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Relation between Fourier series and Wiener algebras

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Abstract. New relations between the Banach algebras of absolutely convergent Fourier integrals of complex-valued measures of Wiener and various issues of trigonometric Fourier series (see classical monographs by A. Zygmund [1] and N. K. Bary [2]) are described. Those bilateral interrelations allow one to derive new properties of the Fourier series from the known properties of the Wiener algebras, as well as new results to be obtained for those algebras from the known properties of Fourier series. For example, criteria, i.e. simultaneously necessary and sufficient conditions, are obtained for any trigonometric series to be a Fourier series, or the Fourier series of a function of bounded variation, and so forth. Approximation properties of various linear summability methods of Fourier series (comparison, approximation of function classes and single functions) and summability almost everywhere (often with the set indication) are considered.

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1. The Banach algebras of Wiener and Beurling

If μ is a finite complex-valued Borel measure on $\mathbb{R}$, and $|\mu|$ is its total variation (see, e.g., [3, chapter XI]), then

$$W = W(\mathbb{R}) = \left\{ f: \ f(x) = \int\limits_{\mathbb{R}} e^{-itx} d\mu(t), \quad \|f\|_W = |\mu|(\mathbb{R}) \right\}.$$

This is a Banach algebra (with pointwise multiplication). Then,

$$W_0 = W_0(\mathbb{R}) = \left\{ f: \ f(x) = \widehat{g}(x) = \int\limits_{\mathbb{R}} g(t) e^{-itx} dt, \quad \|f\|_{W_0} = \int\limits_{\mathbb{R}} |g| \right\}$$

is an ideal in $W(\mathbb{R})$. After joining unity to W_0, we have

$$W_1 = W_1(\mathbb{R}) = \{ f: \ f_0 + c, \quad \|f\|_{W_1} = \|f_0\|_{W_0} + |c|, c \in \mathbb{C} \}.$$

Those algebras have a local property, so functions from different algebras can differ only in a neighborhood of ∞. If, for instance, $f \in W$, $f(\infty) = \lim\limits_{|x|\to\infty} f(x) = 0$ (this is necessary), and f is a function of bounded variation in a neighborhood of ∞, then $f \in W_0(\mathbb{R})$ (see, e.g., [4, 6.1.3c]). In the case of functions of several variables, the Vitali variation is considered.

The properties of the indicated algebras, which were known at that time, were collected in the survey article [5]. It contains theorems of Wiener, Titchmarsh, Beurling, Carleman, Krein, Sz.-Nagy, Stein, etc. The list of references includes 175 items.

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If the measure $\mu \geq 0$ in the definition of $W(\mathbb{R})$, then we have a cone $W^+(\mathbb{R})$ of positive definite functions, with $\|f\|_{W^+} = f(0)$ and $\|fg\| = \|f\| \cdot \|g\|$.

There is also the Beurling algebra

$$W_0^*(\mathbb{R}) = \left\{ f: \ f(x) = \int_{\mathbb{R}} e^{-itx} g(t)dt, \ \|f\|_{W_0^*} = \int_0^\infty \operatorname*{esssup}_{|x|\geq t} |g(x)| \ dt \right\}.$$

(See the theorem in [6], and its generalization in [4, 6.4.9].) For the properties of this algebra, see [7].

Here are some properties of the Wiener algebras, including those that were obtained recently. They will be used below.

A. If $f \in W(\mathbb{R})$, then f is uniformly continuous on $\mathbb{R}$, and $\forall x \in \mathbb{R}$, the improper integral

$$\int_{\to 0}^{\to +\infty} \frac{f(x+t) - f(x-t)}{t} \, dt$$

converges (not necessarily absolutely). If $\tilde{f}$ is trigonometrically conjugate to f (the Hilbert transform), then after its extension by continuity, $\|\tilde{f}\|_{W_0} = \|f\|_{W_0}$.

B. If $\|f_n\|_W \leq 1$ $(n \in N)$ and $\lim\limits_{n\to\infty} f_n(x) = f(x)$, whereas $f \in C(\mathbb{R})$, then $\|f\|_W \leq 1$.

C. (the Wiener $1/f$-theorem). If $f \in W_1(\mathbb{R})$ and $f(x) \neq 0$ for $x \in \mathbb{R} \cup \{\infty\}$, then $\dfrac{1}{f} \in W_1(\mathbb{R})$ as well (see the reference in [5]).

D. If $\in W(\mathbb{R})$ $(W_0(\mathbb{R}),\ W^+(\mathbb{R}))$, and l_f is a piecewise linear continuous function defined by its values $l_f(k) = f(k)$, $k \in \mathbb{Z}$ (broken line), then $\|l_f\|_W \leq \|f\|_W$ $(\|l_f\|_{W_0} \leq \|f\|_{W_0},\ l_f \in W^+(\mathbb{R}))$ [9,10].

The relevant story is as follows. This inequality for W^+ can be found already in Feller's book [8, XIX, 9, Problems 15 and 16] (with reference to Larry A. Shepp). Since any measure $\mu = |\mu| - (|\mu| - \mu)$, then the inequality in question,with 3 as a coefficient rather than 1, is true for any real measure, and with 6 for any complex-valued measure. Goldberg [9] proved this inequality, perhaps not knowing this particular case. At the same time, the authors of paper [10] did not know about paper [9] (the reference to paper [9] was added at the last moment, when it was revealed by a graduate student of DonNU).

For the currently available applications, see Section 2.

E. If $f \in W_0 \cap L_1(\mathbb{R})$, then $\widehat{f} \in L_1(\mathbb{R})$ and $\|f\|_{W_0} = \dfrac{1}{2\pi} \int_{\mathbb{R}} \left|\widehat{f}\right|$ [11, Theorem 8].

F. Let $f \in AC_{loc}(\mathbb{R})$, $f(\infty) = 0$, $f_0(x) = \sup\limits_{|t|\geq|x|} |f(t)|$, and $f_1(x) = \operatorname*{esssup}\limits_{|t|\geq|x|} |f'(t)| < \infty$. If, additionally, $A_1 = \int_0^1 f_1(x) \log \dfrac{2}{x} dx < \infty$ and

$$A_{01} = \int_1^\infty \left[\int_t^\infty f_0(x) f_1(x) dx \right]^{\frac{1}{2}} \frac{dt}{t} < \infty,$$

then $f \in W_0(\mathbb{R})$ and $\|f\|_{W_0(\mathbb{R})} < c(A_1 + A_{01})$ [12].

