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STABILITY OF SYSTEMS COMPOSED OF THE SHELLS OF REVOLUTION WITH VARIABLE GAUSSIAN CURVATURE

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We analyze the stability of elastic systems composed of the shells of revolution with variable curvature and complex structures in the field of conservative axisymmetric loads of different nature. Within the framework of classical and refined theories of shells, we determine the limit and bifurcation critical values of the acting loads based on the geometrically nonlinear statement of the problem and a criterion of dynamic stability. To solve the corresponding nonlinear and eigenvalue problems, we propose to use a numerical-analytic approach based on their rational reduction to one-dimensional linear boundary-value problems in the meridional coordinate and their numerical solution by the discrete-orthogonalization method. We present test examples that confirm the applicability of the proposed procedure to the analyzed class of problems. The limit and bifurcation values of the critical loads in the shell system are analyzed depending on its geometric parameters.

Keywords: compound shell of revolution, axisymmetric loads, bifurcation and limit critical values, numerical-analytic approach, analysis.

The present paper is a generalization of the works [2, 25, 26] devoted to the study of stability of the shell elements aimed at the subsequent extension of the class of analyzed problems, investigation of more complicated objects, and improvement of the corresponding numerical procedures.

As in the works [1–3, 25, 26], we study elastic inhomogeneous anisotropic systems formed by the shells of revolution of different shapes and structures that can be used to model numerous objects of contemporary engineering (spacecrafts and underwater vehicle, protective equipment of nuclear reactors, reservoirs of different destinations, etc.). The analysis of stability of these systems is an important factor in the evaluation of their strength, reliability, and functional efficiency under the actual conditions of operation.

The results of investigations in this field of mechanics of deformable solids are described in numerous fundamental monographs and separate papers, where one can find the foundations of theory of static stability and the analysis of specific features of its investigation, various models of deformation of the shells, theoretical and experimental results of evaluation of the critical loads, the analysis of the influence of the structure of materials and the presence of imperfections or various inclusions, etc., on the stability of shell elements [4, 9, 20, 22, 28, 37]. Until recently, the main attention of the researchers was mainly focused on the shells of simple shapes, i.e., plates, cylinders, cones, and spherical segments, subjected to the action of various kinds of loads (force, temperature, and aerodynamic) for different characteristics of composite materials [16, 19, 23, 24, 30, 32–35]. Thus, the work [16] is devoted to the analysis of buckling of a simply supported rectangular plate with stiffening ribs. The problem of stability of cylindrical and conic shells interacting with a flow of liquid was studied in [19]. In [35], one can find the results of experimental and theoretical investigations of the critical loads of composite

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pipelines under the action of uniform external hydrostatic pressure. The influence of wind loads and temperature fields on the stability of cylinders was studied in [23, 24]. In [34], investigations of the same kind were carried out for functionally graded piezoelectric materials. The detailed analysis of the subcritical and supercritical states mainly of cylindrical shells (open, closed, and with elliptic cross sections) subjected to the action of nonaxisymmetric, cyclically symmetric, or localized loads for various conditions of fastening of the boundary contour was presented in [30]. For spherical shells, similar investigations were carried out in [32].

We especially mention the works devoted to the analysis of stability of the shells made of nanomaterials and their applications in modern structures, which now become quite actual [10, 15, 40].

A separate, relatively insignificant, part of scientific investigations carried out in this field is formed by the works devoted to the analysis of stability of elastic systems formed by mated shells of different shapes. In this case, additional difficulties are caused by the formulation and rational accounting of the conditions of conjugation of separate constitutive elements in the solution of the corresponding problems. Furthermore, in the joints of these shells, we, as a rule, observe the formation of additional stresses that may exert unpredicted influence on the stability of the entire system. For plates and shells, the problem of formulation and conditions of conjugation conditions on the basis of the finite-element method was considered in [29, 42]. In the comprehensive paper [31], one can find a survey of works devoted to the problems of nonlinear (linear) deformation and stability of systems formed by cylinders, cones, and spherical segments (coaxial and out-of-alignment), i.e., systems with zero or constant Gaussian curvature. Further, compound systems with elements of cylindrical and conic shapes were considered, in particular, in [36] and [38]. Moreover, systems with elements of the spherical type in static fields of different kinds were studied for different properties of the materials in [27, 41]. It is worth noting that, for systems of conjugate shells with variable curvatures, investigations of this kind are practically absent despite the fact that these objects are often encountered in building practice (domes of buildings for public worship and government institutions), in space technology (spacecraft bodies), and in the construction of vessels of various destinations (pressure balloons).

The investigations of the static stability of shells are based on the analysis of diagrams of their equilibrium states with determination of the characteristic points, i.e., bifurcation and limit critical values of the acting loads. The determination of the limit critical loads is based on the use of nonlinear statements of the corresponding problems. Moreover, to find the bifurcation values, it is customary to use the energy and static criteria and (much less frequently) the dynamic criterion. The last two criteria in conservative systems are identical from the viewpoint of the mechanics of deformable systems [5]. However, they differ in the form of their computational realizations, which may serve as an additional factor increasing the reliability of the obtained results. The application of these criteria to shells is connected with finding nontrivial solutions of a homogeneous two-dimensional linearized boundary-value problem, which can be reduced by the well-known methods of applied mathematics to the eigenvalue problem for algebraic or ordinary differential equations. In this case, it is customary to use different modifications of the finite-element and finite-difference methods, variational-difference and projective approaches, the methods of reduction of dimensionality, etc. [8, 9, 12, 16, 17, 21, 29, 30, 42].

In the present work, for elastic systems formed by compound shells of revolution, including elements of variable curvature and complex structure in thickness, we analyze the limit (nonlinear statement) and bifurcation (dynamic criterion of stability) critical values of the acting conservative axisymmetric loads within the framework of the classical and refined theories of shells. To solve the corresponding nonlinear boundary-value problems and eigenvalue problems, we use a numerical-analytic approach based on the rational reduction (by analytical methods) of the original problem to one-dimensional linear boundary-value problems along the meridian (generatrix) and their practically exact numerical solution by the orthogonal-sweep method [7, 18]. Thus, in the indicated coordinate direction, the variability of the geometric and stiffness characteristics of the shells is taken into account in the continual form, which is especially important for joining dissimilar elements into a single system.

