed action.

The main goal of this paper is to propose a general approach to finding solutions with prescribed action. We present a new type variational functional associated with Equation (1.1), owing to which we find a ground state $\overline{u}$ of the problem and uniquely determine the corresponding values of mass m, frequency λ, and action level $S_{\lambda,\mu}$ of this state.

We pay a special attention to the standing wave ψ_λ, called the *zero frequency solution*, of Equation (1.1) with $\lambda = 0$. The corresponding equation (1.2) with $\lambda = 0$ for the amplitude function u is called the *zero frequency problem* (also known as the zero mass problem; cf. [4, 9]).

Our approach is based on the nonlinear generalized Rayleigh quotient method introduced in [10]. This method is applied to Equation (1.2) with $\Lambda^S_\mu(u)$ taking for the Rayleigh quotient. A feature of the nonlinear generalized Rayleigh quotient method is that it allows us to find critical values of parameters of the problem and, at the same time, convert the original variational functionals into functionals with simpler geometry (cf. [10]). We note that $u^S \in H^1 \setminus 0$ is a prescribed action solution to Equation (1.2) with action $S > 0$; namely, u^S satisfies (1.3) if and only if u^S is a critical point of $\Lambda^S_\mu(u)$ with critical value λ, i.e., $D\Lambda^S_\mu(u^S) = 0$ and $\lambda = \Lambda^S_\mu(u^S)$. For given $S > 0$ we call a solution $\widehat{u}$ to Equation (1.2) a *fundamental frequency solution* (respectively, $e^{i\widehat{\lambda}t}\widehat{u}$ is called a *fundamental frequency standing wave* of Equation (1.1)) with *fundamental frequency* $\widehat{\lambda}^S_\mu$ if $\widehat{\lambda}^S_\mu = \Lambda^S_\mu(\widehat{u}) \leqslant \Lambda^S_\mu(w)$ for any $w \in H^1\setminus 0$ such that $D\Lambda^S_\mu(w) = 0$. For $S > 0$ we denote by

$$G^S(\mu) := \{u \in H^1 \setminus 0 : \Lambda^S_\mu(u) = \widehat{\lambda}^S_\mu,\ D\Lambda^S_\mu(u) = 0\} \tag{1.5}$$

the set of fundamental frequency solutions to Equation (1.2) with fundamental frequency $\widehat{\lambda}^S_\mu$. We will show that the existence of a fundamental frequency solution implies the existence of a ground state and the converse is also true (cf. Lemma 5.3). In what follows, $G^S(\mu)$ also denotes the set of ground states.

In accordance with the nonlinear generalized Rayleigh quotient method [10] the functional $\Lambda^S_\mu(u)$ corresponds to the following nonlinear generalized Rayleigh quotient:

$$\lambda^S_\mu(u) := \frac{c^S_N\Big(\displaystyle\int |\nabla u|^2\Big)^{\frac{N}{(N-2)}} - \mu\frac{2}{p}\displaystyle\int |u|^p + \frac{2}{q}\displaystyle\int |u|^q}{\displaystyle\int |u|^2}, \quad u \in H^1 \setminus \{0\}, \quad S > 0, \tag{1.6}$$

where

$$c^S_N = \frac{N-2}{N^{N/(N-2)}S^{2/(N-2)}},$$

for $p, q \in (2, 2^*)$ and $\mu > 0$. We will see that any critical point of $\lambda^S_\mu(u)$ in $H^1 \setminus 0$ corresponds, possibly after scaling, to a critical point of $\Lambda^S_\mu(u)$ and, consequently, yields a solution to Equation (1.2) with prescribed action $S > 0$. Moreover, the nonlinear generalized Rayleigh quotient $\lambda^S_\mu(u)$ is characterized by properties similar to the properties of the usual Rayleigh quotient of linear theory. In particular, as in the spectral theory, the critical value

$$\widehat{\lambda}^S_\mu := \min_{u\in H^1\setminus 0} \lambda^S_\mu(u) \tag{1.7}$$

plays a principal role in the study of Equation (1.2). In the case $2 < q < p < 2^*$, we introduce the additional principal critical value

$$\widehat{\mu}^S = \inf_{u\in \mathscr{D}\setminus 0} \mu^S(u), \tag{1.8}$$

where $\mathscr{D} := \mathscr{D}^{1,2}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$,

$$\mu^S(u) := \Big(\frac{c(p,q,N)}{S^{\frac{2(p-q)}{(2^*-q)(N-2)}}}\Big)\frac{\Big(\displaystyle\int |u|^q\Big)^{\frac{2^*-p}{2^*-q}}\Big(\displaystyle\int |\nabla u|^2\Big)^{\frac{2^*(p-q)}{2(2^*-q)}}}{\displaystyle\int |u|^p}, \quad u \in H^1, \tag{1.9}$$

the constant $c(p,q,N)$ is independent of S (cf. (2.4) below).

The nonlinear generalized Rayleigh quotient $\mu^S(u)$ is characterized by the fact that its critical points correspond to the zero frequency solutions to Equation (1.2). We will show that $\widehat{\mu}^S > 0$.

Our first result concerns the existence of a ground state and a fundamental frequency solution to Equation (1.2) in the nonzero frequency case $\lambda < 0$.

Theorem 1.1. *Let $S > 0$.*

1^o *If $2 < p < q < 2^*$, then for any $\mu > 0$ Equation (1.2) has a fundamental frequency solution $\widehat{u}_\mu^S$ with prescribe action S and frequency $\widehat{\lambda}_\mu^S = \Lambda_\mu^S(\widehat{u}_\mu^S) < 0$.*

2^o *If $2 < q < p < 2^*$, then for any $\mu > \widehat{\mu}^S$ Equation (1.2) has a fundamental frequency solution $\widehat{u}_\mu^S$ with prescribe action S and frequency $\widehat{\lambda}_\mu^S = \Lambda_\mu^S(\widehat{u}_\mu^S) < 0$.*

3^o *If $2 < q < p < 2^*$ and $0 \leqslant \mu < \widehat{\mu}^S$, then Equation (1.2) has no weak solution in H^1 with any action such that $\widetilde{S} \leqslant S$ and $\lambda < 0$.*

Furthermore, $\widehat{u}_\mu^S$ in Assertions 1^o and 2^o is a ground state of Equation (1.2) and a global minimum point of $\lambda_\mu^S(u)$ in H^1, $\widehat{u}_\mu^S > 0$ in $\mathbb{R}^N$, and $\widehat{u}_\mu^S \in C^2(\mathbb{R}^N)$.

Under the assumptions of Assertions 1^o and 2^o of Theorem 1.1, the existence of a spherically symmetric ground state of Equation (1.2) decreasing with respect to $r := |x|$ follows from [4, 11]. The main novelty of Theorem 1.1 is that the ground state $\widehat{u}_\mu^S$ is obtained as the global minimum of $\lambda_\mu^S(u)$ in H^1, which helps in further study of Equation (1.2).

Remark 1.1. By [12], Equation (1.2) cannot have weak, spherically symmetric, decreasing in $r := |x|$ positive solutions if $\lambda > 0$ (cf. [4]).

Regarding the existence and nonexistence of a zero frequency solution to Equation (1.2), we will prove the following assertion.

Theorem 1.2. *Let $S > 0$.*

1^o *If $2 < q < p < 2^*$, then for $\mu = \widehat{\mu}^S$ the zero frequency problem (1.2) has a fundamental frequency solution $\widehat{u}_{\widehat{\mu}^S}^S \in \mathscr{D}$ with prescribe action S. Furthermore, $\widehat{u}_{\widehat{\mu}^S}^S$ is a ground state of Equation (1.2) with $\lambda = 0$ and a global minimum point of $\mu^S(u)$ in $\mathscr{D}$, $\widehat{u}_{\widehat{\mu}^S}^S > 0$ in $\mathbb{R}^N$, and $\widehat{u}_{\widehat{\mu}^S}^S \in C^2(\mathbb{R}^N)$.*

2^o *If $2 < p < q < 2^*$, then the zero frequency problem (1.2) has no weak solution in $\mathscr{D}$ for any $\mu > 0$.*

The assumption of Assertion 1^o of Theorem 1.2 corresponds to the sufficient condition in [4] for the existence of spherically symmetric ground states of the zero–mass problem in $\mathscr{D}$. However, it appears that the result on the absence of solutions to problems with zero frequency, as in Assertion 2^o of Theorem 1.2, has not been earlier known. In this paper, we show that this result admits a fairly simple proof. It should be emphasized that the simplicity of the proof is achieved owing to the use of the nonlinear generalized Rayleigh quotient.

Remark 1.2. The existence of spherically symmetric ground states of the zero-mass problem (1.2) including the more general form

$$-\Delta u = g(u), \quad u \in \mathscr{D}^{1,2},$$

was proved in [4] under some assumptions including the sufficient condition

$$\limsup_{s \to 0+} \frac{g(s)}{s^{2^*-1}} \leqslant 0. \tag{1.10}$$

In the case $2 < p < q < 2^*$, we have $\lim\limits_{s \to 0+} g(s)/s^{2^*-1} = +\infty$. Thus, from Assertion 2^o of Theorem 1.2 it follows that (1.10) is also a necessary condition.