It is important here that only the simultaneous decrease of the function and its derivative is taken into account.

G$_1$. If $A_1 < \infty$, $f(x) = O\left(\frac{1}{|x|^\alpha}\right)$ $(\alpha > 0)$, $f'(x) = O\left(\frac{1}{|x|^\beta}\right)$ $(\beta \in \mathbb{R})$, and $\alpha + \beta > 1$, then $f \in W_0(\mathbb{R})$. Generally speaking, one cannot assume that $\alpha + \beta < 1$ [12].

G$_2$**.** If $f \in L_p(\mathbb{R})$ $(p \in (1,\infty))$, $f' \in L_q(\mathbb{R})$ $(q \in (1,\infty))$, and $\dfrac{1}{p}+\dfrac{1}{q} > 1$, then $f \in W_0$. One cannot assume that $\dfrac{1}{p}+\dfrac{1}{q} < 1$ [13].

The simplest sufficient condition consists in that f and $f' \in L_2(\mathbb{R})$ (Titchmarsh–Beurling). In other cases, the condition $\dfrac{1}{p}+\dfrac{1}{q} = 1$ is not sufficient (see an important example in the introduction of survey [5]; this example was examined in [14]).

H. Functions from $W_0^* \subset W_0$ can decrease arbitrarily slowly as $|x| \to \infty$. At the same time, in the case of convexity, the integral condition in item **A** may be sufficient as well if we take into account the asymptotics of the Fourier transform of a convex function. Namely, if f is convex on $[a,+\infty)$ and $f(\infty)=0$, then $\forall x \in \mathbb{R}\setminus[-2,2]$,

$$\int_a^\infty f(t)e^{itx}dt = \frac{i}{x} f\left(a + \frac{\pi}{|x|}\right) e^{iax} + \theta F\left(|x|\right),$$

where F decreases on $[2,+\infty)$, $\int_2^\infty F \le V_a^\infty(f)$ (total variation), and $|\theta| \le c$ (see, e.g., [4, 6.4.7 b]).

Here is an example (see [7]):

$$f_0(x) = \begin{cases} x\sin\frac{\pi}{x}, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases} \in W_0^*.$$

For the application of this algebra, see Section 4.

2. Trigonometric Fourier series

In what follows, all functions are assumed to be 2π-periodic, and $\mathbb{T} = [-\pi,\pi]$.

If $f \in L_1(\mathbb{T})$, then its Fourier series can be written in the form

$$f \sim \sum_{k\in\mathbb{Z}} \widehat{f}_k e_k, \quad e_k = e^{ikt}, \quad \widehat{f}_k = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-ikt}dt \quad (k \in \mathbb{Z}).$$

The Fourier–Stieltjes series (or that of a measure) with the coefficients

$$\int_{\mathbb{T}} e^{-ikt} d\mu(t)$$

is determined analogously.

If the measures of the points π and $-\pi$ are identical, then the measure μ can be considered 2π-periodic.

Theorem 1 [11]. *The series $\sum_k c_k e_k$ is the Fourier series of a measure (function) if and only if there exists $\varphi \in W(\mathbb{R})$ $(\varphi \in W_0(\mathbb{R}))$, with the condition $\varphi(k) = c_k$ $(k \in \mathbb{Z})$.*

In addition, the variation of the measure on $\mathbb{T}$ equals

$$|\mu|(\mathbb{T}) = \int_{\mathbb{T}} d\,|\mu| = \min_{\varphi} \|\varphi\|_W \quad (\varphi(k) = c_k, k \in \mathbb{Z})$$

and the minimum is reached at $\varphi_0(x) = \int_{\mathbb{T}} e^{-itx} d\mu(t)$.

In the class of entire functions of exponential type at most π*, all such extensions* φ *look like*

$$\varphi(x) = \varphi_0(x) + c \sin \pi x.$$

The measure μ *is positive* $\Leftrightarrow \exists \varphi \in W^+(\mathbb{R})$.

Let $\sum_k c_k e_k$ be a trigonometric series. How can we determine from its coefficients whether it is a Fourier series or belongs to a narrower class than $L_1(\mathbb{T})$?

We obtain several criteria.

Criterion 1. *The series* $\sum_k c_k e_k$ *is a Fourier (Fourier–Stieltjes) series if and only if* $\exists \varphi \in W_0(\mathbb{R})$ $(\varphi \in W(\mathbb{R}))$*, with the condition* $\varphi(k) = c_k$ $(k \in \mathbb{Z})$*. It can be verified on the broken line with the nodes* (k, c_k)*. The function increases on* $\mathbb{T}$ *(the measure is positive) if and only if this broken line belongs to* W_0^+.

Proof. Theorem 1 and property **D** are applied. □

Criterion 2. *The series* $\sum c_k e_k$ *is the Fourier series of a function from* $AC(\mathbb{T})$ *if and only if* $\exists \varphi \in W_0(\mathbb{R})$*, with the condition* $\varphi(k) = c_k$ $(k \in \mathbb{Z})$*, for which* $\varphi_1 \in W_0(\mathbb{R})$ *as well, where* $\varphi_1(x) = x\varphi(x)$*. This can be verified on two broken lines.*

Proof. This criterion follows from Criterion 1. □

Criterion 3. *The series* $\sum c_k e_k$ *is the Fourier series of a function of bounded variation on* $\mathbb{T}$ *if and only if* $\varphi \in W_0(\mathbb{R})$ *and* $\varphi_1 \in W(\mathbb{R})$ *(both functions from Criterion 2).*

Proof. It should be taken into account that $f \in V(\mathbb{T})$ if and only if

$$V(f) = \sup_n \int_{\mathbb{T}} \left| \sum_k k \left(1 - \frac{|k|}{n}\right)_+ \widehat{f}_k e^{ikt} \right| dt < \infty$$

(see [2, Chapter 1, Section 60]) and make use of Theorem 1, **D** and **B**. □

If $f \in L_1(\mathbb{T})$, then the conjugate function exists almost everywhere, but it may be not Lebesgue-integrable.

Criterion 4. *If* $f \in L_1(\mathbb{T})$*, then the conjugate function* $\tilde{f} \in L_1(\mathbb{T})$ *iff the broken line with nodes* $(k, \hat{f}_k \text{sign } k)$ $(k \in \mathbb{Z})$ *belongs to* $W_0(\mathbb{R})$.

