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GENERALIZED INTERPOLATION PROBLEM OF THE KOREVAAR–DIXON TYPE

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Abstract. In this paper, we study the generalized interpolation problem in the class of entire functions of exponential type defined by a certain majorant from the convergence class.

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1. Introduction. In this paper, we consider the generalized interpolation problem in the class of entire functions of exponential type, which is determined by a certain majorant from the convergence class. The ordinary interpolation problem of the Korevaar–Dixon type in the class of entire functions determined by a non-quasi-alalytical majorant was examined in [3]. In a narrower class, where the majorant possesses the concavity property, a similar problem with nodes at points of a certain subsequence of natural numbers was earlier considered by Berndtsson. By an ingenious method based on an estimate of a solution of the $\overline{\partial}$-problem obtained by Hörmander, Berndtsson proved a solvability criterion for this interpolation problem. In papers of Pavlov, Korevaar, and Dixon, interpolation sequences were successfully applied in many problems of complex analysis; moreover, a relationship with approximative properties of power systems $\{z^{p_n}\}$ and with well-known Pólya and MacIntyre problems was detected.

In [3], an interpolation criterion in a more general sense was stated for an arbitrary sequence of real numbers; the proof of the main theorem in [3] was based on the modified Berndtsson method. In this paper, we transfer this result to the case where given values of an entire function are constrained by certain minimal restrictions, namely, natural conditions imposed by the convergence class.

Let L be the class of all continuous functions $l = l(x)$ on $\mathbb{R}_+$ such that $0 < l(x) \uparrow \infty$ as $x \to \infty$,

$$W = \left\{ w \in L : \int_1^\infty \frac{w(x)}{x^2} dx < \infty \right\}, \quad \Omega = \left\{ \omega \in W : \frac{\omega(x)}{x} \downarrow, \ x \to \infty \right\}.$$

The set W is called the *convergence class* and functions w from W are called *weights* (*non-quasi-analytic weights*; see [3]).

Definition 1 (see [1]). An increasing sequence $\{p_n\}$ of natural numbers is called a *Pavlov–Korevaar–Dixon interpolation sequence* if there exists a function $\omega \in \Omega$ depending only on the sequence $\{p_n\}$ such that for any sequence $\{b_n\}$ of complex numbers, $|b_n| \le 1$, there exists an entire function f possessing the following properties:

$$(1)\ f(p_n) = b_n, \quad n \ge 1, \qquad\qquad (2)\ M_f(r) = \max_{|z| \le r} |f(z)| \le e^{\omega(r)}.$$

Let $\Lambda = \{\lambda_n\}$ be an arbitrary sequence of real numbers, $0 < \lambda_n \uparrow \infty$. A sequence Λ is called an *interpolation sequence* if there exists a function $w \in W$ depending only on this sequence such that for any sequence $\{b_n\}$ of complex numbers, $|b_n| \le 1$, there exists an entire function f possessing the properties (1) and (2), but with the function w.

Necessary and sufficient conditions under which a sequence $\{p_n\}$ ($p_n \in \mathbb{N}$) is an interpolation sequence were obtained in [1] for the class Ω, and in [3] for the class W.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory, Vol. 162, Complex Analysis. Mathematical Physics, 2019.

Definition 2. Let β be a function from the class W. A sequence $\Lambda = \{\lambda_n\}$ $(0 < \lambda_n \uparrow \infty)$ is called an *interpolation sequence in the wide sense* if there exists a function $w \in W$ depending only on the sequence Λ such that for any sequence $\{b_n\}$ of complex numbers, $|b_n| \le e^{\beta(\lambda_n)}$ $(n \ge 1)$, there exists an entire function f possessing the following properties:

$$(1')\ f(\lambda_n) = b_n, \quad n \ge 1, \qquad\qquad (2')\ M_f(r) = \max_{|z|\le r} |f(z)| \le e^{w(r)}.$$

The problem $(1')$–$(2')$ is called the *generalized Korevaar–Dixon interpolation problem*, whereas the condition

$$|b_n| \le e^{\beta(\lambda_n)} \quad n \ge 1, \tag{1}$$

where β is a fixed function from the class W, is called the *natural condition.*

The aim of this paper is to prove an interpolation criterion for a sequence Λ in the wide sense.

