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SCHLESINGER TRANSFORMATIONS FOR ALGEBRAIC PAINLEVÉ VI SOLUTIONS

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Schlesinger (S) transformations can be combined with a direct rational (R) pull-back of a hypergeometric 2×2 system of ODEs to obtain RS_4^2-pullback transformations to isomonodromic 2×2 Fuchsian systems with 4 singularities. The corresponding Painlevé VI solutions are algebraic functions, possibly in different orbits under Okamoto transformations. The paper demonstrates direct computations (involving polynomial syzygies) of Schlesinger transformations that affect several singular points at once, and presents an algebraic procedure of computing algebraic Painlevé VI solutions without deriving full RS-pullback transformations. Bibliography: 33 titles.

1. Introduction

Given a matrix differential system

$$\frac{d}{dz}\,\Psi(z) = M(z)\Psi(z), \tag{1.1}$$

one often considers its *Schlesinger transformations* [16, 18, 25]. Thereby the local monodromy difference at any z-point is shifted by an integer, and the total shift must be an even integer. Usually only *elementary Schlesinger transformations* [16, 20] are considered explicitly, whereby local monodromy differences at 2 points are shifted by ± 1. General Schlesinger transformations are implicitly handled as composite sequences of elementary Schlesinger transformations. The present paper demonstrates explicit computation of the Schlesinger transformations in a bulk ansatz, without resorting to a chain of elementary transformations. Polynomial syzygies [10] play a prominent role in the bulk computations.

It is convenient to include the Schlesinger transformations in a general definition of a pull-back transformation of differential systems (1.1). Accordingly, a general pull-back transformation has the form

$$z \mapsto R(x), \qquad \Psi(z) \mapsto S(x)\,\Psi(R(x)), \tag{1.2}$$

where $R(x)$ is a rational function of x, and $S(x)$ is (up to a radical normalizing factor) a linear transformation of solution vectors with rational coefficients. The transformed equation is

$$\frac{d\Psi(x)}{dx} = \left(\frac{dR(x)}{dx}\,S^{-1}(x)M(R(x))S(x) - S^{-1}(x)\frac{dS(x)}{dx}\right)\Psi(x). \tag{1.3}$$

The transformation by $S(x)$ is a Schlesinger transformation affecting local behavior at x-points. It is analogous here to the *projective equivalence* transformations $y(x) \to \theta(x)y(x)$ of ordinary differential equations. If $S(x)$ is the identity map, we have a *direct pull-back* of a differential system. The Schlesinger transformations can be designed [18, (16)] to remove apparent singularities of the direct pull-back with respect to $R(x)$.

Pull-back transformations (1.2) change Fuchsian equations to Fuchsian equations. In [19, 20], they are called *RS-transformations* in the context of isomonodromic Fuchsian systems corresponding to algebraic solutions of the sixth Painlevé equation. To merge terminology, we refer to these pull-back transformations as *RS-pullbacks*, or *RS-pullback transformations*.

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Thereby we emphasize the composition of a rational change of independent variable $z \mapsto R(x)$ and the Schlesinger transformation $S(x)$.

The subject of the present paper is a construction of Schlesinger S-transformations for the RS-pullback transformations of 2×2 matrix hypergeometric equations to isomonodromic 2×2 Fuchsian systems with 4 singular points. The corresponding solutions [16] of the sixth Painlevé equation are algebraic functions, since they are determined algebraically by matrix entries of pulled-back equations (1.3) while those entries are algebraic functions in x and the isomonodromy parameter. Algebraic Painlevé VI solutions have applications to Frobenius manifolds [8], WDVV equations, and surface singularities [17]. Characterizing all algebraic Painlevé VI solutions was an active field of research recently [2–4, 9, 20]. The final classification was achieved by Lisovyy and Tykhyy in [22].

The second author conjectured in [19, 20] that all algebraic solutions of the sixth Painlevé equation can be obtained by RS-transformations of matrix hypergeometric equations, up to Okamoto transformations [27]. The classification in [22] validates this conjecture, as explained in [32, Sec. 4.1]. Particularly, quadratic transformations [29] of Painlevé VI solutions can be expressed as RS-transformations of the corresponding Fuchsian systems [18]. Isomonodromic Fuchsian systems with finite (say, icosahedral [3]) monodromy group are always pull-backs of a standard hypergeometric equation with the same monodromy group, as asserted by celebrated Klein's theorem [21]. R. Fuchs [12] soon considered extension of Klein's theorem to algebraic solutions of the Painlevé equations. In particular [26], algebraic solutions of the Painlevé equations from the first to the fifth indeed arise from pull-back transformations of confluent hypergeometric equations. The pull-back method for computing algebraic Painlevé VI solutions is considered in [2, 19, 20], and in the context of the Picard-Fuchs equations [7, 24] where only the direct pull-backs are considered.

The paper is organized as follows. Section 2 presents two parametric rational functions $R(x)$ of considered pull-back transformations (1.2). They are *almost Belyi coverings*, that is, one-dimensional families of finite coverings of $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ with only one simple ramification point. Just as the *Belyi coverings* (i.e., those unramified above $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$) are typical for pull-back transformations between hypergeometric equations [1, 28], the almost Belyi coverings are characteristically suitable for RS-pullbacks from hypergeometric equations to isomonodromic Fuchsian systems [2, 20, 30–32]. We use two coverings of degree 8 and 12 that were previously used in [20]; our full RS-transformations are already implied there. In Theorem 2.1, we recall a method [20] to obtain an algebraic Painlevé VI solution directly from an almost Belyi map.

Section 3 demonstrates two examples of full RS-transformations, both with respect to the degree 8 covering. The Schlesinger transformations $S(x)$ are constructed at once, instead of composing elementary Schlesinger transformations as was done in [1, 2, 16]. Thereby we avoid unnecessary factorization of high degree polynomials when shifting local monodromy differences at conjugate roots by the same integer. A similar approach is adopted in [11, 3F(iii)] for construction of rational solutions of the Painlevé II equation. The degree 8 pull-back transformation is applied to matrix hypergeometric equations with icosahedral monodromy group. Consequently, we recompute the icosahedral Painlevé VI solutions of Boalch [3] types 26, 27 (or solutions 14, 13 in [22], respectively).

In Sec. 4, we formulate basic algebraic facts useful in computations of RS-pullback transformations. Polynomial syzygies [10] are extensively used. Section 5 gives a general formula for some algebraic Painlevé VI solutions, with minimum information from full RS-transformations.

Section 6 presents full RS-transformations with respect to the degree 12 covering. An important demonstration is that the same rational covering $R(x)$ can be used in several RS-transformations, starting from various matrix hypergeometric equations with different monodromy groups. Using the same degree 12 covering, we obtain icosahedral solutions of Boalch [3] types 31, 32 (or solutions 17, 16 in [22], respectively), and customarily unrelated octahedral and Hitchin [15] solutions. Similarly, [31] demonstrates the use of the same degree 10 covering to pull-back 3 hypergeometric equations (with local monodromy differences 1/2, 1/3, $k/7$ for $k = 1, 2$ or 3) to obtain solutions 33, 34, 32 in [22].

The Appendix presents the Jimbo–Miwa [16] correspondence between the solutions of the sixth Painlevé equation and isomonodromic Fuchsian 2×2 systems, and the matrix hypergeometric equation. In particular, the Appendix introduces the notations

$$P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t) \quad \text{and} \quad E(\nu_0, \nu_1, \nu_t, \nu_\infty; y(t); z)$$

for the Painlevé VI equation and corresponding isomonodromic Fuchsian systems.

2. Almost Belyi coverings

Following [20], we introduce notation for ramification patterns and RS-transformations. A ramification pattern for an almost Belyi covering of degree n is denoted by $R_4(P_1|\,P_2|\,P_3)$, where P_1, P_2, P_3 are three partitions of n, specifying the ramification orders above three points. The ramification pattern above the fourth ramification locus is assumed to be $2+1+1+\ldots+1$. By *the extra ramification point*, we refer to the simple ramification point in the fourth fiber. The Hurwitz space for such a ramification pattern is generally one-dimensional [33, Proposition 3.1].

We consider almost Belyi coverings of genus 0 only, and write them as $\mathbb{P}^1_x \to \mathbb{P}^1_z$, meaning that the projective line with the projective coordinate x is mapped to the projective line with coordinate z. Then the total number of parts in P_1, P_2, P_3 must be equal to $n+3$, according to [20, Proposition 2.1]; this is a consequence of the Riemann–Hurwitz formula.

We use almost Belyi coverings with the following ramification patterns:

$$R_4\big(5+1+1+1 \mid 2+2+2+2 \mid 3+3+2\big), \tag{2.1}$$

$$R_4\big(3+3+3+3 \mid 2+2+2+2+2+2 \mid 5+4+1+1+1\big). \tag{2.2}$$

The degrees of the coverings are 8 and 12, respectively. For each covering type, three specified fibers with ramified points can be brought to any three distinct locations by a fractional-linear transformation of $\mathbb{P}^1_z$. We assign the first and the next two partitions to $z = 0$, $z = 1$, and $z = \infty$, respectively. Similarly, by a fractional-linear transformation of $\mathbb{P}^1_x$, we may choose any three x-points[1] as $x = 0$, $x = 1$, and $x = \infty$.

For direct applications to the Painlevé VI equation, it is required to normalize the point above $z = \infty$ with the deviating ramification order 2, 4 (respectively) and three nonramified points above $\{0, 1, \infty\} \subset \mathbb{P}^1_z$ as $x = 0$, $x = 1$, $x = \infty$, and $x = t$. We refer to explicit almost Belyi coverings normalized this way as *properly normalized*.

The properly normalized coverings with ramification patterns (2.1)–(2.2) were first computed in [20]. In computation of RS-transformations, compact expressions for nonnormalized coverings are more convenient to use. The coverings can be computed on modern computers either using the most straightforward method with undetermined coefficients, or an improved method [30] that uses differentiation. Here we present only explicit expressions for the almost Belyi coverings.

[1] Strictly speaking, the x-points in our settings are curves or branches, parametrized by an isomonodromy parameter t or other parameter, since the Hurwitz spaces for the almost Belyi maps are one-dimensional. For simplicity, we ignore the dimensions introduced by such parameters, and consider a one-dimensional Hurwitz space as a generic point.

The degree 8 covering is

$$\varphi_8(x) = \frac{(s+1)^2\, x^5\, \big(9(s+1)^2x^3 - 24s(s+3)x^2 + 8s(11s-1)x + 48s^2\big)}{64\, s\, \left(x^2 - 2sx - s\right)^3}. \tag{2.3}$$

The Hurwitz space is realized here by a projective line with projective parameter s. (In the pulled-back Fuchsian equations, s is the isomonodromy parameter.) One can check that

$$\varphi_8(x) - 1 = \frac{\big(3(s+1)^2x^4 - 4s(s+3)x^3 + 12s(s-1)x^2 + 24s^2x + 8s^2\big)^2}{64\, s\, \left(x^2 - 2sx - s\right)^3}. \tag{2.4}$$

It is evident that the ramification pattern is indeed (2.1). The extra ramification point is $x = 5s$. To get a properly normalized expression, the degree 3 polynomial in the numerator of $\varphi_8(x)$ has to be factorized. We reparametrize

$$s = \frac{2(u-1)}{u^3 + 4u^2 + 2u + 2} \tag{2.5}$$

and make the fractional-linear transformation

$$x \mapsto \frac{2(u+8)w}{3(u+2)^4}(2x-1) - \frac{2(u-1)(u^2-4u-24)}{3(u+2)^4}, \tag{2.6}$$

where $w = \sqrt{u(u-1)(u+3)(u+8)}$. Apparently, the Hurwitz space parametrizing the properly normalized almost Belyi covering has genus 1. We obtain the following properly normalized expression:

$$\widehat{\varphi}_8(x) = \frac{u^5(u+8)^3(u+3)}{8(u^3+4u^2+2u+2)} \frac{x(x-1)(x-t_8)\left(x - \frac{1}{2} - \frac{(u-1)(u^2-4u-24)}{2w(u+8)}\right)^5}{\left(x^2 - (L_1w+1)x + \frac{1}{2}L_1w - L_2\right)^3}, \tag{2.7}$$

where

$$t_8 = \frac{1}{2} + \frac{(u-1)(3u^4 + 12u^3 + 24u^2 + 64u + 32)}{2u^2(u+8)\sqrt{u(u-1)(u+3)(u+8)}}, \tag{2.8}$$

and

$$L_1 = \tfrac{(u-1)(u+4)(u^2-10)}{(u+3)(u+8)^2(u^3+4u^2+2u+2)}, \qquad L_2 = \tfrac{5u^6+40u^5+20u^4-320u^3-40u^2+1216u-192}{8(u+3)(u+8)^2(u^3+4u^2+2u+2)}.$$

To get the degree 8 covering in [20, pp. 11–12], one has to make the substitutions $u \mapsto -8(s+1)^2/(s^2-34s+1)$ or $u \mapsto (8s_1+1)/(1-s_1)$. After the first substitution, the quadratic polynomial in the denominator of (2.7) factors as well.

