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METHODS FOR STUDYING THE STABILITY OF LINEAR PERIODIC SYSTEMS DEPENDING ON A SMALL PARAMETER

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Abstract. In this paper, we consider systems of linear differential equations with periodic coefficients depending on a small parameter. We propose new approaches to the problem of constructing a monodromy matrix that lead to new effective formulas for calculating multipliers of the system studies. We present a number of applications in problems of the perturbation theory of linear operators, in the analysis of stability of linear differential equations with periodic coefficients, in the problem of constructing the stability domains of linear dynamical systems, etc.

***Keywords and phrases*:** differential equation, periodic system, Hamiltonian system, monodromy matrix, multiplier, stability, small parameter.

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1. Introduction and statement of the problem. Many theoretic and applied problems lead to the necessity of the study of the following linear system depending on a scalar or vector parameter ε:

$$\frac{dx}{dt} = \big[A_0 + S(t,\varepsilon)\big]x, \quad x \in \mathbb{R}^N; \tag{1.1}$$

here A_0 is a constant square matrix with real entries and $S(t,\varepsilon)$ is a T-periodic in t matrix (i.e., $S(t+T,\varepsilon) \equiv S(t,\varepsilon)$) with real entries satisfying the condition

$$S(t,0) \equiv 0. \tag{1.2}$$

In the study of systems, the analysis of their stability and, in particular, the search for stability domains in the space of parameters, examining of the order of increasing or decreasing of solutions, etc. are of special interest. Extensive literature is devoted to the study of such problems, and a number of effective research methods were proposed both for a general setting and aimed at studying various versions of a system of the form (1.1) (see, e.g., [1, 8, 10, 13] and the reference therein). The analysis of the unperturbed system

$$\frac{dx}{dt} = A_0 x, \quad x \in \mathbb{R}^N, \tag{1.3}$$

i.e., the system (1.1) as $\varepsilon = 0$, is sufficiently simple. This simplicity is related to the existence of explicit formulas for the fundamental system of solutions and, therefore, explicit formulas for the general solution of the autonomous system (1.3) of the form $x(t) = e^{A_0 t}x_0$. Unfortunately, for the perturbed system (1.1) in the general case (for $N \geq 2$), due to its nonautonomy, there are no such explicit formulas, and this essentially complicates the analysis of the system.

A number of methods have been proposed in the literature for studying systems of the form (1.1); these methods yield certain formulas for an approximate representation of general solutions, allow one to analyze the stability of the system and to examine properties of solutions, etc. The classical Floquet theory (see, e.g., [8, 13] allows one to pass from linear equations with periodic coefficients to linear equations with constant coefficients. However, the use of this theory assumes the knowledge of a fundamental system of solutions of the equation with periodic coefficients, which is possible only in the simplest cases. Therefore, the Floquet theory, which has an important theoretical value, is not effective from a practical point of view in many cases. In a number of works (see, e.g., [1, 8, 13]), methods

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were proposed for an approximate study of systems of the form (1.1) for a number of important cases, in particular, for the case where elements of the matrix $S(t,\varepsilon)$ are finite sums of exponential terms. In [10], a method was proposed that realizes the step-by-step (by powers of the small parameter ε) conversion of entries of the periodic matrix $S(t,\varepsilon)$ into entries of a certain constant matrix. Studies of various problems related to systems of the form (1.1) actively develop in many directions (see, e.g., [3, 5, 9, 12]).

In this paper, we propose new approaches to approximate constructing the fundamental matrix for the system (1.1) and new formulas for approximate calculating its multipliers. Also, we discuss several applications.

2. Calculation of the fundamental matrix.

We assume that the matrix $S(t,\varepsilon)$ is continuously differentiable by ε up to order k ($k \geq 1$) inclusively. For simplicity, we assume that the small parameter ε is scalar.

We denote by $U = X(t,\varepsilon)$ a solution of the following matrix Cauchy problem:

$$U' = \big[A_0 + S(t,\varepsilon)\big]U, \quad U(0) = I, \tag{2.1}$$

where I is the identity matrix. The matrix $X(t,\varepsilon)$ is the *fundamental matrix* (FM) of the system (1.1) and the matrix $V(\varepsilon) = X(T,\varepsilon)$ is its *monodromy matrix*. Eigenvalues of the matrix $V(\varepsilon)$ are called *multipliers* of the system (1.1). The equalities $X(t,0) = e^{A_0 t}$ and $V(0) = e^{A_0 T}$ are obvious.

We propose a scheme of constructing the formula for approximate calculation of the fundamental matrix of the system (1.1). We construct the FM as an expansion by powers of the small parameter ε:

$$X(t,\varepsilon) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 \frac{X_2(t)}{2!} + \ldots + \varepsilon^k \frac{X_k(t)}{k!} + \Psi(t,\varepsilon), \tag{2.2}$$

where $X_0(t) = e^{A_0 t}$ and the matrices $X_1(t), X_2(t), \ldots, X_k(t)$ are unknown; here $\Psi(t,\varepsilon)$ is a continuous in t and continuously differentiable in ε matrix satisfying the condition

$$\big\|\Psi(t,\varepsilon)\big\| = O\big(|\varepsilon|^{k+1}\big) \quad \text{as } \varepsilon \to 0.$$

For simplicity, we introduce the notation

$$S_j(t) = S_\varepsilon^{(j)}(t,0), \quad j = 1, 2, \ldots k, \tag{2.3}$$

i.e., $S_j(t)$ is the jth derivative of the matrix $S(t,\varepsilon)$ by ε for $\varepsilon = 0$.