2. Auxiliary results. Let

$$n(t) = \sum_{\lambda_n \le t} 1$$

be the counting function of a sequence Λ and

$$N(t) = \int_0^t \frac{n(x)}{x} dx.$$

Without loss of generality, we assume that $\lambda_1 = 1$; this allows us to simplify some calculations.

Lemma 1. *Let* $\tau_n = \min\limits_{\substack{k\ne n\\ \ge 1}} |\lambda_n - \lambda_k|$, $h_n = \min(\tau_n, 1)$,

$$K_n = \left\{\xi : \frac{h_n}{4} \le |\xi - \lambda_n| \le \frac{h_n}{2}\right\}, \quad n \ge 1.$$

Then in rings K_n, *the following estimates hold*:

$$(1)\ \sup_{k\ne n} \left|\ln\left|\frac{\lambda_k - z}{\lambda_k - \lambda_n}\right|\right| \le \ln 2; \qquad (2)\ \sup_{k} \left|\ln\left|\frac{\lambda_k + z}{\lambda_k + \lambda_n}\right|\right| \le \ln\frac{4}{3};$$

$$(3)\ \left|\ln\left|1 - \frac{z^2}{\lambda_n^2}\right|\right| \le \ln 10 + |\ln h_n| + \ln \lambda_n.$$

Assume that a sequence Λ has a finite upper density

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = \tau < \infty.$$

Then

$$q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

is an entire function of the exponential type.

Lemma 2 (see [3]). *Let a sequence* $\Lambda = \{\lambda_n\}$ $(1 = \lambda_1 < \lambda_n \uparrow \infty)$ *have a finite upper density,* $h_n = \min\big(\min\limits_{k\ne n} |\lambda_k - \lambda_n|, 1\big)$,

$$q(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right).$$

Then in the rings

$$K_n = \left\{\xi : \frac{h_n}{4} \le |\xi - \lambda_n| \le \frac{h_n}{2}\right\},$$

the following estimate holds:

$$\left|\ln \frac{1}{|q(z)|} - \int_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t} dt\right| \le m(\lambda_n),$$

where $\nu(\lambda_n;\ t)$ *is the number of points* $\lambda_k \ne \lambda_n$ *from the segment* $\{h : |h - \lambda_n| \le t\}$,

$$m(\lambda_n) = \ln 10 + \ln \lambda_n + |\ln h_n| + n(2\lambda_n) \ln 8 + 2N(2\lambda_n) + 20 \ln M_q(\lambda_n).$$

Corollary 1. *If*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \quad \textit{and} \quad |\ln h_n| \le w_1(\lambda_n), \quad n \ge 1,$$

for a certain function $w_1 \in W$, *then for* $z \in K_n$ *we have*

$$\left|\ln \frac{1}{|q(z)|} - \int_0^{r} \frac{\nu(z;\ t)}{t} dt\right| \le w_2(r),$$

where w_2 *is a function from* W.

Corollary 1 easily follows from Lemma 2, if we take into account the fact that the convergence of the series $\sum\limits_{n=1}^{\infty} 1/\lambda_n$ is equivalent to the convergence of the integrals

$$\int_1^{\infty} \frac{n(r)}{r^2} dr, \quad \int_1^{\infty} \frac{N(r)}{r^2} dr, \quad \int_1^{\infty} \frac{\ln M_q(r)}{r^2} dr$$

(see [2, 6]). Further, since

$$-\ln \prod_{\substack{\lambda_n/2 \le \lambda_k \le 2\lambda_n, \\ k \ne n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| = -\sum_{|\lambda_k - \lambda_n| \le \lambda_n} \ln \left|1 - \frac{\lambda_n}{\lambda_k}\right| + \sum_{\lambda_k \le \lambda_n/2} \left|1 - \frac{\lambda_n}{\lambda_k}\right| = -\Sigma_1(\lambda_n) + A,$$

for $\Sigma_1(\lambda_n)$ and A we have

$$0 \le A = \sum_{\lambda_k \le \lambda_n/2} \ln \left(\frac{\lambda_n}{\lambda_k} - 1\right) \le \sum_{\lambda_k \le \lambda_n/2} \ln \left(1 + \frac{\lambda_n^2}{\lambda_k^2}\right) \le \ln M_q(\lambda_n),$$

$$\Sigma_1(\lambda_n) = -\int_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t} dt + N_1(2\lambda_n) - n_1(2\lambda_n) \ln 2.$$

Thus, the following assertion holds.