The degree 12 covering is given by

$$\varphi_{12}(x) = \frac{4}{27(s+4)^3}\frac{F_{12}^3}{x^5G_{12}}, \quad \text{or} \quad \varphi_{12}(x) - 1 = \frac{1}{27(s+4)^3}\frac{P_{12}^2}{x^5G_{12}}, \tag{2.9}$$

where

$$\begin{aligned} F_{12} &= x^4 - 4(s+3)x^3 + (s^2+6s+14)x^2 + 2(s+6)x + 1, \\ G_{12} &= sx^3 - 4(s^2+3s-1)x^2 - 4(2s+11)x - 4, \\ P_{12} &= 2x^6 - 12(s+3)x^5 + 15(s^2+6s+10)x^4 \\ &\qquad + 2s(s^2+9s+15)x^3 + 6(s^2+9s+25)x^2 + 6(s+6)x + 2. \end{aligned} \tag{2.10}$$

The extra ramification point is $x = -5/s$. To get a properly normalized expression, we reparametrize

$$s = \frac{(u^2-5)(u^2+4u-1)(u^2-4u-1)}{8(u+1)^2(u-1)^2}, \tag{2.11}$$

and make the fractional-linear transformation

$$x \mapsto \frac{(u+1)^2(u-1)^2}{2(u^2-5)} - \frac{(u+1)^3(u-3)^3(u^2+3)\,x}{2(u-1)^2(u^2-5)(u^2-4u-1)}. \tag{2.12}$$

The obtained expression is

$$\widehat{\varphi}_{12}(x) = \frac{1024(u+1)^{20}(u-3)^{12}\left(x^4 - \frac{(u^2-4u-1)(3u^6-21u^4+49u^2+33)}{(u+1)^5(u-3)^3}x^3 + L_6\right)^3}{27(u^2+3)^5(u^2-5)^5(u^2+4u-1)(u^2-4u-1)^5\, x(x-1)\,(x-t_{12})\,(x-t_{12}^*)^5},$$

where

$$t_{12} = \frac{(u-1)^5(u+3)^3(u^2-4u-1)}{(u+1)^5(u-3)^3(u^2+4u-1)}, \qquad t_{12}^* = \frac{(u-1)^4(u^2-4u-1)}{(u+1)(u-3)^3(u^2+3)}, \tag{2.13}$$

and

$$\begin{aligned} L_6 = {} & \frac{(u^2-4u-1)^2(49u^{12}-686u^{10}+3895u^8-9700u^6+10575u^4-2446u^2+2409)}{16(u+1)^{10}(u-3)^6}x^2 \\ & - \frac{(u-1)^5(u^2-4u-1)^3(9u^8-144u^6+874u^4-2184u^2+2469)}{8(u+1)^{10}(u-3)^9}x \\ & + \frac{(u-1)^{10}(u+3)^2(u^2-4u-1)^4}{16(u+1)^{10}(u-3)^{10}}. \end{aligned}$$

The Hurwitz space parametrizing this properly normalized almost Belyi covering has still genus 0. To get the degree 12 covering in [20], one has to consider $1/\widehat{\varphi}_{12}(x)$, and substitute $u \mapsto (s-3)/(s+1)$.

In [20], the following symbol is introduced to denote the RS-pullback transformations of $E(e_0, e_1, 0, e_\infty; t; z)$ with respect to a covering with ramification pattern $R_4(P_0|P_1|P_\infty)$:

$$RS_4^2\left(\begin{array}{c|c|c} e_0 & e_1 & e_\infty \\ P_0 & P_1 & P_\infty \end{array}\right), \tag{2.14}$$

where the subscripts 2 and 4 indicate a second order Fuchsian system with 4 singular points after the pull-back. We assume the same assignment of the fibers $z=0$, $z=1$, $z=\infty$ as for the R_4-notation. Location of the x-branches $0, 1, t, \infty$ does not have to be normalized. In Sec. 3, we present explicit computations for

$$RS_4^2\left(\begin{array}{c|c|c} 1/5 & 1/2 & 1/3 \\ 5+1+1+1 & 2+2+2+2 & 3+3+2 \end{array}\right)$$

and

$$RS_4^2\left(\begin{array}{c|c|c} 2/5 & 1/2 & 1/3 \\ 5+1+1+1 & 2+2+2+2 & 3+3+2 \end{array}\right).$$

These RS-pullbacks produce algebraic solutions of $P_{VI}(1/5, 1/5, 1/5, \pm 1/3; t)$, respectively, $P_{VI}(2/5, 2/5, 2/5, \pm 2/3; t)$.

As was noticed in [20] and [7], some algebraic Painlevé VI solutions determined by the RS-pullback transformations $RS_4^2\left(\begin{array}{c|c|c} 1/k_0 & 1/k_1 & 1/k_\infty \\ P_0 & P_1 & P_\infty \end{array}\right)$ with $k_0, k_1, k_\infty \in \mathbb{Z}$, can be calculated from the rational covering alone, without computing any Schlesinger transformation. Here is a general formulation of this situation.

Theorem 2.1. *Let k_0, k_1, k_∞ denote three integers, all ≥ 2. Let $\varphi\colon \mathbb{P}_x^1 \to \mathbb{P}_z^1$ denote an almost Belyi map depending on a parameter t. Suppose that the following conditions are satisfied.*

(i) *The covering $z = \varphi(x)$ is ramified above the points $z=0$, $z=1$, $z=\infty$; there is one simply ramified point $x=y$ above $\mathbb{P}_z^1 \setminus \{0,1,\infty\}$, and there are no other ramified points.*

(ii) *The points* $x=0$, $x=1$, $x=\infty$, $x=t$ *lie above the set* $\{0,1,\infty\}\subset\mathbb{P}^1_z$.

(iii) *The points in* $\varphi^{-1}(0)\setminus\{0,1,t,\infty\}$ *are all ramified of order* k_0. *The points in* $\varphi^{-1}(1)\setminus\{0,1,t,\infty\}$ *are all ramified of order* k_1. *The points in* $\varphi^{-1}(\infty)\setminus\{0,1,t,\infty\}$ *are all ramified of order* k_∞.

Let a_0,a_1,a_t,a_∞ *denote the ramification orders at* $x=0,1,t,\infty$, *respectively. Then the point* $x=y$, *as a function of* $x=t$, *is an algebraic solution of*

$$P_{VI}\left(\frac{a_0}{k_{\varphi(0)}},\frac{a_1}{k_{\varphi(1)}},\frac{a_t}{k_{\varphi(t)}},1-\frac{a_\infty}{k_{\varphi(\infty)}};t\right). \tag{2.15}$$

Proof. See [31, Theorem 3.1]. □

Our two coverings $\widehat{\varphi}_8(x)$ and $\widehat{\varphi}_{12}(x)$ immediately give solutions of $P_{VI}(1/5,1/5,1/5,1/3;t)$ and $P_{VI}(1/5,1/5,1/5,1/5;t)$, respectively. To parametrize the algebraic solutions, it is convenient to parametrize the indeterminant t as, respectively, t_8 in (2.8) or t_{12} in (2.13).

Direct application of Theorem 2.1 to $\widehat{\varphi}_8(x)$ gives the following[2] solution $y_{26}(t_8)$ of $P_{VI}(1/5,1/5,1/5,1/3;t_8)$:

$$y_{26} \;=\; \frac{1}{2}+\frac{(u-1)(u+3)(u^3+4u^2+14u+8)}{2(u^3+4u^2+2u+2)\sqrt{u(u-1)(u+3)(u+8)}}. \tag{2.16}$$

Note that t_8 is the nonramified point of $\widehat{\varphi}_8(x)$ above $z=0$ not equal to $x=0$ or $x=1$, while y_{26} is the extra ramification x-point of $\widehat{\varphi}_8(x)$; it corresponds to the point $x=5s$ in the expression (2.3) of $\varphi_8(x)$. To get the parametrizations in [20], one has to substitute $u\mapsto -8(s+1)^2/(s^2-34s+1)$ or $u\mapsto(8s_1+1)/(1-s_1)$. In Sec. 3, we derive the same algebraic solution by computing the full transformation $RS^2_4\left(\begin{smallmatrix}1/5\\5+1+1+1\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/3\\3+3+2\end{smallmatrix}\right)$.

Similarly, application of Theorem 2.1 to $\widehat{\varphi}_{12}(x)$ gives the following solution $y_{31}(t_{12})$ of $P_{VI}(1/5,1/5,1/5,1/5;t_{12})$:

$$y_{31}=\frac{(u-1)^4(u+3)^2}{(u-3)(u+1)(u^2+3)(u^2+4u+1)}. \tag{2.17}$$

To get the parametrization in [20], one has to substitute $u\mapsto(s-3)/(s+1)$. The implied RS-transformation is $RS^2_4\left(\begin{smallmatrix}1/3\\3+3+3+3\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/5\\5+4+1+1+1\end{smallmatrix}\right)$. As Sec. 6 demonstrates, Theorem 2.1 can be applied to an alternative normalization of $\varphi_{12}(x)$ giving a solution of $P_{VI}(1/4,1/4,1/4,-1/4;t)$.

We note that the genus of algebraic Painlevé VI solutions is not a monotonic function of the minimal genus of the Hurwitz spaces parametrizing the pull-back covering: the degree 8 covering $\widehat{\varphi}_8(x)$ gives a genus 1 solution, while the degree 12 covering $\widehat{\varphi}_{12}(x)$ gives a genus 0 solution. The covering $\varphi_8(x)$ is still parametrized by a projective line, even if its normalization $\widehat{\varphi}_8(x)$ gives an algebraic Painlevé VI solution of genus 1.

3. Computation of Schlesinger transformations

This section starts with construction of $RS^2_4\left(\begin{smallmatrix}1/5\\5+1+1+1\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/3\\3+3+2\end{smallmatrix}\right)$, demonstrating the construction of the S-part of full RS-pullbacks as a single Schlesinger transformation. This gives us the same Painlevé VI solution $y_{26}(t_8)$ as dictated by Theorem 2.1. From the full RS-transformations, we easily derive a solution of $P_{VI}(1/5,1/5,1/5,-1/3;t_8)$ as well. Then we construct an example of $RS^2_4\left(\begin{smallmatrix}2/5\\5+1+1+1\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/3\\3+3+2\end{smallmatrix}\right)$ and derive solutions of $P_{VI}(2/5,2/5,2/5,2/3;t_8)$ and $P_{VI}(2/5,2/5,2/5,-2/3;t_8)$ of Boalch type 27.

[2] Throughout this paper, the indices 26, 27, 31, 32 refer to the Boalch types [3] of icosahedral Painlevé VI solutions.