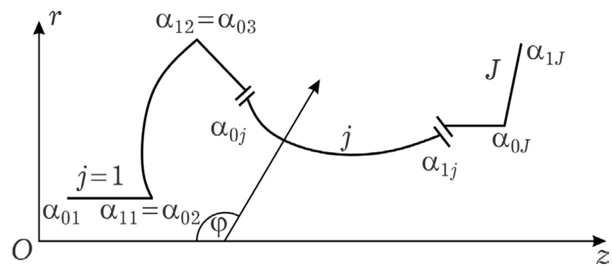


Fig. 1

1. Statement of the Problem and the General Characteristic of the Procedure of Its Solution

In the present work, we study an object traditional for numerous works of the authors [1–3, 25, 26], i.e., an elastic system of coaxial conjugate shells of revolution whose initial surface $\gamma = 0$ across the thickness is referred to an orthogonal curvilinear coordinate system α, θ. Here, α is the coordinate varying along the meridian (generatrix) of the system and, moreover, its certain jth shell can be specified in a local coordinate system $\alpha_j \in [\alpha_{0j}, \alpha_{1j}]$, $j = 1, \ldots, J$, where J is the number of constituent shells in the analyzed system; θ is the central angle in the cross-sectional plane $z = \text{const}$; γ is the coordinate directed along the normal to the chosen initial surface; Oz is the axis of rotation of the generatrix, and φ is the angle between the normal to the generatrix and the Oz-axis (Fig. 1). The physical characteristics of the materials of shell can be homogeneous, inhomogeneous (functionally graded), or discretely inhomogeneous (layered) along the thickness. For layered shells, we assume that the neighboring layers contact without separation and sliding. As for the conditions of fastening of the end faces of the system $\alpha = \alpha_{01}$ and $\alpha = \alpha_{1J}$, we impose no additional restrictions except, possibly, their physical compatibility. In the lines of contact between the neighboring shells $\alpha = \alpha_{1j} = \alpha_{0\,j+1}$, $j = 1, \ldots, J-1$, we impose the conditions of balance of static characteristics of the stress-strain state (SSS) and the conditions of continuity of its kinematic characteristics in the common coordinate system rOz.

The shells are placed in a field of conservative axisymmetric force and/or temperature loads of different kinds: distributed along the meridian or concentrated either in the end-face contours or in certain cross sections $z = \text{const}$.

In the present work, we determine the critical values of the acting fields corresponding to the loss of stability of the described shell systems in the axisymmetric (limit critical values) or nonaxisymmetric (bifurcation critical values) form. The investigations are carried out under the following assumptions:

- the main (subcritical) SSS is determined by the elastic stage of deformation within the range of acting axisymmetric loads up to their limit values;

- the bifurcation critical loads are determined according to the dynamic criterion of stability in which their critical values are specified as the loads for which the minimal frequency of natural vibration of a preliminarily loaded shell system becomes equal to zero;

- vibrations are regarded as small perturbations of the main state of preliminarily loaded shell;

- the mathematical model of stability is based on the quadratic approximation of the geometrically nonlinear theory within the framework of the assumptions of the Kirchhoff–Love classical theory and the Timoshenko refined theory for the entire set of layers as a whole and with regard for temperature loads according to the Duhamel–Neumann hypothesis [7, 13, 14].

Thus, in the case of application of the dynamic criterion of stability, the equilibrium states of conjugate shell systems and the required characteristic points of their diagrams (limit and bifurcation critical values) are determined from a nonlinear two-dimensional problem, which can be conditionally represented in the following vector-matrix form:

– a system of nonlinear partial differential equations

$$\frac{\partial \bar{N}}{A\partial\alpha} = \sum_{q=0}^{qq} A_q \frac{\partial^q \bar{N}}{\partial\theta^q} + \bar{G} + \bar{q}^0 + C\frac{\partial^2 \bar{N}}{\partial t^2}, \quad \alpha = \{\alpha_j \in (\alpha_{0j}, \alpha_{1j})\},$$

$$j = 1, \dots, J, \quad \theta \in [0, 2\pi]; \tag{1}$$

– conjugation conditions on the contact lines

$$S_{1j}\bar{N} = S_{0j+1}\bar{N} + \bar{F}_j^0, \quad \alpha = \alpha_{1j} = \alpha_{0j+1}, \quad j = 1, \dots, J-1; \tag{2}$$

– boundary conditions at end faces of the system

$$B_{01}\bar{N} = \bar{b}_{01}^0, \quad \alpha = \alpha_{01}, \tag{3}$$

$$B_{1J}\bar{N} = \bar{b}_{1J}^0, \quad \alpha = \alpha_{1J}; \tag{4}$$

– periodicity conditions in the circumferential direction

$$\bar{N}(\alpha, \theta + 2\pi, t) = \bar{N}(\alpha, \theta, t). \tag{5}$$

Here, $\bar{N} = \{N_n(\alpha, \theta, t)\}$ is the required vector function whose components are the following static and kinematic characteristics of the SSS according to the accepted theory of shells:

$$\bar{N} = \{N_n(\alpha, \theta, t)\} = \begin{cases} \{N_\alpha, \hat{S}_\alpha, \hat{Q}_\alpha, M_\alpha, u, v, w, \vartheta_\alpha\}, & qq = 4, \\ \{N_\alpha, N_{\alpha\theta}, Q_\alpha, M_\alpha, H, u, v, w, \psi_\alpha, \psi_\theta\}, & qq = 2, \end{cases} \tag{6}$$

where

N_α and Q_α are the tangential and transverse forces; M_α is the bending moment in the section $\alpha = \text{const}$; $N_{\alpha\theta}$ is the tangential shear force; H is the torque;

$\hat{S}_\alpha = N_{\alpha\theta} + k_\theta H$ and $\hat{Q}_\alpha = Q_\alpha + \dfrac{\partial H}{r\partial\theta}$ are, respectively, the reduced shear and transverse forces;

u, v, and w are the meridional, circumferential, and normal displacements, respectively; ϑ_α is the angle of rotation of normal in the plane $\alpha = \text{const}$; ψ_α and ψ_θ are the total angles of rotation of a rectilinear element; $r = r(\alpha)$ is the distance from a point of the coordinate surface to the axis of revolution Oz; $k_\theta = k_\theta(\alpha)$ is the curvature of the coordinate line $\alpha = \text{const}$;

$\sum_{q=0}^{qq} A_q \dfrac{\partial^q \bar{N}}{\partial \theta^q} = L$ is the matrix differential operator of order qq with respect to the variable θ constructed according to the basic relations of the classical $(qq = 4)$ and refined $(qq = 2)$ theories [7];

A is the Lamé coefficient in the coordinate α; the components of the vector $\bar{G} = \{g_n(\alpha)\}$ are quadratic functions of the components of the vector $\bar{N}$ corresponding to the geometrically nonlinear theory of shells in the quadratic approximation; C is the matrix characterizing the inertial properties of the system;