Proof. This criterion follows from Criterion 1. □

The criterion for the convergence of a Fourier series in $L_1(\mathbb{T})$ is formulated in a similar way.

We also apply the properties of Wiener algebras to Fourier series.

Theorem 2. *If* $\sum_k c_k e_k \sim d\mu$ *(the Fourier series of the measure),* $\lim_{|k|\to\infty} c_k = 0$ *(this is necessary), and* $\sum_k |c_k - c_{k+1}| < \infty$*, then* $\sum_k c_k e_k$ *is a Fourier series.*

Proof. Taking the broken line $\varphi \in W(\mathbb{R})$ from Criterion 1 and applying the theorem given above after the definition of W_1, we derive that $\varphi \in W_0(\mathbb{R})$. It remains to apply Criterion 1 once more. □

Theorem 3 (new sufficient conditions).

1). If

$$\sum_{m=1}^{\infty}\frac{1}{m}\left(\sum_{k=m}^{\infty}\sup_{|k|\geq m}|c_k|\sup_{|k|\geq m}|c_k-c_{k+1}|\right)^{\frac{1}{2}}<\infty,$$

then $\sum_k c_k e_k$ *is a Fourier series.*

2). If the sequence $\{c_k\}_{-\infty}^{\infty}\in l_p$ *only for a certain* $p\in(2,+\infty)$*, and* $\{c_k-c_{k+1}\}_{-\infty}^{\infty}\in l_q$ *, where* $\frac{1}{p}+\frac{1}{q}>1$*, then* $\sum_k c_k e_k$ *is a Fourier series (if* $p\leq 2$*,this is a Fourier series in* $L_2(\mathbb{T})$*).*

Proof. 1) We should take a broken line with nodes (k, c_k). According to Property **F**, this broken line belongs to $W_0(\mathbb{R})$, and we should apply Theorem 1.

2) Analogously, we should apply Property $\mathbf{G}_2$. □

Corollary. *The condition*

$$\sup_{|k|\geq m}|c_k|\sup_{|k|\geq m}|c_k-c_{k+1}|=O\left(\frac{1}{m\log^{3+\varepsilon}m}\right)$$

is sufficient if $\varepsilon>0$*, but not if* $\varepsilon=0$*. The latter is confirmed by the counterexample* $\sum_k \frac{sign\ k}{\log(|k|+2)}e_k$.

Now, let us give an example of the motion in the opposite direction (from the Fourier series to the Wiener algebras).

Theorem 4. $\forall f\in W_0(\mathbb{R})$*, there exists such an even function* g *that* $g\nearrow+\infty$ *and* $gf\in W_0(\mathbb{R})$*.*

See also corollary in Section 4.

Proof. We should apply Salem's theorem for Fourier series (see [1, Chapter 1 and remarks to Section 11 of Chapter IV]) and Property **D**. □

3. Linear methods for summability of Fourier series

We are going to discuss various aspects of convergence in the norm of $C(\mathbb{T})$ and $L_p(\mathbb{T})$).

Fejér (1904) was the first who studied the convergence of the arithmetic means of partial Fourier sums (the $(C,1)$-summation method),

$$\sigma_n(f;x)=\frac{1}{n+1}\sum_{k=0}^{n}S_k(f;x)=\sum_k\left(1-\frac{|k|}{n+1}\right)_+\widehat{f}_k e^{ikx}\underset{n\to+\infty}{\to}f(x).$$

In essence, the Abel–Poisson summation method was known earlier,

$$f_r(x)=\sum_k r^{|k|}\widehat{f}_k e^{ikx}\underset{r\nearrow 1}{\to}f(x).$$

These are convolution integral operators with positive Fejér and Poisson kernels, since $(1-|x|)_+\in W_0^+(\mathbb{R})$ and $e^{-|x|}\in W_0^+(\mathbb{R})$.

The general linear summability methods determined by a single function $\varphi:\mathbb{R}\to\mathbb{C}$, which is sometimes called the method generator,

$$\Phi_\varepsilon(f)\sim\sum_k\varphi(k\varepsilon)\widehat{f}_k e_k,\tag{3.1}$$

have been studied for a long time. As early as 1968, there arose a question about a comparison of various methods. Namely, for $r = 1 - \frac{1}{n+1}$ $\forall f \in L_p(\mathbb{T})$, $1 \le p \le \infty$ ($L_\infty = C(\mathbb{T})$), we have

$$\|f - f_r\|_p \asymp \|f - \sigma_n(f)\|_p$$

(two-sided inequality with absolute positive constants [15, 16]). In [16], instead of the $(C,1)$-ones, the (C,α)-means are indicated, for any $\alpha > 0$, with the constants depending on α, of course.

For those means to converge, the norms of the operators Φ_ε must be bounded with respect to ε, whereas, for the convergence on polynomials, it must be $\lim\limits_{x\to 0} \phi(x) = \phi(0) = 1$.

Let us further assume that φ is bounded and continuous almost everywhere. The continuity almost everywhere instead of everywhere was added by the author [4].

Theorem 5 (the general comparison principle).

I. Let φ and $\psi \in C(\mathbb{R})$, from the condition $\psi(x) = 1$ it follows that $\varphi(x) = 1$ (this is also necessary), and the "transition function" $g = \dfrac{1-\varphi}{1-\psi}$ after the extension by continuity belongs to $W(\mathbb{R})$. In this case, if $p \in [1, +\infty]$ and $\varepsilon > 0$, then

$$\|f - \Phi_\varepsilon(f)\|_p \le \|g\|_W \ \|f - \Psi_\varepsilon(f)\|_p .$$

If, in addition, $\psi \in W_0(\mathbb{R})$, then, if $p = \infty$, the factor $\|g\|_{W_0}$ cannot be decreased.

II. The same inequality, but with a certain factor independent of f and ε is also valid if $g \in W_1 \backslash W_0(\mathbb{R})$ and $g(x) \ne g(\infty)$ $\forall x \in \mathbb{R}$.

Proof. I is in [4, 7.1.11], with the transition in the inequality from $p = \infty$ to any $p \ge 1$.

II follows from I and the Wiener $1/f$-theorem. □

The comparison in L_p for $p \in (1, +\infty)$, is considered below (see the lemma and its application below in this section).

The Gagliardo–Nirenberg inequality (1959) for various partial and mixed derivatives of the functions of any number of variables in various L_p-norms is well-known.