Theorem 2.1. *The fundamental matrix of the system* (1.1) *can be represented in the form* (2.2), *where*

$$X_0(t) = e^{A_0 t}, \quad X_1(t) = e^{A_0 t} \int\limits_0^t e^{-A_0 \tau} S_1(\tau) e^{A_0 \tau}\, d\tau, \tag{2.4}$$

and the matrices $X_2(t), \ldots, X_k(t)$ are defined by the recurrent formula

$$X_m(t) = e^{A_0 t} \int\limits_0^t e^{-A_0 \tau} \sum_{j=0}^{m-1} \Big(C_m^j S_{m-j}(\tau) X_j(\tau)\Big) d\tau, \quad m = 2, \ldots, k; \quad C_m^j = \frac{m!}{(m-j)!j!}. \tag{2.5}$$

In particular, the matrix $X_2(t)$ is defined by the formula

$$X_2(t) = e^{A_0 t} \int\limits_0^t e^{-A_0 \tau} \Big(S_2(\tau) e^{-A_0 \tau} + 2 S_1(\tau) X_1(\tau)\Big) d\tau. \tag{2.6}$$

The proofs of this and other assertions are contained in Sec. 4.

Theorem 2.1 implies the following assertion.

Theorem 2.2. *The monodromy matrix $V(\varepsilon)$ of the system* (1.1) *can be represented in the form*

$$V(\varepsilon) = V_0 + \varepsilon V_1 + \varepsilon^2 \frac{V_2}{2!} + \ldots + \varepsilon^k \frac{V_k}{k!} + \tilde{V}(\varepsilon), \tag{2.7}$$

where

$$V_0 = e^{A_0 T}, \quad V_1 = e^{A_0 T} \int_0^T e^{-A_0 \tau} S_1(\tau) e^{A_0 \tau} d\tau, \tag{2.8}$$

the matrices $V_2, \ldots, V_k$ are defined by the recurrent formula

$$V_m = e^{A_0 T} \int_0^T e^{-A_0 \tau} \sum_{j=0}^{m-1} \Big(C_m^j S_{m-j}(\tau) X_j(\tau) \Big) d\tau, \quad m = 2, \ldots, k,$$

and $\tilde{V}(\varepsilon)$ is a continuously differentiable in ε matrix satisfying the following condition:

$$\|\tilde{V}(\varepsilon)\| = O\big(|\varepsilon|^{k+1}\big) \quad \textit{as } \varepsilon \to 0.$$

In particular, the matrix V_2 is defined by the formula

$$V_2 = e^{A_0 T} \int_0^T e^{-A_0 \tau} \Big(S_2(\tau) e^{A_0 \tau} + 2 S_1(\tau) X_1(\tau) \Big) d\tau.$$

3. Applications.

In this section, we consider some applications of Theorems 2.1 and 2.2.

3.1. Formulas of the perturbation theory of linear operators. As the first application, we consider the problem of constructing multipliers of the system (1.1) in the following cases, which are important for applications:

1° the matrix A_0 has a simple eigenvalue 0;
2° the matrix A_0 has a pair of simple eigenvalues $\pm\omega_0 i$, where $\omega_0 > 0$, and $\omega_0 T \neq \pi k$ for integer k;
3° the matrix A_0 has a pair of simple eigenvalues $\pm\omega_0 i$, where $\omega_0 > 0$, and $\omega_0 T = \pi k_0$ for some integer k_0.

In all these cases, we assume that the matrix A_0 has no other eigenvalues with zero real part.

Case 1°. In this case, the monodromy matrix $V_0 = e^{A_0 T}$ of the unperturbed system (1.3) has a simple eigenvalue 1. It is known from the theory of perturbation of linear operators (see [6]) that the monodromy matrix $V(\varepsilon)$ of the perturbed system (1.1) for any small $|\varepsilon|$ has a unique simple eigenvalue $\mu(\varepsilon)$ for which $\mu(0) = 1$, and the function $\mu(\varepsilon)$ is C^k-smooth. We consider the problem on the approximate constricting the function $\mu(\varepsilon)$.

We denote by e and g^* the eigenvectors of the matrices A_0 and A_0^* corresponding to the zero eigenvalue (here and below, we denote by B^* the transpose matrix).

Theorem 3.1. *The vectors e and g^* can be normalized corresponding to the equalities*

$$\|e\| = 1, \quad (e, g^*) = 1. \tag{3.1}$$

Clearly, there exist exactly two variants of such normalization of the vectors e and g^*, which differ only by sign.

Theorem 3.2. *The function $\mu(\varepsilon)$ for small $|\varepsilon|$ can be represented in the form*

$$\mu(\varepsilon) = 1 + \varepsilon \mu_1 + O(\varepsilon^2), \tag{3.2}$$

where

$$\mu_1 = \int_0^T (S_1(t) e, g^*) dt; \tag{3.3}$$

here $S_1(t)$ *is the first of the matrices* (2.3)*.*

Case $2°$**.** In this case, the monodromy matrix $V_0 = e^{A_0 T}$ of the unperturbed system (1.3) has a pair of simple complex conjugate eigenvalues $e^{\pm\omega_0 Ti}$ ($e^{\pm\omega_0 Ti} \neq \pm 1$). It is known from the theory of perturbations of linear operators (see [6]) that the monodromy matrix $V(\varepsilon)$ of the perturbed system (1.1) for any small $|\varepsilon|$ has a unique simple eigenvalue $\mu(\varepsilon)$ such that $\mu(0) = \mu_0 = e^{\omega_0 Ti}$, and the function $\mu(\varepsilon)$ is C^k-smooth. Consider the problem on the approximate construction of the function $\mu(\varepsilon)$.

Since the matrix A_0 has a pair of simple purely imaginary eigenvalues $\pm\omega_0 i$ ($\omega_0 > 0$), there exists nonzero vector $e, g, e^*, g^* \in \mathbb{R}^N$ such that the following equalities hold:

$$A_0(e + ig) = i\omega_0(e + ig), \quad A_0^*(e^* + ig^*) = -i\omega_0(e^* + ig^*). \tag{3.4}$$

The vectors e and g (respectively, e^* and g^*) are linearly independent; however, they are defined not uniquely. Now we are interested in the following normalization of these vectors.

