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ON THE CANONICAL FORMS OF A REGULAR MATRIX PENCIL

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We construct an algorithm for the reduction of a regular matrix pencil to the canonical form.

Introduction

Classical methods for the investigation of the properties of singularly perturbed systems of differential equations

$$\varepsilon\frac{dx}{dt} = A(t,\varepsilon)x, \quad t \in [0;T], \tag{1}$$

are based on the algorithms of reduction of the nonperturbed matrix $A(t,0)$ to certain canonical forms. These forms enable us not only to simplify the problem but also, which is more important, to guarantee the possibility of classification of its cases and efficient determination of its solutions.

Note that the Jordan form of the matrix $A(t,0)$ is extensively used in the theory of differential equations. This form enables one to give a fairly simple description of the asymptotic properties of the fundamental matrix of system (1). If the eigenvalues and the corresponding elementary divisors of the matrix $A(t,0) - \lambda E$, where E is the identity matrix, preserve constant multiplicities on the segment $[0;T]$, then the methods of asymptotic integration of system (1) are, in fact, generalizations of the corresponding methods for the solution of systems of linear differential equations with constant coefficients [1–4]. Moreover, the number of constructed linearly independent asymptotic solutions of system (1) is equal to the number of roots (with regard for their multiplicities) of the characteristic equation

$$\det(A(t,0) - \lambda E) = 0. \tag{2}$$

The solutions corresponding to the simple roots of Eq. (2) are represented in the form of asymptotic expansions in powers of the parameter ε. At the same time, the solutions generated by multiple roots are constructed in the form of asymptotic expansions in the fractional powers of the parameter ε whose exponents are specified by the multiplicity of roots of the characteristic equation, the corresponding elementary divisors, and the perturbed coefficients of system [4].

For singularly perturbed differential-algebraic systems

$$\varepsilon B(t,\varepsilon)\frac{dx}{dt} = A(t,\varepsilon)x, \quad t \in [0;T], \tag{3}$$

similar results were obtained by Samoilenko, Shkil', Yakovets', and their colleagues [5]. It was established that under certain conditions imposed on perturbed matrices, system (3) has two types of formal solutions. Thus,

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solutions of the first type correspond to finite elementary divisors of the limit matrix pencil $A(t,0) - \lambda B(t,0)$, whereas solutions of the second type correspond to infinite divisors. The indicated solutions are represented in the form of asymptotic expansions in powers of the small parameter ε whose exponents depend on the multiplicity of roots of the corresponding characteristic equation

$$\det\big(A(t,0) - \lambda B(t,0)\big) = 0$$

on the corresponding elementary divisors, and on the behavior of perturbed coefficients of the system [5]. Moreover, the case of turning points was not considered.

Another, more general, approach to the asymptotic integration of systems of differential equations with parameter based on the use of the spectral properties of the matrix $A(0,0)$ was proposed in the works by Iwano [6, 7] and Sibuya [8]. In these works, the authors developed an algorithm of asymptotic splitting of system (1). Moreover, the structure of the matrix with variable elements after splitting is specified by the structure of $A(0,0)$. In the case of different eigenvalues of the matrix $A(0,0)$, the proposed procedure of splitting also enables one to construct the fundamental matrix of system (1).

In the present paper, we generalize the results obtained by Iwano and Sibuya to the case of singularly perturbed differential algebraic systems (3). The developed algorithm of reduction of system (3) to the canonical form can be used in the presence of turning points and in the case of possible changes in the rank of the matrix $B(t,\varepsilon)$.

1. The Case of Local Reduction of a Matrix Pencil to the Canonical Form

Assume that the pencil $A(t) - \lambda B(t)$ satisfies the following conditions:

1) $A(0) = \operatorname{diag}\{E_q, J_p\}$ and $B(0) = \operatorname{diag}\{J_q, E_p\}$, $p + q = n$, where E_q is the identity matrix of order q, J_q is a square matrix of order q for which the elements of the upper superdiagonal are equal to 1 and the other elements are equal to 0; the matrices E_p and J_p are specified in a similar way;

2) $\dfrac{d}{dt}\big(\det A(t)\big)\,\Big|_{t=0} \neq 0$ and $\dfrac{d}{dt}\big(\det B(t)\big)\,\Big|_{t=0} \neq 0$.