Remark 1.3. In the case of $\lambda = 0$ in (1.2), the dependence of the problem on μ can be neglected since the change of variables $u = (1/\mu)^{1/(p-q)} v(x/\mu^{(q-2)/2(p-q)})$ transforms Equation (1.2) to the equation

$$-\Delta v - |v|^{p-2}v + |v|^{q-2}u = 0, \quad x \in \mathbb{R}^N,$$

but this is not the case for the nonlinear Schrödinger equation (1.1).

Definition 1.1. We call $u^S \in G^S(\mu)$, $S > 0$, a *physical ground state* of Equation (1.2) with respect to the action value if there exists a sequence $u^{S_m} \in G^{S_m}(\mu)$, $m = 1, \ldots$, such that $\lim\limits_{m \to +\infty} S_m = S$, $S_m \neq S$, $m = 1, \ldots,$, and $u^{S_m} \to u^S$ in H^1 as $m \to +\infty$. The set of physical ground states to Equation (1.2) with respect to the action value is denoted by $\widehat{G}^S(\mu)$.

Remark 1.4. Nonphysical ground states, i.e., those in the residual set $\widehat{G}^{S,c}(\mu) := G^S(\mu) \setminus \widehat{G}^S(\mu)$, can be neglected and should not be encountered in nature. Indeed, from Definition 1.1 it follows that for any ground state $u^S \in \widehat{G}^{S,c}(\mu)$ of Equation (1.2) there exists a neighborhood $U \subset H^1$ of u^S such that for any sufficiently small nonzero perturbation of S the problem (1.2) has no ground states in U.

Since the function $S \mapsto \widehat{\mu}^S$ is invertible, for any $\mu > 0$ we can introduce

$$S(\mu) = \overline{\mu}^{\frac{N}{2}} \left(\frac{c(p,q,N)}{\mu}\right)^{\frac{(N-2)(2^*-q)}{2(p-q)}} \tag{1.11}$$

such that $\widehat{\mu}^{S(\mu)} = \mu$ for any $\mu > 0$, where

$$\widehat{\mu}^S := c(p,q,N) \frac{1}{S^{\frac{2(p-q)}{(2^*-q)(N-2)}}} \overline{\mu}^{p/\rho} \quad \forall\, S > 0. \tag{1.12}$$

Theorem 1.3. *Let $\mu > 0$.*

1^o *If $2 < p < q < 2^*$, then for any $S \in (0, \infty)$ there exists a physical ground state of Equation (1.2), i.e., $\widehat{G}^S(\mu) \neq \varnothing$. Furthermore, there exists a unique mass value m^S such that $m^S := m^S_\mu = Q(\widehat{u}^S_\mu)$ for all $\widehat{u}^S_\mu \in \widehat{G}^S(\mu)$. Moreover, the function $S \mapsto m^S$ is continuous on $(0, \infty)$.*

2^o *If $2 < q < p < 2^*$, then for any $S \in (S(\mu), \infty)$ there exists a physical ground state of Equation (1.2), $\widehat{G}^S(\mu) \neq \varnothing$. Furthermore, there exists a unique mass value m^S such that $m^S := m^S_\mu = Q(\widehat{u}^S_\mu)$ for all $\widehat{u}^S_\mu \in \widehat{G}^S(\mu)$. Moreover the function $S \mapsto m^S$ is continuous on $(S(\mu), \infty)$.*

The paper is organized as follows. In Section 2, we give some preliminary information and introduce nonlinear generalized Rayleigh quotients. In Section 3, we prove that the functional $\lambda^S_\mu(u)$ possesses a global minimizer. In Section 4, we prove that $\mu^S(u)$ attains its global minimum in $\mathscr{D}$. In Section 5, we study the behavior of solutions depending on the main parameters λ, S, and m. In Section 6, we complete the proof of Theorems 1.1–1.3.

2 Nonlinear Generalized Rayleigh Quotients

We denote by $H^1 := H^1(\mathbb{R}^N)$ the Sobolev space of functions equipped with the norm

$$\|u\|_1 = \left(\int (|u|^2 + |\nabla u|^2) \right)^{1/2}$$

and introduce the space $\mathscr{D}^{1,2} := \mathscr{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ with the inner product

$$(u, v) := \int \nabla u \cdot \nabla v \, dx$$

and norm

$$\|u\|_{\mathscr{D}^{1,2}} := \int |\nabla u|^2 \, dx.$$

For the sake of brevity we set

$$\int \cdots := \int_{\mathbb{R}^N} \cdots \, dx.$$

For a Gateaux differentiable functional $F : H^1(\mathscr{D}^{1,2}) \to \mathbb{R}$ we denote by $DF(u)$ the derivative of F at $u \in H^1(\mathscr{D}^{1,2})$. For $u \in H^1$ we set

$$u_\sigma := u(x/\sigma), \quad x \in \mathbb{R}^N, \quad \sigma > 0,$$

$$T(u) := \int |\nabla u|^2, \quad Q(u) := \int |u|^2, \quad A(u) := \int |u|^p, \quad B(u) := \int |u|^q.$$

Then we can write

$$S_{\lambda,\mu}(u) := \frac{1}{2}T(u) - \lambda\frac{1}{2}Q(u) - \mu\frac{1}{p}A(u) + \frac{1}{q}B(u).$$

For $S \geqslant 0$ we introduce the so-called *action-level Rayleigh quotient*

$$\Lambda_\mu^S(u) := \frac{\frac{1}{2}T(u) - \mu\frac{1}{p}A(u) + \frac{1}{q}B(u) - S}{\frac{1}{2}Q(u)}. \tag{2.1}$$

We note that for any $S \in \mathbb{R}$ and $\lambda \in \mathbb{R}$

1) $\Lambda_\mu^S(u) = \lambda$ if and only if $S_{\lambda,\mu}(u) = S$,
2) $D\Lambda_\mu^S(u) = 0$ with $\Lambda_\mu^S(u) = \lambda$ if and only if $DS_{\lambda,\mu}(u) = 0$.

For $u \in H^1 \setminus 0$, $S > 0$, $\sigma > 0$ we consider

$$\Lambda_\mu^S(u_\sigma) = \frac{\sigma^{-2}\frac{1}{2}T(u) - \mu\frac{1}{p}A(u) + \frac{1}{q}B(u) - \sigma^{-N}S}{\frac{1}{2}Q(u)}.$$

Then the following equalities are equivalent:

$$\frac{d}{d\sigma}(\Lambda_\mu^S)(u_\sigma) = 0, \quad \frac{2}{Q(u)}\Big(-\sigma^{-3}T(u) + \frac{NS}{\sigma^{N+1}}\Big) = 0, \quad \sigma = \sigma^S(u) := \Big(\frac{NS}{T(u)}\Big)^{\frac{1}{N-2}}.$$

In accordance with the nonlinear generalized Rayleigh quotient method, we can introduce the nonlinear generalized Rayleigh quotient (cf. (1.6))

$$\lambda^S_\mu(u) := \lambda^S_\mu(u_{\sigma^S(u)}) = \frac{2}{Q(u)}\Big(\frac{c^S_N}{2}T^{\frac{N}{(N-2)}}(u) - \mu\frac{1}{p}A(u) + \frac{1}{q}B(u)\Big), \tag{2.2}$$

where

$$c^S_N = \frac{(N-2)}{N^{\frac{N}{(N-2)}}S^{\frac{2}{(N-2)}}}.$$

We note that $\lambda^S_\mu(u)$ is a 0-homogeneous functional with respect to the scale change $\sigma \mapsto u_\sigma$, i.e., $\lambda^S_\mu(u_\sigma) = \lambda^S_\mu(u)$ for all $\sigma > 0$.

A direct calculation of the derivative $D\lambda^S_\mu(u)$ yields the following assertion.

Lemma 2.1. *We assume that $S > 0$ and $u \in H^1 \setminus 0$. Then $D\lambda^S_\mu(u) = 0$, $\lambda^S_\mu(u) = \lambda$, and $\sigma^S(u) = 1$ if and only if u is a weak solution to Equation* (1.2) *with prescribed action S.*

By the homogeneity of $\lambda^S_\mu(u)$, we can assume that any critical point u of $\lambda^S_\mu(u)$ satisfies the equality $\sigma^S(u) = 1$.

We consider

$$\lambda^S_\mu(tu) = \frac{1}{Q(u)}\Big(t^{\frac{4}{N-2}}c^S_N T^{\frac{N}{(N-2)}}(u) - \mu\frac{2}{p}t^{p-2}A(u) + \frac{2}{q}t^{q-2}B(u)\Big).$$

A point $t_0 > 0$ is a *fibering critical point* of $\lambda^S_\mu(tu)$ if $(d\lambda^S_\mu(tu)/dt)|_{t=t_0} = 0$. We note that if $p, q \in (2, 2^*)$, then $4/(N-2) > \max\{p-2, q-2\}$. Then the following assertions are obvious.

