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AROUND THE BAER–KAPLANSKY THEOREM

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Abstract. Using examples of modules and a number of familiar Abelian groups, we demonstrate the Kaplansky method of proving isomorphism theorems for endomorphism rings.

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Introduction

One of the central problems concerning endomorphism rings is the question of how much the endomorphism ring defines an Abelian group or a module. In the simplest formulation, the results that positively solve this problem (they are usually called the *isomorphism theorems for endomorphism rings*) have the following form: if $\operatorname{End} A \cong \operatorname{End} B$, then $A \cong B$. A stronger formulation of the isomorphism theorem has the following form: a given ring isomorphism $\psi : \operatorname{End} A \to \operatorname{End} B$ is induced by some group or module isomorphism $\varphi : A \to B$, i.e., $\psi(h) = \varphi h \varphi^{-1}$, $h \in \operatorname{End} A$. (In the case of modules, as a rule, semilinear module isomorphisms arise.) Theorems of this kind are related to the following problem: For which modules are all the automorphisms of their endomorphism rings inner automorphisms?

On endomorphism rings, we can define the finite topology and consider continuous isomorphisms of endomorphism rings. They contain more information about the original modules. In this case, we deal with *topological isomorphism theorems.*

This paper is a survey. Its goal is to give a detailed account of several characteristic isomorphism theorems for endomorphism rings of modules and Abelian groups. What is noteworthy, to prove these theorems, one of the modifications of the Kaplansky method is applied.

Sections 1 and 2 contain some necessary information about Abelian groups and their endomorphism rings. Here we accept agreement on notation and terms.

In Sec. 3, we consider the finite topology on the endomorphism rings. In the remaining Secs. 4–7, isomorphism theorems for endomorphism rings are considered. Moreover, in Sec. 4, we consider the classical case of vector spaces over division rings and fields. Here there is the most transparent

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situation for this problem. In Sec. 5, we prove the Baer–Kaplansky theorem for p-groups. It can serve as a standard for theorems of this kind. In Sec. 6, we prove some theorems of topological isomorphism. In the last section, we show that for p-groups with unbounded basic subgroups, in the Baer–Kaplansky theorem, one can confine ourselves to an isomorphism between Jacobson radicals of endomorphism rings.

1. Some Definitions and Notation in Abelian Group Theory

The word "group" means an Abelian group with additive notation, with the exception of the automorphism group. Most often, we denote the group by the letters A or G.

In Abelian group theory, direct sums and direct products play a very important role. The direct sum of the groups $A_1, A_2, \dots, A_n$ is denoted by $A_1 \oplus A_2 \oplus \dots \oplus A_n$ or $\bigoplus_{i=1}^{n} A_i$. We denote by $\bigoplus_{i\in I} A_i$ the direct sum of the groups A_i, $i \in I$.

For an element a of the group A, the least positive integer n with $na = 0$ is called the *order* of a; it is denoted by $o(a)$. If such an integer n does not exist, then we set $o(a) = \infty$ and say that the element a is of *infinite order*.

Each Abelian group belongs to one of the following three classes of groups: periodic groups, torsion-free groups, and mixed groups. In a periodic group, every element is of finite order; in a torsion-free group, on the contrary, all nonzero elements are of infinite order. A mixed group contains both nonzero elements of finite order and elements of infinite order. In a mixed group A, the subgroup $t(A)$ consisting of all elements of finite order is called the *periodic part* or the *periodic subgroup* of the group A.

A group A is called a *primary* group or a *p-group* if the order of any element of A is a power of a fixed prime integer p. A torsion group A is equal to the direct sum of p-groups $t_p(A)$ for various p; the groups $t_p(A)$ are called *p-components* of the group A. If A is a mixed group, then the p-component of its torsion part $t(A)$ is called the *p-component* of A.

There are several known classes of groups which are direct sums of cyclic groups. A group is said to be *bounded* if the orders of all its elements are bounded by some positive integer. A bounded group is the direct sum of cyclic p-groups. If every element of a torsion group A is of order not divisible by a square of a positive integer, then A is called an *elementary* group. An elementary group is the direct sum of cyclic groups of prime orders.

For a group A, a subgroup H of A is said to be *fully invariant* if $\alpha H \subseteq H$ for every endomorphism α of the group A. The torsion part of the group and its p-components are fully invariant subgroups. For every $n \in \mathbb{N}$, we set

$$nA = \{na \mid a \in A\}, \quad A[n] = \{a \in A \mid na = 0\}.$$

The subgroups nA and $A[n]$ are fully invariant in A.

Let p be some prime integer, A be a group, and let $a \in A$. The largest nonnegative integer k such that the equation $p^k x = a$ has a solution in A is called the *p-height* $h_p^A(a)$ of the element a in the group A. If the equation $p^k x = a$ is solvable for any positive integer k, then a is called an element of infinite p-height, $h_p^A(a) = \infty$. If A and p are clear, then we write $h_p(a)$ and $h(a)$ and call $h(a)$ the height of the element a.

One says that a subgroup B of the group A is *pure* (in A) if the equation $nx = b \in B$, which has a solution in the group A, also has a solution in the subgroup B. The subgroup B is pure if and only if $B \cap nA = nB$ for all $n \in \mathbb{Z}$.

We give some properties of free or divisible groups. A free group is the direct sum of some number of copies of the group $\mathbb{Z}$. A *divisible* group D is the direct sum of quasi-cyclic groups $\mathbb{Z}_{p^\infty}$ for various p and copies of the group $\mathbb{Q}$,

$$D = \bigoplus_{p} \bigoplus_{\mathfrak{m}_p} \mathbb{Z}_{p^\infty} \oplus \bigoplus_{\mathfrak{m}_0} \mathbb{Q}.$$

The cardinal numbers $\mathfrak{m}_0$ and $\mathfrak{m}_p$ over all p form a complete independent system of invariants of the divisible group D. A group is said to be *reduced* if it does not have a nonzero divisible subgroup. Every group is representable in the form $A = D \oplus V$, where D is a divisible group and V is a reduced group. The subgroup D is uniquely determined; it is called the *divisible part* of the group A. The subgroup V is called the *reduced part* of the group A. It is uniquely determined up to isomorphism. Every group A can be embedded in a divisible group E as a subgroup, and E does not have a proper divisible subgroup containing A. Such a group E is called the *divisible hull* of the group A. Any two divisible hulls E_1 and E_2 are isomorphic over A, i.e., there exists an isomorphism $E_1 \to E_2$ fixing the elements of A.