Theorem 1. *Suppose that $A(t), B(t) \in C^m[0;T]$ and conditions 1) and 2) are satisfied. Then there exist nonsingular matrices $P(t)$, $Q(t) \in C^m[0;t_0]$, $t_0 \leq T$, such that*

$$P(t)A(t)Q(t) = \Omega(t) \equiv \operatorname{diag}\{E_q, J_p(t)\}, \tag{4}$$

$$P(t)B(t)Q(t) = H(t) \equiv \operatorname{diag}\{J_q(t), E_p\}, \tag{5}$$

where

$$J_p(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_p(t) & a_{p-1}(t) & a_{p-2}(t) & \dots & a_1(t) \end{pmatrix},$$

$$J_q(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ b_q(t) & b_{q-1}(t) & b_{q-2}(t) & \dots & b_1(t) \end{pmatrix},$$

$a_i(t) = t\widetilde{a}_i(t)$, $i = \overline{1, p}$, and $b_i(t) = t\widetilde{b}_i(t)$, $i = \overline{1, q}$; moreover, $\widetilde{a}_p(0) \neq 0$ and $\widetilde{b}_q(0) \neq 0$.

Proof. We set

$$A(t) = A(0) + D(t), \qquad B(t) = B(0) + F(t), \tag{6}$$

$$\Omega(t) = \Omega(0) + U(t), \qquad H(t) = H(0) + V(t). \tag{7}$$

Then

$$D(0) = F(0) = U(0) = V(0) = 0.$$

By construction, we get

$$A(0) = \Omega(0), \qquad B(0) = H(0).$$

We determine the matrices $P_1(t)$ and $Q_1(t)$ from the system of equations

$$P_1(t)A(t)Q_1(t) = \Omega_1(t), \qquad P_1(t)B(t)Q_1(t) = H_1(t), \tag{8}$$

where

$$\Omega_1(t) = \operatorname{diag}\{E_q(t), M_p(t)\}, \qquad H_1(t) = \operatorname{diag}\{M_q(t), E_p(t)\},$$

$$E_q(0) = E_q, \qquad E_p(0) = E_p, \qquad M_q(0) = J_q, \qquad M_p(0) = J_p.$$

Further, we set

$$P_1(t) = E_n + R(t), \qquad Q_1(t) = E_n + S(t), \tag{9}$$

where E_n is the identity matrix of order n. Substituting (6), (7), and (9) in system (8), we obtain

$$\Omega_1(0)S(t) + R(t)\Omega_1(0) + D(t) + D(t)S(t) + R(t)\Omega_1(0)S(t) + R(t)D(t) + R(t)D(t)S(t) - U(t) = 0, \tag{10}$$

$$H_1(0)S(t) + R(t)H_1(0) + F(t) + F(t)S(t) + R(t)H_1(0)S(t) + R(t)F(t) + R(t)F(t)S(t) - V(t) = 0. \tag{11}$$

Let

$$\Omega_1(0) = \begin{pmatrix} \Omega_{11}(0) & 0 \\ 0 & \Omega_{22}(0) \end{pmatrix} \equiv \begin{pmatrix} E_q & 0 \\ 0 & J_p \end{pmatrix},$$

$$H_1(0) = \begin{pmatrix} H_{11}(0) & 0 \\ 0 & H_{22}(0) \end{pmatrix} \equiv \begin{pmatrix} J_q & 0 \\ 0 & E_p \end{pmatrix},$$

$$U(t) = \operatorname{diag}\{U_1(t), U_2(t)\}, \qquad V(t) = \operatorname{diag}\{V_1(t), V_2(t)\},$$

$$D(t) = \begin{pmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{pmatrix}, \qquad F(t) = \begin{pmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t) \end{pmatrix},$$

where, e.g., $U_1(t)$, $V_1(t)$, $D_{11}(t)$, and $F_{11}(t)$ are square matrices of order q.