1. If $2 < p < q < 2^*$, then for any $u \in H^1 \setminus 0$ and $\mu > 0$ the fibering function $\lambda^S_\mu(tu)$ has a unique fibering critical point $t = t(u)$. Furthermore, $\lambda^S_\mu(t(u)u) = \min\limits_{t>0} \lambda^S_\mu(tu) < 0$ for all $u \in H^1 \setminus 0$ and $\mu > 0$.
2. If $2 < q < p < 2^*$, then for $u \in H^1 \setminus 0$ the fibering function $\lambda^S_\mu(tu)$ can have at most two nonzero critical points $t^0_\mu(u)$ and $t^1_\mu(u)$ s.t. $0 < t^0_\mu(u) \leqslant t^1_\mu(u)$.

Thus, the following assertion holds.

Corollary 2.1. *If $2 < p < q < 2^*$, then $\widehat{\lambda}^S_\mu \equiv \inf\limits_{u\in H^1\setminus 0} \lambda^S_\mu(u) < 0$ for any $\mu > 0$.*

In the case $2 < q < p < 2^*$, we need to know the value of μ if the fibering function $\lambda^S_\mu(tu)$ has two distinct critical points $t^0_\mu(u)$ and $t^1_\mu(u)$, i.e., $0 < t^0_\mu(u) < t^1_\mu(u)$. To find such values, we consider the Rayleigh quotient

$$M^S(u) := \frac{\frac{c^S_N}{2}T^{\frac{N}{(N-2)}}(u) + \frac{1}{q}B(u)}{\frac{1}{p}A(u)}.$$

We note that $M^S(u) = \mu$ if and only if $\lambda^S_\mu(u) = 0$.

For every $u \in H^1 \setminus 0$ we consider the corresponding fibering function

$$M^S(su) := \frac{\frac{c^S_N}{2}s^{2^*-p}T^{\frac{N}{(N-2)}}(u) + \frac{1}{q}s^{q-p}B(u)}{\frac{1}{p}A(u)}, \quad s > 0.$$

It is easy to see that the function $s \mapsto M^S(su)$ has a unique global minimum point $s^S(u) > 0$ such that $M^S(su)$ is monotonically decreasing in $(0, s^S(u))$ and monotonically increasing in $(s^S(u), +\infty)$. To find $s^S(u)$, we calculate

$$\frac{d}{ds} M^S(su) = 0 \;\Leftrightarrow\; s^{2^*-q} c_{p,q,N,S} T^{\frac{N}{(N-2)}}(u) = B(u),$$

where $c_{p,q,N,S} = c_N^S q(2^* - p)/2(p - q)$. Thus for every $u \in H^1 \setminus 0$ the function $M^S(su)$ attains its global minimum at the unique point

$$s^S(u) = \Big(\frac{B(u)}{c_{p,q,N,S} T^{\frac{N}{(N-2)}}(u)}\Big)^{1/(2^*-q)}.$$

Hence we can introduce the nonlinear generalized Rayleigh quotient (cf. (1.9))

$$\mu^S(u) := M^S(s_m(u)u) = \min_{s \geqslant 0} M^S(su) = C_{p,q,N,S} \frac{B^{\frac{2^*-p}{2^*-q}}(u) T^{\frac{2^*(p-q)}{2(2^*-q)}}(u)}{A(u)}, \tag{2.3}$$

where

$$C_{p,q,N,S} = \frac{c(p,q,N)}{S^{\frac{2(p-q)}{(2^*-q)(N-2)}}},$$

and

$$c(p,q,N) = \Big(\frac{(N-2)}{N^{\frac{N}{(N-2)}}} \frac{q(2^*-p)}{2(p-q)}\Big)^{\frac{(p-q)}{(2^*-q)}} \frac{p(2^*-q)}{q(2^*-p)}. \tag{2.4}$$

It is easy to see that $\mu^S(u)$ is 0-homogeneous with respect to both actions $t \mapsto tu$ and $\sigma \mapsto u_\sigma \equiv u(\cdot/\sigma)$, i.e.,

$$\mu^S(u_\sigma) = \mu^S(u), \quad \mu^S(su) = \mu^S(u) \quad \forall \sigma > 0,\ s > 0,\ u \in H^1 \setminus 0. \tag{2.5}$$

A direct calculation of the derivative $D\mu^S(u)$ yields the following assertion.

Lemma 2.2. *We assume that $D\mu^S(u_0) = 0$, $\mu^S(u_0) = \mu_0$, $\sigma(u_0) = 1$, and $t^1_{\mu_0}(u_0) = 1$. Then $DS_{\lambda,\mu}(u_0) = 0$ and $S_{\lambda,\mu}(u_0) = S$ with $\lambda = 0$, $\mu = \mu_0 \equiv \mu^S(u_0)$.*

If $2 < q < p < 2^*$, then from the Gagliardo–Nirenberg interpolation inequality it follows that

$$\int |u|^p \leqslant C_{gn} \Big(\int |\nabla u|^2\Big)^{\frac{2^*(p-q)}{2(2^*-q)}} \Big(\int |u|^q\Big)^{\frac{2^*-p}{2^*-q}} \Leftrightarrow A(u) \leqslant C_{gn} (T(u))^{\frac{2^*(p-q)}{2(2^*-q)}} (B(u))^{\frac{2^*-p}{2^*-q}}, \tag{2.6}$$

where the constant C_{gn} is independent of $u \in \mathscr{D}$. Thus, $\mu^S(u)$ can be extended to $\mathscr{D} \setminus 0$.

We consider the principal critical value (1.8), i.e.,

$$\widehat{\mu}^S = \inf_{u \in \mathscr{D} \setminus 0} \mu^S(u) \equiv C_{p,q,N,S} \inf_{u \in \mathscr{D} \setminus 0} \frac{B^{\frac{2^*-p}{2^*-q}}(u) T^{\frac{2^*(p-q)}{2(2^*-q)}}(u)}{A(u)}.$$

By (2.3),

$$\widehat{\mu}^S = \inf_{u \in \mathscr{D} \setminus 0} M^S(u). \tag{2.7}$$

Moreover, (2.6) implies $\widehat{\mu}^S > 0$.

Proposition 2.1. *Let $2 < q < p < 2^*$.*

(i) *If $0 < \mu \leqslant \widehat{\mu}^S$, then $\lambda_\mu^S(t_\mu^i(u)u) \geqslant 0$ for all $u \in H^1 \setminus 0$, $i = 0, 1$.*

(ii) *If $\mu > \widehat{\mu}^S$, then there exists $u \in H^1 \setminus 0$ such that the function $\lambda_\mu^S(tu)$ has two distinct nonzero critical points $t_\mu^0(u)$ and $t_\mu^1(u)$, $0 < t_\mu^0(u) < t_\mu^1(u) < +\infty$. Moreover, $\lambda_\mu^S(t_\mu^0(u)u) > 0$ and $\lambda_\mu^S(t_\mu^1(u)u) < 0$.*

Proof. Assertion (i) immediately follows from the definition of $\widehat{\mu}^S$ in (1.8).

To prove Assertion (ii), we assume that $\mu > \widehat{\mu}^S$. Then from (1.8) it follows that there exists $u \in H^1 \setminus 0$ such that $\widehat{\mu}^S < \mu^S(u) < \mu$. Since $\mu^S(u)$ is a global minimum of $M^S(su)$ and $M^S(su) \to +\infty$ as $s \downarrow 0$ and $s \to +\infty$, we infer that the equation $M^S(su) = \mu$ has two distinct solutions $s^0(u) < s_\mu^1(u)$. Hence $\lambda_\mu^S(s_\mu^0(u)u) = \lambda_\mu^S(s_\mu^1(u)u) = 0$. Since $\lambda_\mu^S(su) < 0$ for $s \in (s_\mu^0(u), s_\mu^1(u))$, $\lambda_\mu^S(su)$ attains its minimum value at a point $t_\mu^1(u)$ in the interval $(s_\mu^0(u), s_\mu^1(u))$, whereas the local maximum point $t_\mu^0(u)$ belongs to $(0, s_\mu^0(u))$. □

Corollary 2.2. *Let $2 < q < p < 2^*$.*

(i) *If $\mu > \widehat{\mu}^S$, then $\widehat{\lambda}_\mu^S = \inf\limits_{u \in H^1 \setminus 0} \lambda_\mu^S(u) < 0$.*

(ii) *If $\mu = \widehat{\mu}^S$, then $\widehat{\lambda}_{\widehat{\mu}^S}^S = \inf\limits_{u \in H^1 \setminus 0} \lambda_{\widehat{\mu}^S}^S(u) \geqslant 0$,*

(iii) *If $\mu < \widehat{\mu}^S$, then $\widehat{\lambda}_\mu^S = \inf\limits_{u \in H^1 \setminus 0} \lambda_\mu^S(u) > 0$.*

Proof. Assertions (i) and (iii) follow from Proposition 2.1. To prove (ii), we assume the contrary: $\widehat{\lambda}_{\widehat{\mu}^S}^S < 0$. By (1.7), there exists $u \in H^1 \setminus 0$ such that $\widehat{\lambda}_{\widehat{\mu}^S}^S < \lambda_{\widehat{\mu}^S}^S(u) < 0$, which implies $M^S(u) < \widehat{\mu}^S$. Then $\mu^S(u) = \min\limits_{s \geqslant 0} M^S(su) < \widehat{\mu}^S$ which contradicts the definition of $\widehat{\mu}^S$. □

3 Existence of Global Minimizer of $\lambda_\mu^S(u)$

We consider the minimization problem (1.7).