Let us consider the following question in the one-dimensional case:

When is the inequality

$$\|Q(D)f\|_q \le a \, \|P(D)f\|_p ,$$

where $D = \frac{d}{dx}$, P is a polynomial of degree $r \in \mathbb{N}$, Q is a polynomial of degree $s \in \mathbb{N} \cup \{0\}$, and the constant a does not depend on $f \in W_p^r$, fulfilled?

Note that if the inequality is valid for all functions satisfying the condition $P(D)f = 0$, then $Q(z) \equiv cP(z)$.

If p and $q \in [1, +\infty]$, three criteria – for the sets $\mathbb{T}$, $\mathbb{R}$, and $\mathbb{R}_+$ – were found in [17].

For instance, for the semiaxis $\mathbb{R}_+ = [0, +\infty)$, the criterion reads as follows.

If $p > 1$ and/or $q < \infty$, then

$$\sup_{z:Re z \le 0} \left| \frac{Q(z)}{P(z)} \right| < \infty.$$

But if $p = 1$ and $q = \infty$, then $s < r$ and $\forall \delta < 0$,

$$\sup_{z:Re z \le \delta} \left| \frac{Q(z)}{P(z)} \right| + \sup_{x \in \mathbb{R}} \frac{|Q(ix)|}{|P(ix)| + |P'(ix)|} < \infty.$$

Here is an example of a sharp inequality for the semiaxis:
If $P(z) = (z - \lambda)Q(z)$ and $\mathrm{Re}\lambda > 0$, then $\forall p \geq 1$,

$$\|Q(D)f\|_p \leq \left(\frac{p-1}{p\,\mathrm{Re}\lambda}\right)^{\frac{p-1}{p}} \|P(D)f\|_p .$$

3.1 Approximation of a class of functions

The question is as follows:

$$\sup_{\|f^{(r)}\| \leq 1} \|f - u_n(f)\| = ?$$

Here, $\{u_n\}$ is a sequence of linear bounded operators in $L_p(\mathbb{T})$.

If, following Marcinkiewicz, we introduce the operators

$$u_{0,n}(f;x) = \frac{1}{2\pi} \int_{\mathbb{T}} u_n(f^\theta;x)d\theta,$$

where $f^\theta(x) = f(x+\theta)$, we obtain

$$u_{0,n}(f) \sim \sum_k \lambda_{k,n} \widehat{f}_k e_k \text{ (multipliers, convolutions).}$$

Then, taking into account that the class of functions satisfying the condition $\left\|f^{(r)}\right\| \leq 1$ is translation invariant with respect to the shift $f \to f^\theta$, we obtain that, for this class,

$$\|f - u_{0,n}(f)\| \leq \|f - u_n(f)\|$$

(for more details, see [4, 7.1.1]).

Let us restrict ourselves to consideration of the operators $\Phi_\varepsilon(f)$, with $\varepsilon = \frac{1}{n}$ (for instance, $n \in \mathbb{N}$).

Theorem 6. *If, for $r > 0$ (r is not necessarily integer), the derivative (in the Weyl sense)*

$$f^{(r)} \sim \sum_k e^{i\frac{r\pi}{2} sign k} |k|^r \widehat{f}_k e_k$$

and

$$\widetilde{f} \sim -i \sum_k sign k \widehat{f}_k e_k,$$

then, if a and $b \in \mathbb{C}$, $a \pm bi \neq 0$, and $p \in [1, \infty]$, we have

$$\left\| f - \sum_k \varphi\left(\frac{k}{n}\right) \widehat{f}_k e_k \right\|_p \leq \frac{1}{n^r} \|g\|_{W_0} \left\| a f^{(r)} + b \widetilde{f}^{(r)} \right\|_p ,$$

where

$$g(x) = \frac{[1 - \varphi(x)] e^{-i\frac{r\pi}{2} sign x}}{(a + bi \ sign x)|x|^r}.$$

Theorem 6 follows from the comparison principle (Theorem 5).

The story is as follows. Bernstein (1911) proved, for the class Lip α ($\alpha \in (0,1]$), that

$$\sup_{f:\omega(f;h)\le h^\alpha} \|f-\sigma_n(f)\|_\infty \asymp \begin{cases} \dfrac{1}{n^\alpha} & \text{if } \alpha \in (0,1), \\ \dfrac{\log n}{n} & \text{if } \alpha = 1. \end{cases} \tag{3.2}$$

Jackson, in his dissertation (1911), constructed polynomials satisfying the condition

$$\|f-\tau_n(f)\|_\infty \le c\omega\left(f;\frac{1}{n}\right)_\infty . \tag{3.3}$$

The first book on approximation theory (de la Vallée Poussin, 1919) included the Weierstrass approximation theorem, the Chebyshev alternance theorem, as well as the Markov, Lebesgue, Fejér, Jackson, Bernstein, and de la Vallée Poussin theorems.

The modulus of smoothness of order r ($\omega_1 = \omega$) and step $h>0$ equals

$$\omega_r(f;h)_p = \sup_{0<\delta\le h} \|\Delta_\delta^r f(\cdot)\|_p ,$$

where $\Delta_\delta^r(f)$ is the r-th difference of the step $\delta > 0$. Those moduli were introduced by Bernstein for $r \ge 2(1913)$, and their main properties were determined by Marchaud (1927). It turned out [18] that for any $r \in \mathbb{N}$,

$$\min_{\tau_n} \|f-\tau_n(f)\|_p \le c(r)\omega_r\left(f;\frac{1}{n}\right)_p . \tag{3.4}$$

This implies both Jackson's theorems; see [4, 19, 20].

For approximation of the Lip 1 class, which is the main case,Jackson makes use of the Dirichlet kernel to the fourth power (with the corresponding normalization) instead of the second power used by Fejér. Then, he approximated any continuous function by broken lines with equidistant nodes $\left\{\dfrac{k\pi}{n}\right\}$, which already belong to Lip 1.

As mentioned in [20, p. 236], "Bernstein almost proved Jackson's theorem".

If we take of the Dirichlet kernel to the third power (with the corresponding norming), then, estimating the kernel moments in a similar way, we obtain $O\left(\omega_2\left(f;\frac{\sqrt{\ln n}}{n}\right)\right)$, and not better. If we apply the comparison principle (dealing with the kernel coefficients, i.e.with the operator spectrum), we obtain the exact order $\omega_2\left(f;\frac{1}{n}\right)$ [21].