We also set

$$S(t) = \begin{pmatrix} 0 & S_{12}(t) \\ S_{21}(t) & 0 \end{pmatrix} \qquad \text{and} \qquad R(t) = \begin{pmatrix} 0 & R_{12}(t) \\ R_{21}(t) & 0 \end{pmatrix},$$

where $S_{12}(t)$, $R_{12}(t)$ and $S_{21}(t)$, $R_{21}(t)$ are $q \times p$ and $p \times q$ rectangular matrices, respectively.

Thus, it follows from system (10), (11) that

$$U_i(t) = D_{ii}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{2} D_{ij}(t) S_{ji}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{2} R_{ij}(t) \Omega_{jj}(0) S_{ji}(t)$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{2} R_{ij}(t) D_{ji}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{2} R_{ij}(t) \sum_{\substack{k=1 \\ k \neq i}}^{2} D_{jk}(t) S_{ki}(t),$$

$$V_i(t) = F_{ii}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{2} F_{ij}(t) S_{ji}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{2} R_{ij}(t) H_{jj}(0) S_{ji}(t)$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{2} R_{ij}(t) F_{ji}(t) + \sum_{\substack{j=1 \\ j \neq i}}^{2} R_{ij}(t) \sum_{\substack{k=1 \\ k \neq i}}^{2} F_{jk}(t) S_{ki}(t), \quad i = 1, 2,$$

and

$$\Omega_{ii}(0) S_{ij}(t) + R_{ij}(t) \Omega_{jj}(0) + D_{ij}(t) + \sum_{\substack{k=1 \\ k \neq j}}^{2} D_{ik}(t) S_{kj}(t)$$

$$+\sum_{\substack{k=1\\k\neq i}}^{2} R_{ik}(t)D_{kj}(t)+\sum_{\substack{k=1\\k\neq i}}^{2} R_{ik}(t)\sum_{\substack{l=1\\l\neq j}}^{2} D_{kl}(t)S_{lj}(t)=0, \quad (12)$$

$$H_{ii}(0)S_{ij}(t)+R_{ij}(t)H_{jj}(0)+F_{ij}(t)+\sum_{\substack{k=1\\k\neq j}}^{2} F_{ik}(t)S_{kj}(t)$$

$$+\sum_{\substack{k=1\\k\neq i}}^{2} R_{ik}(t)F_{kj}(t)+\sum_{\substack{k=1\\k\neq i}}^{2} R_{ik}(t)\sum_{\substack{l=1\\l\neq j}}^{2} F_{kl}(t)S_{lj}(t)=0, \quad (13)$$

$$i\neq j, \quad i,j=1,2.$$

According to [8], the Jacobian of system (12), (13) at the point $t=0$ is not equal to zero. Hence, there exists t_1, $t_1\leq T$, such that system (12), (13) for $S_{ij}(t)$ and $R_{ij}(t)$, $i\neq j$, $i,j=1,2$, is consistent for all $t\in[0;t_1]$. Moreover, $S_{ij}(t),\ R_{ij}(t)\in C^m[0;t_1]$ and $R(0)=S(0)=0$.