A subgroup B of the p-group A is called a *basis* subgroup if the following three conditions hold:

(1) B is the direct sum of cyclic groups;
(2) B is a pure subgroup of the group A;
(3) the factor group A/B is a divisible group.

Every p-group contains basis subgroups. Any two basis subgroups are isomorphic.

Let B be a basis subgroup of the group A. We have

$$B = B_1 \oplus B_2 \oplus \ldots \oplus B_n \oplus \ldots, \quad \text{where} \quad B_n = \bigoplus_{\mathfrak{m}_n} \mathbb{Z}_{p^n}.$$

For every n, there are direct decompositions

$$A = B_1 \oplus B_2 \oplus \ldots \oplus B_n \oplus A_n \quad \text{and} \quad A_n = B_{n+1} \oplus A_{n+1}$$

In addition, the group A_n does not contain a cyclic direct summand of the order $\leqslant p^n$.

For p-groups, the following classical results of Kulikov are of great importance.

Theorem 1.1 (see [11, Theorem 27.5, Corollary 27.2])**.** *Every bounded pure subgroup is a direct summand. In any p-group G, every element of order p and of finite height can be embedded in a cyclic direct summand of G.*

Therefore, if an element x of the p-group A is of finite height, then there exists a decomposition $A = \langle y \rangle \oplus Y$ such that the element x has a nonzero component in $\langle y \rangle$.

In Abelian group theory, the $\mathbb{Z}$-adic topology and the p-adic topology are often used. In the $\mathbb{Z}$-adic topology of the group A, the subgroups nA, $n \in \mathbb{N}$, form a basis of neighborhoods of zero. In the p-adic topology, the subgroups p^kA, $k \geqslant 0$, form a basis. The structure of groups that are complete in these topologies is known (see [11, Chap. 7], [18, Sec. 11]); such groups have a complete independent system of invariants consisting of cardinal numbers. The class of groups that are complete in the $\mathbb{Z}$-adic topology coincides with the class of reduced algebraically compact groups.

For a torsion-free group A, the *rank* $r(A)$ of A is the cardinality of some maximal linearly independent system of elements of the group A, and the *p-rank* $r_p(A)$ of the group A is the dimension of the vector space A/pA over the field $\mathbb{Z}_p$,

$$r_p(A) = \dim_{\mathbb{Z}_p}(A/pA).$$

We have the inequality

$$r_p(A) \leqslant r(A).$$

For a mixed group A, the rank of the factor group $A/t(A)$ is called the *torsion-free rank* of the mixed group A. Usually, one says simply the "rank" instead of the "torsion-free rank." A torsion-free group G is of rank 1 if and only if G is isomorphic to some subgroup of the group $\mathbb{Q}$.

The tensor product of torsion-free groups also is a torsion-free group. For a torsion-free group A, the tensor product $A \otimes \mathbb{Q}$ is a $\mathbb{Q}$-space and

$$r(A) = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}).$$

We have a group embedding $A \to A \otimes \mathbb{Q}$, $a \mapsto a \otimes 1$, $a \in A$. The group $A \otimes \mathbb{Q}$ is called the *divisible hull* of the group A.

For torsion-free groups, there are very useful notions related to divisibility of elements by prime integers. Let $p_1, p_2, \dots, p_n, \dots$ be a sequence of all prime integers which is ordered in ascending order. The sequence of p-heights

$$\chi(a) = \Big(h_{p_1}(a), h_{p_2}(a), \dots, h_{p_n}(a), \dots\Big)$$

is called the *characteristic* of the element a of the torsion-free group A. One writes $\chi_A(a)$ if one wants to specify the group in which the p-height and characteristics are calculated.

Any ordered sequence $(k_1, k_2, \dots, k_n, \dots)$ of nonnegative integers and the symbols ∞ is called a *characteristic*. Characteristics can be compared. Namely, we assume

$$\big(k_1, k_2, \dots, k_n, \dots\big) \leqslant \big(l_1, l_2, \dots, l_n, \dots\big),$$

if $k_n \leqslant l_n$ for all $n \in \mathbb{N}$. The relation $\leqslant$ turns the set of all characteristics into a complete lattice.

Now we introduce an equivalence relation on the set of all characteristics which leads to some basic concept for torsion-free groups: the concept of type. Two characteristics $(k_1, k_2, \dots, k_n, \dots)$ and $(l_1, l_2, \dots, l_n, \dots)$ are said to be *equivalent* if $k_n \neq l_n$ only for a finite number of the subscripts n and, for such n, the symbols k_n and l_n are finite.

In the set of characteristics, equivalence classes are called *types*. If $\chi(a) \in \mathbf{t}$ for some type $\mathbf{t}$, then one says that the element a is of type $\mathbf{t}$ and we write $\mathbf{t}(a) = \mathbf{t}$ or $\mathbf{t}_A(a) = \mathbf{t}$. If $\chi(a) = (k_1, k_2, \dots, k_n, \dots)$, then we write

$$\mathbf{t}(a) = \big[(k_1, k_2, \dots, k_n, \dots)\big] = \big[(k_i)\big],$$

i.e., any type is represented by the characteristic belonging to this type. The ordering on the set of characteristics induces the ordering on the set of types.

A torsion-free group A, in which all nonzero elements the same type $\mathbf{t}$, is said to be *homogeneous of type* $\mathbf{t}$. In addition, we write $\mathbf{t}(A) = \mathbf{t}$. A group of rank 1 is homogeneous. By one of Baer's theorems, two torsion-free groups A, B of rank 1 are isomorphic if and only if $\mathbf{t}(A) = \mathbf{t}(B)$.