Application of Appendix formulas (7.6)–(7.8) to the equation $E(1/5, 1/2,\ 0, 1/3; t; z)$ yields the following leading terms of dominant local solutions at the singular points, up to multiplication by constants:

$$u_0 = \begin{pmatrix} 11 \\ 1 \end{pmatrix} z^{-\frac{1}{10}}, \qquad u_1 = \begin{pmatrix} 11 \\ -19 \end{pmatrix} (1-z)^{-\frac{1}{4}}, \qquad u_\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{\frac{1}{6}}. \tag{3.1}$$

Let $f_1(z), f_2(z)$ denote the normalized basis for solutions of $E(1/5, 1/2, 0,\ 1/3; t; z)$. We have $f_1(z) \sim \binom{1}{0} z^{1/6}$ and $f_2(z) \sim \binom{0}{1} z^{-1/6}$, as $z \to \infty$. Up to scalar multiples, explicit expressions for these solutions can be copied from (7.9)–(7.10).

The Fuchsian system for the equation $P_{VI}(1/5, 1/5, 1/5, 1/3; t)$ must be an RS-pullback $RS_4^2 \left(\begin{smallmatrix} 1/5 \\ 5+1+1+1 \end{smallmatrix} \middle| \begin{smallmatrix} 1/2 \\ 2+2+2+2 \end{smallmatrix} \middle| \begin{smallmatrix} 1/3 \\ 3+3+2 \end{smallmatrix} \right)$ with respect to the covering $z = \widehat{\varphi}_8(x)$. It is preferable to work with the simpler parametrization $z = \varphi_8(x)$, and apply the fractional-linear transformation (2.6) to switch to $z = \widehat{\varphi}_8(x)$ at the last stage. Let us denote

$$\begin{aligned} F_8 &= 9(s+1)^2 x^3 - 24s(s+3)x^2 + 8s(11s-1)x + 48s^2, \\ P_8 &= 3(s+1)^2 x^4 - 4s(s+3)x^3 + 12s(s-1)x^2 + 24s^2 x + 8s^2, \\ G_8 &= x^2 - 2sx - s, \end{aligned} \tag{3.2}$$

so that, copying (2.3) and (2.4), we have

$$\varphi_8(x) = \frac{(s+1)^2}{64s} \frac{x^5 F_8}{G_8^3}, \qquad \varphi_8(x) - 1 = \frac{1}{64s} \frac{P_8^2}{G_8^3}. \tag{3.3}$$

The direct pull-back of $E(1/5, 1/2, 0, 1/3; t, z)$ with respect to $\varphi_8(x)$ is a Fuchsian system with singularities at $x = \infty$ and the roots of $F_8(x)$, and apparent singularities at $x = 0$ and the roots of $G_8(x)$, $P_8(x)$. In particular, the local monodromy exponents at $x = \infty$ are $\pm 1/3$, twice the exponents at $z = \infty$. We have to remove apparent singularities, and choose a solution basis $g_1(x)$, $g_2(x)$ of the pulled-back equation so that, up to constant multiples, $g_1(x) \sim \binom{1}{0} x^{1/10}$ and $g_2(x) \sim \binom{0}{1} x^{-1/10}$. This would allow straightforward normalization[3] of the pulled-back equation for the Jimbo–Miwa correspondence.

Let T_{26} denote the matrix representing the basis $g_1(x)$, $g_2(x)$ in terms of the solution basis $f_1(\varphi_8(x))$, $f_2(\varphi_8(x))$ of the directly pulled-back equation. That is, $\binom{g_1}{g_2} = T_{26} \binom{f_1}{f_2}$. The S-matrix in (1.2)–(1.3) can be taken to be T_{26}^{-1}. It has to shift local exponents at $x = 0$, and the roots of $G_8(x)$ and $P_8(x)$. The local exponents at $x = \infty$ have to be shifted as well, since the shifts of local monodromy differences must add to an even integer. The matrix T_{26} has to satisfy the following conditions.

(i) **Local exponent shifts for general vectors.** For general vectors u, the vector $T_{26} u$ is $O\left(1/\sqrt{x}\right)$ at $x = 0$, $O\left(1/\sqrt{P_8}\right)$ at the roots of $P_8(x)$, $O\left(1/\sqrt{G_8}\right)$ at the roots of $G_8(x)$, and $O(\sqrt{x})$ at infinity.

(ii) **Local exponent shifts for dominant solutions at singular points.** We must have $T_{26} u_0 = O(\sqrt{x})$ at $x = 0$, $T_{26} u_1 = O(\sqrt{P_8})$ at the roots of $P_8(x)$, $T_{26} u_\infty = O(\sqrt{G_8})$ at the roots of $G_8(x)$, and $T_{26} u_\infty = O\left(1/\sqrt{x}\right)$ at infinity.

(iii) **Normalization at infinity.** The positive local monodromy exponent $1/6$ at $z = \infty$ gets transformed to the local monodromy exponent $2 \cdot \frac{1}{6} - \frac{1}{2} < 0$ at $x = \infty$. Hence the dominant solution $\sim \binom{1}{0} z^{1/6}$ should be mapped, up to a constant multiple, to

[3] We may require strict asymptotic behavior for $g_1(x)$, $g_2(x)$ without reference to constant multiples, as in [2] and [20], but this is unnecessary. The Jimbo–Miwa correspondence merely requires existence of a basis with the strict asymptotics. There is no value in controlling strict identification of normalized bases all the way until final fractional-linear normalization, (2.6) in this particular case.

the vanishing solution $\sim \binom{0}{1}x^{-1/6}$, and the vanishing solution $\sim \binom{0}{1}z^{-1/6}$ should be mapped, up to a constant multiple, to the dominant solution $\sim \binom{1}{0}x^{1/6}$.

By the first condition, the matrix T_{26} has the form

$$T_{26} = \frac{1}{\sqrt{x\,G_8 P_8}} \begin{pmatrix} A_{26} & B_{26} \\ C_{26} & D_{26} \end{pmatrix}, \tag{3.4}$$

where the matrix entries A_{26}, B_{26}, C_{26}, D_{26} are polynomials in x of maximal degree 4, with the coefficients being rational functions in s. By the second condition, the expressions $11A_{26}+B_{26}$ and $11C_{26}+D_{26}$ vanish at $x=0$, $11A_{26}-19B_{26}$ and $11C_{26}-19D_{26}$ are divisible by P_8, and A_{26}, C_{26} are divisible by G_8 and have degree at most 3. Let us define a few polynomials:

$$U_1 = 19\cdot\frac{11A_{26}+B_{26}}{x}, \qquad V_1 = \frac{11A_{26}-19B_{26}}{P_8}, \qquad W_1 = -220\cdot\frac{A_{26}}{G_8},$$
$$U_2 = 19\cdot\frac{11C_{26}+D_{26}}{x}, \qquad V_2 = \frac{11C_{26}-19D_{26}}{P_8}, \qquad W_2 = -220\cdot\frac{C_{26}}{G_8}.$$

Then for $i=1,2$ we have

$$x\,U_i + P_8\,V_i + G_8\,W_i = 0. \tag{3.5}$$

In other words, the two polynomial vectors (U_i, V_i, W_i) are syzygies between the three polynomials x, P_8, G_8. The last condition sets up the degrees for the entries of T_{26},

$$\deg A_{26} \le 2, \quad \deg B_{26} = 4, \quad \deg C_{26} = 3, \quad \deg D_{26} \le 3. \tag{3.6}$$

It turns out that the syzygies giving relations (3.5) of degree at most 4 form a linear space of dimension 3. Here is a basis:

$$(G_8, 0, -x), \qquad (xG_8, 0, -x^2), \qquad (L_1,\ -1,\ -8s)\,, \tag{3.7}$$

where $L_1 = 3(s+1)^2x^3 - 4s(s+3)x^2 + 4s(3s-1)x + 8s^2$. The third syzygy gives the entries A_{26}, B_{26} satisfying (3.6). The first syzygy in (3.7) gives the entries C_{26}, D_{26}. For constructing the transformation matrix T_{26}, we multiply the syzygies (or the rows) by constant factors. Here is a suitable transformation matrix:

$$T_{26} = \frac{1}{\sqrt{x\,G_8 P_8}} \begin{pmatrix} 38s\,G_8 & 55P_8 + 22sG_8 \\ 19\,x\,G_8 & 11\,x\,G_8 \end{pmatrix}. \tag{3.8}$$

Using (1.3) with $S = T_{38}^{-1}$, we routinely compute the transformed differential equation

$$\frac{d\Psi}{dx} = \frac{1}{F_8} \begin{pmatrix} K_1 & -\frac{8}{5}s\,K_2 \\ -\frac{2}{5}(x-5s) & -K_1 \end{pmatrix} \Psi, \tag{3.9}$$

where

$$K_1 = \tfrac{3}{2}(s+1)^2x^2 - 2s(s+3)x - \tfrac{4}{5}s, \qquad K_2 = (15s^2+30s+4)x - 7s(5s+4).$$

We note that the x-root of the lower-left entry of the transformed equation is the extra ramification point of the covering $z = \varphi_8(x)$. We can apply the Jimbo–Miwa correspondence after reparametrization (2.5) and fractional-linear transformation (2.6) of (3.9). Then a solution of $P_{VI}(1/5, 1/5, 1/5,\ 1/3; t)$ is equal to the x-root of the lower-left entry, while the independent variable t is parametrized by the singularity t_8 of the transformed Fuchsian equation. We get the same solution $y_{26}(t_8)$ as in (2.16), (2.8), reaffirming Theorem 2.1 for this case.

The full RS-pullback $RS_4^2\left(\begin{smallmatrix}1/5\\5+1+1+1\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}1/3\\3+3+2\end{smallmatrix}\right)$ can provide much more additional results. For instance, the x-root of upper-right entry of T_{26} determines a solution of $P_{VI}(1/5, 1/5,$

$1/5, -1/3; t_8)$. After applying transformations (2.5)–(2.6) to K_2, the x-root gives the following solution $\widetilde{y}_{26}(t_8)$:

$$\widetilde{y}_{26} = \frac{1}{2} + \frac{(u-1)(u+3)(2u^6+28u^5+106u^4+169u^3+274u^2+142u+8)}{4(u^6+8u^5+35u^4+65u^3+5u^2-22u-11)\sqrt{u(u-1)(u+3)(u+8)}}.$$

Alternatively, this solution can be computed from $y_{26}(t_8)$ by applying a few Okamoto transformations.

Now we consider construction of the RS-pullback

$$RS_4^2\left(\begin{array}{c|c|c} 2/5 & 1/2 & 1/3 \\ 5+1+1+1 & 2+2+2+2 & 3+3+2 \end{array}\right)$$

with respect to $\widehat{\varphi}_8(x)$, aiming for a solution of $P_{VI}(2/5, 2/5, 2/5,\ 2/3; t)$. The leading terms of dominant local solutions of $E(2/5, 1/2, 0, 1/3;\ t; z)$ at the singular points are constant multiples of

$$v_0 = \begin{pmatrix} 17 \\ 7 \end{pmatrix} z^{\frac{1}{5}}, \qquad v_1 = \begin{pmatrix} 17 \\ -13 \end{pmatrix} (1-z)^{-\frac{1}{4}}, \qquad v_\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{-\frac{1}{6}}. \tag{3.10}$$

Again, it is preferable to work first with the simpler covering $z = \varphi_8(x)$. The direct pull-back of $E(2/5, 1/2, 0, 1/3; t, z)$ with respect to $\varphi_8(x)$ is a Fuchsian system with the same singularities as in the previous case, but the local monodromy exponents at $x = 0$ and the roots of $F_8(x)$ are multiplied by 2. Hence we have to shift the local exponent difference at $x = 0$ by 2, and we do not shift the local exponents at $x = \infty$. Let T_{27} denote the transition matrix to a basis of Fuchsian solution normalized at $x = \infty$, analogous to T_{26} above. The matrix T_{27} has to satisfy the following conditions.