But in order to apply the comparison principle, it turned out possible to replace $\omega_r(f;h)$ by the equivalent linearized modulus

$$\widetilde{\omega}_r(f;h)_p = \frac{1}{h}\left\|\int_0^h \Delta_\delta^r f(\cdot)\right\|_p ,$$

where sup over $\delta \in (0,h]$ is replaced by the integral mean in δ:

$$\widetilde{\omega}_r(f;h)_p \le \omega_r(f;h)_p \le c(r)\widetilde{\omega}_r(f;h)_p.$$

The left inequality is obvious. To prove the right inequality, the author used the classical Lindemann theorem about the transcendence of the exponential function values [4, 8.3.5 b)].

Note that there are various inequalities of the Marchaud type (for the moduli of smoothness of non-integer order in various metrics, see [22]). In 1957, when nothing was known in Dnipropetrovsk about such inequalities, A.F. Timan deduced the Marshaud inequality from direct and inverse theorems of approximation theory in the following way. The direct theorem gives an estimate from above of the best polynomial approximation via ω_{r_1}. Then, using Bernstein's method and basing on the inequality for the polynomial derivative (it is also possible to do if the metric changes), we arrive at the upper estimate ω_{r_2} for $r_2 < r_1$. A.F. Timan proposed me to give a direct proof of this inequality. But I, a fifth-year student of the DSU at that time, did not succeed. We learned about Marchaud's article (1927) three years later.

Note also that after Kolmogorov's paper (1935), a lot of mathematicians (S.M. Nikolskii, B. Sz.-Nagy, S.A. Telyakovskii, and others [23]) were engaged in the study of asymptotics of approximation of a function class.

3.2 Approximation of individual functions

Quite a long ago, the author found exact two-sided estimates for the approximation of individual functions via the known and new operators [24]. The estimates of approximations from above were known earlier.

Examples. (Rogozinski and Bernstein sums)

$$\left\| f(\cdot) - \frac{1}{2}\left[S_n\left(f; \cdot + \frac{\pi}{2n}\right) + S_n\left(f; \cdot - \frac{\pi}{2n}\right)\right]\right\| \asymp \omega_2\left(f; \frac{1}{n}\right),$$

$$\left\| f(\cdot) - \frac{1}{2}\left[S_n(f; \cdot) + S\left(f; \cdot + \frac{\pi}{n}\right)\right]\right\| \asymp \omega\left(f; \frac{1}{n}\right).$$

In [24], the polynomials of the Rogozinski–Bernstein type determined by the function $\varphi = \varphi_r$ were also indicated. There holds for them, with $\varepsilon = \frac{1}{n}$ (see [1]),

$$\|f - \Phi_\varepsilon(f)\| \asymp \widetilde{\omega}_r(f; \varepsilon) \asymp \omega_r(f; \varepsilon).$$

V. V. Zhuk [25] proved that

$$\|f - \sigma_n(f)\| \asymp \omega_2\left(f; \frac{1}{n}\right) + n\omega_2\left(\widetilde{F}; \frac{1}{n}\right),$$

where $\widetilde{F}$ is the conjugate of the periodic integral $\int\limits_0^x \left(f(t) - \widehat{f}_0\right) dt$. For a simple proof based on the comparison principle, see [4, 5.8.8]. The case $p \in (1, +\infty)$ is considered below in this section.

After the paper by Ditzian and Ivanov [26], such relations are often referred to as "strong converse inequalities". For the Bernstein polynomials used by Stechkin in [18] (see (3.4)), a special modulus of smoothness had to be introduced [27]. Thus, an answer to the question of V. I. Ivanov [28] was obtained.

Let us present only one result for linear means of Fourier series on the torus $\mathbb{T}^d$.

For the Bochner–Riesz means ($|x|$ is the Euclidean norm in $\mathbb{R}^d$, $r \in \mathbb{N}$, $\delta > \dfrac{d-1}{2}$, $\varepsilon > 0$, and $\{e_j\}_1^d$ is a standard basis in $\mathbb{R}^d$), we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \left(1 - \varepsilon^{2r}|k|^{2r}\right)_+^\delta \widehat{f}_k e_k \right\| \asymp \sup_{0<\delta\le\varepsilon} \left\| \left(\Delta_{2,\delta}^+\right)^r f \right\|,$$

where

$$\Delta^+_{2,\delta} f(x) = \sum_{j=1}^{d} \left[f(x - \delta e_j) - 2f(x) + f(x + \delta e_j) \right],$$

or (the answer in another form),

$$\widetilde{\omega}^0_{2r}(f; h) = \left\| \int\limits_{|u|\le 1} \sum_{\nu=0}^{2r} (-1)^\nu \binom{2r}{\nu} f\left(\cdot + (\nu - r)hu\right) du \right\|$$

on the right (see survey [29]).

Let us now consider separately the issue concerning the approximation of functions in $L_p(\mathbb{T})$ norm, for $p \in (1, +\infty)$. According to the M. Riesz projection theorem (see, e.g., [4, 2.4.7]), we have

$$\left\| \sum_{k\ge 0} \widehat{f}_k e_k \right\|_p + \left\| \sum_{k<0} \widehat{f}_k e_k \right\|_p \le c_0(p) \|f\|_p \quad \left(c_0(p) = \frac{2p^2}{p-1} \right).$$

It immediately follows from this that for $\widetilde{f} \sim -i \sum\limits_k \widehat{f}_k \text{sign } k e_k$, there holds

$$\|\widetilde{f}\|_p \le c_0(p) \|f\|_p \quad [p \in (1, +\infty)]. \tag{3.5}$$

It is easy to verify (see also [2, chapter VIII, section 20] that, for $f_1(x) = f(x) \sin nx$ and $f_2(x) = f(x) \cos nx$, we get

$$S_n(f; x) = \widetilde{f}_1(x) \cos nx - \widetilde{f}_2 \sin nx + \frac{1}{2\pi} \int\limits_{\mathbb{T}} f(x+t) \cos nt dt.$$

Therefore, with regard for inequality (3.5) for any n, we obtain

$$\begin{aligned} \|S_n(f)\|_p &\le \|\widetilde{f}_1\|_p + \|\widetilde{f}_2\|_p + (2\pi)^{-\frac{1}{p}} \|f\|_p \le \\ &\le c_0(p) \left(\|f_1\|_p + \|f_2\|_p \right) + \|f\|_p \le \\ &\le c_1(p) \|f\|_p \quad (c_1(p) = 2c_0(p) + 1). \end{aligned} \tag{3.6}$$

By virtue of the Lebesgue inequality, we get

$$\|f - S_n(f)\|_p \le [c_1(p) + 1]\, E^{\mathbb{T}}_n(f)_p, \qquad E^{\mathbb{T}}_n(f)_p = \min_{\tau_n} \|f - \tau_n\|_p,$$

and convergence of S_n as $n \to \infty$ follows.