Lemma 3.1. *We assume that $S > 0$ and $2 < p < q < 2^*$, $\mu > 0$ or $2 < q < p < 2^*$, $\mu > \widehat{\mu}^S$. Then the following assertions hold:*

(1) *$\widehat{\lambda}_\mu^S < 0$ and there exists a minimizer $\widehat{u}_\mu^S$ of (1.7), i.e., $\widehat{\lambda}_\mu^S = \lambda_\mu^S(\widehat{u}_\mu^S)$,*

(2) *$\widehat{u}_\mu^S$ is a fundamental frequency solution to Equation (1.2) with prescribe action S. Moreover, $\widehat{u}_\mu^S > 0$ in $\mathbb{R}^N$ and $\widehat{u}_\mu \in C^2(\mathbb{R}^N)$.*

Proof. We simultaneously consider both cases $2 < p < q < 2^*$, $\mu > 0$, and $2 < q < p < 2^*$, $\mu \geqslant \widehat{\mu}^S$. Corollaries 2.1 and 2.2 imply $\widehat{\lambda}_\mu^S < 0$. We consider a minimizing sequence (u_n) of (1.7), i.e., $\lambda_\mu^S(u_n) \to \widehat{\lambda}_\mu^S$ as $n \to +\infty$. Let us show that (u_n) is bounded in H^1. Since the functional $\lambda_\mu^S(u)$ is 0-homogenous, we can assume that $\|u_n\|_{L^2} = 1$, $n = 1, 2, \ldots$. Let $\|\nabla u_n\|_{L^2} \to +\infty$. By the Hölder and Sobolev inequalities,

$$\int |u|^p \leqslant C\|u\|_{L^2}^{\varkappa} \|\nabla u\|_{L^2}^{\frac{2^*(2-\varkappa)}{2}} = C\|\nabla u\|_{L^2}^{\frac{2^*(2-\varkappa)}{2}}, u \in H^1, \tag{3.1}$$

where $\varkappa = 2(2^* - p)/(2^* - 2)$ and $0 < C < +\infty$ is independent of $u \in H^1$. Since $2^* > 2^*(2-\varkappa)/2$, we get

$$\begin{aligned}\lambda_\mu^S(u_n) &\geqslant \frac{c_N^S}{2}\|\nabla u_n\|_{L^2}^{2^*} - \mu\frac{1}{p}\int |u_n|^p \\ &\geqslant \frac{c_N^S}{2}\|\nabla u_n\|_{L^2}^{2^*} - \mu C\frac{1}{p}\|\nabla u_n\|_{L^2}^{\frac{2^*(2-\varkappa)}{2}} \to +\infty, \quad \|\nabla u_n\|_{L^2} \to +\infty,\end{aligned} \tag{3.2}$$

and arrive at a contradiction. Thus, (u_n) is bounded in H^1. By the Sobolev inequality, the norms $\|u_n\|_{L^p}$ and $\|u_n\|_{L^q}$ are also bounded. By the Banach–Alaoglu and Sobolev embedding theorems, there exists a subsequence, still denoted by (u_n), such that $u_n \rightharpoonup \widehat{u}_\mu^S$ in $H^1(\mathbb{R}^N)$, $u_n \to \widehat{u}_\mu^S$ in $L^\gamma_{\text{loc}}(\mathbb{R}^N)$, $1 \leqslant \gamma < 2^*$, and $u_n \to \widehat{u}_\mu^S$ almost everywhere on $\mathbb{R}^N$ for some $\widehat{u}_\mu^S \in H^1$. Let us show that $\widehat{u}_\mu^S \neq 0$. We note that the sequence $\|u_n\|_{L^p}^p \equiv A(u_n)$ is separated from zero. Indeed, if $A(u_n) \to 0$ as $n \to +\infty$, then

$$\widehat{\lambda}_\mu^S = \lim_{n\to+\infty} \frac{2}{Q(u_n)}\Big(\frac{c_N^S}{2}\|\nabla u_n\|_{L^2}^{2^*} - \mu\frac{1}{p}\int |u_n|^p + \frac{1}{q}\int |u_n|^q\Big) \geqslant 0.$$

However, $\widehat{\lambda}_\mu^S < 0$, and we arrive at a contradiction.

Lemma 3.2 (cf. [13, Lemma I.1]). *We assume that $r > 0$, $1 \leqslant \gamma < 2^*$, (u_n) is a bounded in $L^\gamma(\mathbb{R}^N)$, $|\nabla u_n|$ is bounded in $L^2(\mathbb{R}^N)$, and*

$$\sup_{y\in\mathbb{R}^N} \int\limits_{B(y;r)} |u_n|^\gamma \to 0, \quad n \to \infty.$$

Then $u_n \to 0$ in $L^l(\mathbb{R}^N)$ for any $l \in (\gamma, 2^*)$.

Let $r > 0$. We note that

$$\delta := \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B(y;r)} |u_n|^\gamma > 0, \quad 1 \leqslant \gamma < p.$$

Indeed, if this is not the case, then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ in view of Lemma 3.2. However, this is impossible by the above. Thus, passing to a subsequence if necessary, we infer that there exists $(y_n) \subset \mathbb{R}^N$ such that the integral

$$\int\limits_{B(y_n;r)} |u_n|^\gamma > \delta/2, \quad n = 1, \dots,$$

exists. Hence we can assume, setting $u_n := u_n(\cdot + y_n)$ if necessary, that

$$\int\limits_{B(0;r)} |u_n|^\gamma > \delta/2, \; n = 1, \dots, \quad 1 \leqslant \gamma < p.$$

Thus, $\widehat{u}_\mu^S \neq 0$.

Lemma 3.3 (cf. [13]). *We assume that (u_n) is bounded in $L^\gamma(\mathbb{R}^N)$, $1 \leqslant \gamma < +\infty$, and $u_n \to u$ almost everywhere on $\mathbb{R}^N$. Then*

$$\lim_{n\to+\infty} \|u_n\|_{L^\gamma}^\gamma = \|u\|_{L^\gamma}^\gamma + \lim_{n\to+\infty} \|u_n - u\|_{L^\gamma}^\gamma. \tag{3.3}$$

From Lemma 3.3 it follows that

$$\lim_{n\to+\infty} \|\nabla u_n\|_{L^2}^2 = \|\nabla \widehat{u}_\mu^S\|_{L^2}^2 + \lim_{n\to+\infty} \|\nabla(u_n - \widehat{u}_\mu^S)\|_{L^2}^2,$$

$$\lim_{n\to+\infty} \|u_n\|_{L^p}^p = \|\widehat{u}_\mu^S\|_{L^p}^p + \lim_{n\to+\infty} \|u_n - \widehat{u}_\mu^S\|_{L^p}^p,$$

$$\lim_{n\to+\infty} \|u_n\|_{L^q}^q = \|\widehat{u}_\mu^S\|_{L^q}^q + \lim_{n\to+\infty} \|u_n - \widehat{u}_\mu^S\|_{L^q}^q,$$

$$\overline{Q} := \lim_{n\to+\infty} \|u_n\|_{L^2}^2 = \|\widehat{u}_\mu^S\|_{L^2}^2 + \lim_{n\to+\infty} \|u_n - \widehat{u}_\mu^S\|^2.$$

Since

$$(\|\nabla \widehat{u}_\mu^S\|_{L^2}^2 + \|\nabla(u_n - \widehat{u}_\mu^S)\|_{L^2}^2)^{\frac{N}{(N-2)}} \geqslant \|\nabla \widehat{u}_\mu^S\|_{L^2}^{2^*} + \|\nabla(u_n - \widehat{u}_\mu^S)\|_{L^2}^{2^*},$$

we have

$$\widehat{\lambda}_\mu^S = \lim_{n\to+\infty} \lambda_\mu^S(u_n) \geqslant \frac{c_N^S\|\nabla \widehat{u}_\mu^S\|_{L^2}^{2^*} - \mu\frac{2}{p}\|\widehat{u}_\mu^S\|_{L^p}^p + \frac{2}{q}\|\widehat{u}_\mu^S\|_{L^q}^q}{\overline{Q}}$$
$$+ \frac{c_N^S \lim_{n\to+\infty} \|\nabla(u_n - \widehat{u}_\mu^S)\|_{L^2}^{2^*} - \mu\frac{2}{p} \lim_{n\to+\infty} \|u_n - \widehat{u}_\mu^S\|_{L^p}^p + \frac{2}{q} \lim_{n\to+\infty} \|u_n - \widehat{u}_\mu^S\|_{L^q}^q}{\overline{Q}}.$$