A torsion-free group A is said to be *fully decomposable* if it is the direct sum of groups of rank 1. Any two decompositions of the fully decomposable group into the direct sum of groups of rank 1 are isomorphic. Let $A = \bigoplus\limits_{i \in I} A_i$ be a fully decomposable torsion-free group, $r(A_i) = 1$, $i \in I$. For every type $\mathbf{t}$, we denote by $A_{\mathbf{t}}$ the direct sum of all groups A_i of type $\mathbf{t}$; if A does not have such $A_{\mathbf{t}}$, then $A_{\mathbf{t}} = 0$. The ranks $r(A_{\mathbf{t}})$, where $\mathbf{t}$ runs over the set of all types, form a complete independent system of invariants of the group A. We set $\Omega(A) = \{\mathbf{t}(A_i) \mid i \in I\}$. The decomposition

$$A = \bigoplus_{\mathbf{t} \in \Omega(A)} A_{\mathbf{t}}$$

is called the *canonical decomposition* of the group A.

In comparison with completely decomposable groups, separable groups form a broader class. A torsion-free group A is said to be *separable* if every finite subset of elements of A is contained in some fully decomposable direct summand of the group A.

Abelian groups and modules over the ring of integers $\mathbb{Z}$ cannot be distinguished; they are the same objects. The Abelian group theory can be considered a branch of module theory in which the specificity of the ring $\mathbb{Z}$ is used. All concepts and constructions of the theory of modules are applicable to Abelian groups.

In the theory of Abelian groups, various other rings also occur, in addition to the ring $\mathbb{Z}$. A glance at Abelian groups as modules over some rings is sometimes useful. Thus, divisible torsion-free groups are precisely vector spaces over the field $\mathbb{Q}$. The modules over the ring of residue classes $\mathbb{Z}_{p^n}$ are bounded p-groups whose orders do not exceed p^n.

From the point of view of the Abelian group theory, the most important ring is the ring of p-adic integers $\widehat{\mathbb{Z}}_p$. It is known that p-groups and primary $\widehat{\mathbb{Z}}_p$-modules coincide. The groups complete in the

p-adic topology are naturally $\widehat{\mathbb{Z}}_p$-modules. The ring $\widehat{\mathbb{Z}}_p$ is an example of a complete discrete valuation domain. Theories of abelian groups and modules over discrete valuation domains are related theories. The latter is described in some detail in the book [18].

2. Primary Properties of Endomorphisms of Abelian Groups

If A and B are two Abelian groups, then $\mathrm{Hom}(A, B)$ is the group of all homomorphisms from A into B, and $\mathrm{End}\, A$ (sometimes $\mathrm{End}(A)$) is the endomorphism ring of the group A.

We give some elementary properties of endomorphism rings. All of them are valid for endomorphism rings of arbitrary modules.

Let $A = B \oplus C$ be some direct decomposition of the group A. The *projection* of the group A onto the direct summand B with kernel C is the homomorphism $\pi : A \to B$, which is defined as follows. If $a \in A$ and $a = b + c$, where $b \in B$ and $c \in C$, then $\pi(a) = b$. We denote by $i : B \to A$ the embedding of the group B in the group A. Then $i\pi \in \mathrm{End}\, A$ and $(i\pi)^2 = i\pi$, i.e., $i\pi$ is an idempotent of the ring $\mathrm{End}\, A$. It is called an *idempotent endomorphism* of the group A. We set $\varepsilon = i\pi$ and identify ε with π. Thus, we assume that the projection π is an endomorphism of A acting on B identically and annihilating C. It is clear that $1 - \varepsilon$ also is an idempotent orthogonal to ε. In addition, $B = \varepsilon A$ and $C = (1 - \varepsilon)A = \ker \varepsilon$, whence $A = \varepsilon A \oplus (1 - \varepsilon)A$. The obtained decomposition holds for any idempotent ε of the ring $\mathrm{End}\, A$.

More generally, if $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$ is some direct decomposition of the group A, then we denote the projection $A \to A_i$ with kernel $\bigoplus_{j \neq i} A_j$ by ε_i and obtain $A_i = \varepsilon_i A$ $(i = 1, 2, \ldots, n)$; in addition, $\{\varepsilon_i \mid i = 1, 2, \ldots, n\}$ is a complete orthogonal system of idempotent endomorphisms of the group A.

Proposition 2.1. *There is a bijective correspondence*

$$A = \varepsilon_1 A \oplus \ldots \oplus \varepsilon_n A \mapsto \mathrm{End}\, A = (\mathrm{End}\, A)\varepsilon_1 \oplus \ldots \oplus (\mathrm{End}\, A)\varepsilon_n$$

between finite direct decompositions of the group A and decompositions of the ring $\mathrm{End}\, A$ *into direct sums of left ideals, where $\{\varepsilon_i \mid i = 1, 2, \ldots, n\}$ is a complete orthogonal system of idempotents of the ring* $\mathrm{End}\, A$.

Proof. We have already proved that for a given direct decomposition $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$, there exists a complete system $\{\varepsilon_i \mid i = 1, 2, \ldots, n\}$ of orthogonal idempotents of the ring $\mathrm{End}\, A$ such that $A_i = \varepsilon_i A$ for all i. This system leads to the decomposition $\mathrm{End}\, A = (\mathrm{End}\, A)\varepsilon_1 \oplus (\mathrm{End}\, A)\varepsilon_2 \oplus \ldots \oplus (\mathrm{End}\, A)\varepsilon_n$ of the ring $\mathrm{End}\, A$ into the direct sum of left ideals.

Conversely, if $\mathrm{End}\, A = L_1 \oplus L_2 \oplus \ldots \oplus L_n$, where L_i are left ideals of the ring $\mathrm{End}\, A$, then we write $1 = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$, $\varepsilon_i \in L_i$, and obtain a complete orthogonal system $\{\varepsilon_i \mid i = 1, 2, \ldots, n\}$ of idempotents of the ring $\mathrm{End}\, A$. It is easy to verify that we have the decomposition $A = \varepsilon_1 A \oplus \varepsilon_2 A \oplus \ldots \oplus \varepsilon_n A$. The constructed correspondence is bijective. □

We consider several standard relations between a group and its endomorphism ring related to idempotent endomorphisms. The following properties directly follow from Proposition 2.1.