Lemma 3. *The following estimate is valid*:

$$\left|-\ln \prod_{\substack{k \ne n \\ \lambda_n/2 \le \lambda_k \le 2\lambda_n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| - \int_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t} dt\right| \le n(2\lambda_n) + N(2\lambda_n) + \ln M_q(\lambda_n),$$

where $\nu(\lambda_n; t)$ *is the number of points* $\lambda_k \ne \lambda_n$ *from the segment* $\{h : |h - \lambda_n| \le t\}$.

Lemma 4 (see [5]). *Let* $w \in W$. *Then*

$$v(z) = w^*(|z|),$$

where

$$w^*(r) = \int_1^\infty \ln\left(1 + \frac{r^2}{t^2}\right) dw(t), \quad r = |z|,$$

is a subharmonic function in $\mathbb{C}$.

3. Solvability criterion for the generalized interpolation problem. Let

$$\Lambda = \{\lambda_n\}, \quad 0 < \lambda_n \uparrow \infty, \quad \overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = \tau < \infty.$$

Theorem 1. *A sequence* Λ *is an interpolation sequence in the wide sense if and only if there exists a function* $w \in W$ *such that*

$$\text{(a)} \ \sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty; \qquad \text{(b)} \ -\ln \prod_{\substack{\lambda_n/2<\lambda_k<2\lambda_n \\ k\neq n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| \le w(\lambda_n), \quad n \ge 1.$$

Note that Lemma 3 and the conditions (a) and (b) imply that

$$\ln \frac{1}{h_n} \le w_0(\lambda_n), \quad n \ge 1,$$

where $h_n = \min\left(\min_{1\le k\neq n} |\lambda_n - \lambda_k|, 1\right)$ and w_0 is a function from the class W.

The proof of sufficiency in Theorem 1 is based on a certain existence theorem for $\overline{\partial}$-equations obtained by Hörmander. We recall this theorem.

Theorem 2 (see [1]). *Let* $\varphi = \varphi(z)$ *be a subharmonic function in* $\mathbb{C}$ *and* $g \in C^\infty(\mathbb{C})$. *Then there exists a solution* $u \in C^\infty(\mathbb{C})$ *of the equation* $\partial u/\partial \overline{z} = g$ *satisfying the condition*

$$\int_{\mathbb{C}} |u|^2 e^{-\varphi}\left(1 + |z|^2\right)^{-2} d\lambda \le \int_{\mathbb{C}} |g|^2 e^{-\varphi} d\lambda, \tag{2}$$

under the condition that the right-hand side is finite (here λ *is the Lebesgue measure).*

Proof of Theorem 1. *Sufficiency.* We choose a function $\psi \in C^\infty$ such that $0 \le \psi(z) \le 1$, $\psi(z) = 1$ for $|z| < 1/4$, and $\psi(z) = 0$ for $|z| > 1/2$. We set

$$A(z) = \sum_{n=1}^\infty b_n \Psi_n(z - \lambda_n), \quad \Psi_n(z) = \psi\left(\frac{z}{h_n}\right)$$

where $\{b_n\}$ is an arbitrary given sequence of complex numbers satisfying the natural condition (1). Since $A(z) = b_k \Psi_k(z - \lambda_k)$ for $z \in B_k = \{z : |z - \lambda_k| < h_k/2\}$ and $A(z) = 0$ if z belongs to the exterior of the union of the disks B_n, $n \ge 1$, we obviously conclude that $A \in C^\infty$. Further, since $|\lambda_k - \lambda_n| \ge h_n$ for $k \neq n$, we obtain $A(\lambda_k) = b_k\psi(0) = b_k$, $k \ge 1$.

Let

$$\varphi(z) = 2\ln \prod_{n=1}^\infty \left|1 - \frac{z^2}{\lambda_n^2}\right| + v(z),$$

where v is a subharmonic function to be specified below. Since the sequence Λ has a finite upper density, we see that

$$q(z) = \prod_{n=1}^\infty \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

is an entire function of the exponential type and φ is a subharmonic function. We have

$$M_\varphi(r) = \max_{|z|=r} |\varphi(z)| \le 2\ln \prod_{n=1}^\infty \left(1 + \frac{r^2}{\lambda_n^2}\right) + M_v(r).$$

Further,

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{r^2}{\lambda_n^2}\right) = \int_0^{\infty} \ln\left(1+\frac{r^2}{t^2}\right) dn(t). \tag{3}$$

Integrating by parts the Stiltjes integral (3), we obviously obtain

$$\int_0^{\infty} \ln\left(1+\frac{r^2}{t^2}\right) dn(t) = 2r^2 \int_1^{\infty} \frac{n(t)}{t(t^2+r^2)} dt \equiv w_1(r).$$

The fact $w_1 \in W$ is proved immediately.