Consequently, $\widehat{\lambda}_\mu^S \neq 0$ which implies

$$\widehat{\lambda}_\mu^S \geqslant \frac{1}{\overline{Q}}\widehat{\lambda}_\mu^S \cdot \|\widehat{u}_\mu^S\|_{L^2}^2 + \frac{1}{\overline{Q}} \lim_{n\to\infty} \Big(c_N^S\|\nabla(u_n - \widehat{u}_\mu^S)\|_{L^2}^{2^*} - \mu\frac{2}{p}\|u_n - \widehat{u}_\mu^S\|_{L^p}^p + \frac{2}{q}\|u_n - \widehat{u}_\mu^S\|_{L^q}^q\Big)$$
$$\geqslant \frac{1}{\overline{Q}}(\widehat{\lambda}_\mu^S \cdot \|\widehat{u}_\mu^S\|_{L^2}^2 + \widehat{\lambda}_\mu^S \lim_{n\to\infty} \|u_n - \widehat{u}_\mu^S\|_{L^2}^2) = \widehat{\lambda}_\mu^S \frac{\lim_{n\to+\infty} \|u_n\|_{L^2}^2}{\overline{Q}} = \widehat{\lambda}_\mu^S.$$

However, this is possible only in the case of equality. Hence $u_n \to \widehat{u}_\mu^S$ strongly in H^1 and $\widehat{u}_\mu^S$ is a minimizer of (1.7).

By the homogeneity of $\lambda_\mu^S(u)$, we can assume that $\sigma^S(\widehat{u}_\mu^S) = 1$. By Lemma 2.1, we have $DS_{\lambda,\mu}(\widehat{u}_\mu^S) = 0$ and $S_{\lambda,\mu}(\widehat{u}_\mu^S) = S$, where $\lambda = \widehat{\lambda}_\mu^S$. Since $\lambda_\mu^S(|u|) = \lambda_\mu^S(u)$ for $u \in H^1 \setminus 0$, we can assume that $\widehat{u}_\mu^S \geqslant 0$ in $\mathbb{R}^N$.

Since $\widehat{u}_\mu \in H^1(\mathbb{R}^N)$, from the Brézis–Kato theorem [14] and L^γ estimates for elliptic problems [15] it follows that $\widehat{u}_\mu^S \in W_{\text{loc}}^{2,\gamma}(\mathbb{R}^N)$ for any $\gamma \in (1, +\infty)$. By the regularity theory for solutions to elliptic problems, $\widehat{u}_\mu \in C^2(\mathbb{R}^N)$. By the Harnack inequality, $\widehat{u}_\mu^S > 0$ in $\mathbb{R}^N$. □

Proposition 3.1. *Let $S > 0$.*

1^o *If $2 < p < q < 2^*$, then $\widehat{\lambda}_\mu^S \to 0$ as $\mu \to 0$.*

2^o *IF $2 < q < p < 2^*$, then $\widehat{\lambda}_\mu^S \to 0$ as $\mu \downarrow \widehat{\mu}^S$.*

Proof. We first prove Assertion 2^o. We fix $S > 0$. By Lemma 3.1, for any $\mu > \widehat{\mu}^S$ there exists a minimizer $\widehat{u}^S_\mu$ of (1.7) and $\lambda^S_\mu(\widehat{u}^S_\mu) < 0$. By the homogeneity of $\lambda^S_\mu(u)$, we can assume that $Q(\widehat{u}^S_\mu) = 1$ for all $\mu > \widehat{\mu}^S$. By (2.2),

$$\lambda^S_\mu(\widehat{u}^S_\mu) - \lambda^S_{\widehat{\mu}^S}(\widehat{u}^S_\mu) = -2(\mu - \widehat{\mu}^S)\frac{1}{p}A(\widehat{u}^S_\mu)$$

Arguing as in the proof of Lemma 3.1, we see that the set $(\widehat{u}^S_\mu)$ is bounded in H^1 for μ sufficiency close to $\widehat{\mu}^S$. By (3.1), the sequence $A(\widehat{u}^S_\mu)$ is also bounded. Hence $\lambda^S_\mu(\widehat{u}^S_\mu) - \lambda^S_{\widehat{\mu}^S}(\widehat{u}^S_\mu) \to 0$ as $\mu \downarrow \widehat{\mu}^S$. By Corollary 2.2, $\lambda^S_{\widehat{\mu}^S}(\widehat{u}^S_\mu) \geqslant 0$, whereas $\lambda^S_\mu(\widehat{u}^S_\mu) < 0$, which implies $\widehat{\lambda}^S_\mu \to 0$ as $\mu \downarrow \widehat{\mu}^S$.

The proof of Assertion 1^o is similar. We only note that, in this case, $\lambda^S_{\widehat{\mu}^S}(u)$ is replaced by $\lambda^S_0(u) := \lambda^S_\mu(u)|_{\mu=0}$. We note that $\inf\limits_{u\in H^1\setminus 0} \lambda^S_0(u) = 0$ in view of (2.2). □

4 Existence of Solution to Zero Frequency Problem in $\mathscr{D}$

In this section, we establish the existence of a fundamental frequency solution to the zero frequency problem (1.2) by using the minimization problem (1.8). We denote

$$\beta := \frac{2q(2^* - p)}{2^*(p-q)}, \quad \rho := \frac{2p(2^* - q)}{2^*(p-q)},$$

$$\mu(u) := \frac{\|u\|^\beta_{L^q}\|\nabla u\|^2_{L^2}}{\|u\|^\rho_{L^p}(u)} \equiv (S^{\frac{2(p-q)}{(2^*-q)(N-2)}}\mu^S(u)/c(p,q,N))^{\rho/p}, \quad u \in H^1 \setminus 0,$$

and consider

$$\overline{\mu} = \inf_{u\in\mathscr{D}\setminus 0} \mu(u). \tag{4.1}$$

Then

$$\widehat{\mu}^S := c(p,q,N)\frac{1}{S^{\frac{2(p-q)}{(2^*-q)(N-2)}}}\overline{\mu}^{p/\rho} \quad \forall\ S > 0. \tag{4.2}$$

Lemma 4.1. *Let $2 < q < p < 2^*$. Then there exists a minimizer $\widehat{u}^S_{\widehat{\mu}^S} \in \mathscr{D}$ of (1.8) such that $\widehat{u}^S_{\widehat{\mu}^S}$ weakly satisfies to Equation (1.2) with $\lambda = 0$ and $\mu = \widehat{\mu}^S$. Moreover, $\widehat{u}^S_\mu > 0$ in $\mathbb{R}^N$, $\widehat{u}_\mu \in C^2(\mathbb{R}^N)$, and $S_{0,\widehat{\mu}^S}(\widehat{u}^S_{\widehat{\mu}^S}) = S$.*

Proof. Let (v_i) be a minimizing sequence of (4.1), i.e., $\mu(v_i) \to \overline{\mu}$ as $i \to \infty$. Set $u_i = t_i(v_i)_{\sigma_i}$, $i = 1, 2, \ldots$, where $t_i = (\|v_i\|^q_{L^q}/\|v_i\|^p_{L^p})^{1/(p-q)}$ and $\sigma_i = (\|v_i\|^{pq}_{L^q}/\|v_i\|^{qp}_{L^p})^{1/N(p-q)}$. Then $\|u_i\|_{L^p} = 1$ and $\|u_i\|_{L^q} = 1$, $i = 1, \ldots$. By the homogeneity of $\mu(u)$, (u_i) is a minimizing sequence of (4.1). Since $\overline{\mu} < +\infty$, $(\|\nabla u_i\|_{L^2})$ is bounded. Thus, (u_i) is bounded in $\mathscr{D}$ and in H^1_{loc}. By the Banach–Alaoglu theorem and the Sobolev embeddings, there exists a subsequence, still denoted by (u_i), such that $u_i \rightharpoonup \widehat{u}^*$ in $\mathscr{D}$, $u_i \to \widehat{u}^*$ in L^γ_{loc}, $1 \leqslant \gamma < 2^*$, $u_i \to \widehat{u}^*$ almost everywhere on $\mathbb{R}^N$ for some $\widehat{u}^* \in \mathscr{D}$. Since the sequence $\|u_i\|_{L^q} = B(u_i)$ is bounded, we can apply Lemma 3.2. Arguing as in the proof of Lemma 3.1, we see that for any fixed $r > 0$ there exists $(y_i) \subset \mathbb{R}^N$ such that

$$\int\limits_{B(y_i;r)} |u_i|^q > \delta/2, \quad i = 1, \ldots.$$

Setting $u_i := u_i(\cdot + y_n)$, if necessary, we get

$$\int\limits_{B(0;r)} |u_i|^q > \delta/2, \quad i = 1, \dots,$$

which implies $\widehat{u}^* \neq 0$. By Lemma 3.3, we have

$$\|\nabla \widehat{u}^*\|_{L^2}^2 = \lim_{i\to\infty} \|\nabla u_i\|_{L^2}^2 - \lim_{i\to\infty} \|\nabla (u_i - \widehat{u}^*)\|_{L^2}^2, \tag{4.3}$$

$$\|\widehat{u}^*\|_{L^p}^p = \lim_{i\to\infty} \|u_i\|_{L^p}^p - \lim_{i\to\infty} \|u_i - \widehat{u}^*\|_{L^p}^p, \tag{4.4}$$

$$\|\widehat{u}^*\|_{L^q}^q = \lim_{i\to\infty} \|u_i\|_{L^q}^q - \lim_{i\to\infty} \|u_i - \widehat{u}^*\|_{L^q}^q. \tag{4.5}$$