$S_{1j} = \{s_{ni}^{1j}\}$ and $S_{0\,j+1} = \{s_{ni}^{0\,j+1}\}$ are the matrices that form the conjugation conditions in the sections $\alpha = \alpha_j$, $j = 1, \ldots, J-1$;

$B_{01} = \{b_{ni}^{01}\}$ and $B_{1J} = \{b_{ni}^{1J}\}$ are the matrices of boundary conditions on the contours $\alpha = \alpha_{01}$ and $\alpha = \alpha_{1J}$, respectively;

$\bar{q}^0 = \{q_n^0\}$, $\bar{F}_j^0 = \{f_{nj}^0\}$ and $\bar{b}_{01}^0 = \{b_{01n}^0\}$, $\bar{b}_{1J}^0 = \{b_{1Jn}^0\}$ are the vectors characterizing the axisymmetric distributed loads and temperature fields, concentrated forces (moments) in the section $\alpha = \alpha_j$, $j = 1, \ldots, J-1$, and contour loads in the end faces $\alpha = \alpha_{01}$ and $\alpha = \alpha_{1J}$, and t is time.

The expressions for the elements of the matrices $A_q = \{a_{ik}^q\}$, $i, k = 1, \ldots, n$, and the components of the vector $\bar{G} = \{g_n(\alpha)\}$ can be found in [7].

In analyzing the stability of these systems, we consider the action of one or several components of given fields

$$\bar{q}^0 = \{q_n^0\}, \quad \bar{F}_j^0 = \bar{F}_j^0 = \{f_{nj}^0\}, \quad \bar{b}_{01}^0 = \{b_{01n}^0\}, \quad \text{and} \quad \bar{b}_{1J}^0 = \{b_{1Jn}^0\}$$

whose variations are proportional to the variations of the same parameter δ. In this case, the problem is reduced to finding a value of this parameter for which the loss of stability is realized in the axisymmetric (limit critical value δ^{lim}) or nonaxisymmetric (bifurcation critical value δ^{bif}) mode. In this case, according to the dynamic criterion, δ^{bif} is determined from the condition $\omega_{\min}(\delta) = 0$, where $\omega_{\min}$ is the lowest frequency of natural vibration of the shell system with regard for the preliminary action of a given load with parameter δ.

The solution of problem (1)–(5) under the accepted assumptions admits a physically substantiated decomposition of the required solution $\bar{N}$ into the following two components: the main component $\bar{N}^0$ that determines the subcritical state and the dynamic component $\bar{N}^{\text{d}}$ corresponding to the vibration of the shell system about the ground state:

$$\bar{N} \approx \bar{N}^0 + \bar{N}^{\text{d}}, \qquad \bar{N}^0 \gg \bar{N}^{\text{d}}.$$

Hence, problem (1)–(5) can be also approximately reduced to the following two problems:

– a problem of ground state of conjugate shells under given axisymmetric loads, which is a one-dimensional nonlinear boundary-value problem formulated for the vector function $\bar{N}^0 = \{N_n^0(\alpha)\}$ as follows:

$$\frac{1}{A}\frac{d\bar{N}^0}{d\alpha} = L^0\bar{N}^0 + \bar{G}(\alpha, \bar{N}^0, \ldots) + \bar{q}^0, \qquad \alpha = \{\alpha_j \in (\alpha_{0j}, \alpha_{1j})\}, \quad j = 1, \ldots, J, \tag{7}$$

$$S_{1j}\bar{N}^0 = S_{0j+1}\bar{N}^0 + \bar{F}_j^0, \qquad \alpha = \alpha_j, \quad j = 1, \ldots, J-1, \tag{8}$$

$$B_{01}\bar{N}^0 = \bar{b}_{01}^0, \quad \alpha = \alpha_{01}, \tag{9}$$

$$B_{1J}\bar{N}^0 = \bar{b}_{1J}^0, \quad \alpha = \alpha_{1J}, \tag{10}$$

where L^0 is an algebraic operator;

– a problem of small undamped vibration of the shells about the ground state obtained as a result of linearization of the initial problem (1)–(5) for the vector function $\bar{N}^{\mathrm{d}} = \{N_n^{\mathrm{d}}(\alpha, \theta, t)\}$:

$$\frac{1}{A}\frac{\partial\bar{N}^{\mathrm{d}}}{\partial\alpha} = \tilde{L}\bar{N}^{\mathrm{d}} + C\frac{\partial^2\bar{N}^{\mathrm{d}}}{\partial t^2}, \qquad \alpha = \{\alpha_j \in (\alpha_{0j}, \alpha_{1j})\},$$
$$j = 1, \ldots, J, \quad \theta \in [0, 2\pi], \tag{11}$$

$$S_{1j}\bar{N}^{\mathrm{d}} = S_{0j+1}\bar{N}^{\mathrm{d}}, \qquad \alpha = \alpha_{1j} = \alpha_{0j+1}, \quad j = 1, \ldots, J-1, \tag{12}$$

$$B_{01}\bar{N}^{\mathrm{d}} = 0, \quad \alpha = \alpha_{01}, \tag{13}$$

$$B_{1J}\bar{N}^{\mathrm{d}} = 0, \quad \alpha = \alpha_{1J}, \tag{14}$$

$$\bar{N}^{\mathrm{d}}(\alpha, \theta + 2\pi, t) = \bar{N}^{\mathrm{d}}(\alpha, \theta, t), \tag{15}$$

where $\tilde{L} = L + \frac{\partial\bar{G}}{\partial\bar{N}^0}$ is a differential matrix operator containing (as parameters) the components of the vector function of the ground stressed state $\bar{N}^0 = \{N_n^0(\alpha, \delta)\}$ depending on the parameter of loading δ.

To solve the one-dimensional nonlinear boundary-value problem (7)–(10), we use the procedure of linearization in the Newton–Kantorovich–Raphson form (the quasilinearization method) together with the orthogonal-sweep method whose algorithm is described in [3, 7, 18]. The process of linearization ($m = 1, 2, \ldots$) is convergent in the domain of convexity of the operator L^0 of the problem (for this class of shells, it coincides with the domain of subcritical loads) and not convergent outside this domain [6]. This fact serves as a computational criterion for the evaluation of the ultimate critical value of parameter $\delta^{\lim}$: $\delta^- \le \delta^{\lim} \le \delta^+$, where δ^- is the maximal value of loading for which the process of quasilinearization is monotonically convergent and δ^+ is the minimal value of δ for which this process is not convergent or its monotonic convergence is violated.