We set

$$P_2(t)=\begin{pmatrix}E_q^{-1}(t) & 0\\ 0 & E_p\end{pmatrix}, \qquad Q_2(t)=\begin{pmatrix}E_q & 0\\ 0 & E_p^{-1}(t)\end{pmatrix}.$$

This yields

$$P_2(t)\Omega_1(t)Q_2(t)=\begin{pmatrix}E_q & 0\\ 0 & M_p(t)E_p^{-1}(t)\end{pmatrix}, \qquad P_2(t)H_1(t)Q_2(t)=\begin{pmatrix}E_q^{-1}(t)M_q(t) & 0\\ 0 & E_p\end{pmatrix}.$$

We now choose functions $b_i(t)$, $i=\overline{1,q}$, for which the matrices $E_q^{-1}(t)M_q(t)-\lambda E_q$ and $J_q(t)-\lambda E_q$ have identical characteristic polynomials. Thus, in view of the structure of the matrices $E_q^{-1}(0)M_q(0)$ and $J_q(0)$, we conclude that the matrices $E_q^{-1}(t)M_q(t)-\lambda E_q$ and $J_q(t)-\lambda E_q$ on the segment $[0;t_0]$, $t_0\leq t_1$, have the same invariant polynomials. Hence, there exists a nonsingular sufficiently smooth matrix $T_q(t)$ [9, 10] such that

$$T_q^{-1}(t)E_q^{-1}(t)M_q(t)T_q(t)=J_q(t).$$

Note that

$$b_q(t)=(-1)^{q+1}\det(E_q^{-1}(t)M_q(t)).$$

Thus, according to conditions 1) and 2), we get $b_q(t)=t\widetilde{b}_q(t)$ and, moreover, $\widetilde{b}_q(0)\neq 0$. The functions $a_i(t)$, $i=\overline{1,p}$, are defined in a similar way.

Let

$$T(t)=\operatorname{diag}\{T_q(t),T_p(t)\},$$

where

$$T_p^{-1}(t)M_p(t)E_p^{-1}(t)T_p(t) = J_p(t).$$

Setting $P(t) = T^{-1}(t)P_2(t)P_1(t)$ and $Q(t) = Q_1(t)Q_2(t)T(t)$, we conclude that relations (4) and (5) are true.

Theorem 1 is proved.

Theorem 1 can be generalized to the case of a matrix pencil $A(t,\varepsilon) - \lambda B(t,\varepsilon)$ whose elements are defined on the set

$$\overline{K} = \{(t,\varepsilon)\colon 0 \le t \le T,\ \ 0 \le \varepsilon \le \varepsilon_0\}.$$

Assume that the pencil $A(t,\varepsilon) - \lambda B(t,\varepsilon)$ satisfies the following conditions:

3) $A(0,0) = \operatorname{diag}\{E_q, J_p\}$, $B(0,0) = \operatorname{diag}\{J_q, E_p\}$, $p+q=n$;

4) the inequalities

$$\frac{\partial}{\partial t}(\det A(t,\varepsilon)) \,\Big|\, _{(t,\varepsilon)=(0,0)} + \frac{\partial}{\partial \varepsilon}(\det A(t,\varepsilon)) \,\Big|\, _{(t,\varepsilon)=(0,0)} \neq 0,$$

$$\frac{\partial}{\partial t}(\det B(t,\varepsilon)) \,\Big|\, _{(t,\varepsilon)=(0,0)} + \frac{\partial}{\partial \varepsilon}(\det B(t,\varepsilon)) \,\Big|\, _{(t,\varepsilon)=(0,0)} \neq 0$$

are true.

Theorem 2. *Suppose that $A(t,\varepsilon)$, $B(t,\varepsilon) \in C^m(\overline{K})$ and conditions 3) and 4) are satisfied. Then there exist nonsingular matrices $P(t,\varepsilon)$, $Q(t,\varepsilon) \in C^m(\overline{K}_1)$ $((0,0) \in \overline{K}_1 \subset \overline{K})$ such that*

$$P(t,\varepsilon)A(t,\varepsilon)Q(t,\varepsilon) = \Omega(t,\varepsilon) \equiv \operatorname{diag}\{E_q, J_p(t,\varepsilon)\}, \tag{14}$$