Let $\lim\limits_{i\to\infty} \|\nabla (u_i - \widehat{u}^*)\|_{L^2}^2 > 0$, and let $\lim\limits_{i\to\infty} \|\nabla (u_i - \widehat{u}^*)\|_{L^2}^2 > 0$. Then

$$\begin{aligned} \overline{\mu} &= \lim_{i\to\infty} \|\nabla u_i\|_{L^2}^2 = \|\nabla \widehat{u}^*\|_{L^2}^2 + \lim_{i\to\infty} \|\nabla (u_i - \widehat{u}^*)\|_{L^2}^2 \\ &\geqslant \overline{\mu}\Big(\frac{\|\widehat{u}^*\|_{L^p}^\rho}{\|\widehat{u}^*\|_{L^q}^\beta} + \lim_{i\to\infty} \frac{\|u_i - \widehat{u}^*\|_{L^p}^\rho}{\|u_i - \widehat{u}^*\|_{L^q}^\beta}\Big) = \overline{\mu}\Big(\frac{\|\widehat{u}^*\|_{L^p}^\rho}{\|\widehat{u}^*\|_{L^q}^\beta} + \frac{(1 - \|\widehat{u}^*\|_{L^p}^p)^{\rho/p}}{(1 - \|\widehat{u}^*\|_{L^q}^q)^{\beta/q}}\Big) > \overline{\mu}, \end{aligned} \tag{4.6}$$

and we arrive at a contradiction. Hence

$$\|\nabla \widehat{u}^*\|_{L^2}^2 = \lim_{i\to\infty} \|\nabla u_i\|_{L^2}^2 = \overline{\mu}, \quad \|\widehat{u}^*\|_{L^p}^p = \lim_{i\to\infty} \|u_i\|_{L^p}^p, \quad \|\widehat{u}^*\|_{L^q}^q = \lim_{i\to\infty} \|u_i\|_{L^q}^q.$$

Consequently, $\widehat{u}^*$ is a minimizer of (4.1). By the homogeneity of $\mu^S(u)$, any function $s\widehat{u}^*_\sigma$ with $s > 0$, $\sigma > 0$ is also a minimizer of (1.8). Hence we can find a minimizer $\widehat{u}^S_{\widehat{\mu}^S} \in \mathscr{D}$ of (1.8) satisfying $\sigma^S(\widehat{u}^S_{\widehat{\mu}^S}) = 1$ and $s^S(\widehat{u}^S_{\widehat{\mu}^S}) = 1$. Since $D\mu^S(\widehat{u}^S_{\widehat{\mu}^S}) = 0$, we have $DS_{0,\mu}(\widehat{u}^S_{\widehat{\mu}^S}) = 0$. The further argument is the same as in the proof of Lemma 3.1. □

5 Further Properties of Solutions

In this section, we study the behavior of solutions depending on the main parameters λ, S, and m of the problem. In Sections 2–4, for every given value of S Equation (1.2) was considered with respect to the parameter $\mu > 0$. In this section, for a given $\mu > 0$ we consider Equation (1.2) with respect to S.

From Corollary 2.2 and Lemma 3.1 we obtain the following assertion.

Corollary 5.1. 1^o *If* $2 < p < q < 2^*$ *and* $\mu > 0$, *then* $G^S(\mu) \neq \varnothing$ *for any* $S > 0$.

2^o *If* $2 < q < p < 2^*$ *and* $\mu > 0$, *then*

(i) $G^S(\mu) \neq \varnothing$ *for any* $S \geqslant S(\mu)$,

(ii) *Equation* (1.2) *has no solution with* $\lambda \leqslant 0$ *and* $S \in (0, S(\mu))$.

Proof. Assertions 1^o and 2^o (i) immediately follow from Lemma 3.1. Let us prove Assertion 2^o (ii). Indeed, if there exists a solution u_0 to Equation (1.2) with $S_{\lambda,\mu}(u_0) = S \in (0, S(\mu))$, then $\mu < \widehat{\mu}^S$ in view of (1.11). By Corollary 2.2, we get $\lambda = \Lambda^S(u_0) = \lambda^S(u_0) > 0$, which contradicts the assumption $\lambda \leqslant 0$. □

Proposition 5.1. *Let $p,q \in (2,2^*)$. If $\mu>0$, $S_2>S_1>0$, and $G^{S_j}(\mu)\neq\varnothing$, $j=1,2$, then*

$$-2\frac{(S_2-S_1)(S_2/S_1)^{N/(N-2)}}{Q(\widehat{u}_\mu^{S_2})}<\widehat{\lambda}_\mu^{S_2}-\widehat{\lambda}_\mu^{S_1}<-2\frac{(S_2-S_1)(S_1/S_2)^{N/(N-2)}}{Q(\widehat{u}_\mu^{S_1})} \tag{5.1}$$

for all $\widehat{u}_\mu^{S_j}\in G^{S_j}(\mu)$, $j=1,2$.

Proof. We note that for $\widehat{u}_\mu^{S_j}\in G^{S_j}(\mu)$ we have $\sigma^{S_j}(\widehat{u}_\mu^{S_j})=1$, $j=1,2$. Hence

$$\widehat{\lambda}_\mu^{S_2}=\lambda_\mu^{S_2}(\widehat{u}_\mu^{S_2})\leqslant\lambda_\mu^{S_2}(\widehat{u}_\mu^{S_1})=\Lambda_\mu^{S_2}((\widehat{u}_\mu^{S_1})_{\sigma^{S_2}(\widehat{u}_\mu^{S_1})})=\Lambda_\mu^{S_1}((\widehat{u}_\mu^{S_1})_{\sigma^{S_2}(\widehat{u}_\mu^{S_1})})-2\frac{S_2-S_1}{Q((\widehat{u}_\mu^{S_1})_{\sigma^{S_2}((\widehat{u}_\mu^{S_1})})}.$$

Since $\sigma^{S_1}(\widehat{u}_\mu^{S_1})$ is a global maximum point of $\mathbb{R}^+\ni\sigma\mapsto\Lambda^{S_1}((\widehat{u}_\mu^{S_1})_\sigma)$, we have

$$\widehat{\lambda}_\mu^{S_1}=\Lambda_\mu^{S_1}((\widehat{u}_\mu^{S_1})_{\sigma^{S_1}(\widehat{u}_\mu^{S_1})})>\Lambda_\mu^{S_1}((\widehat{u}_\mu^{S_1})_{\sigma^{S_2}(\widehat{u}_\mu^{S_1})}).$$

Hence

$$\widehat{\lambda}_\mu^{S_2}-\widehat{\lambda}_\mu^{S_1}<-2\frac{S_2-S_1}{Q((\widehat{u}_\mu^{S_1})_{\sigma^{S_2}(\widehat{u}_\mu^{S_1})})}.$$

Taking into account that

$$\frac{1}{Q((\widehat{u}_\mu^{S_1})_{\sigma^{S_2}(\widehat{u}_\mu^{S_1})})}=\frac{T(\widehat{u}_\mu^{S_1})^{\frac{N}{(N-2)}}}{(NS_2)^{N/(N-2)}Q(\widehat{u}_\mu^{S_1})},\quad \sigma^{S_1}(\widehat{u}^{S_1})=\Big(\frac{NS_1}{T(\widehat{u}_\mu^{S_1})}\Big)^{\frac{1}{N-2}}=1,$$

we obtain the second inequality in (5.1). The first one is proved in the same way. □

Corollary 5.2. *We assume that $\mu>0$ and $2<p<q<2^*$ $(2<q<p<2^*)$. Then the function $S\mapsto\widehat{\lambda}_\mu^S$ is continuous and monotonically decreases on $(0,\infty)$ ($(S(\mu),+\infty)$). Furthermore, $\widehat{\lambda}_\mu^S\to 0$ as $S\to 0$ $(S\to S(\mu))$.*

Proof. Let $S_0\in(0,\infty)$ $((S(\mu),+\infty))$. By (5.1),

$$\frac{S_2^{2^*}}{Q(\widehat{u}_\mu^{S_2})}>\frac{S_1^{2^*}}{Q(\widehat{u}_\mu^{S_1})}\quad\forall\ S_2>S_1>0,\quad \widehat{u}_\mu^{S_j}\in G^{S_j}(\mu),\quad j=1,2,$$

which implies that the set $(Q(\widehat{u}_\mu^S))_{S\in(S_0-\varepsilon,S_0+\varepsilon)}$ is bounded and separated from zero for any $S_0>0$ and $\varepsilon>0$ such that $S_0>\varepsilon$ $(S_0>S(\mu)+\varepsilon)$. By (5.1), $\widehat{\lambda}_\mu^S\to\widehat{\lambda}_\mu^{S_0}$ as $S\to S_0$. By (5.1), $\widehat{\lambda}_\mu^S$ is monotonically decreasing on $(0,\infty)$ $((S(\mu),+\infty))$. The further consideration follows Proposition 3.1. □

Lemma 5.1. *We assume that $\mu>0$ and $2<p<q<2^*$ $(2<q<p<2^*)$. Then $\widehat{G}^S(\mu)\neq\varnothing$ for all $S\in(0,+\infty)$ $(S\in(S(\mu),+\infty))$.*

Proof. We note that for $2<p<q<2^*$ from Corollary 5.1 it follows that $G^S(\mu)\neq\varnothing$ for any $\mu>0$ and $S\in(0,+\infty)$, whereas for $2<q<p<2^*$ the inequality $S>S(\mu)$ implies $\widehat{\mu}^S<\mu$ in view of (1.11). Then Lemma 3.1 yields $G^S(\mu)\neq\varnothing$.