The two-dimensional boundary-value problem (11)–(15) of small undamped vibrations of a preliminarily loaded shell system, after separation of the time factor $e^{i\omega t}$ in the components of the required solution $\bar{N}^d$ and their representation, with regard for the conditions of periodicity (15), in the form of trigonometric series in the circumferential coordinate θ:

$$\bar{N}^{\mathrm{d}} = \left\{ N_n^{\mathrm{d}}(\alpha,\theta,t) = \sum_{k=0,1,2,\ldots} N_{nk}^{\mathrm{d}}(\alpha) \right\} \begin{bmatrix} \sin k\theta \\ \cos k\theta \end{bmatrix} e^{i\omega t}, \tag{16}$$

is reduced to the following sequence of one-parameter homogeneous boundary-value problems for the functional coefficients

$$\bar{N}_k^{\mathrm{d}} = \left\{ N_{nk}^d(\alpha) \right\}$$

appearing in relation (16):

$$\frac{1}{A}\frac{d\bar{N}_k^{\mathrm{d}}}{d\alpha} = (A_k - \lambda C)\bar{N}_k^{\mathrm{d}}, \qquad \alpha = \left\{ \alpha_j \in (\alpha_{0j}, \alpha_{1j}) \right\}, \quad j = 1, \ldots, J, \tag{17}$$

$$S_{1j}\bar{N}_k^{\mathrm{d}} = S_{0j+1}\bar{N}_k^{\mathrm{d}}, \qquad \alpha = \alpha_{1j} = \alpha_{0j+1}, \quad j = 1, \ldots, J-1, \tag{18}$$

$$B_{01}\bar{N}_k^{\mathrm{d}} = 0, \quad \alpha = \alpha_{01}, \tag{19}$$

$$B_{1J}\bar{N}_k^{\mathrm{d}} = 0, \quad \alpha = \alpha_{1J}. \tag{20}$$

Here, A_k is a quadratic matrix of the eighth or tenth order obtained from the operator $\tilde{L}$ in (11) by using expansion (16), $\lambda = \omega^2$, ω is the natural frequency of vibration of the shell system, and k is a parameter characterizing the form of wave formation in the circumferential direction. The expression $\begin{bmatrix} \sin k\theta \\ \cos k\theta \end{bmatrix}$ in equality (16) means that the odd components of the vector function $\bar{N}^{\mathrm{d}}$ $\left(\{\hat{S}_\alpha, v\}\right.$ corresponds to the classical theory and $\{N_{\alpha\theta}, H, v, \psi_\theta\}$ corresponds to the refined theory$\left.\right)$ are taken with $\sin k\theta$, whereas the even components are taken with $\cos k\theta$.

To find the unknown numerical factor $\lambda = \omega^2$ for which the homogeneous boundary-value problem (17)–(20) has a nontrivial solution for each value of the harmonic k in (16), it is customary to use well-known methods, namely, the method of successive approximations in the modification of inverse iteration and the step-by-step search method in combination with the orthogonal-sweep method. The algorithms of these methods were described in detail in [1, 7, 11].

We now present the general scheme of the procedure of evaluation of the parameters of limit δ^{lim} and bifurcation δ^{bif} critical values of the applied load. We find these values step-by-step for a sequence $\delta_0 = 0$, $\delta_i = \delta_{i-1} + \Delta_i$, $i = 1, 2, \ldots$.

In the ith intermediate step of the process, if $\delta_{i-1} \neq \delta^{\text{lim}}$, then we realize the following operations:

1°. Solving the nonlinear axisymmetric problem (7)–(10) with an aim to determine the ground state of the shell system $\bar{N}^0(\delta_i)$ by the method of quasilinearization $(m = 1, 2, \ldots)$ and according to the Kirchhoff–Love classical theory $(n = 8,\ qq = 4)$ (6).

2°. Refinement of the obtained solution according to the refined theory $(n = 10,\ qq = 2)$ in (6).

3°. If the process of quasilinearization is not convergent, then we return to the previous $(i-1)$th step with $\delta = \delta_{i-1}$ and a lower value of Δ_{i-1}.

4°. If the process of quasilinearization monotonically converges and

4.1. $\delta_{i-1} = \delta^{\text{bif}}$, then we pass to the next $(i+1)$th step of the process;

4.2. $\delta_{i-1} \neq \delta^{\text{bif}}$, then we go further (to ***5°***).

5°. Computation of natural frequencies by the method of successive approximations for $k = 0, 1, 2, \ldots, K$ in (16) according to the classical theory of shells $(n = 8,\ qq = 4)$ in (6).

6°. Refinement of the obtained frequencies by the $\Delta(\lambda)$-method according to the Timoshenko refined theory $(n = 10,\ qq = 2)$ in (6).

7°. If, for some $k = k^*$, we get

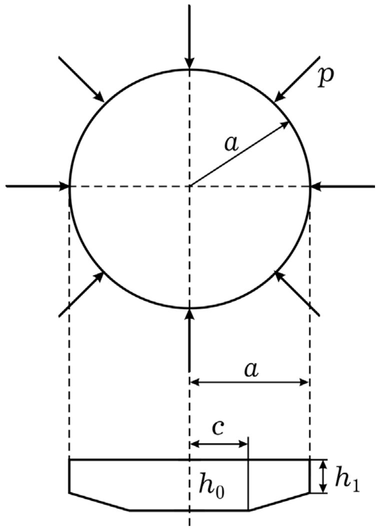
$$\omega_{k^*}(\delta_i) = 0 \qquad (\omega_{k^*}(\delta_i) \ll \omega_\ell(\delta_i),\ \ \ell \neq k^*),$$

then, according to the dynamic criterion of stability, we set $\delta^{\text{bif}} = \delta_i$, pass to the next $(i+1)$th step, and then repeat all operations ***1°–7°***.

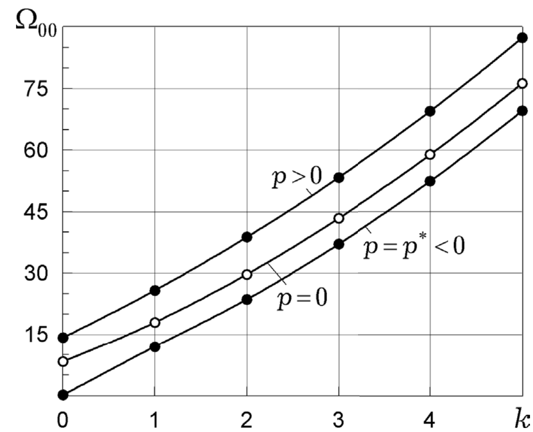
The attainment of required accuracy of evaluation of the limit δ^{lim} and bifurcation δ^{bif} critical values of the acting loads is realized by decreasing the value of Δ_i.