Theorem 3.3. *The vectors* e, g, e^*, *and* g^* *can be normalized corresponding to the equalities*

$$(e, e^*) = (g, g^*) = 1, \quad (e, g^*) = (g, e^*) = 0. \tag{3.5}$$

We assume that this normalization of the vectors e, g, e^*, and g^* has been performed.

Theorem 3.4. *The function* $\mu(\varepsilon)$ *for small* $|\varepsilon|$ *can be represented in the form*

$$\mu(\varepsilon) = \mu_0 + \varepsilon\mu_1 + O(\varepsilon^2), \tag{3.6}$$

where $\mu_0 = e^{\omega_0 Ti}$,

$$\mu_1 = \frac{\mu_0}{2}(\gamma_1 + i\gamma_2);$$

here

$$\gamma_1 = \int_0^T \Big[\big(S_1(t)e, e^*\big) + \big(S_1(t)g, g^*\big)\Big]dt, \quad \gamma_2 = \int_0^T \Big[\big(S_1(t)g, e^*\big) - \big(S_1(t)e, g^*\big)\Big]dt,$$

and $S_1(t)$ *is the first of the matrices* (2.3)*.*

Case $3°$**.** In this case, the monodromy matrix $V_0 = e^{A_0 T}$ of the unperturbed system (1.3) has a semisimple eigenvalue μ_0 of multiplicity 2, where $\mu_0 = 1$ for even k_0 or $\mu_0 = -1$ for odd k_0. It follows from the theory of perturbations of linear operators (see [6]) that the monodromy matrix $V(\varepsilon)$ of the perturbed system (1.1) for any small $|\varepsilon|$ has a pair of eigenvalues $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ for which $\mu_1(0) = \mu_2(0) = \mu_0$ and the functions $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ are C^k-smooth. Consider the problem on approximate construction of these functions.

Since the matrix A_0 has a pair of simple, purely imaginary eigenvalues $\pm\omega_0 i$ ($\omega_0 > 0$), there exist nonzero vectors $e, g, e^*, g^* \in \mathbb{R}^N$ such that Eqs. (2.8) are fulfilled. We also assume that Eqs. (3.5) are valid.

Below, we will need the following assertion (see, e.g., [11]).

Theorem 3.5. *The functions* $\mu_1(\varepsilon)$ *and* $\mu_2(\varepsilon)$ *for small* $|\varepsilon|$ *can be represented in the form*

$$\mu_1(\varepsilon) = \mu_0 + \mu_1\varepsilon + o(\varepsilon), \quad \mu_2(\varepsilon) = \mu_0 + \mu_2\varepsilon + o(\varepsilon), \tag{3.7}$$

where μ_1 *and* μ_2 *are eigenvalues of the matrix*

$$B = \begin{bmatrix} (V_1 e, e^*) & (V_1 g, e^*) \\ (V_1 e, g^*) & (V_1 g, g^*) \end{bmatrix}; \tag{3.8}$$

here V_1 *is the matrix defined by the second equality in* (2.8)*.*

Elements of the matrix (3.8) can be calculated by the formulas (2.8) and (3.4). As a result, we obtain the equalities

$$(V_1e, e^*) = \mu_0 \int_0^T \Big\{ \cos^2(\omega_0 t)\big(S_1(t)e, e^*\big) + \sin^2(\omega_0 t)\big(S_1(t)g, g^*\big) - \frac{1}{2}\sin(2\omega_0 t)\Big[\big(S_1(t)e, g^*\big) + \big(S_1(t)g, e^*\big)\Big]\Big\} dt, \quad (3.9)$$

$$(V_1g, e^*) = \mu_0 \int_0^T \Big\{ \cos^2(\omega_0 t)\big(S_1(t)g, e^*\big) - \sin^2(\omega_0 t)\big(S_1(t)e, g^*\big) + \frac{1}{2}\sin(2\omega_0 t)\Big[\big(S_1(t)e, e^*\big) + \big(S_1(t)g, g^*\big)\Big]\Big\} dt, \quad (3.10)$$

$$(V_1e, g^*) = \mu_0 \int_0^T \Big\{ \cos^2(\omega_0 t)\big(S_1(t)e, g^*\big) - \sin^2(\omega_0 t)\big(S_1(t)g, e^*\big) + \frac{1}{2}\sin(2\omega_0 t)\Big[\big(S_1(t)e, e^*\big) - \big(S_1(t)g, g^*\big)\Big]\Big\} dt, \quad (3.11)$$

$$(V_1g, g^*) = \mu_0 \int_0^T \Big\{ \cos^2(\omega_0 t)\big(S_1(t)g, g^*\big) + \sin^2(\omega_0 t)\big(S_1(t)e, e^*\big) + \frac{1}{2}\sin(2\omega_0 t)\Big[\big(S_1(t)e, g^*\big) - \big(S_1(t)g, e^*\big)\Big]\Big\} dt. \quad (3.12)$$

3.2. Stability of linear periodic systems. As the second application, we examine the stability of the linear system (1.1) for small $|\varepsilon|$.

This problem is analyzed relatively simply in the following two cases (see, e.g., [10, 13]). In the first case, all eigenvalues of the matrix A_0 have negative real parts; then for all small $|\varepsilon|$, the zero solution of the system (1.1) is asymptotically stable. In the second case, at least one eigenvalue of the matrix A_0 has positive real parts; then for all small $|\varepsilon|$, the zero solution of the system (1.1) is unstable.

Stability of solutions in critical cases. The critical case where the matrix A_0 has one or several eigenvalues with zero real parts and has no eigenvalues with positive real part is significantly more complicated. We consider three main variants of the critical case, namely, the cases 1°–3° indicated in the previous section. We assume that the other eigenvalues of the matrix A_0 have negative real parts.