We construct a subharmonic function v such that the value of $M_v(r)$ (the maximum of the modulus of v) admits an upper estimate in terms of a certain function from the class W and, moreover, the right-hand side in (2) for the function $g = \partial A/\partial \overline{z}$ is finite. Let

$$K_n = \left\{\xi : \frac{h_n}{4} < |\xi - \lambda_n| < \frac{h_n}{2}\right\}, \quad n \geq 1.$$

Note that the rings K_n, $n \geq 1$, do not intersect pairwise. This fact follows from the inequality

$$\frac{h_n}{2} + \frac{h_{n+1}}{2} \leq \lambda_{n+1} - \lambda_n, \quad n \geq 1.$$

Hence we can write

$$\int_{\mathbb{C}} \left|\frac{\partial A}{\partial \overline{\xi}}\right|^2 e^{-\varphi} d\lambda = \int_{\bigcap_n \{\xi:|\xi-\lambda_n|>h_n/2\}} \left|\frac{\partial A}{\partial \overline{\xi}}\right|^2 e^{-\varphi} d\lambda$$

$$+ \sum_{n=1}^{\infty} \int_{K_n} \left|\frac{\partial A}{\partial \overline{\xi}}\right|^2 e^{-\varphi} d\lambda + \sum_{n=1}^{\infty} \int_{\{\xi:|\xi-\lambda_n|<h_n/4\}} \left|\frac{\partial A}{\partial \overline{\xi}}\right|^2 e^{-\varphi} d\lambda. \tag{4}$$

Obviously, the first and last integrals on the right-hand side of Eq. (4) vanish. Further,

$$A(\xi) = b_n \psi\left(\frac{\xi - \lambda_n}{h_n}\right), \quad \xi \in K_n.$$

Assuming that $\psi = \psi(w, \overline{w})$, where $w = x + iy = (\xi - \lambda_n)/h_n$, we obtain

$$\frac{\partial \psi}{\partial \overline{\xi}} = \frac{\partial \psi}{\partial \overline{w}} \overline{\left(\frac{\partial w}{\partial \xi}\right)} = \frac{\partial \psi}{\partial \overline{w}} \frac{1}{h_n}.$$

This implies

$$\left|\frac{\partial \psi}{\partial \overline{\xi}}\right| = \frac{1}{2h_n} \left|\frac{\partial \psi}{\partial x} + i\frac{\partial \psi}{\partial y}\right| \leq \frac{1}{h_n} \left|\frac{\partial \psi}{\partial x}\right|, \quad \xi \in K_n.$$

Since $|b_n| \leq e^{\beta(\lambda_n)}$, $n \geq 1$, where β is a function from the class W, we have

$$\int_{\mathbb{C}} \left|\frac{\partial A}{\partial \overline{\xi}}\right|^2 e^{-\varphi} d\lambda \leq C_1 \sum_{n=1}^{\infty} T_n,$$

where

$$T_n = \frac{e^{2\beta(\lambda_n)}}{h_n^2} \int_{K_n} e^{-v(\xi)} \prod_{k=1}^{\infty} \left|1 - \frac{\xi^2}{\lambda_k^2}\right|^{-2} d\lambda(\xi),$$

$\beta \in W$ and $C_1 = \max\limits_{|x|\leq 1/2} |\partial \psi/\partial x|^2$.

For each fixed n and $\xi \in K_n$ we have

$$p(\xi) = \prod_{k=1}^{\infty} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| = \prod_{\lambda_k \le \lambda_n/2} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| \prod_{\substack{\lambda_n/2 < \lambda_k < 2\lambda_n \\ k \ne n}} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| \prod_{\lambda_k \ge 2\lambda_n} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| \left|1 - \frac{\xi^2}{\lambda_n^2}\right|.$$

Since $\operatorname{Re} \xi > 0$ for $\xi \in K_n$, $n \ge 1$, we have

$$\left|1 + \frac{\xi}{\lambda_k}\right| \ge 1. \tag{5}$$

Further, for $\lambda_k \le \lambda_n/2$ and $n \ge n_0$ we have

$$\left|1 - \frac{\xi^2}{\lambda_k^2}\right| \ge \frac{|\xi|^2}{\lambda_k^2} - 1 \ge 4\left[1 - \frac{1}{2\lambda_n}\right]^2 - 1 \ge 1. \tag{6}$$