2. Practical Substantiation of the Procedure (Testing)

The stage of testing is required to substantiate the possibility of application of the developed procedure to the analyzed class of problems. In our works, it has been carried out inductively on the basis of the well-known methods of applied mathematics, e.g., by comparing with the results of solution of some problems obtained analytically, experimentally, or numerically by using other methods. Thus, in [2], testing of this kind was carried out



(a)



(b)

Fig. 2

Table 1

	$c/a = 0.5$			$c/a = 0.75$		
	KIR	TIM	[39]	KIR	TIM	[39]
Ω_{00}	8.518	8.516	8.519	8.847	8.839	8.882
$\tilde{N}_{cr}$	10.86	10.86	10.78	11.82	11.82	11.89

on examples of some specific shells, namely, a gently sloping spherical dome, a truncated isotropic cone, and a cylinder–hemisphere system made of glass-reinforced plastic. The comparison with the experimental data and the results obtained by the finite-difference method revealed good agreement of the obtained limit and bifurcation critical values of the applied loads. In what follows, we present two additional examples of solution of the problem of stability for a system of conjugate shells taken from [21, 39].

Consider a circular plate of radius a formed by two elements: a circular plate of radius c and constant thickness h_0 and an annular plate with inner radius c and outer radius a whose thickness linearly changes from h_0 to h_1 (Fig. 2 a). The material of the plate is isotropic with elasticity modulus E, Poisson's ratio μ, and density ρ. The outer contour of the conjugate system of plates $r = a$ is simply supported and subjected to the action of compressive tangential forces p.

For this system, by using the developed procedure, we determine the minimal natural frequency $\omega_{\min}$ and the critical values of the contour forces p^* according to the Kirchhoff–Love (KIR) and Timoshenko (TIM) theories and compared these values with the results obtained in [39]. The corresponding data are presented in Table 1 in the form of the dimensionless minimal natural frequency

$$\Omega_{00} = a^2 \sqrt{\frac{\rho h_0}{D_0}}\, \omega_{\min},$$

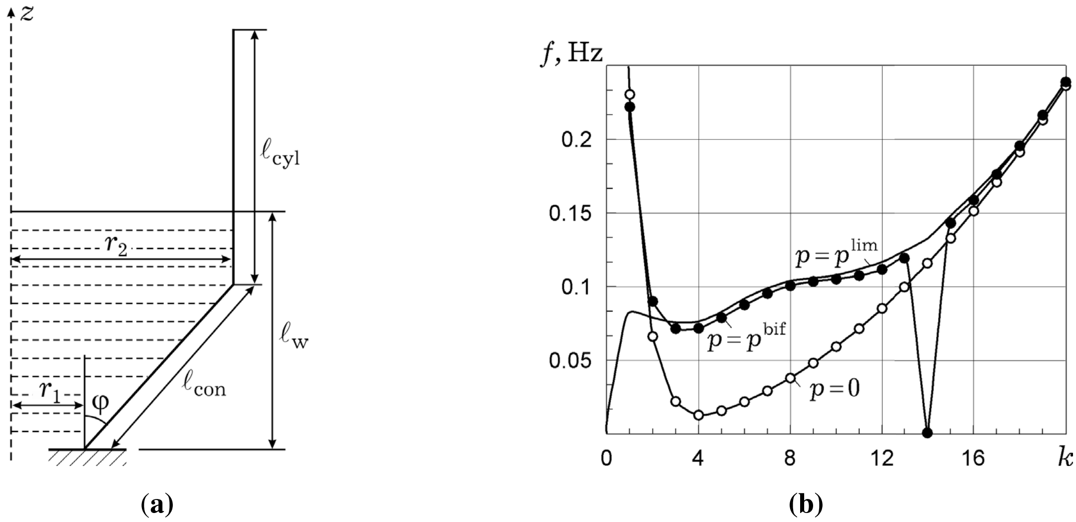


Fig. 3

where $D_0 = \dfrac{Eh_0}{12(1-\mu^2)}$, and the dimensionless tangential force

$$\tilde{N}_{\text{cr}} = \frac{a^2}{D_0} p^*$$

for two types of circular plates with variable thickness, namely, $c/a = 0.5$ and $c/a = 0.75$, and the following values of the other geometric parameters: $a = 1.2$ m, $h_0 = 0.016$ m, and $h_1 = 0.012$ m.

As follows from Table 1, the computations performed according to the Timoshenko refined theory practically do not change the values of critical loads and refine the values of natural frequencies by less than 0.1%.

Here, the difference between the results obtained by using the proposed procedure and the data from [39], obtained by the Rietz method in the two-term approximation does not exceed 1%.

To illustrate the application of the dynamic criterion of stability for the evaluation of the critical values of acting loads, in Fig. 2 b, we present the dependences $\Omega_{00} = \Omega_{00}(k)$ in the form of conditional curves for different values of the contour tangential force: $p = 0$ (absence of load), $p = p^* < 0$ (critical compressive force), and $p > 0$ (stretching force). The analysis of these dependences enables us to predict that these conjugate plates do not lose their stability in the bifurcation mode with formation of bulges and dents in the circumferential direction ($\Omega_{00} \neq 0$ at any $k > 0$) and lose it only in the axisymmetric more ($\Omega_{00} = 0$ only for $k = 0$). It is also clear that, under the action of stretching forces $p > 0$, the plates do not lose their stability in the subcritical stage of deformation.

Consider the problem of stability of a conic vessel filled with water whose computational scheme is shown in the form of two conjugate shells in Fig. 3 a: a conic shell (r_1, r_2, and ℓ_{con} are the initial and final radii of the cone and its length, respectively) and a cylindrical shell (r_2 and ℓ_{cyl} are the radius and length of the cylinder, respectively). Both parts of the vessel have the same thickness and are made of the isotropic material (Mylar) and subjected to the action of normal pressure with intensity p. The conic and cylindrical parts are filled with water up to a height ℓ_{w}. The lower end face of the cone is rigidly fixed and its upper end face is free.