(i) **Local exponent shifts for general vectors.** For general vectors u, the vector $T_{27}u$ is $O(1/x)$ at $x = 0$, $O\left(1/\sqrt{P_8}\right)$ at the roots of P_8, $O\left(1/\sqrt{G_8}\right)$ at the roots of G_8, and $O(1)$ at infinity. Hence, the matrix T_{27} has the form

$$T_{27} = \frac{1}{x\sqrt{G_8 P_8}} \begin{pmatrix} A_{27} & B_{27} \\ C_{27} & D_{27} \end{pmatrix}, \tag{3.11}$$

where A_{27}, B_{27}, C_{27}, D_{27} are polynomials in x of maximal degree 4.

(ii) **Local exponent shifts for dominant solutions at singular points.** We must have $T_{27}v_0 = O(x)$ at $x = 0$, $M_{27}v_1 = O(\sqrt{P_8})$ at the roots of $P_8(x)$, and $M_{27}v_\infty = O(\sqrt{G_8})$ at the roots of $G_8(x)$. This means that the following are triples of polynomials in x:

$$\left(13 \cdot \frac{17A_{27}+7B_{27}}{x^2},\ 7 \cdot \frac{17A_{27}-13B_{27}}{P_8},\ -340 \cdot \frac{A_{27}}{G_8}\right),$$
$$\left(13 \cdot \frac{17C_{27}+7D_{27}}{x^2},\ 7 \cdot \frac{17C_{27}-13D_{27}}{P_8},\ -340 \cdot \frac{C_{27}}{G_8}\right),$$

and the polynomial triples are syzygies between x^2, P_8, G_8.

(iii) **Normalization at infinity.** The local exponents at $x = \infty$ are not shifted by the Schlesinger transformation. Hence the dominant solution $\sim \binom{1}{0} z^{1/6}$ is mapped, up to a constant multiple, to the dominant solution $\sim \binom{1}{0} x^{1/3}$, and the vanishing solution $\sim \binom{0}{1} z^{-1/6}$ is mapped, up to a constant multiple, to the vanishing solution $\sim \binom{0}{1} x^{-1/3}$. This sets up the degrees for the entries of T_{27}:

$$\deg A_{27} = 4, \quad \deg B_{27} \le 3, \quad \deg C_{27} \le 3, \quad \deg D_{27} \le 4. \tag{3.12}$$

It turns out that the syzygies relations of degree at most 4 form a linear space of dimension 2. Here is a syzygy basis:

$$\begin{aligned} S_1 &= \left(G_8, 0, -x^2\right), \\ S_2 &= \left((s+1)\left(3(s+1)x^2 - 4sx - 4s\right),\ -1,\ -8s\,(x+1)\right). \end{aligned} \tag{3.13}$$

To determine the entries C_{27}, D_{27}, we take the syzygy S_2. To determine the entries A_{27}, B_{27}, we take the syzygy $60(s+1)^2 S_1 - 7S_2$. Up to multiplication of the two rows by scalar factors, we obtain

$$A_{27} = -\frac{1}{17}\left(15(s+1)^2x^2 - 14sx - 14s\right)G_8, \qquad B_{27} = \tfrac{17}{13}A_{27} + \tfrac{5}{13}P_8,$$
$$C_{27} = \frac{14}{17}\,s\,(x+1)\,G_8, \qquad D_{27} = \tfrac{17}{13}C_{27} + \tfrac{5}{13}P_8.$$

The transformed differential equation is

$$\frac{d\Psi}{dz} = \frac{1}{F_8}\begin{pmatrix} K_3 & -2sK_4 \\ \frac{14}{5}s\left(x + \frac{8(5s-1)}{15(s+1)}\right) & -K_3 \end{pmatrix}\Psi \tag{3.14}$$

where

$$K_3 = 3(s+1)^2x^2 - \frac{2s(25s+54)}{5}x + \frac{8s(150s^2+235s+1)}{75(s+1)},$$
$$K_4 = (15s^2+20s-8)x + \frac{16(75s^2+100s+4)}{75(s+1)}.$$

To get a solution of $P_{VI}(2/5, 2/5, 2/5, 2/3; t)$ by the Jimbo–Miwa correspondence, we have to apply reparametrization (2.5) and fractional-linear transformation (2.6) to the lower-left entry of the differential equation, and write down the x-root. We get the following solution $y_{27}(t_8)$:

$$y_{27} = \frac{1}{2} + \frac{(u+3)(4u^3 - 7u^2 + 4u + 8)}{10u\sqrt{u(u-1)(u+3)(u+8)}}. \tag{3.15}$$

To get the same parametrization of this solution as in [3], one has to substitute $u \mapsto -\frac{6s}{(2s+1)}$.

In the same way, the x-root of upper-right entry $-3sK_4$ determines a solution of the equation $P_{VI}(2/5, 2/5, 2/5, -2/3; t)$. The solution $\widetilde{y}_{27}(t_8)$ is the following:

$$\widetilde{y}_{27} = \frac{1}{2} + \frac{(u+3)(8u^9 + 90u^8 + 216u^7 + 670u^6 + 2098u^5 - 571u^4 - 850u^3 - 7u^2 - 140u - 56)}{50u(2u^6 + 16u^5 + 30u^4 + 10u^3 + 45u^2 + 46u + 13)\sqrt{u(u-1)(u+3)(u+8)}}.$$

4. Syzygies for RS-pullback transformations

As we saw in the previous section, computation of Schlesinger transformations for full RS-pullback transformations leads to computation of syzygies between three polynomials in one variable x. Recently, this syzygy problem got a lot of attention in computational algebraic geometry of rational curves [5,6]. It was successfully considered by Franz Meyer [23] already in 1887. David Hilbert famously extended Meyer's results in [13].

Here are basic facts regarding the homogeneous version of the syzygy problem.

Theorem 4.1. *Let $\mathbb{K}$ be a field and n an integer. Let $P(u,v)$, $Q(u,v)$, $R(u,v)$ be homogeneous polynomials in $\mathbb{K}[u,v]$ of degree n. We assume that these polynomials have no common factors. Let Z be the graded $\mathbb{K}[u,v]$-module of syzygies between $P(u,v)$, $Q(u,v)$, $R(u,v)$.*

Then the module Z is free of rank 2. If (p_1,q_1,r_1), (p_2,q_2,r_2) is a homogeneous basis for Z, then

$$\deg(p_1,q_1,r_1) + \deg(p_2,q_2,r_2) = n, \tag{4.1}$$

and the polynomial vector (P, Q, R) *is a* $\mathbb{K}$*-multiple of*

$$(q_1r_2 - q_2r_1,\ p_2r_1 - p_1r_2,\ p_1q_2 - p_2q_1)\,. \tag{4.2}$$

Proof. See [6], or even [23]. The form (4.2) is a special case of the Hilbert–Burch theorem [10, Theorem 3.2]. □

In our situation, $\mathbb{K}$ is a function field on a Hurwitz curve. For our applications, $\mathbb{K} = \mathbb{C}(s)$. But we rather consider syzygies between univariate nonhomogeneous polynomials. Here are the facts we use.

Theorem 4.2. *Suppose that* $P(x)$, $Q(x)$, $R(x)$ *are polynomials in* $\mathbb{K}[x]$ *without common factors. Let* Z *be the* $\mathbb{K}[x]$*-module of syzygies between* $P(x)$, $Q(x)$, $R(x)$*. Then*

(i) *the module* Z *is free of rank* 2*,*
(ii) *for any two syzygies* (p_1, q_1, r_1), (p_2, q_2, r_2)*, expression* (4.2) *is a* $\mathbb{K}[x]$*-multiple of* (P, Q, R)*,*
(iii) *there exist syzygies* $(p_1, q_1, r_1), (p_2, q_2, r_2)$*, such that expression* (4.2) *equals* (P,Q,R)*,*
(iv) *two syzygies* (p_1, q_1, r_1), (p_2, q_2, r_2) *form a basis for* Z *if and only if expression* (4.2) *is a nonzero* $\mathbb{K}$*-multiple of* (P, Q, R)*,*
(v) *if the syzygies* (p_1, q_1, r_1), (p_2, q_2, r_2) *form a basis for* Z*, and* α_1, β_1, γ_1, α_2, β_2, $\gamma_2 \in \mathbb{K}$*, then*

$$\det\begin{pmatrix} \alpha_1p_1P + \beta_1q_1Q + \gamma_1r_1R & \alpha_2p_1P + \beta_2q_1Q + \gamma_2r_1R \\ \alpha_1p_2P + \beta_1q_2Q + \gamma_2r_1R & \alpha_2p_2P + \beta_2q_2Q + \gamma_2r_2R \end{pmatrix} \tag{4.3}$$

is a $\mathbb{K}$*-multiple of* $P\,Q\,R$*.*

Proof. The module is free, because $\mathbb{K}[x]$ is a principal ideal domain. The rank is determined by the exact sequence $0 \to Z \to K^3 \to K \to 0$ of free $\mathbb{K}$-modules, where the map $K^3 \to K$ is defined by $(p, q, r) \mapsto pP + qQ + rR$.

Statement (ii) holds, because either expression (4.2) is the zero vector, or the $\mathbb{K}[x]$-module of simultaneous syzygies between the two triples p_1, q_1, r_1 and p_2, q_2, r_2 is free of rank 1. The triple (P, Q, R) is a generator of this module (since P, Q, R have no common factors), while triple (4.2) belongs to the module.

For statement (iii), let $D = \gcd(P, Q)$, $\widetilde{P} = P/D$, and $\widetilde{Q} = Q/D$. Then $\widetilde{P} = P/D$ and $\widetilde{Q} = Q/D$ are coprime, and there exist polynomials A, B such that $R = A\widetilde{P} + B\widetilde{Q}$. Then $(\widetilde{Q}, -\widetilde{P}, 0)$ and $(A, B, -D)$ are the two required syzygies.

Now assume that (4.2) is a nonzero scalar multiple of (P, Q, R). Let $u_1 = (p_1, q_1, r_1)$ and $u_2 = (p_2, q_2, r_2)$. If $u_3 = (p_3, q_3, r_3)$ is a syzygy in Z, then

$$\det\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix} = 0, \tag{4.4}$$

because the syzygy condition gives a $\mathbb{K}(x)$-linear relation between the rows. A $\mathbb{K}[x]$-linear relation between the 3 syzygies is determined by the minors, say,

$$(p_2q_3 - p_3q_2)u_1 + (p_3q_1 - p_1q_3)u_2 + (p_1q_2 - p_2q_1)u_3 = 0. \tag{4.5}$$

By the second part, each coefficient here is a polynomial multiple of R. By our assumption, $p_1q_2 - p_2q_1$ is a nonzero constant multiple of R. After dividing (4.5) by R, we get an expression of u_3 as a $\mathbb{K}[x]$-linear combination of u_1 and u_2, proving that the latter two syzygies form a basis for Z. On the other hand, suppose that (4.2) is equal to (fP, fQ, fR), where either $f = 0$ or the degree of f in x is positive. In the former case, the syzygies u_1 and u_2 are linearly dependent over $\mathbb{K}(x)$, and hence they cannot form a basis for Z. In the latter case, one can see that for any two $\mathbb{K}[x]$-linear combinations of u_1 and u_2, the expression analogous to (4.2) is a multiple of (fP, fQ, fR), and hence the syzygy referred to in part (iii) is not in the module generated by u_1, u_2.

In the last claim (v), we can eliminate the terms with r_1R and r_2R in all matrix entries thanks to the syzygy condition. Hence we consider, for some scalars $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, $\widetilde{\beta}_1$, $\widetilde{\beta}_2$,

$$\det\begin{pmatrix}\widetilde{\alpha}_1p_1P+\widetilde{\beta}_1q_1Q & \widetilde{\alpha}_2p_1P+\widetilde{\beta}_2q_1Q\\ \widetilde{\alpha}_1p_2P+\widetilde{\beta}_1q_2Q & \widetilde{\alpha}_2p_2P+\widetilde{\beta}_2q_2Q\end{pmatrix}=(\widetilde{\alpha}_1\widetilde{\beta}_2-\widetilde{\alpha}_2\widetilde{\beta}_1)(p_1q_2-q_2p_1)PQ. \tag{4.6}$$

By the previous statement, $p_1q_2 - q_2p_1$ is a scalar multiple of R. □

In the application to RS-transformations, we start with matrix hypergeometric equation $E(e_0, e_1, 0, e_\infty; t; z)$ and its direct pull-back with respect to a covering $z = \varphi(x)$. After this, we have to shift local monodromy differences at some points of the fiber $\{0, 1, \infty\} \subset \mathbb{P}^1_z$. Let k denote the order of the pole $x = \infty$ of the rational function $\varphi(x)$, or the difference between the degrees of its numerator and denominator.