First, we consider the case 1°, i.e., let the matrix A_0 have a simple eigenvalue 0. Theorem 3.2 implies the following assertion.

Theorem 3.6. *For all small* $|\varepsilon|$ *satisfying the condition* $\varepsilon\mu_1 < 0$ (*respectively,* $\varepsilon\mu_1 > 0$), *the solution* $x = 0$ *of Eq.* (1.1) *is asymptotically stable* (*respectively, unstable*).

Now we consider the case 2°, i.e., let the matrix A_0 have a pair of simple eigenvalues $\pm\omega_0 i$, where $\omega_0 > 0$, and $\omega_0 T \neq \pi k$ for integer k. Theorem 3.4 implies the following assertion.

Theorem 3.7. *For all small* $|\varepsilon|$ *satisfying the condition* $\varepsilon\gamma_1 < 0$ (*respectively,* $\varepsilon\gamma_1 > 0$), *the solution* $x = 0$ *of Eq.* (1.1) *is asymptotically stable* (*respectively, unstable*).

In the case 3°, the problem on the stability of the system (1.1) is reduced to calculating eigenvalues of the matrix (3.8) and analyzing the formula (3.7) for small $|\varepsilon|$. Here we restrict ourselves by an example.

Example. Consider the system (see, e.g., [13])

$$x' = (A_0 + \varepsilon A_1(t))x, \quad x \in \mathbb{R}^3, \tag{3.13}$$

where ε is a small parameter,

$$A_0 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\cos t & -1 & -1 \end{pmatrix}.$$

Eigenvalues of the matrix A_0 are $\lambda_{1,2} = \pm i$ and $\lambda_3 = -1$. We examine the stability of the solution $x = 0$ of the system (3.13) for small ε.

Since the matrix A_0 has a pair of simple eigenvalues $\lambda_{1,2} = \pm i$ and the period T of the right-hand side of the system (3.13) is equal to $T = 2\pi$, in the example considered the conditions of the case 3° for $k_0 = 2$ are fulfilled.

To examine the stability of the system (3.13), we construct the matrix (3.8) whose elements can be calculated by the formulas (3.9)–(3.12). We find the eigenvectors e, g and e^*, g^* of the matrices A_0 and A_0^* corresponding to the eigenvalues i and $-i$, respectively, and satisfying the relations (3.5):

$$e = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad e^* = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}, \quad g^* = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}.$$

By easy calculations we obtain the matrix (3.8) in the form

$$B = \pi \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Then Theorem 3.5 and the formula (3.7) imply that the system (3.13) has a pair of multipliers, which can be represented in the form $\mu_{1,2}(\varepsilon) = 1 - \pi\varepsilon \pm \pi i\varepsilon + o(\varepsilon)$. Obviously, the third multiplier has the form $\mu_3(\varepsilon) = e^{-2\pi} + O(\varepsilon)$. This implies that for small $\varepsilon > 0$, the solution $x = 0$ of the system (3.13) is asymptotically stable and for small $\varepsilon < 0$ is unstable.

Stability of linear Hamiltonian systems. Consider the case where the system (1.1) is Hamiltonian, namely, consider the linear system

$$\frac{dx}{dt} = \bigl[A_0 + S(t, \varepsilon)\bigr]x, \quad x \in \mathbb{R}^{2N}, \tag{3.14}$$

under the same conditions as for the initial system (1.1). In addition, we assume that the system (3.14) is Hamiltonian (see, e.g., [13]). Then the product of its multipliers is equal to 1 and the Poincaré mapping $U(T)$ of the system for the time T preserves the phase volume.

Since the system (3.14) is a particular case of the system (1.1), the constructions performed above and the results obtained remain valid for the system (3.14). However, the system (3.14) has its own specifics, contained, in particular, in the following properties of linear Hamiltonian systems with periodic coefficients (see, e.g., [13]):

(i) if the system (3.14) has a multiplier μ_1, then the number $\mu_2 = 1/\mu_1$ is also its multiplier of the same multiplicity;

(ii) if the system (3.14) has a multiplier $\mu = 1$ or $\mu = -1$, then its multiplicity is even;

(iii) the system (3.14) is stable in the Lyapunov sense if and only if all its multipliers μ satisfy the equality $|\mu| = 1$ and are semisimple.

Since the multipliers μ of the unperturbed system $x' = A_0 x$ are related to the eigenvalues λ of the matrix A_0 by the equality $\mu = e^{\lambda T}$, the system (3.14) can be stable in the Lyapunov sense for small $|\varepsilon|$ if and only if all eigenvalues of the matrix A_0 have zero real parts.

Now we discuss stability conditions for the Hamiltonian system (3.14) in situations similar to the cases 1°–3° (see the previous section). First, we note that the case 1° for the Hamiltonian system (3.14) cannot be realized since the matrix A_0 cannot have a simple eigenvalue 0.

Consider an analog of the case 2° under the condition that all eigenvalues of the matrix A_0 are simple and purely imaginary, $\pm\omega_0 i$, where $\omega_0 > 0$ and $\omega_0 T \neq \pi k$ for integer k. Let the following condition be fulfilled:

(a) for any two distinct pairs of eigenvalues $\pm\omega_1 i$ and $\pm\omega_2 i$ for integer k, the relation $\omega_1 - \omega_2 \neq 2\pi k/T$ holds.

In this case, for all small $|\varepsilon|$, the system (3.14) is stable in the Lyapunov sense (but is not asymptotically stable).

Now we consider an analog of the case 3°. Let the matrix A_0 have exactly one pair of simple eigenvalues $\pm\omega_0 i$ with $\omega_0 > 0$ and $\omega_0 T = \pi k_0$ for some integer k_0. Let all other eigenvalues of the matrix A_0 are simple (naturally, they have zero real parts) and let the condition (a) be fulfilled. Under these conditions, we can construct the matrix B defined by Eq. (3.8). The following assertion holds.