Taking into account the estimates (5) and (6) we obtain

$$p(\xi) \ge \prod_{\substack{\lambda_n/2 < \lambda_k < 2\lambda_n \\ k \ne n}} \left|1 - \frac{\xi}{\lambda_k}\right| \prod_{\lambda_k \ge 2\lambda_n} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| \left|1 - \frac{\xi^2}{\lambda_n^2}\right| \tag{7}$$

for $\xi \in K_n$, $n \ge n_0$. Applying Lemma 1, for $\xi \in K_n$ $(n \ge 1)$ we have

$$\left|1 - \frac{\xi}{\lambda_k}\right| = \left|1 - \frac{\lambda_n}{\lambda_k}\right| \frac{|\xi - \lambda_k|}{|\lambda_n - \lambda_k|} \ge \frac{1}{2}\left|1 - \frac{\lambda_n}{\lambda_k}\right|, \quad k \ne n. \tag{8}$$

Now we estimate the value $\left|1 - \xi^2/\lambda_n^2\right|$ for $\xi \in K_n$:

$$\left|1 - \frac{\xi^2}{\lambda_n^2}\right| \ge \frac{h_n}{4} \frac{|\xi + \lambda_n|}{\lambda_n^2} \ge \frac{h_n}{4\lambda_n}.$$

As was noted above, the conditions (a) and (b) imply that

$$\frac{1}{h_n} \le e^{w_0(\lambda_n)}, \quad n \ge 1,$$

where w_0 is a function from the class W. Therefore, for $n \ge 1$ we have the following estimate for $\xi \in K_n$:

$$\left|1 - \frac{\xi^2}{\lambda_n^2}\right| \ge e^{-w_2(\lambda_n)}, \quad w_2 \in W. \tag{9}$$

The required estimate for the first product in (7) in terms of a function from W easily follows from the conditions (a) and (b) if we take (8) into account. It remains to estimate the product

$$\prod_{\lambda_k \ge 2\lambda_n} \left|1 - \frac{\xi^2}{\lambda_k^2}\right|.$$

Since

$$\frac{|\xi|^2}{t^2} \le \left(\frac{\lambda_n + 1/2}{2\lambda_n}\right)^2 \le \left(\frac{1}{2} + \frac{1}{4\lambda_1}\right)^2 < \frac{2}{3},$$

we have

$$\ln\left|1 - \frac{\xi^2}{t^2}\right| \ge \ln\left(1 - \frac{|\xi|^2}{t^2}\right) \ge -3\frac{|\xi|^2}{t^2},$$

as the function $\varphi(\alpha) = \ln(1-\alpha) + 3\alpha$ increases for $\alpha < 2/3$. Since $|\xi|/\lambda_n \le 3/2$ for $\xi \in K_n$, $n \ge 1$, we have

$$\ln\left|1 - \frac{\xi^2}{t^2}\right| \ge -C_2\frac{\lambda_n^2}{t^2}, \quad C_2 = \frac{27}{4}.$$

Hence we have

$$\ln \prod_{\lambda_k \geq 2\lambda_n} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| = \int_{2\lambda_n}^{\infty} \ln\left|1 - \frac{\xi^2}{t^2}\right| dn(t) \geq -C_2 \int_{2\lambda_n}^{\infty} \frac{\lambda_n^2}{t^2} dn(t) \geq -2C_2\lambda_n^2 \int_{2\lambda_n}^{\infty} \frac{n(t)}{t^3} dt. \tag{10}$$

Let

$$w_3(r) \equiv r^2 \int_{2r}^{\infty} \frac{n(t)}{t^3} dt = \int_{2}^{\infty} \frac{n(sr)}{s^3} ds.$$

One can easily verify that $w_3 \in W$.