Let $F(x)$ be the polynomial whose roots are the points above $z = 0$ where local monodromy differences have to be shifted, with root multiplicities equal to the corresponding shifts of local monodromy differences. Let $G(x)$ and $H(x)$ be similar polynomials whose roots are the finite points above $z = 1$, respectively, $z = \infty$, where the local monodromy differences have to be shifted, with corresponding multiplicities. We set

$$\Delta := \deg F(x) + \deg G(x) + \deg H(x). \tag{4.7}$$

Suppose that the point $x = 0$ is above $z = 0$, and the local monodromy difference at $x = \infty$ has to be shifted by δ. The sum $\Delta + \delta$ must be even.

Local exponent shifts for general asymptotic solutions imply the following form of the inverse Schlesinger matrix:

$$S^{-1} = \frac{1}{\sqrt{F\,G\,H}}\begin{pmatrix}A & B\\ C & D\end{pmatrix}. \tag{4.8}$$

Local exponent shifts for dominant solutions at singular points require that the following are triples of polynomials in x:

$$\left(\tfrac{(e_0+e_1-e_\infty)A+(e_0-e_1+e_\infty)B}{F},\ \tfrac{(e_0+e_1-e_\infty)A-(e_1-e_0+e_\infty)B}{G},\ \tfrac{A}{H}\right), \tag{4.9}$$

$$\left(\tfrac{(e_0+e_1-e_\infty)C+(e_0-e_1+e_\infty)D}{F},\ \tfrac{(e_0+e_1-e_\infty)C+(e_0-e_1-e_\infty)D}{G},\ \tfrac{C}{H}\right). \tag{4.10}$$

These polynomial triples are syzygies between

$$(e_1 - e_0 + e_\infty)F, \quad (e_0 - e_1 + e_\infty)G, \quad -2e_\infty(e_0 + e_1 - e_\infty)H. \tag{4.11}$$

More conveniently, the following polynomial triples are syzygies between F, G, H:

$$\left(\tfrac{\left((e_0-e_\infty)^2-e_1^2\right)A+\left((e_0-e_1)^2-e_\infty^2\right)B}{2e_\infty F},\ \tfrac{\left((e_1-e_\infty)^2-e_0^2\right)A+\left(e_\infty^2-(e_0-e_1)^2\right)B}{2e_\infty G},\ \tfrac{(e_0+e_1-e_\infty)A}{H}\right), \tag{4.12}$$

$$\left(\tfrac{\left((e_0-e_\infty)^2-e_1^2\right)C+\left((e_0-e_1)^2-e_\infty^2\right)D}{2e_\infty F},\ \tfrac{\left((e_1-e_\infty)^2-e_0^2\right)C+\left(e_\infty^2-(e_0-e_1)^2\right)D}{2e_\infty G},\ \tfrac{(e_0+e_1-e_\infty)C}{H}\right). \tag{4.13}$$

Normalization at infinity sets up the degrees for the entries of S^{-1} if $\delta < k$, as we show in the following lemma.

If F is a Laurent polynomial or Laurent series in $1/x$, we denote by $\{F\}$ the polynomial in the x part of F. In particular, $\{Fx^{-j}\}$ for an integer $j > 0$ is equal to the polynomial quotient of the division of F by x^j.

Lemma 4.3. *Let $f_1(z) \sim \binom{1}{0} z^{\frac12 e_\infty}$, $f_2(z) \sim \binom{0}{1} z^{-\frac12 e_\infty}$ be the normalized basis for solutions for $E(e_0, e_1, 0, e_\infty; t; z)$, like in Sec. 3. Suppose that the Schlesinger transformation S maps $f_1(\varphi(x))$, $f_2(\varphi(x))$ to solutions (of the pulled-back equation) asymptotically proportional to, respectively,*

$$\sim \begin{pmatrix}1\\0\end{pmatrix} x^{\frac12 k e_\infty + \frac12 \delta}, \qquad \sim \begin{pmatrix}0\\1\end{pmatrix} x^{-\frac12 k e_\infty - \frac12 \delta}. \tag{4.14}$$

(i) *If $\delta = 0$, we have these degree bounds for the entries of S^{-1}:*

$$\deg A = \tfrac{\Delta}{2}, \qquad \deg B < \tfrac{\Delta}{2}, \qquad \deg C < \tfrac{\Delta}{2}, \qquad \deg D = \tfrac{\Delta}{2}. \tag{4.15}$$

(ii) *If $\delta > 0$, let $\Delta^* = \frac12(\Delta - \delta)$ and $f_2(\varphi(x)) = \theta(x)\begin{pmatrix}h_1\\h_2\end{pmatrix}$, where $\theta(x)$ is a power function, and h_1, h_2 are power series in $1/x$ with $h_1 \to 0$, $h_2 \to 1$ as $x \to \infty$. Then $\deg A = \frac12(\Delta+\delta)$, the other three entries of S^{-1} have lower degree, and*

$$\begin{pmatrix}\{A\,x^{-\Delta^*-1}\} & \{B\,x^{-\Delta^*}\}\\ \{C\,x^{-\Delta^*-2}\} & \{D\,x^{-\Delta^*-1}\}\end{pmatrix}\begin{pmatrix}\{x^\delta\,h_1\}\\ \{x^{\delta-1}\,h_2\}\end{pmatrix} \tag{4.16}$$

gives a polynomial vector of degree $\le \delta - 2$ in x.

(iii) *If $\delta > 1$, then $\deg D < \frac12(\Delta+\delta) - 1$.*

(iv) *$\deg(AD - BC) \le \Delta$.*

(v) *If $\delta < k$, then the degree bounds for the entries of S^{-1} are*

$$\deg A = \tfrac{\Delta+\delta}{2}, \quad \deg B < \tfrac{\Delta-\delta}{2}, \quad \deg C < \tfrac{\Delta+\delta}{2}, \quad \deg D = \tfrac{\Delta-\delta}{2}. \tag{4.17}$$

(vi) *If $\delta \le \max(2,k)$, then $\deg C < \frac{\Delta+\delta}{2}$ and $\deg D \le \frac{\Delta-\delta}{2}$.*

Proof. The first statement is straightforward. In part (ii), the degree bounds on A, C follow from the action S^{-1} on $f_1(\varphi(x))$, that increases the local exponent $\frac12 k e_\infty$. The prescribed action on $f_2(\varphi(x))$ should cancel the terms of A, B, C, D of degree greater than roughly Δ^*. More precisely, that action of S^{-1} can be explicitly written as follows:

$$\frac{1}{\sqrt{FGH}}\begin{pmatrix}x^{\Delta^*} & 0\\ 0 & x^{\Delta^*+1}\end{pmatrix}\begin{pmatrix}A\,x^{-\Delta^*-1} & B\,x^{-\Delta^*}\\ C\,x^{-\Delta^*-2} & D\,x^{-\Delta^*-1}\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\theta(x)h_1\\ \theta(x)h_2\end{pmatrix}$$
$$= \frac{x^{2-\delta}\,\theta(x)}{\sqrt{FGH}}\begin{pmatrix}x^{\Delta^*-1} & 0\\ 0 & x^{\Delta^*}\end{pmatrix}\begin{pmatrix}A\,x^{-\Delta^*-1} & B\,x^{-\Delta^*}\\ C\,x^{-\Delta^*-2} & D\,x^{-\Delta^*-1}\end{pmatrix}\begin{pmatrix}x^\delta\,h_1\\ x^{\delta-1}h_2\end{pmatrix}.$$

The entries of the last product (or a matrix and the vector) have degree at most $\delta - 2$. The coefficients at greater powers of x depend on the truncated entries in (4.16) only. This completes the proof of part (ii). Note that

$$\deg\{A\,x^{-\Delta^*-1}\} = \delta - 1, \qquad \deg\{B\,x^{-\Delta^*}\} \le \delta - 1,$$
$$\deg\{C\,x^{-\Delta^*-2}\} \le \delta - 3, \qquad \deg\{D\,x^{-\Delta^*-1}\} \le \delta - 2,$$
$$\deg\{x^\delta\,h_1\} \le \delta - k, \qquad \deg\{x^{\delta-1}\,h_2\} = \delta - 1.$$

If $\delta \ge 2$, then $\deg\{D\,x^{-\Delta^*-1}\} \le \delta - 3$ as well, giving part (iii).

Part (iv) is immediate if $\delta = 0$. Otherwise $\deg(AD - BC) < \Delta + \delta$ as a first estimate. The matrix in (4.16) is formed by the leading terms contributing to the terms in $AD - BC$ greater than Δ; its columns are linear dependent modulo division by $x^{\delta-1}$, and that translates to the claim of part (iv).

If $\delta < k$, then the vector in (4.16) is simply $\binom{0}{1}$, and this gives trivial restrictions on the coefficients of B and D to the powers $\ge \Delta^*$ of x, giving part (v). For the last part, we have to consider additionally $\delta \le 2$ and $\delta = k$. If $\delta = k > 2$, then $\{x^\delta h_1\}$ is a constant and the second row of the matrix in (4.16) has degree at most $\delta - 3$. Hence the constant $\{x^\delta h_1\}$ does not influence the conditions on C and D. □

Explicit expressions for the solutions $f_1(z)$, $f_2(z)$ can be obtained from (7.9)–(7.10). If the Schlesinger transformation increases the local exponent of $-\frac{1}{2}ke_\infty$ by $\delta/2$, rather than the local exponent $\frac{1}{2}ke_\infty$ as in (4.14), then the specifications of Lemma 4.3 for the diagonal entries A, D and for the off-diagonal entries B, C should be pairwise interchanged, and the function f_2 in part (ii) should be replaced by f_1. If the Schlesinger transformation maps $f_1(\varphi(x))$, $f_2(\varphi(x))$ to functions proportional to

$$\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} x^{\frac{1}{2}ke_\infty \pm \frac{1}{2}\delta}, \qquad \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} x^{-\frac{1}{2}ke_\infty \mp \frac{1}{2}\delta},$$

respectively, then the rows of S^{-1} must be interchanged. Normalization of the pulled-back solutions to the $\binom{0}{1}$ and $\binom{1}{0}$ leading terms can be softened by allowing the leading terms of the transformed solutions $f_1(z)$, $f_2(z)$ to be scalar nonzero multiples of the two basis vectors. Then the rows of S^{-1} are determined up to scalar multiples, and we do not have any further conditions on the Schlesinger transformation. In particular, the rows of S^{-1} can be computed independently, from syzygy (4.12) or (4.13) between F, G, H satisfying the extra conditions of Lemma 4.3. The two syzygies ought to be defined uniquely by Lemma 4.3, up to a constant multiple. (One can check that the linear problems with undetermined coefficients have one variable more than the number of linear relations from the syzygy and Lemma 4.3 conditions. Multiple solutions would give low degree syzygies F, G, H; generically, the two degrees in (4.1) are equal or differ by merely 1.)

Theorem 4.4.

(i) *The lower-left entry of the pulled-back Fuchsian equation depends only on the syzygy* (4.13) *alone; that is, it does not depend on the syzygy* (4.12). *Similarly, the upper-right entry of the pulled-back equation depends only on the syzygy* (4.12).

(ii) *The required syzygies* (4.12)–(4.13) *form a basis for the* $\mathbb{C}(t)[x]$*-module of syzygies* K *between the polynomials* F, G, H.

(iii) *The determinant* $AD - BC$ *is a* $\mathbb{C}(t)$*-multiple of* $F\,G\,H$.