Let $S\in(0,+\infty)$ $(S\in(S(\mu),+\infty))$. We assume that the sequence (S_m) is such that $S_m\to S$. We fix an arbitrary $\widehat{u}_\mu^{S_m}\in G^{S_m}(\mu)$, $m=1,\ldots$. By the homogeneity of $\lambda_\mu^S(u)$, we can assume that $Q(\widehat{u}_\mu^{S_m}) = 1$, $m = 1, \ldots$. Arguing as in the proof of Lemma 3.1, we conclude that $(\widehat{u}_\mu^{S_m})$ is bounded in H^1. By (2.2), $\lambda_\mu^{S_m}(\widehat{u}_\mu^{S_m}) - \lambda_\mu^S(\widehat{u}_\mu^{S_m}) = (c_N^{S_m} - c_N^S)T^{\frac{N}{(N-2)}}(\widehat{u}_\mu^{S_m})$, $m = 1, \ldots$. By the continuity of $\widehat{\lambda}_\mu^{(\cdot)}$, the last equality implies $\lambda_\mu^S(\widehat{u}_\mu^{S_m}) \to \widehat{\lambda}_\mu^S$, i.e., $(\widehat{u}_\mu^{S_m})$ is a minimizing sequence of $\lambda_\mu^S(u)$. Arguing as in the proof of Lemma 3.1, we infer that there exists a subsequence, still denoted by $(\widehat{u}_\mu^{S_m})$, such that $\widehat{u}_\mu^{S_m} \to \widehat{u}_\mu^S$ strongly in H^1 for some $\widehat{u}_\mu^S \in G_\mu^S$. By Definition 1.1, this means that $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$. Hence $\widehat{G}^S(\mu) \neq \varnothing$. □

Lemma 5.2. *Let $\mu > 0$.*

1^o *If $2 < p < q < 2^*$, then the function $\widehat{\lambda}_\mu^S$ is differentiable at every $S \in (0, +\infty)$ and*

$$\frac{d}{dS}\widehat{\lambda}_\mu^S = -2\frac{1}{Q(\widehat{u}_\mu^S)} \tag{5.2}$$

for all $S \in (0, +\infty)$, $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$. Furthermore, for every $S \in (0, +\infty)$ there is a constant $m^S > 0$ such that

$$m^S := m_\mu^S = Q(\widehat{u}_\mu^S) \quad \forall \widehat{u}_\mu^S \in \widehat{G}^S(\mu); \tag{5.3}$$

moreover, $S \mapsto m^S$ is continuous on $(0, \infty)$.

2^o *If $2 < q < p < 2^*$, then the function $\widehat{\lambda}_\mu^S$ is differentiable at every $S \in (S(\mu), +\infty)$ and* (5.2) *holds for all $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$ and $S \in (S(\mu), +\infty)$. Furthermore, for every $S \in (S(\mu), +\infty)$ there is a constant $m^S > 0$ such that* (5.3) *holds; moreover, $S \mapsto m^S$ is continuous on $(0, \infty)$.*

Proof. We will prove both assertions 1^o and 2^o simultaneously.

We assume that $\mu > 0$ and $2 < p < q < 2^*$ $(2 < q < p < 2^*)$. For every $S \in (0, +\infty)$ $(S \in (S(\mu), +\infty))$ we take an arbitrary $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$ so that we have a uniquely defined branch of the solution $(\widehat{u}_\mu^S)$, $S \in (0, +\infty)$ $(S \in (S(\mu), +\infty))$.

We fix $S \in (0, +\infty)$ $(S \in (S(\mu), +\infty))$ and consider a sequence (S_m) such that $S_m \to S$ as $m \to +\infty$. From the proof of Corollary 5.2 we know that the sequence $(Q(\widehat{u}_\mu^{S_m}))$ is bounded and separated from zero. Since $\sigma^{S_m}(\widehat{u}_\mu^{S_m}) = 1$, i.e., $\|\nabla \widehat{u}_\mu^{S_m}\|_{L^2}^2 = NS_m$, $m = 1, \ldots$, we conclude that the set $(\widehat{u}_\mu^{S_m})$ is bounded in H^1. Arguing as in the proof of Lemma 3.1, we conclude that there exists a limit point $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$ such that $\widehat{u}_\mu^{S_{m_k}} \to \widehat{u}_\mu^S$ in H^1 for some subsequence (m_k) such that $m_k \to +\infty$ as $k \to +\infty$. Therefore, (5.1) implies the existence of the derivative $\frac{d}{dS}\widehat{\lambda}_\mu^S$ satisfying (5.2) with some $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$. Thus, the derivative $\frac{d}{dS}\widehat{\lambda}_\mu^S$ exists for any $S \in (0, +\infty)$ $(S \in (S(\mu), +\infty))$. We take an arbitrary $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$. By Definition 1.1, there exists a sequence $u^{S_n} \in G^{S_n}(\mu)$, $S_n \neq S$, $n = 1, 2, \ldots$, such that $\lim_{n \to +\infty} S_n = S$ and $u^{S_n} \to \widehat{u}_\mu^S$ in H^1 as $n \to +\infty$. As above, (5.1) implies (5.2). Thus, the equality (5.2) holds for any $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$.

Since $\widehat{\lambda}_\mu^{(\cdot)}$ is well defined, the map $S \mapsto \frac{d}{dS}\widehat{\lambda}_\mu^S$ is uniquely defined and, consequently, the right-hand side of (5.2) is also uniquely defined. Thus, for every $S \in (0, +\infty)$ $(S \in (S(\mu), +\infty))$ there exists a unique constant $m^S > 0$ such that $m_\mu^S = Q(\widehat{u}_\mu^S)$ for every $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$. From the convergence $\widehat{u}_\mu^{S_{m_k}} \to \widehat{u}_\mu^S$ in H^1 it follows that the function $S \mapsto m^S$ is continuous on $(0, +\infty)$ $((S(\mu), +\infty))$. □

Corollary 5.3. *We assume that* $\mu > 0$ *and* $2 < p < q < 2^*$. *Then* $\lim\limits_{S\to+\infty} \widehat{\lambda}_\mu^S = -\infty$. *Moreover, there exists the inverse* $\lambda \mapsto S_\lambda$ *of* $\widehat{\lambda}_\mu^S$ *such that* $\widehat{\lambda}_\mu^{S_\lambda} = \lambda$ *for all* $\lambda \in (-\infty, 0)$. *Moreover,* S_λ *is continuous and monotonically increases on* $(-\infty, 0)$.

Proof. Since $S \mapsto \widehat{\lambda}_\mu^S$ is continuous and monotonically decreases on $(0,\infty)$ $((S(\mu), +\infty))$, we can conclude that $\lim\limits_{S\to+\infty} \widehat{\lambda}_\mu^S = \overline{\lambda}_\mu \geqslant -\infty$ exists. Assume, contrary to our claim, that $\overline{\lambda}_\mu > -\infty$.

We take $\lambda < \overline{\lambda}_\mu$. Since $2 < p < q < 2^*$, there exists a solution $\widehat{u}_\lambda \in H^1 \setminus 0$ to Equation (1.2) in view of the Berestycki–Lions theorem [4]. We denote $S = S_{\lambda,\mu}(\widehat{u}_\lambda)$. Then $\Lambda_\mu^S(\widehat{u}_\lambda) = \lambda$. Since $\sigma^S(\widehat{u}_\lambda) = 1$, we have $\Lambda_\mu^S(\widehat{u}_\lambda) = \lambda_\mu^S(\widehat{u}_\lambda) = \lambda$. Hence $\lambda \geqslant \widehat{\lambda}_\mu^S > \overline{\lambda}_\mu$, which contradicts the assumption $\lambda < \overline{\lambda}_\mu$. The rest of the proof immediately follows from Corollary 5.2. □

Lemma 5.3. 1^o *We assume that* $\mu > 0$, $p, q \in (2, 2^*)$, *and* $S > 0$ *is such that there exists a fundamental frequency solution* $\widehat{u}_\mu^S$ *to Equation* (1.2) *with fundamental frequency level* $\lambda := \widehat{\lambda}_\mu^S = \lambda_\mu^S(\widehat{u}_\mu^S) < 0$. *Then* $\widehat{u}_{\lambda,\mu} := \widehat{u}_\mu^S$ *is a ground state of Equation* (1.2) *with ground level* S.