(a) *If ε is an idempotent of the ring* $\mathrm{End}\, A$, *then εA is an indecomposable direct summand of the group A if and only if ε is a primitive idempotent.*
(b) *Let ε and ω be two idempotents of the ring* $\mathrm{End}\, A$. *There exist canonical group isomorphisms* $\mathrm{Hom}(\omega A, \varepsilon A) \cong \varepsilon(\mathrm{End}\, A)\omega$ *and canonical ring isomorphisms* $\mathrm{End}(\varepsilon A) \cong \varepsilon(\mathrm{End}\, A)\varepsilon$.

Indeed, let $\varphi : \omega A \to \varepsilon A$ be some homomorphism. It can be extended to an endomorphism $\overline{\varphi}$ of the group A such that $\overline{\varphi}$ annihilates the complement summand $(1 - \omega)A$ of ωA. We obtain the required isomorphism $f : \mathrm{Hom}(\omega A, \varepsilon A) \to \varepsilon(\mathrm{End}\, A)\omega$ from the correspondence $\varphi \xmapsto{f} \varepsilon\overline{\varphi}\omega$. Indeed, if $\varepsilon\psi\omega \in \varepsilon(\mathrm{End}\, A)\omega$ for some $\psi \in \mathrm{End}\, A$, then $\varepsilon\psi\omega|_{\omega A}$ is a homomorphism $\omega A \to \varepsilon A$ and $f : \varepsilon\psi\omega\big|_{\omega A} \mapsto \varepsilon\psi\omega$.

For $\varepsilon = \omega$, we have the isomorphism $\mathrm{End}(\varepsilon A) \cong \varepsilon(\mathrm{End}\, A)\varepsilon$; this is a ring isomorphism.

Let $A = B \oplus C$ and $\varepsilon : A \to B$ be the projection with kernel C. We can assume that $\operatorname{End} B$ is a subring of the ring $\operatorname{End} A$ if we identify $\operatorname{End} B$ with $\varepsilon(\operatorname{End} A)\varepsilon$ by the use of the isomorphism constructed in the property (b).

We consider two primary facts on relations between isomorphisms of groups and isomorphisms of their endomorphism rings.

(c) *If two groups A and C are isomorphic, then their endomorphism rings are isomorphic. More precisely, every group isomorphism $\varphi : A \to C$ induces the ring isomorphism $\psi : \operatorname{End} A \to \operatorname{End} C$ which acts by the rule $\psi : \eta \mapsto \varphi\eta\varphi^{-1}, \eta \in \operatorname{End} A$.*

For $\eta_1, \eta_2 \in \operatorname{End} A$, we have the relations

$$\psi(\eta_1 + \eta_2) = \psi(\eta_1) + \psi(\eta_2), \quad \psi(\eta_1\eta_2) = \psi(\eta_1)\psi(\eta_2), \quad \psi(id_A) = id_C.$$

Consequently, ψ is a ring homomorphism. Further, if $0 \neq \eta \in \operatorname{End} A$, then it is clear that $\varphi\eta\varphi^{-1} \neq 0$, i.e., $\ker \psi = 0$. Now let $\xi \in \operatorname{End} C$; then $\psi(\varphi^{-1}\xi\varphi) = \xi$; therefore, ψ is a ring isomorphism.

(d) *Let $A = A_1 \oplus A_2$ and C be groups. If $\psi : \operatorname{End} A \to \operatorname{End} C$ is a ring isomorphism, then the group C has the decomposition $C = C_1 \oplus C_2$, where ψ induces isomorphisms* $\operatorname{End} A_i \to \operatorname{End} C_i$, *$i = 1, 2$.*

We denote by ε the projection $A \to A_1$ with kernel A_2. Then $\omega = \psi(\varepsilon)$ is an idempotent of the ring $\operatorname{End} C$. We have the relation $C = C_1 \oplus C_2$, where $C_1 = \omega C$ and $C_2 = \ker \omega$. The isomorphism ψ induces the ring isomorphism $\varepsilon(\operatorname{End} A)\varepsilon \to \omega(\operatorname{End} C)\omega$ and, therefore, the ring isomorphism $\operatorname{End} A_1 \to \operatorname{End} C_1$; see property (b). The second isomorphism $\operatorname{End} A_2 \to \operatorname{End} C_2$ can be proved similarly.

In some cases, the endomorphism ring can be easily calculated. For example, we have $\operatorname{End} \mathbb{Z} \cong \mathbb{Z}$, $\operatorname{End} \mathbb{Z}_n \cong \mathbb{Z}_n$, $\operatorname{End} \mathbb{Q}_p \cong \mathbb{Q}_p$, where $\mathbb{Q}_p = \{s/t \in \mathbb{Q} \mid (t, p) = 1\}$, $\operatorname{End} \mathbb{Q} \cong \mathbb{Q}$, $\operatorname{End} \mathbb{Z}_{p^\infty} \cong \operatorname{End} \widehat{\mathbb{Z}}_p \cong \widehat{\mathbb{Z}}_p$.

By considering direct sums of groups, we can obtain examples of endomorphism rings in the matrix form. First, we consider the corresponding construction.

Let us have the direct sum of groups $A = \bigoplus_{i=1}^{n} A_i$. We construct the square matrix

$$(\alpha_{ji}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{pmatrix}$$

with elements $\alpha_{ji} \in \operatorname{Hom}(A_i, A_j)$. For such matrices, one can define ordinary matrix operations of addition and multiplication. It is easy to see that addition and multiplication of matrices are always feasible and lead to matrices of the same form. As a result, we obtain a ring of matrices of the indicated type; such rings are called *formal matrix rings* or *generalized matrix rings*, (see [19–21]). The standard isomorphism from the ring of operators of a finite-dimensional vector space onto the matrix ring in the case of Abelian groups (and modules) takes the following form.