Theorem 3.8. *If* $\det B < 0$, *then for all small* $|\varepsilon|$ *the system* (3.14) *is unstable, and if* $\det B > 0$, *then it is stable, but not asymptotically stable.*

This theorem follows from the fact that the eigenvalues μ_1 and μ_2 of the matrix B are solutions of the equation

$$\lambda^2 + \det B = 0;$$

this can be easily proved, for example, by using Eqs.(3.9)–(3.12).

3.3. Construction of boundaries of stability domains for dynamical systems. As the third application, we consider the problem on the construction of boundaries of stability domains of the linear system

$$x' = \big[A_0 + S(t,\alpha,\beta)\big]x, \quad x \in \mathbb{R}^N, \tag{3.15}$$

involving two scalar parameters α and β. Such problems appear in many theoretic and applied aspects of the theory of differential equations (see, e.g., [2, 4, 15]).

We assume that the matrix $S(t,\alpha,\beta)$ is T-periodic, i.e., $S(t+T,\alpha,\beta) \equiv S(t,\alpha,\beta)$; moreover, $S(t,\alpha_0,\beta_0) \equiv 0$.

A system can be stable in the Lyapunov sense for some values of parameters and unstable for other. The set G in the plane Π of parameters (α,β) is called the stability domain (respectively, instability domain) of the system (3.15) if for any $(\alpha,\beta) \in G$ the system (3.15) is stable in the Lyapunov sense (respectively, unstable).

A point $(\alpha_0,\beta_0) \in \Pi$ is called a *boundary point* of the stability domain G of the system (3.15) if any neighborhood of it contains both points of the stability domain G and points of the in stability domain. Obviously, a point $(\alpha_0,\beta_0) \in \Pi$ is a boundary point for the stability domain G if the matrix A_0 has at least one imaginary eigenvalue and has no eigenvalues with positive real parts.

The set of boundary points of the set G is called the boundary Γ of the set G. If a point $(\alpha_0,\beta_0) \in \Pi$ is a boundary point of the stability domain of the system (3.15), then, as a rule, one or several smooth boundary curves pass through this point.

Let (α_0,β_0) be a boundary point of the stability domain of the system (3.15). We discuss the construction of a boundary curve (or several such curves) passing through the point (α_0,β_0).

An equation describing the required boundary curve can be constructed by various methods. Assuming for definiteness that the corresponding function is monotonic in α, we construct it in the parametric form:

$$\begin{cases} \alpha(\delta) = \alpha_0 + \delta, \\ \beta(\delta) = \beta_0 + \beta_1\delta + \beta_2\delta^2 + \ldots + \beta_k\delta^k + \psi(\delta), \end{cases} \tag{3.16}$$

where δ is a small parameter, $\beta_1,\ldots,\beta_k$ are unknown coefficients, and $\|\psi(\delta)\| = o(|\delta|^k)$ as $\delta \to 0$.

Substituting (3.16) into (3.15), we obtain a system of the form (1.1) depending on the small parameter δ. Applying the schemes described in Theorems 2.1 and 2.2, we choose the coefficients $\beta_1,\ldots,\beta_k$ such that the multipliers of the corresponding systems lie on the unit circle of the complex plane.

We illustrate this scheme by the Mathieu equation (see, e.g., [1]):

$$u'' + (\alpha + \beta \cos 2t)u = 0, \tag{3.17}$$

where α and β are real-valued parameters. We must take into account the fact that (3.17) is a Hamiltonian equation.

We find boundary points of the stability domain of Eq. (3.17) lying on the positive semiaxis α of the plane Π of parameters (α, β) and construct boundary curves passing through these points. The properties of Hamiltonian systems indicated above imply that the required boundary points have the coordinates $(n^2, 0)$, where n is an integer.

For simplicity, we consider the case $n = 1$, i.e., construct boundary curves of the stability domain of Eq. (3.17) passing through the point $(1, 0)$ of the plane (α, β). This means that we consider the dynamics of Eq. (3.17) for values α and β close to $\alpha_0 = 1$ and $\beta_0 = 0$, respectively.

By the standard change of variables $x_1 = u$ and $x_2 = u'$, we reduce Eq. (3.17) to the linear system

$$x' = A(t, \alpha, \beta)x, \quad x \in \mathbb{R}^2, \tag{3.18}$$

where

$$A(t, \alpha, \beta) = \begin{bmatrix} 0 & 1 \\ -(\alpha + \beta \cos 2t) & 0 \end{bmatrix}.$$

The system (3.18) is a linear system with periodic coefficients with the period $T = \pi$.

We search for a curve Υ_0, which bounds the stability and instability domains of Eq. (3.17) and passing through the point $(1, 0)$ in the plane of parameters (α, β) in the form of a function defined parametrically:

$$\begin{cases} \alpha(\delta) = 1 + \delta, \\ \beta(\delta) = \beta_1\delta + \beta_2\delta^2 + \ldots + \beta_k\delta^k + o(\delta^k), \end{cases} \tag{3.19}$$

where β_j are unknown coefficients. We find the first coefficient β_1; the other coefficients in (3.19) can be found similarly. Substituting (3.19) into (3.18), we obtain the system

$$x' = \big[A_0 + \delta S(t, \beta_1)\big]x + \tilde{S}(t, \delta)x, \tag{3.20}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad S(t, \beta_1) = \begin{bmatrix} 0 & 0 \\ -(1 + \beta_1 \cos 2t) & 0 \end{bmatrix},$$

and the matrix $\tilde{S}(t, \delta)$ satisfies the relation $\|\tilde{S}(t, \delta)\| = o(\delta)$ as $\delta \to 0$ uniformly with respect to t.

