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ON THE LAW OF THE ITERATED LOGARITHM FOR SUMS OF DEPENDENT RANDOM VARIABLES

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Sufficient conditions are found for the applicability of the generalized law of the iterated logarithm for sums of dependent random variables in the case where the sequence of normalizing constants is not necessarily nondecreasing. Bibliography: 7 titles.

Let $\{X_n;\ n = 1, 2, \ldots\}$ be a sequence of random variables on a probability space and let $\{a_n;\ n = 1, 2, \ldots\}$ be a sequence of positive numbers such that $a_n \to \infty$ as $n \to \infty$. Set $S_n = \sum_{k=1}^{n} X_k$. The paper [1] contains new conditions for the applicability of the generalized law of the iterated logarithm in the form

$$\limsup S_n/a_n \le 1 \text{ a.s.} \tag{1}$$

under the additional assumption on nondecreasing of the number sequence $\{a_n\}$. In this note, we obtain sufficient conditions under which inequality (1) holds if one waives the condition formulated above. The following condition D had been introduced in [1]: For any positive ε and $\varepsilon_0 < \varepsilon$ there exists a number $\gamma > 0$ such that

$$\mathbf{P}(\max_{1\le k\le n} S_k > (1+\varepsilon)a_n) \le \gamma \mathbf{P}(S_n > (1+\varepsilon_0)a_n) \tag{2}$$

for all n large enough. The origin of this condition is the classical Kolmogorov inequality:

$$\mathbf{P}(\max_{1\le k\le n} S_k \ge x) \le 2\mathbf{P}(S_n \ge x - (2B_n)^{1/2}) \tag{3}$$

for any x and for a sum S_n of independent random variables with zero expectations and finite variances, where $B_n = \mathbf{E}{S_n}^2$ [2] (see also [3, Theorem 7.1 and inequality (2.12)]).

The paper [1] contains sufficient conditions for inequality (1) in the case where the random variables of the initial sequence are not assumed to be independent and no conditions on the existence of any moments are imposed. At the same time, it is additionally assumed that the number sequence $\{a_n\}$ is nondecreasing. The last condition is not satisfied, for example, for sequences of m-dependent random variables with zero expectations and finite variances in the case of the classical choice of the sequence $\{a_n\}$:

$$a_n = (2B_n \log\log B_n)^{1/2}, \;\; B_n = \mathbf{E}{S_n}^2. \tag{4}$$

Sequences of m-dependent random variables had been introduced by Hoeffding and Robbins in [4]; these authors also obtained first limit theorems for sums of m-dependent random variables. At present, there exists a vast literature on limit theorems for sums of m-dependent random variables.

In the study of sufficient conditions for the applicability of the law of the iterated logarithm in the form (1) to sequences of m-dependent random variables in the case where the numbers a_n are defined by equalities (4), one compensates the violation of the assumption of nondecreasing of the number sequences $\{B_n\}$ and $\{a_n\}$ by introduction of additional conditions on the behavior of the sequence $\{B_n\}$.

In the paper [6], the notion of a sequence of m-orthogonal random variables had been introduced. Let m be a nonnegative integer. We say that a sequence of random variables $\{X_n;\ n = 1, 2, \ldots\}$ on a probability space is a sequence of m-orthogonal random variables if

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$\mathbf{E}{X_n}^2 < \infty$ for any n and $\mathbf{E}(X_k X_j) = 0$ for $|k-j| > m$. In particular, a sequence of 0-orthogonal random variables is simply a sequence of orthogonal random variables. If $\{X_n\}$ is a sequence of m-dependent random variables with zero expectations and finite variances, then this sequence is a sequence of m-orthogonal random variables. The last statement remains valid if the condition of m-dependence is replaced by a weaker condition of pairwise m-dependence.

The study of limit theorems for sequences of m-orthogonal random variables is of real interest since much attention is given in the literature to limit theorems for sums of orthogonal random variables and sums of m-dependent random variables.

The papers [5–7] contain conditions for the applicability of the law of the iterated logarithm in the form (1), where the a_n are defined by equalities (4), to sequences of m-dependent and m-orthogonal random variables. For such sequences, the condition of nondecreasing of the number sequence $\{a_n\}$ is violated; in [5], the following condition had been introduced: For any $\varepsilon > 0$, the inequalities $a_{n+r} \geq a_n(1-\varepsilon)$ hold for all $r \geq 1$ and all n large enough.

In contrast to [1], the following result does not contain the condition of nondecreasing of the number sequence $\{a_n\}$.

Let $\{a_n\}$ be a sequence of positive numbers such that

$$a_n \to \infty \quad \text{and} \quad a_{n+1}/a_n \to 1 \quad \text{as} \quad n \to \infty. \tag{5}$$

Let $c > 1$. There exists a nondecreasing sequence of integers $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$ and the inequalities

$$a_{n_k - 1} \leq c^k < a_{n_k} \tag{6}$$

are valid for all k large enough. Due to (5),

$$a_{n_k} \sim c^k, \quad k \to \infty. \tag{7}$$

Theorem. *Let the condition D be satisfied with a_n replaced in inequality* (2) *by c^n for any $c > 1$. Let also*

$$\sum_{k=1}^{\infty} \mathbf{P}(S_{n_k} > (1+\varepsilon)c^k) < \infty \tag{8}$$

for the introduced sequence $\{n_k\}$ and for any $\varepsilon > 0$ and $c > 1$. Then inequality (1) *is valid.*

Proof. Fix an arbitrary positive ε. It follows from (5)–(7) that

$$\mathbf{P}(S_n > (1+\varepsilon)a_n \text{ i.o.}) \leq \mathbf{P}\Big(\max_{1 \leq j \leq n_k} S_j > (1+\varepsilon/2)c^k \text{ i.o.}\Big) \tag{9}$$

for any $c > 1$ that is close enough to 1. By the Borel–Cantelli lemma, to prove (1), it is enough to show that

$$\sum_{k=1}^{\infty} \mathbf{P}\Big(\max_{1 \leq j \leq n_k} S_j > (1+\varepsilon)c^k\Big) < \infty$$

for any $\varepsilon > 0$ and $c > 1$. The last statement follows from (8) and from the condition D in which a_n are replaced by c^n. □

The paper [7] contains conditions on a sequence of m-orthogonal random variables which guarantee inequality (1) for numbers a_n defined by equalities (4).

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