Since $p(\xi) \geq \beta > 0$ on $\bigcup_{n \leq n_0} K_n$, due to the estimates (7)–(10) and the conditions (a) and (b) of the theorem we finally conclude that there exists a function $w_4 \in W$ such that for all $n \geq 1$ we have

$$p(\xi) \geq e^{-w_4(\lambda_n)}, \quad \xi \in K_n. \tag{11}$$

We set

$$w^*(r) = \int_1^{\infty} \ln\left(1 + \frac{r^2}{t^2}\right) dw_4^*(t) + \left(w_4^*(1) + 1\right) \ln(1 + r^2),$$

where $w_4^* = w_4 + \beta$. Then $v(z) = Cw^*(|z|)$ is the required function for some $C > 0$. Indeed, by Lemma 4, v is a subharmonic function in $\mathbb{C}$, whereas $M_v(r) = Cw^*(r)$ is a function of the class W (cf. above arguments concerning the function w_1).

It remains to prove that $\sum_{n=1}^{\infty} T_n < \infty$. Taking into account the estimate (11) and the definition of the function v, we have

$$T_n \leq \frac{e^{2\beta(\lambda_n)}}{h_n^2} \int_{K_n} e^{-Cw^*\left(|\xi|\right) + 2w_4(\lambda_n)} d\lambda(\xi) \leq C_3 \exp\left[2\beta(\lambda_n) + 2w_4(\lambda_n) - Cw^*\left(\lambda_n - \frac{1}{2}\right)\right],$$

$C_3 = 3/16\pi$. Note that

$$w^*(r) = 2r^2 \int_1^{\infty} \frac{w_4^*(t)}{t(t^2 + r^2)} dt + \ln(1 + r^2) \geq 2r^2 w_4^*(r) \int_r^{\infty} \frac{dt}{t(t^2 + r^2)} \geq \frac{1}{2} w_4^*(r),$$

and also

$$\frac{w^*(\lambda_n)}{w^*(\lambda_n - \frac{1}{2})} \leq M, \quad n \geq 1.$$

Therefore,

$$\sum_{n=1}^{\infty} T_n \leq C_3 \sum_{n=1}^{\infty} e^{-\frac{C}{M} w^*(\lambda_n) + 2w_4^*(\lambda_n)} \leq C_3 \sum_{n=1}^{\infty} e^{(-C/M + 4) w^*(\lambda_n)}.$$

The definition of the function $w^*(r)$ implies that $w^*(r) \geq (w_4^*(1) + 1) \ln(1 + r^2)$; therefore,

$$\sum_{n=1}^{\infty} T_n \leq C_3 \sum_{n=1}^{\infty} \frac{1}{(1 + \lambda_n^2)^{C_4}},$$

where

$$C_4 = \left(\frac{C}{M} - 4\right) \left(w_4^*(1) + 1\right) > \frac{1}{2}$$

due to the choice of the constant C from the definition of the function v (it suffices to set $C > 5M$). Then, obviously, the last series converges.

As was stated above, $M_v(r) = Cw^*(r)$, $w^* \in W$. Therefore,

$$M_{\varphi}(r) \leq w_5(r), \tag{12}$$

where $w_5 = 2w_1 + Cw^*$ is a function of the class W.

Now we apply Theorem 2 to $g = \partial A/\partial\overline{z}$. Since the function φ is chosen so that $e^{-\varphi}$ has a nonintegrable singularity at each point λ_n, we have $u(\lambda_n) = 0$, $n \geq 1$.

Consider the equation

$$\frac{\partial u}{\partial \overline{z}} = \frac{\partial A}{\partial \overline{z}}, \quad u(\lambda_n) = 0, \quad n \geq 1. \tag{13}$$

We set $f = A - u$, where u is a solution of Eq. (13) (it exists due to Hörmander's theorem). Clearly, f is an entire function and $f(\lambda_n) = b_n$, $n \geq 1$.

Since $|f|^2$ is subharmonic in the whole plane, for any $\rho > 0$, in particular, for $1 \leq \rho = r$ (see [4]) we have

$$|f(z)|^2 \leq \frac{1}{\pi\rho^2} \int\limits_{|\xi - z| \leq \rho} |f(\xi)|^2 d\lambda(\xi) \leq \frac{1}{\pi r^2} \int\limits_{|\xi| \leq 2r} |f(\xi)|^2 d\lambda(\xi), \quad r = |z|.$$

Since $|f|^2 \leq 2\big(|A|^2 + |u|^2\big)$, we have

$$\frac{1}{\pi r^2} \int\limits_{|\xi| \leq 2r} |f|^2 d\lambda(\xi) \leq \frac{2}{\pi r^2} \int\limits_{|\xi| \leq 2r} |A|^2 d\lambda(\xi) + \frac{2}{\pi r^2} \int\limits_{|\xi| \leq 2r} |u|^2 d\lambda(\xi).$$