Proposition 2.2. *The endomorphism ring of the group $A = \bigoplus_{i=1}^{n} A_i$ is isomorphic to the ring of all matrices $\big(\alpha_{ji}\big)$ of order n, where $\alpha_{ji} \in \operatorname{Hom}(A_i,\, A_j)$.*

Now we can continue the list of endomorphism rings. Namely, there are isomorphisms

$$\operatorname{End}(\mathbb{Z} \oplus \mathbb{Q}) \cong \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}, \qquad \operatorname{End}(\mathbb{Z}_n \oplus \mathbb{Z}) \cong \begin{pmatrix} \mathbb{Z}_n & \mathbb{Z}_n \\ 0 & \mathbb{Z} \end{pmatrix},$$

$$\operatorname{End}(\mathbb{Z}_{p^n} \oplus \mathbb{Z}_{p^m}) \cong \begin{pmatrix} \mathbb{Z}_{p^n} & \mathbb{Z}_{p^n} \\ \mathbb{Z}_{p^n} & \mathbb{Z}_{p^m} \end{pmatrix} \ (n < m), \quad \operatorname{End}(\mathbb{Z}_{p^\infty} \oplus \mathbb{Q}) \cong \begin{pmatrix} \widehat{\mathbb{Z}}_p & \widehat{\mathbb{Q}}_p \\ 0 & \mathbb{Q} \end{pmatrix},$$

where $\widehat{\mathbb{Q}}_p$ is the field of p-adic numbers.

The part of algebra that studies endomorphism rings of Abelian groups can also be referred to the theory of Abelian groups and to the theory of endomorphism rings of arbitrary modules. The monograph [17] is entirely devoted to various aspects of the theory of endomorphism rings of Abelian groups.

3. Finite Topology

On the ring of endomorphisms of any module, we can define one very useful topology. This is the so-called *finite topology*, which is an example of a linear topology. A *linear topology* on a module (ring) is a topology for which there is a basis of neighborhoods of zero consisting of submodules (left ideals) and the corresponding residue classes form a basis of open sets. Examples of linear topologies are the $\mathbb{Z}$-adic topology and the p-adic topology on Abelian groups.

Let M be a right module over by some ring and $\operatorname{End} M$ its endomorphism ring. The *finite topology* is defined on the ring $\operatorname{End} M$ with the use of the following subbasis of neighborhoods of zero:

$$U_x = \{\alpha \in \operatorname{End} M \mid \alpha(x) = 0\},$$

where x runs over all the elements of the module M. It is clear that U_x are left ideals of the ring $\operatorname{End} M$. The ideals

$$U_X = \{\alpha \in \operatorname{End} M \mid \alpha X = 0\}$$

form a basis of neighborhoods of zero where X runs over all finite subsets of the module M. Since we have $U_X = \bigcap\limits_{x \in X} U_x$, the residue classes $\alpha + U_X$, for all finite subsets X of the module M, form a basis of neighborhoods of the element $\alpha \in \operatorname{End} M$. The finite topology is always a Hausdorff topology. We formulate the main theorem about the finite topology.

Theorem 3.1. *The endomorphism ring of any module M is a topological ring which is complete in the finite topology.*

Proof. Since U_x are left ideals, it is obvious that the addition and the subtraction are continuous in the ring $\operatorname{End} M$. Now we verify that the multiplication is continuous. We take arbitrary endomorphisms $\alpha, \beta \in \operatorname{End} M$, and let $\alpha\beta + U_x$ be a neighborhood of the element $\alpha\beta$. Since $U_{\beta(x)}\beta \subseteq U_x$, we have

$$(\alpha + U_{\beta(x)})(\beta + U_x) \subseteq \alpha\beta + U_x,$$

which implies that the multiplication is continuous.

Thus, $\operatorname{End} M$ is a topological ring. We prove this ring is complete. We assume that $\{\alpha_i\}_{i \in I}$ is a Cauchy sequence in ring $\operatorname{End} M$. By the definitions of the finite topology and a Cauchy sequence, the set of subscripts I is ordered with respect to the order which is dual to the order on the finite subsets of the module M. For a given element $x \in M$, there exists a subscript $i_0 \in I$ such that $\alpha_i - \alpha_j \in U_x$ for all i, $j > i_0$. This means that $\alpha_i(x) = \alpha_j(x)$ for quite large subscripts i and j. There exists an endomorphism α of the module M such that $\alpha(x)$ the common value of all such $\alpha_i(x)$. Then $\alpha - \alpha_i \in U_x$ for $i > i_0$. Thus, α is the limit of this Cauchy sequence $\{\alpha_i\}_{i \in I}$. We obtain that every Cauchy sequence converges in the ring $\operatorname{End} M$; this means that $\operatorname{End} M$ is complete. □

When applying the finite topology, the most important are the completeness of the ring endomorphisms and continuous isomorphisms between endomorphism rings.

Sometimes, the finite topology can be defined in terms of the endomorphism ring itself.

Proposition 3.2. *The following assertions hold.*

1. *Let V be a vector space over some division ring. The finite topology of the operator ring* $\operatorname{End} V$ *of the space V will be defined if we take the set of left annihilators of primitive idempotents as a subbasis of neighborhoods of zero.*

2. *Let G be a reduced p-group. The finite topology of the ring* $\operatorname{End} G$ *can be defined if we take the set of left annihilators of elements $\alpha\varepsilon$ as a subbasis neighborhoods of zero, where $\alpha \in \operatorname{End} G$ and ε is a primitive idempotent. If the group G does not have elements of infinite height, then it suffices to take only left annihilators of primitive idempotents.*
3. *If G is a separable torsion-free group, then the finite topology of the ring* $\operatorname{End} G$ *can be similarly defined by taking the set of left annihilators of primitive idempotents as a subbasis of neighborhoods of zero.*

Proof. 1. Let x be an arbitrary vector of the space V, X be the subspace in V generated by the vector x, and let $\varepsilon : V \to X$ be the projection. The relation $U_x = (\operatorname{End} V)(1-\varepsilon)$ means that U_x is a left annihilator of the primitive idempotent ε.