(iv) *The Schlesinger transformation* S *can be assumed to have the form*

$$S = \frac{1}{\sqrt{F\,G\,H}} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}, \qquad S^{-1} = \frac{1}{\sqrt{F\,G\,H}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.18}$$

where the polynomial entries A, B, C, D *are determined by syzygies* (4.12)–(4.13).

Proof. The first statement can be seen directly, by checking off-diagonal entries of the matrices $S^{-1}MS$ and $S^{-1}S'$ in expression (1.3) for the pulled-back equation. The lower-left entry is determined by the second row of S^{-1} and the first columns of MS and S'; these all depend on C, D, but not on A, B. We have the reverse situation for the upper-right entry.

Let (U_1, V_1, W_1) and (U_2, V_2, W_2) be the 2 syzygies in (4.12)–(4.13), respectively. The syzygies are linearly independent, since they give different degree of A or C. The expression $V_1W_2 - V_2W_1$ is a $\mathbb{C}(t)$ multiple of $(AD - BC)/GH$; part (iii) of Lemma 4.3 implies that

$$\deg(V_1W_2 - V_2W_1) \le \Delta - \deg G - \deg H = \deg F.$$

We conclude that $V_1W_2 - V_2W_1$ is a $\mathbb{C}(t)$ multiple of F by part (ii) of Theorem 4.2. The two syzygies form a module basis by part (iv) of the same theorem.

Part (iii) follows, since $AD - BC$ is divisible by each F, G, H, and has degree $\le \Delta$. We divide one of the rows by that scalar multiple and make the determinant precisely equal to FGH. Then S and S^{-1} have the form (4.18). □

5. General expression in terms of syzygies

By the Jimbo–Miwa correspondence, a Painlevé VI solution is determined by the lower-left entry of a pulled-back Fuchsian system. By the third part of Theorem 4.4, that lower-left entry is determined by one syzygy (4.13) between F, G, H. In general, that syzygy depends on the first coefficients of the solution $f_2(z) \sim \binom{0}{1} z^{-\frac{1}{2}e_\infty}$. But if $\delta \le \max(2,k)$, then we have only the degree bounds of part (vi) of Lemma 4.3. Then we do not need to know coefficients in the expansion of $f_2(z)$ at $z = \infty$ in order to determine the syzygy (and, eventually, the Painlevé VI solution).

If $\delta > 0$, then we prefer to assume that the direct pull-back solutions $f_1(\varphi(x))$ and $f_2(\varphi(x))$ are mapped into solutions (4.14) in the opposite order than in Lemma 4.3. The reason is that in applications we usually apply integer shifts that change the sign of local monodromies $\pm\frac{1}{2}ke_\infty$, while we wish to keep the positive local monodromy for the $\binom{1}{0}$ solution. Respectively, if $\delta > 0$, then the degree bounds in parts (v), (vi) of Lemma 4.3 on the entries of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ change columnwise. In particular,

$$\deg C = \tfrac{\Delta-\delta}{2}, \quad \deg D < \tfrac{\Delta+\delta}{2}. \tag{5.1}$$

Taking only small shifts $\delta \le \max(2,k)$ at $x = \infty$ is enough to generate interesting solutions of the sixth Painlevé equation. Checking the classification of the algebraic Painlevé VI solutions in [22], we conclude that all "seed" algebraic solutions (with respect to Okamoto transformations) can be generated in this way; see [32, Sec. 4.1]. Formula (5.4) in the following theorem is valid for any δ if the syzygy (U_2, V_2, W_2) is right; however, we specify the syzygy only if $\delta \le \max(2,k)$.

Theorem 5.1. *Let $z = \varphi(x)$ be a rational covering, and let $F(x)$, $G(x)$, $H(x)$ be polynomials in x. Let k be the order of the pole of $\varphi(x)$ at $x = \infty$. Suppose that the direct pull-back of $E(e_0, e_1, 0, e_\infty; t; z)$ with respect to $\varphi(x)$ is a Fuchsian equation with the following singularities.*

- *Four singularities are $x = 0$, $x = 1$, $x = \infty$, and $x = t$, with the local monodromy differences d_0, d_1, d_t, and d_∞, respectively. The point $x = \infty$ lies above $z = \infty$.*
- *All other singularities in $\mathbb{P}^1_x \setminus \{0, 1, t, \infty\}$ are apparent singularities. The apparent singularities above $z = 0$ (respectively, above $z = 1$, $z = \infty$) are the roots of $F(x) = 0$ (respectively, $G(x) = 0$, $H(x) = 0$). Their local monodromy differences are equal to the multiplicities of those roots.*

Let $\Delta = \deg F + \deg G + \deg H$, and let $\delta \le \max(2,k)$ be a nonnegative integer such that $\Delta + \delta$ is even. Suppose that (U_2, V_2, W_2) is a syzygy between the three polynomials F, G, H such that

$$\deg U_2 = \tfrac{\Delta}{2} - \deg F, \quad \deg V_2 = \tfrac{\Delta}{2} - \deg G, \quad \deg W_2 < \tfrac{\Delta}{2} - \deg H \tag{5.2}$$

for $\delta = 0$, and

$$\deg U_2 < \tfrac{\Delta+\delta}{2} - \deg F, \quad \deg V_2 < \tfrac{\Delta+\delta}{2} - \deg G, \quad \deg W_2 = \tfrac{\Delta-\delta}{2} - \deg H \tag{5.3}$$

if $\delta > 0$. Then the numerator of the (simplified) rational function

$$\frac{U_2W_2}{G}\left(\frac{(e_0 - e_1 + e_\infty)}{2}\frac{\varphi'}{\varphi} - \frac{(FU_2)'}{FU_2} + \frac{(HW_2)'}{HW_2}\right) + \frac{(e_0 - e_1 - e_\infty)}{2}\frac{V_2W_2}{F}\frac{\varphi'}{\varphi - 1} + \frac{(e_0 + e_1 - e_\infty)}{2}\frac{U_2V_2}{H}\frac{\varphi'}{\varphi(\varphi - 1)}, \tag{5.4}$$

has degree 1 in x, and the x-root of it is an algebraic solution of $P_{VI}(d_0, d_1, d_t, d_\infty + \delta; t)$.

Proof. We use a Schlesinger transformation that removes the apparent singularities and shifts the local monodromy difference at $x = \infty$ by δ. The matrix for its inverse has the form (4.8) with entry degrees given by (4.15) or (5.1). The syzygy (U_2, V_2, W_2) is identified as (4.13). Let (U_1, V_1, W_1) be the syzygy in (4.12). We have

$$A = \frac{HW_1}{e_0 + e_1 - e_\infty}, \qquad B = \frac{2e_\infty FU_1 + (e_1 - e_0 + e_\infty)HW_1}{(e_0 - e_1)^2 - e_\infty^2}, \tag{5.5}$$

$$C = \frac{HW_2}{e_0 + e_1 - e_\infty}, \qquad D = \frac{2e_\infty FU_2 + (e_1 - e_0 + e_\infty)HW_2}{(e_0 - e_1)^2 - e_\infty^2}. \tag{5.6}$$

Let h be a constant,

$$h = \frac{2e_\infty}{(e_0 + e_1 - e_\infty)(e_0 - e_1 + e_\infty)(e_0 - e_1 - e_\infty)}. \tag{5.7}$$

Then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = h\, F\, H\, (U_2W_1 - U_1W_2). \tag{5.8}$$

By the second part of Theorem 4.4, we may assume that this determinant is equal to $F\,G\,H$; this would affect the lower-left entry only by a $\mathbb{C}(t)$-multiple. With this assumption, $U_2W_1 - U_1W_2 = G/h$. Put $K = F\,G\,H$.

Using the form (4.18), we have

$$S' = \frac{1}{\sqrt{K}} \begin{pmatrix} D' & -B' \\ -C' & A' \end{pmatrix} - \frac{K'}{2\sqrt{K^3}} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \tag{5.9}$$

and

$$S^{-1}S' = \frac{1}{K} \begin{pmatrix} AD' - BC' & BA' - AB' \\ CD' - DC' & DA' - CB' \end{pmatrix} - \frac{K'}{2K} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.10}$$

The lower-left entry of $S^{-1}S'$ is equal to

$$h\, \frac{(FU_2)'HW_2 - FU_2(HW_2)'}{K}. \tag{5.11}$$

Let M be the 2×2 matrix on the right-hand side of formula (7.4) in the Appendix. The entries on the second row of $S^{-1}M$ are the following, from left to right:

$$\frac{2e_\infty}{\sqrt{K}} \left(FU_2 + \frac{e_0 - e_\infty\, \varphi}{e_0 + e_1 - e_\infty} HW_2 \right), \tag{5.12}$$

$$\frac{2e_\infty}{(e_0 - e_1 + e_\infty)\sqrt{K}} \left(\frac{2e_\infty^2 \varphi - e_0^2 + e_1^2 - e_\infty^2}{e_0 - e_1 - e_\infty} FU_2 - (e_0 + e_\infty \varphi) HW_2 \right). \tag{5.13}$$

The lower-left entry of the matrix $S^{-1}MS$ is the following:

$$2e_\infty h\, \frac{(e_0 + e_1 - e_\infty)F^2U_2^2 + 2(e_0 - e_\infty\varphi)FHU_2W_2 + (e_0 - e_1 - e_\infty)\,\varphi\, H^2W_2^2}{K}. \tag{5.14}$$

We use the syzygy relation to rewrite this expression in an attractive form. Here is a symmetric expression equal to (5.14):

$$2e_\infty h \left((e_0 - e_1 - e_\infty) \frac{V_2W_2}{F} \varphi + (e_0 - e_1 + e_\infty) \frac{U_2W_2}{G} (\varphi - 1) + (e_0 + e_1 - e_\infty) \frac{U_2V_2}{H} \right). \tag{5.15}$$

According to (1.3), the lower left-entry of the pulled-back Fuchsian system is equal to (5.15) times $\varphi'/4e_\infty\varphi(1-\varphi)$, minus (5.11). Up to the constant multiple $-h$, we get expression (5.4) for the lower-left entry of the pulled-back Fuchsian equation. By the Jimbo–Miwa correspondence, this entry must determine the Painlevé VI solution. □

We give now alternative forms of expression (5.4). Let us introduce the following notation:

$$f_0 = \frac{e_1 - e_0 + e_\infty}{2}, \quad f_1 = \frac{e_0 - e_1 + e_\infty}{2}, \quad f_\infty = \frac{e_0 + e_1 - e_\infty}{2}. \tag{5.16}$$

Besides, for a function ψ of x, let $[\psi]$ denote the logarithmic derivative ψ'/ψ of ψ. The expression (5.4) can be written as follows:

$$\frac{U_2W_2}{G}\left(f_1\,[\varphi] - \left[\frac{FU_2}{HW_2}\right]\right) - f_0\,\frac{V_2W_2}{F}\,[\varphi-1] - f_\infty\frac{U_2V_2}{H}\left[\frac{\varphi}{\varphi-1}\right]. \tag{5.17}$$

Thanks to the syzygy relation, we have

$$(FU_2)'HW_2 - FU_2(HW_2)' = FU_2(GV_2)' - (FU_2)'GV_2 = GV_2(HW_2)' - (GV_2)'HW_2,$$

hence these alternative expressions expressions for (5.17) hold:

$$f_1\,\frac{U_2W_2}{G}\,[\varphi] - \frac{V_2W_2}{F}\left(f_0\,[\varphi-1] - \left[\frac{GV_2}{HW_2}\right]\right) - f_\infty\frac{U_2V_2}{H}\left[\frac{\varphi}{\varphi-1}\right], \tag{5.18}$$

$$f_1\,\frac{U_2W_2}{G}\,[\varphi] - f_0\frac{V_2W_2}{F}\,[\varphi-1] - \frac{U_2V_2}{H}\left(f_\infty\left[\frac{\varphi}{\varphi-1}\right] - \left[\frac{FV_2}{GW_2}\right]\right). \tag{5.19}$$

Besides, these expressions for (5.17) can be derived:

$$f_1\tfrac{U_2W_2}{G}\left([\varphi] - \tfrac{1}{e_\infty}\left[\tfrac{FU_2}{HW_2}\right]\right) - f_0\tfrac{V_2W_2}{F}\left([\varphi-1] - \tfrac{1}{e_\infty}\left[\tfrac{GV_2}{HW_2}\right]\right) - f_\infty\tfrac{U_2V_2}{H}\left[\tfrac{\varphi}{\varphi-1}\right], \tag{5.20}$$

$$f_1\,\tfrac{U_2W_2}{G}\,[\varphi] - f_0\,\tfrac{V_2W_2}{F}\left([\varphi-1] - \tfrac{1}{e_1}\left[\tfrac{GV_2}{HW_2}\right]\right) - f_\infty\tfrac{U_2V_2}{H}\left(\left[\tfrac{\varphi}{\varphi-1}\right] - \tfrac{1}{e_0}\left[\tfrac{FU_2}{GV_2}\right]\right), \tag{5.21}$$

$$f_1\,\tfrac{U_2W_2}{G}\left([\varphi] - \tfrac{1}{e_0}\left[\tfrac{FU_2}{HW_2}\right]\right) - f_0\,\tfrac{V_2W_2}{F}\,[\varphi-1] - f_\infty\tfrac{U_2V_2}{H}\left(\left[\tfrac{\varphi}{\varphi-1}\right] - \tfrac{1}{e_0}\left[\tfrac{FU_2}{GV_2}\right]\right). \tag{5.22}$$

All these expressions are supposed to simplify greatly to a rational function with the denominator of degree 3 in x, and the numerator linear in x. The root of the numerator determines a Painlevé VI solution.