Since $A(\xi) = 0$ outside the circles B_k, $k \geq 1$, the first integral in the right-hand side is really taken over the set

$$B(r) = \Big(\bigcup_k B_k\Big) \cap \big\{\xi : |\xi| \leq 2r\big\}, \quad \text{where} \quad B_k = \Big\{\xi : |\xi - \lambda_k| \leq \frac{h_k}{2}\Big\},$$

where B_k do no intersect pairwise. But for $\xi \in B_k$, $k \geq 1$, we have

$$|A(\xi)| = |b_k| \left|\psi\left(\frac{\xi - \lambda_k}{h_k}\right)\right| \leq e^{\beta(\lambda_k)}, \quad \beta \in W,$$

so that this integral does not exceed $8e^{\beta(2r+1)} \leq 8e^{\beta(3r)}$, $r \geq 1$. Therefore,

$$|f(z)|^2 \leq 8e^{\beta(3r)} + \int\limits_{|\xi| \leq 2r} |u|^2 \frac{e^{-\varphi}}{\big(1 + |\xi|^2\big)^2} \big(1 + |\xi|^2\big)^2 e^{\varphi} d\lambda(\xi), \quad r = |z|.$$

Applying to the last integral the estimate (2) from Hörmander's theorem, we obtain

$$|f(z)|^2 \leq 8e^{\beta(3r)} + \exp\big\{2\ln(1 + 4r^2) + M_\varphi(2r)\big\} \int\limits_{\mathbb{C}} |g|^2 e^{-\varphi} d\lambda.$$

Taking into account the convergence of the last integral and the estimate (12), we conclude that $|f(z)| \leq e^{w_6(|z|)}$, where $w_6 \in W$. This means that the function $f = A - u$ is a solution of the generalized interpolation problem. The sufficiency is proved.

Necessity. Let $\Lambda = \{\lambda_n\}$ be an interpolation (in the wide sense) sequence and $\tilde{w}$ be a function from the class W (its existence is stated in Definition 2). Therefore, there exists an entire function f, which is a solution of the generalized interpolation problem for $b_1 = 1$ and $b_n = 0$, $n > 1$. From the Jensen inequality, taking into account the property $(2')$ from the definition of an interpolation (in the wide sense) sequence (Definition 2), we obtain

$$n(r) \leq \ln M_f(er) \leq \tilde{w}(er).$$

As was said above, the following integral and series converge simultaneously:

$$\int\limits_1^\infty \frac{n(r)}{r^2} dr, \quad \sum_{n=1}^\infty \frac{1}{\lambda_n}.$$

To prove the condition (b), we fix n and choose an entire function f, which solves the generalized interpolation problem for $b_n = 1$ and $b_k = 0$, $k \neq n$.

The following representation holds:

$$f(z) = \prod_{\substack{\lambda_n/2<\lambda_k<2\lambda_n,\\ k\neq n}} \left(1 - \frac{z}{\lambda_k}\right) G(z), \tag{14}$$

where G is an entire function (if neither of λ_k, $k \neq n$, lies in the interval $(\lambda_n/2, 2\lambda_n)$, then we take $G = f$). For $\lambda_n/2 < \lambda_k < 2\lambda_n$ we have

$$\left|1 - \frac{z}{\lambda_k}\right| \geq \left|1 - \frac{4\lambda_n}{\lambda_k}\right| \geq 1, \quad |z| = 4\lambda_n.$$

This implies that $|G(z)| \leq |f(z)|$, $|z| = 4\lambda_n$. By the maximum principle for the modulus we have

$$\bigl|G(\lambda_n)\bigr| \leq M_G(4\lambda_n) \leq M_f(4\lambda_n) \leq e^{\tilde{w}(4\lambda_n)}. \tag{15}$$

On the other hand, from (14) we have

$$G(\lambda_n) = \prod_{\substack{\lambda_n/2<\lambda_k<2\lambda_n,\\ n\neq k}} \left(1 - \frac{\lambda_n}{\lambda_k}\right)^{-1}, \tag{16}$$

since $f(\lambda_n) = 1$. The relations (15) and (16) imply

$$-\ln \prod_{\substack{\lambda_n/2<\lambda_k<2\lambda_n,\\ k\neq n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| \leq \tilde{w}(4\lambda_n), \quad n \geq 1,$$

where $\tilde{w}$ is a function from the class W.

The theorem is proved. □

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