$$P(t,\varepsilon)B(t,\varepsilon)Q(t,\varepsilon) = H(t,\varepsilon) \equiv \operatorname{diag}\{J_q(t,\varepsilon), E_p\}, \tag{15}$$

where

$$J_p(t,\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ a_p(t,\varepsilon) & a_{p-1}(t,\varepsilon) & a_{p-2}(t,\varepsilon) & \dots & a_1(t,\varepsilon) \end{pmatrix},$$

$$J_q(t,\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ b_q(t,\varepsilon) & b_{q-1}(t,\varepsilon) & b_{q-2}(t,\varepsilon) & \dots & b_1(t,\varepsilon) \end{pmatrix},$$

$a_i(0,0) = 0$, $i = \overline{1, p}$, *and* $b_i(0,0) = 0$, $i = \overline{1, q}$; *moreover,*

$$\frac{\partial}{\partial t}(a_p(t,\varepsilon))\,|_{(t,\varepsilon)=(0,0)} + \frac{\partial}{\partial \varepsilon}(a_p(t,\varepsilon))\,|_{(t,\varepsilon)=(0,0)} \neq 0,$$

$$\frac{\partial}{\partial t}(b_q(t,\varepsilon))\,|_{(t,\varepsilon)=(0,0)} + \frac{\partial}{\partial \varepsilon}(b_q(t,\varepsilon))\,|_{(t,\varepsilon)=(0,0)} \neq 0.$$

Proof. We use the same scheme as in the proof of Theorem 1. Let

$$A(t,\varepsilon) = A(0,0) + D(t,\varepsilon), \qquad B(t,\varepsilon) = B(0,0) + F(t,\varepsilon),$$

$$\Omega(t,\varepsilon) = \Omega(0,0) + U(t,\varepsilon), \qquad H(t,\varepsilon) = H(0,0) + V(t,\varepsilon).$$

Then $D(0,0) = F(0,0) = U(0,0) = V(0,0) = 0$. By construction,

$$A(0,0) = \Omega(0,0) \qquad \text{and} \qquad B(0,0) = H(0,0).$$

As above, we determine the matrices $P_1(t,\varepsilon)$ and $Q_1(t,\varepsilon)$ from the system of equations

$$P_1(t,\varepsilon)A(t,\varepsilon)Q_1(t,\varepsilon) = \Omega_1(t,\varepsilon), \qquad P_1(t,\varepsilon)B(t,\varepsilon)Q_1(t,\varepsilon) = H_1(t,\varepsilon), \tag{16}$$

where

$$\Omega_1(t,\varepsilon) = \operatorname{diag} E_q(t,\varepsilon), M_p(t,\varepsilon)\}, \qquad H_1(t,\varepsilon) = \operatorname{diag}\{M_q(t,\varepsilon), E_p(t,\varepsilon)\},$$

$$E_q(0,0) = E_q, \qquad E_p(0,0) = E_p, \qquad M_q(0,0) = J_q, \qquad M_p(0,0) = J_p.$$

Setting

$$P_1(t,\varepsilon) = E_n + R(t,\varepsilon), \qquad Q_1(t,\varepsilon) = E_n + S(t,\varepsilon)$$

and assuming that the matrices $R(t,\varepsilon)$ and $S(t,\varepsilon)$ have the same structure as in Theorem 1, we prove that system (16) is consistent on the set $\overline{K}_1$. The matrices $P_2(t,\varepsilon)$, $Q_2(t,\varepsilon)$, and $T(t,\varepsilon)$ are determined in exactly the same way as in Theorem 1.

Theorem 2 is proved.

We now replace condition 4) by the following condition:

5) $\dfrac{d}{dt}(\det A(t,0))\,\Big|_{\,t=0} \neq 0$ and $\dfrac{d}{dt}(\det B(t,0))\,\Big|_{\,t=0} \neq 0$.