2^o *We assume that* $\mu > 0$, $2 < q < p < 2^*$, *and* $\lambda \in (-\infty, 0)$ *and* $\widehat{u}_{\lambda,\mu}$ *is a ground state of Equation* (1.2) *with ground level* $S = S_{\lambda,\mu}(\widehat{u}_\lambda)$. *Then* $\widehat{u}_\mu^S := \widehat{u}_{\lambda,\mu}$ *is a fundamental frequency solution to Equation* (1.2) *with fundamental frequency level* $\lambda = \widehat{\lambda}_\mu^S$.

Proof. We assume that Assertion 1^o fails. Then there exists a solution w to Equation (1.2) with $\lambda = \widehat{\lambda}_\mu^S$ such that $S_1 := S_{\widehat{\lambda}_\mu^S,\mu}(w) < S_{\widehat{\lambda}_\mu^S,\mu}(\widehat{u}^S) = S$. We note that $\Lambda_\mu^{S_1}(w) = \widehat{\lambda}_\mu^S$ and $D\Lambda_\mu^{S_1}(w) = 0$. Hence $\sigma^{S_1}(w) = 1$. Therefore, $\lambda_\mu^{S_1}(w) = \Lambda_\mu^{S_1}(w)$. Hence

$$\widehat{\lambda}_\mu^{S_1} = \min_{u\in H^1(\mathbb{R}^N)\setminus 0} \lambda_\mu^{S_1}(u) \leqslant \lambda_\mu^{S_1}(w) = \Lambda_\mu^{S_1}(w) = \widehat{\lambda}_\mu^S,$$

which contradicts the fact that $\widehat{\lambda}_\mu^S$ is monotonically decreasing and $S_1 < S$ by Lemma 5.2.

To prove Assertion 2^o, we assume that $2 < q < p < 2^*$, $\mu > 0$, $\lambda \in (-\infty, 0)$, and $\widehat{u}_{\lambda,\mu}$ is a ground state of Equation (1.2) with ground level $S = S_{\lambda,\mu}(\widehat{u}_{\lambda,\mu})$. Since $\sigma^S(\widehat{u}_{\lambda,\mu}) = 1$, we infer that $\Lambda_\mu^S(\widehat{u}_{\lambda,\mu}) = \lambda_\mu^S(\widehat{u}_{\lambda,\mu}) = \lambda$. Thus, $\lambda \geqslant \widehat{\lambda}_\mu^S$. By Corollary 5.3, there exists $S_\lambda > S(\mu)$ and a fundamental frequency solution $\widehat{u}_\mu^{S_\lambda}$ such that $\lambda = \widehat{\lambda}_\mu^{S_\lambda}$ and $S_{\lambda,\mu}(\widehat{u}_\mu^{S_\lambda}) = S_\lambda$. Then $S_\lambda \geqslant S = S_{\lambda,\mu}(\widehat{u}_{\lambda,\mu})$ since $\widehat{u}_{\lambda,\mu}$ is a ground state of Equation (1.2). Consequently, $\widehat{\lambda}_\mu^S \geqslant \widehat{\lambda}_\mu^{S_\lambda} = \lambda$ by Proposition 5.1. At the same time, by the above, $\lambda \geqslant \widehat{\lambda}_\mu^S$ and, consequently, $\lambda = \widehat{\lambda}_\mu^S$, i.e., $\widehat{u}_{\lambda,\mu}$ is a fundamental frequency solution to Equation (1.2) with fundamental frequency λ. □

Corollary 5.4. *We assume that* $2 < q < p < 2^*$ *and* $\mu > 0$. *For any given* $\lambda \in (-\infty, 0)$ *there exists a ground state* $\widehat{u}_\lambda$ *of Equation* (1.2); *moreover,* $\widehat{u}_\lambda \in \widehat{G}^S(\mu)$ *with* $S := S_{\lambda,\mu}(\widehat{u}_\lambda)$.

Proof. Let $\lambda \in (-\infty, 0)$. By Corollary 5.3, there exists a unique $S := S_\lambda \in (S(\mu), +\infty)$ such that $\lambda = \widehat{\lambda}_\mu^S$. By Lemma 3.1, there exists a fundamental frequency solution $\widehat{u}_\mu^S \in \widehat{G}^S(\mu)$ to Equation (1.2). By Assertion 1^o of Lemma 5.3, we conclude that $\widehat{u}_\lambda := \widehat{u}_\mu^S$ is a ground state of Equation (1.2) with frequency λ and action level S. □

6 Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. We assume that $2 < p < q < 2^*$, $\mu > 0$ or $2 < q < p < 2^*$, $\mu > \widehat{\mu}^S$. By Lemma 3.1, there exists a fundamental frequency solution $\widehat{u}^S_\mu$ to Equation (1.2) with prescribe action S and frequency $\lambda = \widehat{\lambda}^S_\mu < 0$. Moreover, $\widehat{u}^S_\mu > 0$ in $\mathbb{R}^N$ and $\widehat{u}_\mu \in C^2(\mathbb{R}^N)$. By Lemma 5.3, $\widehat{u}^S_\mu$ is a ground state of Equation (1.2).

It remains to prove Assertion 3^o. We assume that $2 < q < p < 2^*$, $0 \leqslant \mu < \widehat{\mu}^S$. Assume, contrary to our claim, that there exists a weak solutions $\widetilde{u} \in H^1(\mathbb{R}^N)$ to Equation (1.2) such that $\Lambda^{\widetilde{S}}_\mu(\widetilde{u}) =: \lambda < 0$ with $\widetilde{S} \leqslant S$. Then $0 > \Lambda^{\widetilde{S}}_\mu(\widetilde{u}) = \lambda^{\widetilde{S}}_\mu(\widetilde{u}) \geqslant \lambda^S_\mu(\widetilde{u})$, and thus $M^S(\widetilde{u}) < \mu < \widehat{\mu}^S$ which contradicts (2.3). □

Proof of Theorem 1.2. To prove Assertion 1^o, we assume that $2 < q < p < 2^*$, $S > 0$. By Lemma 4.1, there exists a minimizer $\widehat{u}^S_{\widehat{\mu}^S} \in \mathscr{D}$ of (1.8) such that $\widehat{u}^S_{\widehat{\mu}^S}$ weakly satisfies to Equation (1.2) with $\lambda = 0$ and $\mu = \widehat{\mu}^S$. Moreover, $\widehat{u}^S_\mu > 0$ in $\mathbb{R}^N$, $\widehat{u}^S_\mu \in C^2(\mathbb{R}^N)$, and $S_{0,\widehat{\mu}^S}(\widehat{u}^S_{\widehat{\mu}^S}) = S$.

Let us show that $\widehat{u}^S_{\widehat{\mu}^S}$ is a ground state of Equation (1.2). Assume the contrary, i.e., there exists a weak solution $v \in \mathscr{D}\backslash 0$ to Equation (1.2) such that $DS_{0,\widehat{\mu}^S}(v) = 0$ and $\widetilde{S} := S_{0,\widehat{\mu}^S}(v) < S$. Then $M^{\widetilde{S}}(v) = \widehat{\mu}^S$ and $M^S(v) < M^{\widetilde{S}}(v) = \widehat{\mu}^S$ since $\widetilde{S} < S$, which contradicts (2.3).

To show that $\widehat{u}^S_{\widehat{\mu}^S}$ is a fundamental frequency solution, it suffices to note that Equation (1.2) with $\mu = \widehat{\mu}^S$ cannot have solution with frequency $\lambda < 0$ in view of Corollary 2.2.

To prove Assertion 2^o, we assume that $2 < p < q < 2^*$ and $\mu > 0$. Assume the contrary, i.e., Equation (1.2) with $\lambda = 0$ has a weak solution $\overline{u}_\mu \in \mathscr{D}$. Then $0 < S_{0,\mu}(\overline{u}_\mu) < +\infty$, and for $S := S_{0,\mu}(\overline{u}_\mu)$ we have $M^S(\overline{u}_\mu) = \mu$, $DM^S(\overline{u}_\mu) = 0$. Hence $\dfrac{d}{ds}M^S(s\overline{u}_\mu)|_{s=1} = 0$. However, in the case $2 < p < q < 2^*$, the function $s \mapsto M^S(s\overline{u}_\mu)$ cannot have nonzero critical points. □

Proof of Theorem 1.3. The required assertion follows from Lemmas 5.1 and 5.2. □

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