We denote by $V(\delta)$ the monodromy matrix of the system (3.20). Since the matrix A_0 has simple eigenvalues $\pm i$ and the period of the right-hand side of the system (3.18) is $T = \pi$, the matrix $V(0)$ has a semisimple eigenvalue $\mu_0 = -1$.

According to the scheme described above, we calculate the first derivative of the monodromy matrix $V(\delta)$ at the point $\delta = 0$ by the formula (2.8):

$$V'(0) = V_1 = \frac{1}{4}\pi \begin{bmatrix} 0 & \beta_1 - 2 \\ \beta_1 + 2 & 0 \end{bmatrix}.$$

Eigenvalues of the matrix V_1 are equal to $\pm\pi\sqrt{\beta_1^2 - 4}/4$. Therefore, for $\beta_1 = \pm 2$, the multipliers $\mu(\delta)$ of the system (3.20) have the form $\mu(\delta) = -1 + O(\delta^2)$, i.e., they are equal to -1 with accuracy δ. Thus, the required coefficient β_1 may take two values: $\beta_1 = 2$ or $\beta_1 = -2$. This confirms the fact (see, e.g., [1]) that through the point $(1, 0)$ in the plane of parameters (α, β), pass exactly two boundary curves, which bound the stability and instability domains of the Mathieu equation (3.17); moreover, the slopes of the tangents of these curves are equal to 2 and -2.

3.4. Derivative of the matrix exponential. As the fourth application, we consider an interesting problem, which formally is not formally related to periodic systems of differential equations; however, it can be solved by using approaches presented in this paper. Let $A(\mu)$ be a square real-valued matrix of order N, which is continuously differentiable with respect to a scalar parameter μ. We calculate the derivative of the matrix exponential $e^{A(\mu)}$ with respect to μ, i.e., find $\left(e^{A(\mu)}\right)'_{\mu}$. The necessity of calculating this derivative naturally appears in many theoretical and applied problems (see, e.g., [7, 14]).

Since, by the definition,

$$e^{A(\mu)} = I + A(\mu) + \frac{1}{2!}(A(\mu))^2 + \ldots + \frac{1}{n!}(A(\mu))^n + \ldots,$$

the calculation of the derivative by this formula leads to summing of a complicated matrix series:

$$\left(e^{A(\mu)}\right)'_{\mu} = A' + \frac{1}{2!}(A'A + A'A) + \frac{1}{3!}(A'A^2 + AA'A + A^2A') + \ldots,$$

where we have introduced the notation $A = A(\mu)$ and $A' = (A(\mu))'$. The main difficulty is due to the noncommutativity of the matrix product, i.e., in general, $AA' \neq A'A$.

Approaches proposed above allow one to prove the following theorem.

Theorem 3.9. *The derivative of the matrix exponential* $e^{A(\mu)}$ *is equal to*

$$\left(e^{A(\mu)}\right)'_{\mu} = e^{A(\mu)} \int\limits_0^1 e^{-A(\mu)\tau} A'(\mu) e^{A(\mu)\tau} d\tau.$$

4. Proofs of the basic assertions.

Proof of Theorem 2.1. The matrices $X_j(t)$ involved in the formula (2.2) are the derivatives of the fundamental matric $X(t, \varepsilon)$ of the system (1.1) with respect to ε at the point $\varepsilon = 0$:

$$X_1(t) = X'_{\varepsilon}(t, 0), \quad X_2(t) = X''_{\varepsilon}(t, 0), \ldots. \tag{4.1}$$

In other words, the formula (2.2) is the Taylor formula for $X(t, \varepsilon)$, whose existence follows from general theorems on the C^k-smoothness of the solution to the Cauchy problem (2.1) with respect to the parameter ε (we recall that the matrix $S(t, \varepsilon)$ is assumed to be continuously differentiable with respect to ε up to order k inclusively). Thus, it remains to calculate the derivatives of the matrix $X(t, \varepsilon)$ with respect to ε up to the kth order inclusively.

The matrix $X(t, \varepsilon)$ is a solution of the Cauchy problem (2.1) and, therefore,

$$X(t, \varepsilon) = e^{A_0 t} + e^{A_0 t} \int\limits_0^t e^{-A_0 \tau} S(\tau, \varepsilon) X(\tau, \varepsilon) d\tau.$$

Differentiating both sides of this equality by ε, we obtain

$$X'_{\varepsilon}(t, \varepsilon) = e^{A_0 t} \int\limits_0^t e^{-A_0 \tau} \Big[S'_{\varepsilon}(\tau, \varepsilon) X(\tau, \varepsilon) + S(\tau, \varepsilon) X'_{\varepsilon}(\tau, \varepsilon) \Big] d\tau. \tag{4.2}$$

Setting here $\varepsilon = 0$ and taking into account Eqs. (1.2), (2.3), and (4.1), we obtain the second equality in (2.4).

To find $X_2(t)$, we differentiate Eq. (4.2) by ε. Then, setting $\varepsilon = 0$ in the equality obtained and taking into account Eqs. (1.2), (2.3), and (4.1), we obtain Eq. (2.6).

The general formula (2.5) can be obtain similarly by using the method of mathematical induction. □

Proof of Theorem 3.1. We prove this theorem in a more general setting. Namely, let H be a real Hilbert space or a finite-dimensional linear space. Let $A : H \to H$ be a linear compact operator having a simple, real, isolated eigenvalue μ_0. Here and below, we say that an eigenvalue μ_0 of the operator A is said to be isolated if there exists a neighborhood of the point μ_0 (on the complex plane) that does not contain points of the spectrum of the operator A distinct from μ_0; in particular, in the case where H is an infinite-dimensional Hilbert space, we obtain $\mu_0 \neq 0$.