Theorem 3. *Suppose that* $A(t,\varepsilon)$, $B(t,\varepsilon) \in C^m(\overline{K})$ *and conditions 3) and 5) are satisfied. Then there exist nonsingular matrices* $P(t,\varepsilon)$, $Q(t,\varepsilon) \in C^m(\overline{K}_1)$ *for which equalities (14) and (15) with* $a_i(0,0) = 0$, $i = \overline{1, p}$, *and* $b_i(0,0) = 0$, $i = \overline{1, q}$, *are true and, in addition,*

$$\frac{d}{dt}(a_p(t,0))\,\Big|_{\,t=0} \neq 0, \qquad \frac{d}{dt}(b_q(t,0))\,\Big|_{\,t=0} \neq 0.$$

2. Nonlocal Case

Suppose that the Kronecker structure of the pencil $A(t,0) - \lambda B(t,0)$ does not change on the segment $[0; T]$ [5].

Assume that the matrix pencil $A(t,\varepsilon) - \lambda B(t,\varepsilon)$ satisfies the condition

6) $A(t,0) = \operatorname{diag}\{E_q, J_p\}$ and $B(t,0) = \operatorname{diag}\{J_q, E_p\}$, $p + q = n$.

Theorem 4. *Suppose that $A(t,\varepsilon),\ B(t,\varepsilon) \in C^m(\overline{K})$ and conditions 5) and 6) are satisfied. Then there exist nonsingular matrices $P(t,\varepsilon),\ Q(t,\varepsilon) \in C^m(\overline{K}_2)$,*

$$\overline{K}_2 = \big\{(t,\varepsilon)\colon\ 0 \le t \le T,\ \ 0 \le \varepsilon \le \varepsilon_1\big\}, \quad \varepsilon_1 \le \varepsilon_0,$$

such that equalities (14) and (15) with $J_p(t,0) = J_p$ and $J_q(t,0) = J_q$ are true.

Proof. We set

$$A(t,\varepsilon) = A(t,0) + D(t,\varepsilon), \qquad B(t,\varepsilon) = B(t,0) + F(t,\varepsilon),$$

$$\Omega(t,\varepsilon) = \Omega(t,0) + U(t,\varepsilon), \qquad H(t,\varepsilon) = H(t,0) + V(t,\varepsilon).$$

Then

$$D(t,0) = F(t,0) = U(t,0) = V(t,0) = 0.$$

By construction,

$$A(t,0) = \Omega(t,0) \qquad \text{and} \qquad B(t,0) = H(t,0).$$

Further, the proof of the theorem is similar to the proofs of Theorems 2 and 1.

Theorem 4 is proved.

Theorem 5. *Suppose that $A(t,\varepsilon),\ B(t,\varepsilon) \in C^m(\overline{K})$, that the elements $A(t,\varepsilon)$ and $B(t,\varepsilon)$ admit uniform asymptotic expansions*

$$A(t,\varepsilon) = \sum_{s\ge 0} A_s(t)\varepsilon^s, \qquad B(t,\varepsilon) = \sum_{s\ge 0} B_s(t)\varepsilon^s$$

on the set $\overline{K}$, and that conditions 5) and 6) are satisfied. Then there exist nonsingular matrices $P(t,\varepsilon),\ Q(t,\varepsilon) \in C^m(\overline{K}_2)$ for which equalities (14) and (15) are true with $J_p(t,0) = J_p$ and $J_q(t,0) = J_q$. Moreover, on the set $\overline{K}_2$, the elements $P(t,\varepsilon)$ and $Q(t,\varepsilon)$ admit the following uniform asymptotic expansions:

$$P(t,\varepsilon) = \sum_{s\ge 0} P_s(t)\varepsilon^s \qquad \textit{and} \qquad Q(t,\varepsilon) = \sum_{s\ge 0} Q_s(t)\varepsilon^s.$$

Note that the validity of Theorem 5 follows from the consistency of systems similar to systems (12) and (13). In the analyzed case, these systems are obtained by comparing the coefficients of the same powers of the parameter ε in system (16).

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