Table 2

Critical load	k	Cone–cylinder			Cone
		[21]	[2]	ε, %	[2]
p^{lim}	0	0.993	0.999	0.6	1.001
p^{bif}	14	0.943	0.940	0.3	0.9402

The limit p^{lim} and bifurcation p^{bif} critical values of the load were computed for the following initial data (the sizes are given in centimeters, as in [2]):

$$r_1 = 13 \text{ cm}, \quad r_2 = 32.173 \text{ cm}, \quad \ell_{\text{con}} = 39.16 \text{ cm}, \quad \ell_{\text{cyl}} = 40 \text{ cm}, \quad \ell_{\text{w}} = 8 \text{ cm},$$

$$E = 5.0285 \cdot 10^5 \text{ N/cm}^2, \quad \mu = 0.33.$$

To estimate the influence of some elements of the analyzed system on its stability, we additionally considered one its elements, namely, a cone of the same geometric sizes under the same acting load in the case where its upper end is free.

The values of the limit load coefficient p^{lim} corresponding to the axisymmetric loss of stability (snap) and the bifurcation coefficient p^{bif} corresponding to the nonaxisymmetric loss of stability accompanied by the formation of k bulges and dents in the circumferential direction are presented in Table 2 for the connected (cone–cylinder) system and for a separate element (cone). These results were obtained by using the developed procedure [2] and by the finite-difference method with the help of the BOSOR 4 program [21].

As follows from Table 2, the procedure of evaluation of the critical values of limit and bifurcation loads by the developed method for cone–cylinder system is confirmed by the results obtained in [21] with a difference smaller than 1%. Note that the refinement of the results by the Timoshenko theory almost does not affect the critical values of these loads $(< 0.05\%)$.

In Fig. 3 b, as in the previous example, in order to illustrate the application of the dynamic criterion of stability, we present conditional curves $f = f(k)$ characterizing the dependence of natural frequencies of the shell system on the parameter of wave formation k for different intensities of the acting load: $p = 0$ (absence of loading), $p \simeq p^{\text{bif}}$ (internal pressure in the vicinity of the bifurcation value), and $p \simeq p^{\text{lim}}$ (internal pressure in the vicinity of the limiting value). It follows from the analysis of these dependences that, under the internal pressure, the cone–cylinder system loses its stability in the bifurcation mode with 14 bulges and dents. For the load value $p = p^{\text{lim}}$, the system loses its stability in the axisymmetric mode.

It is worth noting that the comparison of the obtained critical values of loads obtained for the system as a whole and for a separate cone (the last column in Table 2) reveals their practical coincidence (with an accuracy of about 0.5%). This means that the presence of cylindrical element in the analyzed system almost does not affect the values of critical loads. Hence, in this specific case, the analysis of stability of the system as a whole can be performed by considering its separate element, namely, the cone as the most sensitive object from the viewpoint of the loss of stability.

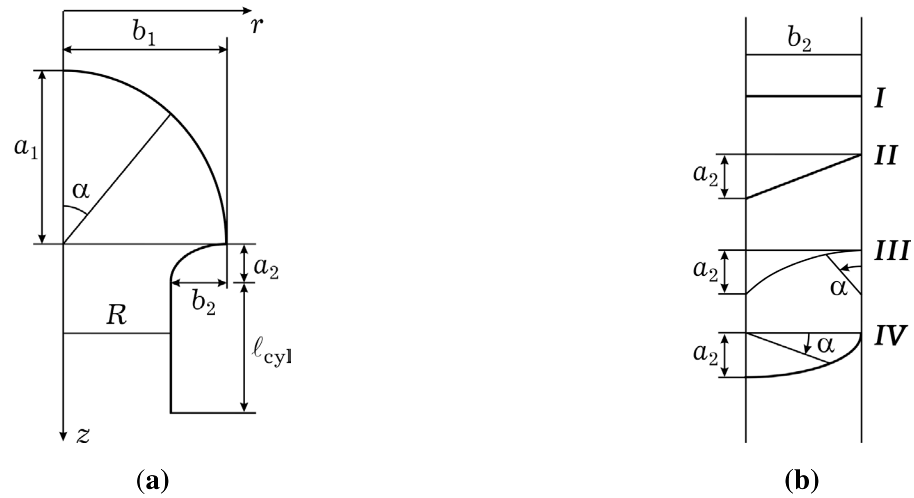


Fig. 4

4. Numerical Results

Consider an elastic system formed by elliptic (EL^+) and cylindrical (CYL) shells connected via a transition element S of different geometric shape (Fig. 4 a).

The elliptic shell with its center on the axis of revolution Oz and semiaxes a_1 (along the Oz-axis) and b_1 (along the Or-axis) has a constant thickness h_E. We specify the meridian (generatrix) of the middle surface of the shell in the rOz coordinate system in the following parametric form:

$$r(\alpha) = b_1 \sin\alpha, \quad z(\alpha) = a_1 \cos\alpha, \quad \alpha \in [0, \pi/2],$$

where α is the central angle of the ellipse that characterizes the current state of a point of the generatrix and measured clockwise from the axis of revolution Oz.

Let h_{cyl} be the radius R of the cylindrical shell of constant thickness and let ℓ_{cyl} be its length. The equation of its generatrix has the form

$$r(s) = R, \quad s \in [0, \ell_{cyl}],$$

where s is the distance between the current point of this generatrix and a certain initial position.

For the transition element S of thickness h_S, we consider the following possibilities of selection of its shape (Fig. 4 b):

I — an annular plate of length b_2 with radii $R + b_2$ and R and the equation of generatrix

$$r(s) = R + b_2 - s, \quad s \in [0, b_2],$$

where s is the distance from the current point on the generatrix of the plate;

II — a conic shell with the same radii $R+b_2$, R, length ℓ_{con}, and the equation of generatrix

$$r(s) = R+b_2 - s\frac{b_2}{\ell_{\text{con}}}, \quad s\in[0,\ell_{\text{con}}], \quad \ell_{\text{con}} = \sqrt{a_2^2+b_2^2},$$

where s is the distance from the current point on the generatrix of the cone and a_2 is the length of the transition element along the Oz-axis;

III — a part of a toroidal elliptic shell of negative curvature (TE^-) with semiaxes a_2 (along the Oz-axis) and b_2 (along the Or-axis); the distance between the center of ellipse and the axis of revolution is equal to $r^- = R+b_2 = b_1$ and the equation of generatrix takes the following parametric form:

$$r(\alpha) = r^- - b_2\sin\alpha, \quad z(\alpha) = -a_2\cos\alpha, \quad \alpha\in[0,\pi/2],$$

where α is the central angle of ellipse measured in this case from the Oz-axis counterclockwise;

IV — a part of a toroidal elliptic shell of positive curvature (TE^+) with the same semiaxes as in case ***III;*** the distance between the center of ellipse and the axis of revolution $r^+ = R$, and the equation of generatrix takes the following parametric form:

$$r(\alpha) = r^+ + b_2\cos\alpha, \quad z(\alpha) = a_2\sin\alpha, \quad \alpha\in[0,\pi/2],$$

where α is the central angle of the ellipse measured from the Or-axis clockwise.