Due to the assumption made, there exist nonzero vectors $e, g^* \in H$ satisfying the equalities

$$Ae = \mu_0 e, \quad A^* g^* = \mu_0 g^*; \tag{4.3}$$

here $A^* : H \to H$ is the adjoint operator. The vectors e and g^* are not uniquely defined: if e and g^* satisfy Eqs. (4.3), then the vectors $e_1 = C_1 e$ and $g_1^* = C_2 g^*$ also satisfy Eqs. (4.3) for any constants C_1 and C_2. Therefore, the vectors e and g^* can be normalized, for example, by the equalities $\|e\| = \|g^*\| = 1$. However, we are more interested in the normalization of these vectors by Eqs. (3.1): $\|e\| = 1$ and $(e, g^*) = 1$.

Since μ_0 is an isolated eigenvalue of the operator A, the space H can be represented in the form $H = H_0 \oplus H^0$, where H_0 is a one-dimensional subspace containing the vector e and H^0 is the subspace complement to H_0 and invariant for A.

Theorem 4.1. *For any $u \in H^0$, the equality $(u, g^*) = 0$ holds.*

Proof. We denote by $\sigma(B)$ the spectrum of the linear operator B. Since $\mu_0 \notin \sigma(A : H^0 \to H^0)$, there exists a bounded inverse operator $(A - \mu_0 I)^{-1} : H^0 \to H^0$ (see, e.g., [6]). Therefore, for any $u \in H^0$, the equation $(\mu_0 I - A)v = u$ has a unique solution $v \in H^0$: $v = (\mu_0 I - A)^{-1} u$. This and the obvious equality $((\mu_0 I - A)x, g^*) = 0$, which is valid for any $x \in H$, imply the required relation $(u, g^*) = 0$. □

Theorem 4.2. *The inequality $(e, g^*) \neq 0$ holds.*

Proof. Indeed, assume the contrary, i.e., $(e, g^*) = 0$. Then $g^* \perp H_0$. By Theorem 4.1 we have $g^* \perp H^0$. This and the equality $H = H_0 \oplus H^0$ imply $g^* = 0$. This contradicts the assumption that the vector g^* is nonzero. □

Note that Theorems 4.1 and 4.2 imply that the space H^0 can be defined by the equality

$$H^0 = \{x : \ x \in H, \ (x, g^*) = 0\}.$$

Now Theorem 3.1 follows from Theorem 4.2: first, we normalize the eigenvector e so that $\|e\| = 1$ and then (already having the normalized vector e) normalize the vector g by the condition $(e, g^*) = 1$. □

Proof of Theorem 3.2. By Theorem 2.2, the matrix $V(\varepsilon)$ can be represented in the form

$$V(\varepsilon) = V_0 + \varepsilon V_1 + o(\varepsilon), \tag{4.4}$$

where V_0 and V_1 are defined by Eqs. (2.8) (see the formula (2.7)). It is known in the theory of perturbations of linear operators (see, e.g., [6, 7]) that, since the operator V_0 has a simple eigenvalue 1, the operator (4.4) has a continuous branch of simple eigenvalues $\mu(\varepsilon)$, $\mu(0) = 1$. Moreover, if the corresponding eigenvectors e and g^* are chosen according to Eqs. (3.1), then the function $\mu(\varepsilon)$ can be represented in the form (3.2), where the coefficient μ_1 is defined by the equality $\mu_1 = (V_1 e, g^*)$.

To complete the proof of Theorem 3.2, it remains to show that the number $\mu_1 = (V_1 e, g^*)$ coincides with the number (3.3). Indeed, due to (2.8) we have

$$(V_1 e, g^*) = \left(e^{A_0 T} \int_0^T e^{-A_0 \tau} S_1(\tau) e^{A_0 \tau} d\tau \, e, \ g^* \right) = \int_0^T \Big(S_1(\tau) e, \ g^* \Big) d\tau,$$

which is what was required; here we take into account the equalities $e^{At} e = e$ and $e^{A^* t} g^* = g^*$, which are valid for all t. □

Proof of Theorem 3.3. Similarly to Theorem 3.1, we prove Theorem 3.3 in a more general setting. For definiteness, we prove Theorem 3.3 for the case 3° considered in Theorem 3.5; the proof for the case 2° considered in Theorem 3.4 is similar.

Let H be a real Hilbert space or a finite-dimensional linear space. Let $A : H \to H$ be a linear compact operator possessing a semisimple real eigenvalue μ_0 of multiplicity 2. Then there exist nonzero vectors $e, g, e^*, g^* \in H$ such that the following equalities hold:

$$Ae = \mu_0 e, \quad Ag = \mu_0 g, \quad A^* e^* = \mu_0 e^*, \quad A^* g^* = \mu_0 g^*; \tag{4.5}$$

moreover, the vectors e and g (respectively, the vectors e^* and g^*) are linearly independent. The vector e, g, e^*, g^* are not uniquely defined.

We prove a more strong assertion than Theorem 3.3. Namely, we prove that the vectors e, g, e^*, and g^* can be normalized according to the equalities

$$\|e\| = \|g\| = 1, \tag{4.6}$$

$$(e, e^*) = (g, g^*) = 1, \quad (e, g^*) = (g, e^*) = 0. \tag{4.7}$$

In other words, we prove that, in addition to the normalization (3.5) described in Theorem 3.3, one can ensure the normalization (4.6).

Below, we use the following two auxiliary assertions, which can be proved by a direct calculation.

Let e, g, e^*, and g^* be eigenvectors of the matrices A_0 and A_0^* satisfying the equalities (4.5).