Lemma. *If $v(\lambda) = \sum\limits_{k\in\mathbb{Z}} |\lambda_k - \lambda_{k+1}| < \infty$, then for $p \in (1, +\infty)$, we have*

$$\left\| \sum_{k\in\mathbb{Z}} \lambda_k \widehat{f}_k e_k \right\|_p \le c(p) \left[v(\lambda) + |\lambda_0| \right] \|f\|_p.$$

Proof. Assuming $S_{-1}(f) = \widetilde{S}_{-1}(f) = 0$, we obtain

$$\sum_{k=0}^{n} \lambda_k \widehat{f}_k e_k = \sum_{k=0}^{n-1} \Delta\lambda_k (S_k(f) + i\widetilde{S_k(f)}) + \lambda_n (S_n(f) + i\widetilde{S_n(f)}),$$

where $\Delta\lambda_k = \lambda_k - \lambda_{k+1}$. Taking into account (3.6) and the inequality $|\lambda_k| \le v(\lambda) + |\lambda_0|$, we get

$$\left\| \sum_{k=0}^{n} \lambda_k \widehat{f}_k e_k \right\|_p \le v(\lambda) c_1(p) \left(\|f\|_p + \|\widetilde{f}\|_p \right) + [v(\lambda) + |\lambda_0|] \left(\|f\|_p + \|\widetilde{f}\|_p \right).$$

Now, applying (3.6), we have that, as $n \to \infty$,

$$\left\| \sum_{k=0}^{\infty} \lambda_k \widehat{f}_k e_k \right\|_p \le c_2(p) \|f\|_p.$$

But in this case,

$$\left\| \sum_{-\infty}^{-1} \lambda_k \widehat{f}_k e_k \right\|_p = \left\| \sum_{k=1}^{\infty} \lambda_{-k} \widehat{f}_{-k} e_{-k} \right\|_p \le [c_2(p) + 2v(\lambda) + 2|\lambda_0|] \, \|f\|_p,$$

which completes the proof. □

Thus, we have the following estimate for the multiplier norm in $L_p(\mathbb{T})$:

$$\Lambda f \sim \sum_{k \in \mathbb{Z}} \lambda_k \widehat{f}_k e_k \quad \Rightarrow \quad \|\Lambda\|_{L_p \to L_p} \le c(p) \, [v(\lambda) + |\lambda_0|] \, .$$

Let us consider $\sigma_n(f)$ as the only example.

Proposition. *If $p \in (1, +\infty)$, then*

$$\|f - \sigma_n(f)\|_p \asymp \omega \left(f; \frac{1}{n} \right)_p .$$

Proof. For comparison, let us take the polynomials

$$\tau_n(f) = \sum_k \varphi \left(\frac{k}{n} \right) \widehat{f}_k e_k, \qquad \varphi(x) = \left(1 - x^2\right)_+ + ix \, (1 - |x|)_+ \, ,$$

for which

$$\|f - \tau_n(f)\|_p \asymp \omega \left(f, \frac{1}{n} \right)_p$$

for any $p \in [1, +\infty]$ (see [4, p. 362]). Taking into account that if $\lambda \ne 0$, then $V_{-\infty}^{\infty}(f(\lambda \cdot)) = V_{-\infty}^{\infty}(f)$, we have to check, by the lemma, that

$$g(x) = \frac{1 - (1 - |x|)_+}{1 - \varphi(x)} \quad \text{and} \quad \frac{1}{g(x)} \in V(\mathbb{R}).$$

This is obvious if $g(0) = g(+0)$, since $g(x) = 1$ for $|x| \ge 1$ and g' is bounded for 0 $0 < |x| < 1$. □

Let us also present a new asymptotic formula for the approximation of individual functions, with the error $\omega_{2m}\left(f;\dfrac{1}{n}\right)$, $m\in\mathbb{N}$.

Theorem 7 ([30]). *Let $\alpha>0$, $2m>\alpha$, and $m_\alpha=\max\{k:\ k\alpha<2m\}$. If $\varphi(0)=1$, $\varphi^{(2m)}\in V\cap Lip\ \delta$ $(\exists\delta>0)$ on $\left[0,\dfrac{2}{3}\right]$, $\varphi\in V\cap Lip\,\delta$ (locally) for $x\geq\dfrac{1}{3}$, and $|\varphi(x)|+|\varphi'(x)|=O\left(\frac{1}{x}\right)$ as $x\to+\infty$, then $\forall f\in C(\mathbb{T})$ as $\varepsilon\to+0$, we have*

$$f(x)-\sum_{k\in\mathbb{Z}}\varphi\left(\varepsilon^\alpha|k|^\alpha\right)\widehat{f}_k e^{ikx}==-2\sum_{k=1}^{m_\alpha}\frac{\varphi^{(k)}(0)}{k!}\frac{1}{C_{1+k\alpha}}\int_1^\infty\frac{\dot{\Delta}^{2m}f(x)}{u^{1+k\alpha}}\,du+O\left(\omega_{2m}(f;\varepsilon)\right),$$

where

$$\dot{\Delta}^{2m}f(x)=\sum_{\nu=0}^{k}\binom{k}{\nu}(-1)^\nu f\left(x+(k-2\nu)h\right)u$$

is the symmetric difference, and, for $q>1$,

$$C_q=C_q(m)=(-1)^m 2^{2m+1}\int_0^\infty\frac{\sin^{2m}t}{t^q}\,dt.$$

The theorem is applicable, e.g., to the Gauss–Weierstrass means ($\varphi(x)=e^{-|x|}$), to the Picard means $\left[\varphi(x)=\left(\frac{1}{1+x}\right)^\beta,\beta\geq1\right]$, and to the Riesz means.

Let us also note the relation between trigonometric series and Fourier integrals due to the author.