2. For every element $x \in G$, there exists of cyclic direct summand $\langle y\rangle$ of the group G with $o(x) \leqslant o(y)$. Consequently, there exists an endomorphism $\alpha \in \operatorname{End} G$ which maps y onto x. Let $\varepsilon : G \to \langle y\rangle$ be the projection. Then U_x coincides with the left annihilator of the element $\alpha\varepsilon$ and ε is a primitive idempotent. If the group G does not have elements of infinite height, then for the finite topology, the set of left ideals U_x is a subbasis of neighborhoods of zero, where element x runs over only elements such that $\langle x\rangle$ is a direct summand of the group G. If $\varepsilon : G \to \langle x\rangle$ is the projection, then $U_x = (\operatorname{End} V)(1-\varepsilon)$, similarly to item 1.

3. Similarly to item 2, a subbasis of neighborhoods of zero can be defined as the set of left ideals U_x, where x runs over only elements such that $\langle x\rangle_*$ is a direct summand of the group G, where $\langle x\rangle_*$ is a pure subgroup generated by the element x. ☐

4. The Case of Vector Spaces

Of course, isomorphic groups have isomorphic endomorphism rings. In general, the converse problem is much more difficult. In its most general form, it can be formulated as follows: how are two groups connected if their endomorphism rings are isomorphic to each other? For example, will these groups be isomorphic? The natural formulation of this problem is suggested by the property (c) in Sec. 2. Will a given ring isomorphism $\psi : \operatorname{End} A \to \operatorname{End} B$ be induced by some group isomorphism $\varphi : A \to B$, i.e., whether the formula $\psi(\eta) = \varphi\eta\varphi^{-1}$ holds for all $\eta \in \operatorname{End} A$?

When passing to modules, new versions of the formulation of the problem arise. Thus, we can consider modules over different rings. In this situation, semilinear isomorphisms of modules are used.

Let R and S be two rings, A be a right R-module, and B be a right S-module. An additive isomorphism $\varphi : A \to B$ is called a *semilinear isomorphism* of the modules A and B if there is a ring isomorphism $\tau : R \to S$ such that $\varphi(ar) = \varphi(a)\tau(r)$ for all $a \in A$ and $r \in R$. In addition, one says that a ring isomorphism (or an algebra isomorphism) of endomorphism rings $\psi : \operatorname{End}_R A \to \operatorname{End}_S B$ is induced by a semilinear isomorphism $\varphi : A \to B$ if $\psi(\eta) = \varphi\eta\varphi^{-1}$, $\eta \in \operatorname{End}_R A$.

This section and the following two sections are directly related to the definability problem of a group (module) by its endomorphism ring. First of all, we turn to the case of vector spaces and their operator rings. By virtue of the simple (from the point of view of module theory) structure of vector spaces, a number of ideas and methods, which we will continue to apply here, appear in a rather simple and direct form.

We will consider the right vector spaces over division rings.

Theorem 4.1 (Baer [2])**.** *Let V and W be two vector spaces over division rings D and F, respectively. Then every ring isomorphism from the operator ring* $\operatorname{End}_D V$ *onto* $\operatorname{End}_F W$ *is induced by some semilinear isomorphism from V onto W.*

Proof. Let $\psi : \operatorname{End}_D V \to \operatorname{End}_F W$ be a ring isomorphism. Further, for convenience, we write α^* instead of $\psi(\alpha)$, where $\alpha \in \operatorname{End}_D V$. We fix a nonzero vector $a \in V$. Let $\varepsilon : V \to aD$ be the projection, where aD is the one-dimensional subspace generated by the vector a. Then $aD = \varepsilon V$. Since ε is an idempotent of the ring $\operatorname{End}_D V$, we have that ε^* is an idempotent of the ring $\operatorname{End}_F W$. Consequently, $\varepsilon^* W$ is an one-dimensional direct summand of the space W. Let b be some nonzero vector in $\varepsilon^* W$. Then $bF = \varepsilon^* W$. By the property (b) from Sec. 2, we identify the rings $\operatorname{End}_D(aD)$ and $\varepsilon(\operatorname{End}_D V)\varepsilon$; we also identify the rings $\operatorname{End}_F(bF)$ and $\varepsilon^*(\operatorname{End}_F W)\varepsilon^*$. Consequently, ψ induces the ring isomorphism $\operatorname{End}_D(aD) \to \operatorname{End}_F(bF)$.

Note the following fact. For a fixed element $d \in D$, there exists a unique operator $\sigma_d \in \operatorname{End}_D(aD)$ which acts by the rule $\sigma_d(ad') = add'$, where $d' \in D$. In particular, $\sigma_d(a) = ad$. The correspondence $d \mapsto \sigma_d$ defines a ring isomorphism $D \to \operatorname{End}_D(aD)$. There exists a similar isomorphism $F \to \operatorname{End}_F(bF)$. We denote by τ the composition of isomorphisms

$$D \to \operatorname{End}_D(aD) \xrightarrow{\psi} \operatorname{End}_F(bF) \to F$$

In addition, the relation $\sigma_d^*(b) = b\tau(d)$ holds.

We construct the required semilinear isomorphism from V onto W. For an arbitrary vector $x \in V$ we take an operator α of the space V such that $x = \alpha(a)$. We define the mapping $\varphi : V \to W$ by setting $\varphi(x) = \alpha^*(b)$. The mapping φ is well defined, i.e., it is independent of the choice of operator α. Indeed, if $x = \alpha_1(a)$, $\alpha_1 \in \operatorname{End}_D(V)$, then $(\alpha - \alpha_1)(a) = 0$ and $(\alpha - \alpha_1)\varepsilon = 0$. Therefore,

$$\big((\alpha - \alpha_1)\varepsilon\big)^* = (\alpha^* - \alpha_1^*)\varepsilon^* = 0;$$

consequently, $(\alpha^* - \alpha_1^*)(b) = 0$ and $\alpha^*(b) = \alpha_1^*(b)$.

We take another vector $y \in V$ and take $\beta \in \operatorname{End}_D V$ with $y = \beta(a)$. Then $x + y = (\alpha + \beta)(a)$; therefore,

$$\varphi(x + y) = (\alpha + \beta)^*(b) = \alpha^*(b) + \beta^*(b) = \varphi(x) + \varphi(y).$$

Thus, φ is an additive homomorphism.