Theorem 4.3. *The vectors e, g, e^*, and g^* satisfy the relations*

$$(e, e^*) = (g, g^*), \quad (e, g^*) = -(g, e^*), \quad (e, e^*)^2 + (e, g^*)^2 > 0.$$

For arbitrary real numbers a, b, α, and β, we set

$$e_1 = ae + bg, \quad g_1 = ag - be, \quad e_1^* = \alpha e^* + \beta g^*, \quad g_1^* = \alpha g^* - \beta e^*. \tag{4.8}$$

Theorem 4.4. *For any a, b, α, and β, the vectors* (4.8) *satisfy the equalities* (4.5)*, in which one must substitute e_1, g_1, e_1^*, and g_1^* instead of e, g, e^*, and g^*. Conversely, if certain vectors e_1, g_1, e_1^*, and g_1^* satisfy Eqs.* (4.5)*, then these vectors can be represented in the form* (4.8)*.*

First, we give the formulas that ensure Eqs. (4.6). It is natural to assume that

$$\big((e, e) - 1\big)^2 + \big((g, g) - 1\big)^2 > 0.$$

We set

$$e_1 = r\big(e \cos\varphi + g \sin\varphi\big), \quad g_1 = r(g \cos\varphi - e \sin\varphi). \tag{4.9}$$

These vectors coincide with the corresponding vectors in (4.8) for $a = r\cos\varphi$ and $b = r\sin\varphi$ and, therefore, they obey Theorem 4.4. We choose $r > 0$ and φ so that the equalities

$$(e_1, e_1) = (g_1, g_1) = 1 \tag{4.10}$$

hold and thus prove that the vectors e and g can be chosen so that Eqs. (4.6) holds.

The following cases are possible:

S1: $(e, g) = 0$, $(e, e) = (g, g)$;
S2: $(e, g) = 0$, $(e, e) \neq (g, g)$;
S3: $(e, g) \neq 0$.

In the case S1, we set in (4.9) $r = 1/\sqrt{(e, e)}$ and the angle φ can be taken arbitrarily. Therefore, the vectors

$$e_1 = \frac{e\cos\varphi + g\sin\varphi}{\sqrt{(e, e)}}, \quad g_1 = \frac{g\cos\varphi - e\sin\varphi}{\sqrt{(e, e)}}$$

satisfy Eqs. (4.10) for any value of φ.

In the case S2, we set in (4.9)

$$r = \sqrt{\frac{2}{(e, e) + (g, g)}} \tag{4.11}$$

and $\varphi = \pi/4 + n\pi/2$, where n is an integer number.

In the case S3, we choose r as in (4.11) and

$$\varphi = \frac{1}{2}\arctan\frac{(g,g)-(e,e)}{2(e,g)} + \frac{n\pi}{2},$$

where n is an integer number.

Assume that in Eqs. (4.9) the values r and φ have been chosen so that the vectors e_1 and g_1 satisfy Eqs. (4.10). Now we choose the vectors e_1^* and g_1^* according to Eqs. (4.8) so that Eqs. (4.7) hold:

$$(e_1, e_1^*) = (g_1, g_1^*) = 1, \quad (e_1, g_1^*) = (g_1, e_1^*) = 0. \tag{4.12}$$

Note that due to Theorem 4.3, the validity of these four equalities is guaranteed by the following two relations:

$$(e_1, e_1^*) = 1, \quad (e_1, g_1^*) = 0.$$

We set

$$\alpha = \frac{(e,e^*)\cos\varphi - (e,g^*)\sin\varphi}{rC_0}, \quad \beta = \frac{(e,g^*)\cos\varphi + (e,e^*)\sin\varphi}{rC_0},$$

where $C_0 = (e,e^*)^2 + (e,g^*)^2 > 0$ (see Theorem 4.3). Then the vectors $e_1^* = \alpha e^* + \beta g^*$ and $g_1^* = \alpha g^* - \beta e^*$ satisfy Eqs. (4.12). □

Proof of Theorem 3.9. Consider the auxiliary linear system

$$\frac{dx}{dt} = A(\varepsilon)x, \quad x \in \mathbb{R}^N, \tag{4.13}$$

where $A(\varepsilon)$ is a real-valued constant square matrix smoothly (continuously differentiable) depending on the scalar parameter ε. The matrix exponential $X(t,\varepsilon) = e^{A(\varepsilon)t}$ is the fundamental matrix of the system (4.13).

The system (4.13) can be considered as a linear system with periodic in t coefficients, where the period T is an arbitrary positive number. Setting for definiteness $T = 1$, we obtain than the monodromy matrix of this system has the form $V(\varepsilon) = X(1,\varepsilon) = e^{A(\varepsilon)}$; in particular, $V_0 = V(0) = e^{A(0)}$. Theorem 3.9 will be proved if we verify that

$$V_\varepsilon'(\varepsilon)|_{\varepsilon=0} = \left(e^{A(\varepsilon)}\right)_\varepsilon'\Big|_{\varepsilon=0} = e^{A_0}\int\limits_0^1 e^{-A_0\tau}A'(0)e^{A_0\tau}\,d\tau. \tag{4.14}$$

Setting $A_0 = A(0)$, we represent the system (4.13) in the form

$$\frac{dx}{dt} = \Big[A_0 + \big(A(\varepsilon) - A_0\big)\Big]x, \quad x \in \mathbb{R}^N, \tag{4.15}$$

i.e., in the form of the system (1.1) with $S(t,\varepsilon) = A(\varepsilon) - A_0$; then, in particular, from the formula (2.3) we obtain the equality

$$S_1(t) = S_\varepsilon'(t,0) = A_\varepsilon'(0).$$

This and the formula (2.8) imply Eq. (4.14) for the system (4.15). □

5. Conclusion. In this paper, we consider systems of linear differential equations with periodic coefficients depending on a small parameter under the assumption that the unperturbed system is autonomous. We propose formulas for constructing the fundamental matrix and the monodromy matrix of the system in the form of an expansion in powers of the small parameter. We obtain new effective formulas for calculating the coefficients of the corresponding expansions and new formulas for calculating multipliers of the linear system considered. We discuss some applications in the theory of perturbations of linear operators, the stability of linear differential equations with periodic coefficients, the constructing stability domains for linear dynamical systems, and others.

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