Remark 5.2. If one of the components of (U_2, V_2, W_2) is equal to zero, then expression (5.4) simplifies to a single multiplicative term, and the extra ramification point of $\varphi(x)$ is a zero of the numerator. For example, if $(U_2, V_2, W_2) = (H, 0, -F)$, then expression (5.4) becomes

$$-\frac{(e_0 - e_1 + e_\infty)}{2}\,\frac{\varphi'}{\varphi}\,\frac{FH}{G}. \tag{5.23}$$

The extra ramification point is a zero of $\varphi'(x)$. If the covering $z = \varphi(x)$ satisfies specifications of Theorem 2.1, then the last numerator FH simplifies out for $e_0 = 1/k_0$, $e_\infty = 1/k_\infty$; similarly, the last denominator G simplifies out for $e_1 = (k_1 - 1)/k_1$. For comparison, in the proof of Theorem 2.1 we assumed that $e_0 = 1/k_0$, $e_1 = 1/k_1$, $e_\infty = (k_\infty - 1)/k_\infty$; the composite Schlesinger transformation there corresponds to a syzygy with the third component equal to zero.

In the context of Theorem 5.1, there is a syzygy with zero component satisfying degree specifications (5.2) or (5.3) if and only if one of the following conditions holds:

$$\delta = 0, \quad \deg F + \deg G = \deg H, \tag{5.24}$$

$$\delta > 0, \quad \deg F + \deg H = \deg G - \delta, \tag{5.25}$$

$$\delta > 0, \quad \deg G + \deg H = \deg F - \delta. \tag{5.26}$$

Remark 5.3. As mentioned with examples in Sec. 3, the upper-right entry of the transformed equation determined in a similar way a solution of another Painlevé VI equation. Accordingly, if we have a proper syzygy (U_1, V_1, W_1) determining the upper row of the Schlesinger matrix, we can use the same formula (5.4) with (U_2, V_2, W_2) replaced by (U_1, V_1, W_1) to compute a solution of $P_{VI}(d_0, d_1, d_t, -d_\infty - \delta)$. Note that this Painlevé VI equation is the same as $P_{VI}(d_0, d_1, d_t, d_\infty + \delta + 2)$. Particularly, it is contiguous (and related by Okamoto transformations) to $P_{VI}(d_0, d_1, d_t, d_\infty + \delta)$.

Moreover, an algebraic solution of $P_{VI}(d_0, d_1, d_t, -d_\infty - \delta')$ can be obtained by using Theorem 5.1 with its δ replaced by $\delta' + 2$, that is, using the lower-right entry of a pull-back by contiguous Schlesinger transformation. It appears that the same algebraic solution is obtained regardless whether the lower-left entry or the upper-right entry of appropriately contiguous Schlesinger transformations is used.

6. More algebraic Painlevé VI solutions

Here we apply Theorem 5.1 to compute a few algebraic Painelevé VI solutions. Implicitly, we employ RS-transformations of the hypergeometric equation $E(1/3, 1/2, 0, 2/5; t; z)$ with respect to the covering $z = \widehat{\varphi}_{12}(x)$ Additionally, we note that a fractional-linear version of $\widehat{\varphi}_{12}(x)$ can be used to pull-back $E(1/3, 1/2, 0, 1/4; t; z)$ and $E(1/3, 1/2, 0, 1/2; t; z)$.

The implied RS-pullback transformation for the equation $E(1/3, 1/2, 0,\ 2/5; t; z)$ is $RS^2_4\left(\begin{smallmatrix}1/3\\3+3+3+3\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}2/5\\5+4+1+1+1\end{smallmatrix}\right)$. We work mainly with the covering $z = \varphi_{12}(x)$ rather than with the normalized $z = \widehat{\varphi}_{12}(x)$, and apply reparametrization (2.11) and normalizing fractional-linear transformation (2.12) at the latest. Theorem 5.1 has to be applied with $(F, G, H) = \left(F_{12}, P_{12}, x^2\right)$ and $\delta = 0$. The Painlevé solution must solve $P_{VI}(2/5, 2/5, 2/5, 8/5; t)$, which is the same equation as $P_{VI}(2/5, 2/5, 2/5, 2/5; t)$. The degree specifications (5.2) are $\deg U_2 = 2$, $\deg V_2 = 0$, $\deg W_2 < 4$. Up to a constant multiple, there is one syzygy satisfying these bounds,

$$\left(x^2 + (s+6)x + 1,\ -\tfrac{1}{2},\ -3(s+4)\left(x^3 - \left(\tfrac{7}{2}s + 11\right)x^2 + (s+7)x + 1\right)\right). \tag{6.1}$$

With this syzygy, expression (5.4) is equal to $-3(s+4)(3sx+8s+20)/10G_{12}$. After reparametrization (2.11) and normalizing fractional-linear transformation (2.12), the x-root gives the following solution $y_{32}(t_{12})$ of $P_{VI}(2/5,\ 2/5, 2/5, 2/5; t_{12})$:

$$y_{32} = \frac{(u-1)^2(u+3)^2(3u^2+1)}{3(u+1)^3(u-3)(u^2+4u-1)}. \tag{6.2}$$

The solutions $y_{31}(t_{12})$ and $y_{32}(t_{12})$ are presented in [29, Sec. 7] as well, but reparametrized $u \mapsto -(s+3)/(s-1)$. With the Okamoto transformations, these two solutions can be transformed to, respectively, the Great Icosahedron and Icosahedron solutions of Dubrovin–Mazzocco [9].

The full RS-transformation $RS^2_4\left(\begin{smallmatrix}1/3\\3+3+3+3\end{smallmatrix}\middle|\begin{smallmatrix}1/2\\2+2+2+2+2+2\end{smallmatrix}\middle|\begin{smallmatrix}2/5\\5+4+1+1+1\end{smallmatrix}\right)$ with $\delta = 0$ gives us a solution of $P_{VI}(2/5, 2/5, 2/5, -8/5; t)$ via the upper-right entry of the pulled-back Fuchsian equation. As mentioned at the end of previous section, we can use the same expression (5.4) with appropriate syzygy (U_1, V_1, W_1) for the triple $\left(F_{12}, P_{12}, x^2\right)$ to compute the Painlevé VI solution. The degree constraints are the following:

$$\deg U_1 = 2,\ \ \deg V_1 = 0,\ \ \deg W_1 = 4,\ \ \deg(17U_1F_{12} + 7V_1G_{12}) < 6. \tag{6.3}$$

Let S_3 be the syzygy $\left(-x^2, 0, F_{12}\right)$. One can take (U_1, V_1, W_1) to be equal to syzygy (6.1) plus $\frac{24}{7}S_3$. Here is the final expression for a solution $\widehat{y}_{32}(t_{12})$ of $P_{VI}(2/5, 2/5, 2/5, -8/5; t_{12})$, obtained after application of reparametrization (2.11) and normalizing fractional-linear transformation (2.12):

$$\widehat{y}_{32} = \frac{(u-1)^2(u+3)^2(13u^4-2u^2+5)(9u^6-55u^4+195u^2+299)}{13(u+1)^3(u-3)(u^2+3)(u^2+4u-1)(9u^6-47u^4+499u^2+115)}. \tag{6.4}$$

It is instructive to observe that to get a solution of $P_{VI}(2/5, 2/5, 2/5, -2/5; t_{12})$, we have to consider a Schlesinger transformation with $\delta = 2$. Then we have the following degree constraints for the two syzygies:

$$\deg U_1 = 3,\ \deg V_1 = 1,\ \deg W_1 < 3,\ \deg U_2 < 3,\ \deg V_2 < 1,\ \deg W_2 = 3.$$

We can take the same syzygy (6.1) for (U_2, V_2, W_2), and derive the same solution (6.2) of $P_{VI}(2/5, 2/5, 2/5, 2/5; t_{12})$. We can take (U_1, V_1, W_1) to be equal to the syzygy (6.1) times $x - s/2 - 1$, plus the syzygy $3(s + 4)S_3$. Application of expression (5.4) to this syzygy gives the following solution $\widetilde{y}_{32}(t_{12})$ of $P_{VI}(2/5, 2/5, 2/5, -2/5; t_{12})$:

$$\widetilde{y}_{32} = \frac{(u-1)^2(u+3)^2(3u^2+1)(7u^8-108u^6+314u^4-588u^2+119)}{7(u+1)^3(u-3)(u^2+3)(u^2+4u-1)(3u^6-37u^4+209u^2+17)}. \tag{6.5}$$

The same covering $z = \varphi_{12}(x)$ can be applied to pull-back the Fuchsian equations $E(1/3, 1/2, 0, 1/4; t; z)$ and $E(1/3, 1/2, 0, 1/2; t; z)$ to isomonodromic matrix equations with four singular points. Put

$$\lambda(x) = \frac{t_{12}^* x}{x + t_{12}^* - 1}. \tag{6.6}$$

The fractional-linear transformation $\lambda(x)$ fixes the points $x = 0$ and $x = 1$, and moves $x = \infty$ to $x = t_{12}^*$. Theorem 2.1 can be applied to $\widehat{\varphi}_{12}(\lambda(x))$ with $k_0 = 3$, $k_1 = 2$, $k_\infty = 4$. Put $t_{60} = \lambda(t_{12})$ and $y_{61} = \lambda(y_{26})$. Explicitly, we have

$$t_{60} = \frac{(u-1)(u+3)^3}{(u+1)(u-3)^3}, \qquad y_{61} = \frac{(u+3)^2(u^2-5)}{5(u+1)(u-3)(u^2+3)}. \tag{6.7}$$

In the current application of Theorem 2.1, the branches $x = t$ and $x = y$ are given by, respectively, $x = t_{60}$ and $x = y_{61}$. We conclude that $y_{61}(t_{60})$ is a solution of $P_{VI}(1/4, 1/4, 1/4, -1/4; t_{60})$. The same solution is given in [20, p. 25], reparametrized with $u \mapsto (s-3)/(s+1)$.