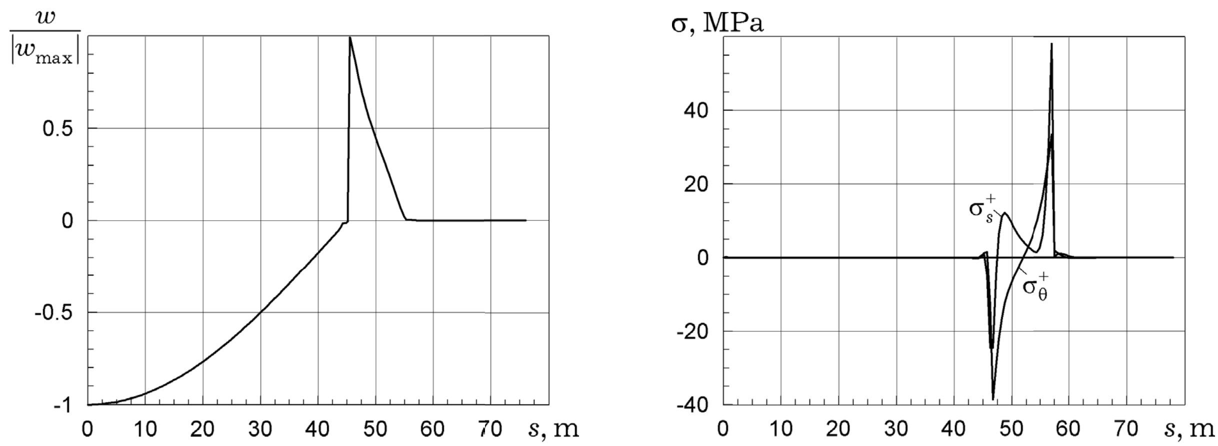
Note that conjugate systems of this kind can be used to model round architectural structures of various destinations (entertaining, religious, defensive, etc.) with dome-shaped roofs of different curvatures.

The constituent shells are isotropic and made of materials with different moduli of elasticity and Poisson's ratios: E^+ and μ^+ correspond to the elliptic part, E^S and μ^S correspond to the transition element, and E^{cyl} and μ^{cyl} correspond to the cylindrical part. We assume that the end face of the cylindrical shell is rigidly fixed. The conditions of symmetry are imposed at the vertex of the elliptic (dome-shaped) part. The system is subjected to the action of axisymmetric external pressure with intensity q uniformly distributed over the main elliptic shell. Both the transition element and the cylindrical part are free of any loads. Thus, the entire system suffers the action of pressure nonuniformly distributed over the generatrix.

We now analyze the problem of stability of this system depending on the changes in the geometric parameters of the loaded elliptic part characterized by the parameter of ellipticity

$$\beta = \frac{a_1}{b_1}$$

equal to the ratio of the semiaxes for $b_1 = \text{const}$ and the variations of height a_1 within the range $\beta\in[1/6,2]$.



(a) (b)

Fig. 5

We study the dependences of critical values of the limit $q^{\text{lim}} = q^{\text{lim}}(\beta)$ and bifurcation $q^{\text{bif}} = q^{\text{bif}}(\beta)$ loads for the presented four versions of the shape of transition element ***I–IV***.

The subsequent calculations are performed for the following initial data:

$$R = 20 \text{ m}, \quad b_1/R = 1.5, \quad a_2/R = 0.25, \quad b_2/R = 0.5,$$

$$\ell_{\text{cyl}}/R = 1.25 \quad \text{in case } \boldsymbol{I},$$

$$\ell_{\text{cyl}}/R = 1.0 \quad \text{in cases } \boldsymbol{II}, \boldsymbol{III}, \boldsymbol{IV},$$

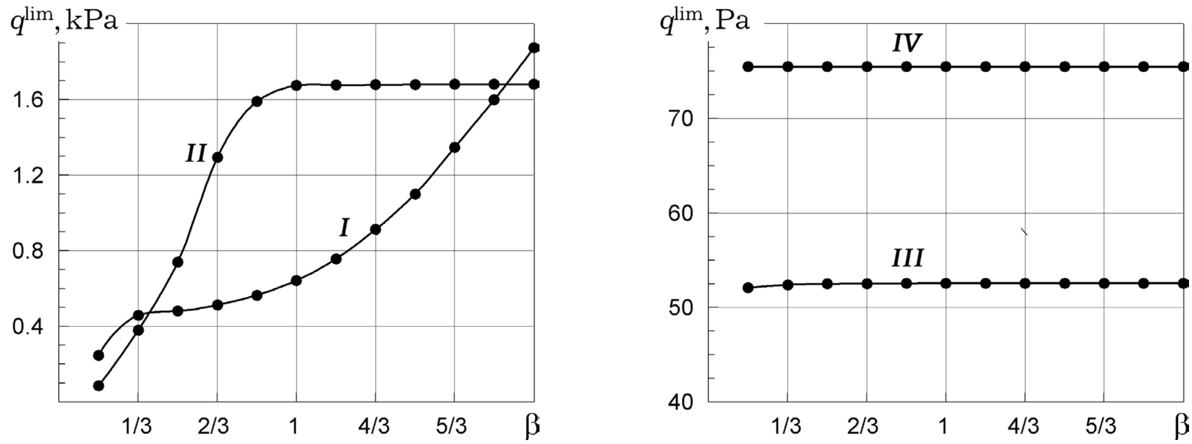
$$h_E/R = h_S/R = 0.5 \cdot 10^{-3}, \quad h_{\text{cyl}}/R = 10^{-2},$$

$$E^+ = 2.5 \cdot 10^{10} \text{ Pa}, \quad \mu^+ = 0.35 \quad \text{(shale–natural slate)},$$

$$E^S = 2.0 \cdot 10^{11} \text{ Pa}, \quad \mu^S = 0.24 \quad \text{(carbon steel)},$$

$$E^{\text{cyl}} = 2.0 \cdot 10^{10} \text{ Pa}, \quad \mu^{\text{cyl}} = 0.16 \quad \text{(concrete)}.$$