Let $n\in\mathbb{Z}$ and let both f and $f^{(r)}\in V(n,\infty)$, with $r\geq0$ integer, and let $f^{(\nu)}(\infty)=0$ $for\nu\in[0,r]$). Then, for $0<|x|\leq\pi$, we have

$$\sum_{k=n}^{\infty}f(k)e^{ikx}=\int_n^\infty f(t)e^{itx}dt+\frac{1}{2}f(n)e^{inx}$$
$$+e^{inx}\sum_{\nu=0}^{r-1}\frac{(-i)^{\nu+1}}{\nu!}h^{(\nu)}(x)f^{(\nu)}(n)+\frac{\theta}{\pi^r}V_n^\infty\left(f^{(r)}\right)$$
$$\left(h(x)=\frac{1}{x}-\frac{1}{2}\operatorname{cotan}\frac{x}{2},\ |\theta|\leq3\right).$$

One arrives at the classical Euler–Maclaurin formula as $x\to0$ (see [4, 4.5.1]). For $r=1$, when the sum on the right is absent, a similar formula was obtained earlier by E.S. Belinsky (see [4, 4.5.1]).

For a generalization to functions of any number of variables, see [31].

4. Summability almost everywhere. A non-linear summation method. Some open problems

Kolmogorov [1, 2] gave an example of a Fourier series diverging everywhere. Luzin (1913) conjectured that the Fourier series of a function from $L_2(\mathbb{T})$ converges almost everywhere [2]. This conjecture was proved by Carleson (a report at the ICM in Moscow in 1966). When presenting this report, Kolmogorov said that this was the best result in the analysis within the past ten years. Hunt immediately strengthened the Carleson theorem for functions from $L_p(\mathbb{T})$ with $p > 1$ [32].

A question: For what functions φ

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(f;x) = f(x)$$

almost everywhere, for any $f \in L_1(\mathbb{T})$?

Immediately after the appearance of Fejér's theorem (see Section 3), Lebesgue proved that

$$\lim_{n\to\infty} \sigma_n(f;x) = f(x)$$

almost everywhere, i.e. at those points x where

$$\lim_{|h|\to 0} \frac{1}{h} \int\limits_0^h \big|f(x+t) - f(x)\big|dt = 0$$

(the so-called Lebesgue points or l-points).

For general singular integrals, the criterion of summability at all Lebesgue points was proved by D. K. Faddeev (1936). For the convolution operators $\Phi_\varepsilon(f)$, this criterion has a simple form (see [7] or [4, 8.1.3]):

$$\varphi(0) = 1, \qquad \varphi \in W_0^*(\mathbb{R}). \tag{4.7}$$

The set of the points of differentiability of the function $F(x) = \int\limits_0^x f$ (d-points) is wider than the set of l-points. Hahn proved [33] that the means $\sigma_n(f;x)$ may diverge at d-points. The means (C,α), for $\alpha > 1$, already converge at all d-points (Hardy).

Theorem 8. (Criterion for σ_n) [34]
For

$$\lim_{n\to\infty} \sigma_n(f;x) = F'(x)$$

to be valid for the function $f \in L_1(\mathbb{T})$ at its d-point x, it is necessary and sufficient that the Fourier series of the continuous function

$$F_x(t) = \frac{1}{t} \int\limits_x^{x+t} f(u)\,du$$

converge at $t = 0$.

Making use of one of the examples of divergent Fourier series of a continuous function, we obtain the Hahn theorem [34].

Here is a general sufficient condition for summability at d-points.

Theorem 9. [35] *Let $\varphi \in W_0(\mathbb{R})$ (this is necessary). If also $\varphi(0) = 1$ and $\varphi_1' \in W_0(\mathbb{R})$, where $\varphi_1(x) = x\varphi(x)$, then $\forall f \in L_1(\mathbb{T})$, we have*

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(f;x) = F'(x)$$

at all its d-points. By this, the condition $\varphi_1' \in W_0(\mathbb{R})$ is not necessary, while the condition $\varphi_1' \in W(\mathbb{R})$ is not sufficient.

Examples. Classical Riesz methods: $\varphi(x) = \big(1-|x|^\alpha\big)_+^\beta$, $\alpha > 0$, $\beta > 1$; Gauss–Weierstrass methods: $\varphi(x) = e^{-|x|^\alpha}$, $\alpha > 0$; and Picard methods: $\varphi(x) = \frac{1}{\big(1+|x|^\alpha\big)^\beta}$, $\alpha > 0$, $\beta > 0$. At the same time, the Rogozinski and Bernstein means may diverge at d-points.

Note that summability at d-points gives rise to summability at l-points, and summability at l-points gives rise to convergence on $C(\mathbb{T})$ and, further, on $L_1(\mathbb{T})$.

Form Theorem 9 and Criterion (4.7), it follows that

$$f, f_1' \in W_0(\mathbb{R}) \Rightarrow f \in W_0^*(\mathbb{R}),$$

where $f_1(x) = xf(x)$.

Marcinkiewicz (1938) proved the uniform convergence of the arithmetic means of square partial sums for continuous periodic functions of two variables as well as the convergence almost everywhere for $f \in L_1 \log_+ L_1(\mathbb{T}^2)$. L. V. Zhizhiashvili [36], using the maximal Hardy–Littlewood function, proved the convergence of the indicated means almost everywhere for all $f \in L_1(\mathbb{T}^2)$ (for more general theorems, see survey [37]). In [38], it was proved that at the Lebesgue points, unlike the one-dimensional case, summability may fail for sums of the Marcinkiewicz type.

Note that, for $f \in L_1(\mathbb{T}^d)$, where $d \geq 2$, there may exist Lebesgue points of two different types (for the criteria for functions φ with compact support, see [39]).

Now, let us consider a nonlinear summability method, namely, strong summability introduced by Hardy and Littlewood [1, 2]:

$$\rho_n(f;x) = \frac{1}{n+1}\sum_{k=0}^{n}\big|f(x) - S_k(f;x)\big| \geq \big|f(x) - \sigma_n(f;x)\big|. \tag{4.8}$$

Let $\{n_k\}_1^\infty$ be a sequence of natural numbers with $n_{k+1} > n_k$. When $\forall f \in C(\mathbb{T})$

$$\lim_{m\to\infty}\frac{1}{m}\Big\|\sum_{k=1}^{m}\big|f(\cdot) - S_{n_k}(f;x)\big|\Big\|_\infty = 0 \quad ?$$

Salem (1955) proved the sufficiency of the power growth for n_m (see [2, Chapter VII, Section 8]) and the general necessary condition $\log n_m = O\big(\sqrt{m}\big)$. It turned out that given the convexity of $\{n_k\}$, this is a necessary and sufficient condition. It was proved independently in [40, 41] (Carleson talked about that at a conference in Budapest in 1979). See also [42]. For generalizations to the multiple case (partial sums over cubes, etc.), see [43].