We show that φ is a semilinear mapping, i.e., we verify that the relation $\varphi(xd) = \varphi(x)\tau(d)$ holds for any $x \in V$ and $d \in D$. Let $x = \alpha(a)$, as above. Then $xd = \alpha(a)d = \alpha(ad)$ Since $\varphi(x) = \alpha^*(b)$, we have

$$\varphi(x)\tau(d) = \alpha^*(b)\tau(d) = \alpha^*\big(b\tau(d)\big).$$

On the other hand, since $xd = \alpha(ad) = \alpha\sigma_d(a)$, we have

$$\varphi(xd) = (\alpha\sigma_d)^*(b) = \alpha^*(\sigma_d^*(b)) = \alpha^*\big(b\tau(d)\big).$$

Thus, $\varphi(xd) = \varphi(x)\tau(d)$.

If $\varphi(x) = \alpha^*(b) = 0$ for some $x \in V$, then $(\alpha\varepsilon)^* = \alpha^*\varepsilon^* = 0$. This implies that $\alpha\varepsilon = 0$ and $x = \alpha\varepsilon(a) = 0$, i.e., $\ker\varphi = 0$. For every vector $z \in W$, there exists an operator $\gamma \in \operatorname{End}_F W$ with $z = \gamma(b)$. Let $\gamma = \alpha^*$ for some $\alpha \in \operatorname{End}_D V$. Then $z = \gamma(b) = \alpha^*(b) = \varphi(x)$, where $x = \alpha(a)$. We obtain that φ is a bijection; in other words, φ is a semilinear isomorphism.

We take an arbitrary operator $\mu \in \operatorname{End}_D V$ and a vector $z \in W$. Then we take a vector $x \in V$ with $z = \varphi(x)$ and an operator $\alpha \in \operatorname{End}_D V$ with $x = \alpha(a)$. Then $z = \varphi(x) = \alpha^*(b)$. Now we have the relations

$$\mu^*(z) = (\mu\alpha)^*(b) = \varphi\big(\mu\alpha(a)\big) = \varphi\mu(x) = (\varphi\mu\varphi^{-1})(z).$$

Thus, $\psi(\mu) = \mu^* = \varphi\mu\varphi^{-1}$, i.e., the isomorphism ψ is induced by the isomorphism φ. □

If the space V is of finite dimension m, then the ring $\operatorname{End}_D V$ is isomorphic to the matrix ring $M_m(D)$ of order m over the division ring D. Then we have the following partial case of Theorem 4.1.

Corollary 4.2. *If the rings $M_m(D)$ and $M_n(F)$ are isomorphic, then $m = n$ and the division rings D and F are isomorphic.*

We consider the situation where D and F are fields. We identify any element $d \in D$ with its action on V. In short, we assume that D is the center of the ring $\operatorname{End}_D V$. Similarly, we identify the field F with the center of the ring $\operatorname{End}_F W$. Under ring isomorphisms, the center passes onto the center. Therefore, under the conditions of Theorem 4.1, we can assume that V and W are spaces over the same field, say F. However, in this case we can only prove that the ring isomorphism $\psi : \operatorname{End}_F V \to \operatorname{End}_F W$ is induced by some semilinear F-space isomorphism $V \to W$.

Since the rings $\operatorname{End}_F V$ and $\operatorname{End}_F W$ are F-algebras, it is natural to go further and assume that ψ is an F-algebra isomorphism, i.e., $\psi(s\alpha) = s\psi(\alpha)$ for all $s \in F$ and $\alpha \in \operatorname{End}_F V$. Under this assumption, taking into account the equalities $\psi(s \cdot id_V) = s\psi(id_V) = s \cdot id_W$, we obtain that the isomorphism of F-algebras $\operatorname{End}_F V$, $\operatorname{End}_F W$ acts identically on F. In general, the isomorphisms between the endomorphism algebras $\operatorname{End}_F V$ and $\operatorname{End}_F W$ are precisely ring isomorphisms that leave the elements of the center in place. Taking this into account, we can write down the following result.

Corollary 4.3. *Let V and W be two vector spaces over the field F. Then every isomorphism of endomorphism algebras $\operatorname{End}_F V$ and $\operatorname{End}_F W$ is induced by some isomorphism of spaces V and W.*

Isomorphism theorems have one important application. Let ψ be an automorphism of the F-algebra $\operatorname{End}_F V$. By Corollary 4.3, there exists an automorphism (invertible operator) φ of the space V such that $\psi(\alpha) = \varphi\alpha\varphi^{-1}$ for every $\alpha \in \operatorname{End}_F V$. Since automorphisms of the space V coincide with invertible elements of the algebra $\operatorname{End}_F V$, the last relation means that ψ is an inner automorphism of the algebra $\operatorname{End}_F V$.

Corollary 4.4. *The following assertions hold.*

1. *Every automorphism of the algebra endomorphisms of the vector space over the field is inner.*
2. (Skolem and Noether). *Every automorphism of the matrix algebra $M_n(F)$ is an inner automorphism.*

We might say that Baer's proof of Theorem 4.1 in [2] is of a geometric character. In the proof of Theorem 4.1, we used the method which is called the Kaplansky method. Relevant reasoning first appeared in his book [15]. The essence of Kaplansky's method is as follows. The primitive idempotents of the operator ring correspond to the indecomposable subspaces; they are one-dimensional in this case. In order to construct an isomorphism from the space V onto the space W, we transfer the properties of such summands by means of operators in order to obtain necessary elements of the space W. In one or another form, the Kaplansky method will be applied several times in the remaining sections of this paper.

5. Baer–Kaplansky Theorem

We state and prove, perhaps, the most famous result on the definability of Abelian groups or modules by their endomorphism rings. We are talking about the following remarkable theorem.