The picture of subcritical state typical of this system is presented in Fig. 5 in the form of distributions, along the generatrix, of the relative deflection w/w_{max} (Fig. 5 a) and the maximal meridional σ_s^+ and circumferential σ_θ^+ stresses on the outer surface (Fig. 5 b) for the transition element in case ***I*** with $\beta = 1$ near the critical value of the limit load $q^{\text{lim}} = 0.6$ (w_{max} is the value of deflection at the tip of the elliptic shell). As follows from Fig. 5 a, the action of external pressure is responsible for the maximum displacements into the interior of the shell $(w < 0)$ in the loaded elliptic part $(s \in [0, 46])$. At the same time, in the transition element $(s \in [46, 56])$, we observe the formation of quite large deflection of the opposite sign, whereas the displacements



(a) (b)

Fig. 6

of the cylinder $(s \in [56, 76])$ are practically absent (they are lower than the maximal displacements by several orders of magnitude). Unlike deflections, the maximal stresses are completely concentrated in the zone of transition element. Moreover, the highest compressive stresses (both meridional and circumferential responsible for the loss of stability) are concentrated in the zone of conjugation of the elliptic shell with the transition element. At the same time, the tensile stresses are formed in the zone of conjugation of the transition element with the cylinder. As compared with these stresses, the stresses formed in the elliptic shell and in the cylinder are lower by more than three orders of magnitude.

The dependences of the critical values of the limit loads $q^{\lim} = q^{\lim}(\beta)$ are shown in Figs. 6 a and 6b for the shapes of transition element in cases ***I***, ***II*** and ***III, IV***, respectively. As follows from Fig. 6, the variations of the parameter β of the elliptic part of the system exert qualitatively different effects on the presented dependences. Thus, the most expected and natural dependence is observed for the transition element in the form of an annular plate (case ***I***) and has the form of a monotonically increasing function. For the toroidal elliptic transition elements both with negative and positive curvatures (cases ***III*** and ***IV***), the increase in β in the entire range of its changes $\beta \in [1/6, 2]$ almost does not affect the level of limit loads and their quantitative values are lower than in cases ***I*** and ***II*** by an order of magnitude. For the conic transition element (case ***II***), the curve $q^{\lim} = q^{\lim}(\beta)$ monotonically increases within the range $\beta \in [1/6, 1]$ as in case ***I*** and remains practically constant for $\beta \in [1, 2]$ as in cases ***III*** and ***IV*** ($\beta = 1$ corresponds to a spherical dome).

Thus, the mutual influence of the parameter of ellipticity β of the loaded part of the system and the shape of the transition element in cases ***I–IV*** is observed only for the transition element in the form of an annular plate (***I***) and (partly) for the conic element (***II***). In the other cases, the picture of critical loads is determined solely by the shape of the transition element.

Similar dependences of the critical values of loads for the bifurcation mode of the loss of stability $q^{\text{bif}} = q^{\text{bif}}(\beta)$ with indication of the number of bulges and dents in the circumferential direction are presented in Fig. 7. Thus, the shapes of transition element in cases ***I*** and ***II*** are illustrated in Fig. 7 a, whereas cases ***III*** and ***IV*** are illustrated in Fig. 7 b (the dashed line corresponds to the critical values of the limit load). The cases of toroidal elliptic transition elements (Fig. 4 b, ***III***, ***IV***) and annular plates (Fig. 4 b, ***I***) are not interesting for the evaluation of the critical values of loads because their values are lower than the corresponding values for

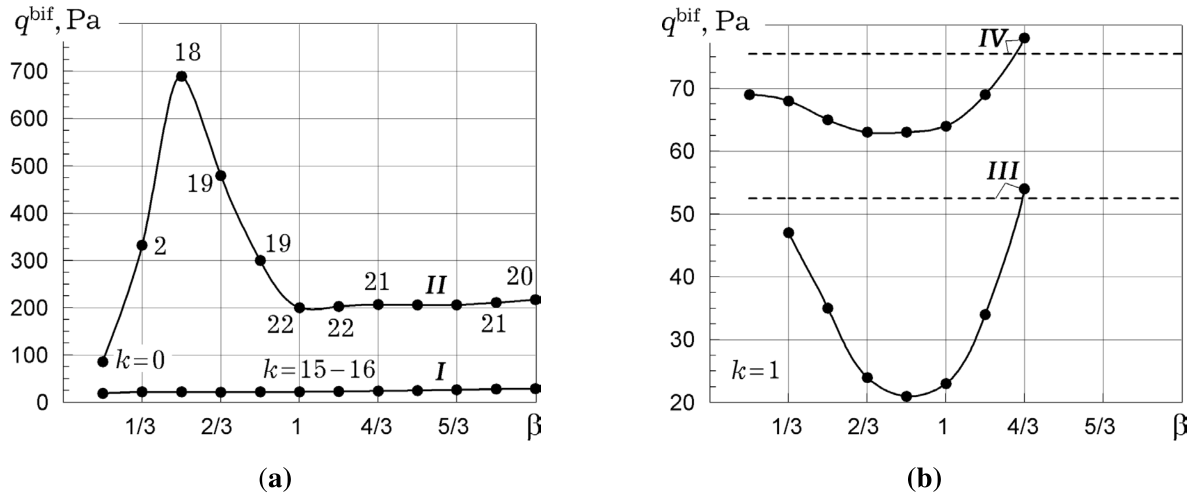


Fig. 7

the conic transition element (***II***) by an order of magnitude. In this last case, the curve $q^{\text{bif}} = q^{\text{bif}}(\beta)$ consists of three branches: increasing for $\beta \in [1/6, 1/2]$ with two deflections $(k = 2)$ in the circumferential direction, decreasing for $\beta \in (1/2, 1]$ with $k = 18, 19, 22$, and weakly increasing for $\beta \in (1, 2]$ with $k = 22, 21, 20$.

The highest critical values of loads for the bifurcation mode of the loss of stability observed as the parameter β changes in all cases ***I–IV*** are attained for the conic transition element and $\beta = 1/2$.

Thus, for the compound elastic systems of the shells of revolution of different curvatures, we investigated their static stability in the field of conservative axisymmetric loads.

The critical values of the limit and bifurcation loads are determined by using the classical and Timoshenko-type refined models of shells based on the quadratic approximation of the geometrically nonlinear theory and the dynamic criterion of stability.

For the solution of the corresponding nonlinear problems and the eigenvalue problem, we propose a unique numerical-analytic procedure based on the rational reduction of these problems to linear one-dimensional boundary-value problems and numerical solution of these problems by the orthogonal-sweep method.

The verification of this procedure confirms the possibility of its application for the solution of the problems of stability for the analyzed class of shell systems.

We study the problem of stability of compound systems with elements of different Gaussian curvatures and reveal the relationship between all its components in the evaluation of the critical bifurcation and limit loads.

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