Currently, the implied RS-transformation is

$$RS_4^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/4 \\ 3+3+3+3 & 2+2+2+2+2+2 & 5+4+1+1+1 \end{array}\right).$$

To get a solution of $P_{VI}(1/4, 1/4, 1/4, 1/4; t)$, we use the upper-right entry of the pulled-back equation. In order to apply Theorem 5.1, we substitute $x \mapsto 1/x$ in expression (2.9) of $\varphi_{12}(x)$. Accordingly, let $\widetilde{F}_{12}(x)$ and $\widetilde{P}_{12}(x)$ denote the polynomials $x^4 F_{12}(1/x)$ and $x^6 P_{12}(1/x)$, respectively. A suitable syzygy between $\left(\widetilde{F}_{12}(x), \widetilde{P}_{12}(x), x\right)$ is the same as in (6.1) except for that the coefficients to x^2 and x of the third component have to be interchanged. The expression as in (5.14) is $(s + 3)/2$. After application of back substitutions $x \mapsto 1/x$, (2.11) and fractional-linear transformation λ^{-1} (2.12), we get the following solution of $P_{VI}(1/4, 1/4, 1/4, 1/4; t_{62})$:

$$y_{62} = -\frac{(u+3)^2}{3(u+1)(u-3)}. \tag{6.8}$$

The parametrization in [14, p.588] and [4, (10)] is related by $u \mapsto -3/(2s - 1)$. Boalch notes that this solution is equivalent to [8, (E.29)].

We may also consider the RS-transformations $RS_4^2\left(\begin{array}{c|c|c} 1/3 & 1/2 & 1/2 \\ 3+3+3+3 & 2+2+2+2+2+2 & 5+4+1+1+1 \end{array}\right)$. We have to compute syzygies between $\left(\widetilde{F}_{12}(x), \widetilde{P}_{12}(x), x^2\right)$. The "lower" syzygy (U_2,V_2,W_2) gives a solution of $P_{VI}(1/2, 1/2, 1/2, 5/2; t)$, or equivalently, $P_{VI}(1/2, 1/2, 1/2, -1/2; t)$. Incidentally, we get the same function $y_{62}(t_{60})$ as the z-root of the lower-left entry, although the syzygy (U_2, V_2, W_2) is different,

$$\left(x^2 - 2(s+3)x + 1,\ -\frac{1}{2},\ 3(s+4)\left(x^3 + (2s+11)x^2 + (s^2+5s+7)x - \frac{1}{2}s - 1\right)\right). \tag{6.9}$$

Hence, $y_{62}(t_{60})$ is a solution of $P_{VI}(1/2, 1/2, 1/2, -1/2; t_{60})$ as well. As for the syzygies (U_1, V_1, W_1) for the upper row of the Schlesinger matrix, we take $\delta = 0$ or $\delta = 2$, and get the syzygies

$$\left(-\frac{2}{3}x^2 - \frac{2}{3}(s+3)x + \frac{1}{3}, -\frac{1}{6}, x^4 + (3s+16)x^3 + (3s^2+25s+58)x^2 + \ldots\right),$$
$$\left(x^3 + (s+7)x^2 - (2s+5)x + 1, -\frac{1}{2}(x+1), \frac{3}{2}(s+4)^2(2x^2+(2s+9)x-1)\right).$$

Eventually, we derive these solutions $y_{63}(t_{60})$ and $y_{64}(t_{60})$ of

$$P_{VI}(1/2, 1/2, 1/2, -5/2; t_{60}) \quad \text{and} \quad P_{VI}(1/2, 1/2, 1/2, 1/2; t_{60}),$$

respectively,

$$y_{63} = -\frac{(u+3)^2(u^2+7)}{7(u+1)(u-3)(u^2+3)}, \qquad y_{64} = \frac{(u-1)(u+3)^2}{(u-3)(u^2+3)}. \tag{6.10}$$

Algebraic solutions of $P_{VI}(1/2, 1/2, 1/2, 1/2; t)$ are investigated in [15] and [14]. In particular, the solution t_{60}/y_{64} is presented in [15, 6.4] and [14, p. 598], reparametrized by $u \mapsto 3(s+1)/(s-1)$. The equation $P_{VI}(1/2, 1/2, 1/2, 1/2; t)$ is related to Picard's equation $P_{VI}(0, 0, 0, 1; t)$ via an Okamoto transformation.

7. Appendix

Recall that the sixth Painlevé equation is, canonically,

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right), \tag{7.1}$$

where α, β, γ, $\delta \in \mathbb{C}$ are parameters. As is well-known [16], its solutions define isomonodromic deformations (with respect to t) of the 2×2 matrix Fuchsian equation with 4 singular points ($\lambda = 0, 1, t$, and ∞),

$$\frac{d}{dz}\Psi = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right)\Psi, \quad \frac{d}{dz}A_k = 0 \ \text{ for } k \in \{0, 1, t\}. \tag{7.2}$$

The standard correspondence is due to Jimbo and Miwa [16]. We choose the traceless normalization of (7.2), so we assume that the eigenvalues of A_0, A_1, A_t are, respectively, $\pm\theta_0/2$, $\pm\theta_1/2$, $\pm\theta_t/2$, and that the matrix $A_\infty := -A_1 - A_2 - A_3$ is diagonal with the diagonal entries $\pm\theta_\infty/2$. Then the corresponding Painlevé equation has the parameters

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1-\theta_t^2}{2}. \tag{7.3}$$

We refer to the numbers θ_0, θ_1, θ_t and θ_∞ as *local monodromy differences.*

For any numbers $\nu_1, \nu_2, \nu_t, \nu_\infty$, we denote by $P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t)$ the Painlevé VI equation for the local monodromy differences $\theta_i = \nu_i$ for $i \in \{0, 1, t, \infty\}$, via (7.3). Note that changing the sign of ν_0, ν_1, ν_t or $1 - \nu_\infty$ does not change the Painlevé equation. Fractional-linear transformations for the Painlevé VI equation permute the 4 singular points and the numbers $\nu_0, \nu_1, \nu_t, 1 - \nu_\infty$.

Similarly, for any numbers $\nu_1, \nu_2, \nu_t, \nu_\infty$ and a solution $y(t)$ of $P_{VI}(\nu_0, \nu_1, \nu_t, \nu_\infty; t)$, we denote by $E(\nu_0, \nu_1, \nu_t, \nu_\infty; y(t); z)$ a Fuchsian equation (7.2) corresponding to $y(t)$ by the Jimbo–Miwa correspondence. The Fuchsian equation is determined uniquely up to conjugation of A_0, A_1, A_t by a diagonal matrix (dependent on t only). In particular, $y(t) = t$ can be considered as a solution of $P_{VI}(e_0, e_1, 0, e_\infty; t)$. The equation $E(e_0, e_1, 0, e_\infty; t; z)$ is a Fuchsian equation with 3 singular points, actually without the parameter t. Its solutions can be expressed in terms of Gauss hypergeometric series with local exponent differences e_0, e_1, and $e_\infty \pm 1$. We refer to $E(e_0, e_1, 0, e_\infty; t; z)$ as a *matrix hypergeometric equation*, and treat it as a matrix form of Euler's ordinary hypergeometric equation. In particular, the monodromy group of $E(1/3, 1/2, 0, 1/5; t; z)$ or $E(1/3, 1/2, 0, 2/5; t; z)$ is the icosahedral group.

The following matrix form of the hypergeometric equation is considered within the Jimbo–Miwa correspondence [16]:

$$\frac{d}{dz}\Psi = \frac{1}{4e_\infty z\,(1-z)} \begin{pmatrix} e_0^2 - e_1^2 + e_\infty^2 - 2e_\infty^2 z & e_\infty^2 - (e_0+e_1)^2 \\ (e_0-e_1)^2 - e_\infty^2 & 2e_\infty^2 z - e_0^2 + e_1^2 - e_\infty^2 \end{pmatrix} \Psi. \tag{7.4}$$

When considered as a "constant" isomonodromic system, this equation corresponds to the function $y(t) = t$ as a solution of the equation $P_{VI}(e_0, e_1, 0,\ e_\infty; t)$ within the Jimbo–Miwa correspondence. The function $y(t) = t$ solves $P_{VI}(e_0, e_1, 0, e_\infty; t)$ in the following sense: if we multiply both sides of (7.1) by $y - t$ and simplify each fractional term, the nonmultiples of $y - t$ on the right-hand side form the expression $\frac{1}{2}\left(\frac{dy}{dt}\right)^2 - \frac{dy}{dt} + \frac{1}{2}\frac{y(y-1)}{t(t-1)}$.

Here is a solution of (7.4), well defined if e_0 is not a positive integer,

$$z^{-\frac{1}{2}e_0}(1-z)^{-\frac{1}{2}e_1} \begin{pmatrix} (e_0+e_1-e_\infty)\ {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}(-e_0-e_1-e_\infty), 1+\frac{1}{2}(-e_0-e_1+e_\infty) \\ 1-e_0 \end{matrix} \right| z \right) \\ (e_0-e_1+e_\infty)\ {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}(-e_0-e_1+e_\infty), 1+\frac{1}{2}(-e_0-e_1-e_\infty) \\ 1-e_0 \end{matrix} \right| z \right) \end{pmatrix}. \tag{7.5}$$

If $e_0 \neq 0$, then an independent solution can be obtained by flipping the sign of e_0 and e_1 in this expression. (If we would flip the sign of e_0 only, the Fuchsian equation would be different.) Up to constant multiples, local solutions at singular points have the following asymptotic first terms:

$$\text{at } z = 0: \begin{pmatrix} e_0+e_1-e_\infty \\ e_0-e_1+e_\infty \end{pmatrix} z^{-\frac{1}{2}e_0} \quad \text{or} \quad \begin{pmatrix} e_0+e_1+e_\infty \\ e_0-e_1-e_\infty \end{pmatrix} z^{\frac{1}{2}e_0}; \tag{7.6}$$

$$\text{at } z = 1: \begin{pmatrix} e_0+e_1-e_\infty \\ e_0-e_1-e_\infty \end{pmatrix} (1-z)^{-\frac{1}{2}e_1} \text{ or } \begin{pmatrix} e_0+e_1+e_\infty \\ e_0-e_1+e_\infty \end{pmatrix} (1-z)^{\frac{1}{2}e_1}. \tag{7.7}$$

Hypergeometric solutions at $z = 1$ can be obtained from (7.5) by the substitutions $z \mapsto 1 - z$, $e_0 \leftrightarrow e_1$ and applying the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to the solution vector. Due to the normalization, at $z = \infty$ we have a basis of solutions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} z^{\frac{1}{2}e_\infty} + O\left(z^{\frac{1}{2}e_\infty - 1}\right), \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} z^{-\frac{1}{2}e_\infty} + O\left(z^{-\frac{1}{2}e_\infty - 1}\right). \tag{7.8}$$

Explicitly, a hypergeometric basis at $z = \infty$ is

$$z^{\frac{1}{2}(e_1+e_\infty)}(1-z)^{-\frac{1}{2}e_1} \begin{pmatrix} 4e_\infty(e_\infty-1)\ {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}(-e_0-e_1-e_\infty), \frac{1}{2}(e_0-e_1-e_\infty) \\ -e_\infty \end{matrix} \right| \frac{1}{z} \right) \\ \frac{e_\infty^2-(e_0-e_1)^2}{z}\ {}_2F_1\left(\left. \begin{matrix} 1+\frac{1}{2}(-e_0-e_1-e_\infty), 1+\frac{1}{2}(e_0-e_1-e_\infty) \\ 2-e_\infty \end{matrix} \right| \frac{1}{z} \right) \end{pmatrix}, \tag{7.9}$$

$$z^{\frac{1}{2}(e_1-e_\infty)}(1-z)^{-\frac{1}{2}e_1} \begin{pmatrix} \frac{e_\infty^2-(e_0+e_1)^2}{z}\ {}_2F_1\left(\left. \begin{matrix} 1+\frac{1}{2}(-e_0-e_1+e_\infty), 1+\frac{1}{2}(e_0-e_1+e_\infty) \\ 2+e_\infty \end{matrix} \right| \frac{1}{z} \right) \\ 4e_\infty(e_\infty+1)\ {}_2F_1\left(\left. \begin{matrix} \frac{1}{2}(-e_0-e_1+e_\infty), \frac{1}{2}(e_0-e_1+e_\infty) \\ e_\infty \end{matrix} \right| \frac{1}{z} \right). \end{pmatrix}. \tag{7.10}$$

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