Unlike $(C, 1)$-means, strong means can diverge at Lebesgue points (Hardy and Littlewood, 1913), but they always converge almost everywhere (Marcinkiewicz–Zygmund, see [1, 2]).

O. D. Gabisoniya [44] determined a set of total measure on $\mathbb{T}$, for which the convergence takes place:

$$\lim_{n\to\infty}\sum_{k=1}^{[2\pi n]}\left\{\frac{n}{k}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\big|f(x+t) - f(x)\big|dt\right\}^2 = 0.$$

We note now that σ_n, and the more so strong means, possess a saturation order:

$$\varliminf_{n\to\infty} n\big\|f - \sigma_n(f)\big\|_\infty = 0 \quad \Rightarrow \quad f = \text{const},$$

and a saturation class (Aleksits, 1941),

$$\|f-\sigma_n(f)\|_\infty = O\Big(\frac{1}{n}\Big) \quad (n\to\infty) \quad \Leftrightarrow \quad \omega(\widetilde{f},h)=O(h) \quad (h\to+0).$$

A formula like the Voronovskaya one and its inversion can also be written. If $\widetilde{f}'\in C(\mathbb{T})$, then

$$f(x)-\sigma_n(f;x)=\frac{\widetilde{f}'(x)}{n}+o\Big(\frac{1}{n}\Big)$$

uniformly in $x\in\mathbb{T}$. If

$$f(x)-\sigma_n(f;x)=\frac{h(x)}{n}+o\Big(\frac{1}{n}\Big)$$

for a certain function $h\in C(\mathbb{T})$, then $\widetilde{f}'(x)=h(x)$ (see [24]).

Theorem 10. *If $p=1$ or $p=\infty$, then*

1) $$\sup_{f:\omega(f;h)_p\le h^\alpha,\alpha\in(0,1]}\|\rho_n(f)\|_p \asymp \begin{cases}\dfrac{1}{n^\alpha} & \text{if } \alpha\in(0,1);\\[2mm] \dfrac{\ln n}{n} & \text{if } \alpha=1;\end{cases}$$

2) $$\sup_{f:\omega_2(f;h)_p\le h}\|\rho_n(f)\|_p \asymp \frac{\ln n}{n}.$$

Lemma. *Let $E_n^{\mathbb{T}}(f)_p=\min\limits_{\tau_n}\|f-\tau_n\|_p$.*

1) If $p=1$, then $\rho_n(f)_1\asymp \dfrac{1}{n+1}\sum\limits_{k=0}^{n}E_k(f)_1$;

2) If $p=+\infty$ and $\forall\varepsilon_n\searrow 0$, then

$$\sup_{f:E_k(f)\le\varepsilon_k,k\in\mathbb{N}\cup\{0\}}\rho_n(f)_\infty \asymp \frac{1}{n+1}\sum_{k=0}^{n}\varepsilon_k.$$

Proof. For the case $p=+\infty$, the inequality

$$\|\rho_n(f)\|_p \le c\frac{1}{n+1}\sum_{k=0}^{n}E_k^{\mathbb{T}}(f)_p \tag{4.9}$$

was proven in [45]. Adding the generalized Minkowski inequality, we can use the same proof for $p\in[1,+\infty)$ as well (the constant c does not change).

If $p=1$, then

$$\rho_n(f)_1=\frac{1}{n+1}\sum_{k=0}^{n}\|f-S_k(f)\|_1 \ge \frac{1}{n+1}\sum_{k=0}^{n}E_k^{\mathbb{T}}(f)_1.$$

If $p=\infty$, we should apply inequalities (4.9) and (4.8), as well as an example of the function from [47] for which

$$f_1(x)=\sum_{k=1}^{\infty}(\varepsilon_{k-1}-\varepsilon_k)\cos kx,$$

$$E_n^T(f_1)\le\varepsilon_n, \rho_n(f_1)\ge|f_1(0)-\sigma_n(f_1;0)|=\frac{1}{n+1}\sum_{k=0}^{n}\varepsilon_k.$$

as $\varepsilon_n\searrow 0$. □

We have also to take into account the classical Bernstein theorem [19]:

For any sequence $\varepsilon_n \searrow 0$*, there exists a function* f *satisfying the condition* $E_k^{\mathbb{T}}(f) = \varepsilon_k$, $k \ge 0$.

Proof. 1) The upper bound for the approximation in Theorem 10 follows from the above lemma and Jackson's theorem, and the lower bound follows from the corresponding result for σ_n (see (3.2)).

2) From Lemma and Zygmund's theorem [19], we get:

$$\omega_2(f;h) = O(h) \quad \Leftrightarrow \quad E_n^{\mathbb{T}}(f) = O\Big(\frac{1}{n}\Big).$$

□

In the same way, one can study the approximation of the class of functions with the condition $\omega_r(f;h) \le \psi(h)$ (or $\omega_r(\widetilde{f},h) \le \psi(h)$), where $\psi \searrow 0$ as $h \searrow 0$, and $r \in (0,2]$.

Here are some more open problems.

a) What are the saturation order and the saturation class for $\rho_n(f)$?

b) What is the special modulus of continuity ω^*, for which

$$\big\|\rho_n(f)\big\|_\infty \asymp \omega^*(f;\varepsilon_n),$$

where $\varepsilon_n \searrow 0$ and does not depend on f?

c) D. Gát [47] proved that if $n_{k+1} > n_k\left(1 + \frac{1}{k^\delta}\right)$, where $k \in \mathbb{N}$ and $\delta \in \left(0, \frac{1}{2}\right)$, then almost everywhere

$$\lim_{m\to\infty} \frac{1}{m+1}\bigg|f(x) - \frac{1}{m+1}\sum_{k=0}^{m} S_{n_k}(f;x)\bigg| = 0.$$

This is a good result. However, a problem about such a convergence at all Lebesgue points has been posed long ago (Zalcwasser, 1936).

See also the theorems about grouped series in [48, 49].

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