Theorem 5.1 (Baer [1], Kaplansky [15])**.** *If A and C are torsion groups with isomorphic endomorphism rings, then every ring isomorphism $\operatorname{End} A \to \operatorname{End} C$ is induced by some group isomorphism $A \to C$.*

Proof. We can restrict ourself to the case of p-groups. Indeed, we have

$$A = \bigoplus_{p \in P} t_p(A), \quad C = \bigoplus_{p \in P} t_p(C),$$

where $t_p(A)$ and $t_p(C)$ are the p-components of the groups A and C, respectively. Then

$$\operatorname{End} A = \prod_{p \in P} \operatorname{End} t_p(A), \quad \operatorname{End} C = \prod_{p \in P} \operatorname{End} t_p(C).$$

Since

$$\operatorname{End} t_p(A) = \bigcap_{(n,\,p)=1} n \operatorname{End} A, \quad \operatorname{End} t_p(C) = \bigcap_{(n,\,p)=1} n \operatorname{End} C,$$

every ring isomorphism $\operatorname{End} A \to \operatorname{End} C$ must map $\operatorname{End} t_p(A)$ onto $\operatorname{End} t_p(C)$. Therefore, it suffices to assume that A and C are p-groups.

Then we proceed with a fixed ring isomorphism $\psi : \operatorname{End} A \to \operatorname{End} C$. For every $\eta \in \operatorname{End} A$, we write $\psi(\eta) = \eta^*$.

If A is a cyclic or quasi-cyclic p-group, then it is easy to see that $A \cong C$ (see Section 2 about the structure of endomorphism rings of such groups). Then we divide the proof into three cases.

Case 1: A is a bounded group. Then A is the direct sum of cyclic p-groups. Let g be one of the generators of some cyclic direct summand of the group A of the largest order p^k. If $\varepsilon : A \to \langle g \rangle$ is the projection, then ε is an idempotent of the ring $\operatorname{End} A$ and ε^* is an idempotent of the ring $\operatorname{End} C$. Consequently, $\varepsilon^* C$ is a direct summand of the group C. By the property (d) from Sec. 2, the isomorphism ψ induces the ring isomorphism $\operatorname{End}\langle g \rangle \to \operatorname{End}(\varepsilon^* C)$. Therefore, $\varepsilon^* C$ is a cyclic group $\langle h \rangle$ of order p^k. Now we can construct the required isomorphism $\varphi : A \to C$. For any element $a \in A$, we take an endomorphism $\eta \in \operatorname{End} A$ with $a = \eta(g)$ and define a mapping $\varphi : A \to C$ such that $\varphi(a) = \eta^*(h)$. Similarly to the proof of Theorem 4.1, we can verify that the mapping φ is well defined, φ is an isomorphism and φ induces ψ.

Case 2: $A = B \oplus D$, where B is a bounded group and D is a nonzero divisible group. Let $\langle g \rangle$ be a cyclic direct summand of maximal order p^k in the group B, E be a direct summand of the group D which is isomorphic to the group $\mathbb{Z}_{p^\infty}$, and let

$$E = \langle d_1, d_2, \ldots, d_n, \ldots \rangle, \quad pd_1 = 0, \quad pd_{n+1} = d_n \quad \text{for} \quad n \geqslant 1.$$

We denote by $\varepsilon : A \to \langle g \rangle$ and $\pi : A \to E$ the corresponding projections. Similarly to Case 1, we obtain that $\varepsilon^* C$ is a cyclic direct summand of the group C and $\pi^* C$ is a direct summand of the group C which is isomorphic to the group $\mathbb{Z}_{p^\infty}$. We define two groups $\varepsilon^* C$ and $\pi^* C$ with the use of their generators:

$$\varepsilon^* C = \langle h \rangle, \quad \pi^* C = \langle e_1, e_2, \ldots, e_n, \ldots \rangle, \quad pe_1 = 0, \quad pe_{n+1} = e_n \quad \text{for} \quad n \geqslant 1.$$

We represent an arbitrary element $a \in A$ in the form $a = a_1 + a_2$, where $a_1 \in B$, $a_2 \in D$, and take an endomorphism $\eta \in \operatorname{End} A$ such that $\eta(g) = a_1$, $\eta(d_n) = a_2$ for some n. We construct a mapping $\varphi : A \to C$ by setting $\varphi(a) = \eta^*(h + e_n)$. First, we show that φ does not depend on the choice of η and n. We take $\eta_1 \in \operatorname{End} A$ such that $\eta_1(g) = a_1$ and $\eta_1(d_m) = a_2$, and we can assume that $m \geqslant n$. Then we obtain the relations

$$(\eta - \eta_1)(g) = 0, \quad (p^{m-n}\eta - \eta_1)(d_m) = 0.$$

Therefore, $(\eta - \eta_1)\varepsilon = 0$ and the endomorphism $(p^{m-n}\eta - \eta_1)\pi$ annihilates $E[p^m]$. Therefore, the endomorphism $(p^{m-n}\eta - \eta_1)\pi$ is divided by p^m. Then the endomorphism $\big((p^{m-n}\eta - \eta_1)\pi\big)^*$ is also divided by p^m; consequently, it annihilates the element e_m. Thus, we obtain that $\eta^*(h) = \eta_1^*(h)$ and $\eta^*(e_n) = p^{m-n}\eta^*(e_m) = \eta_1^*(e_m)$, whence $\eta^*(h + e_n) = \eta_1^*(h + e_m)$.

Similarly to Case 1, we refer to the proof of Theorem 4.1 for the verification of the property that φ is an isomorphism inducing the isomorphism ψ.

Case 3: A has an unbounded basis subgroup. It follows from properties of basis subgroups (see Sec. 1) that there exist decompositions

$$A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \ldots \oplus \langle a_k \rangle \oplus A_k, \quad k \in \mathbb{N},$$

such that $A_k = \langle a_{k+1} \rangle \oplus A_{k+1}$ and $o(a_k) = p^{n_k}$, where $1 \leqslant n_1 < n_2 < \ldots < n_k < \ldots$. Let $\varepsilon_k : A \to \langle a_k \rangle$ be the projection. For distinct subscripts j and k, we define an endomorphism γ_{jk} of the group A as follows. The endomorphism γ_{jk} maps the direct summand, which is complementary to $\langle a_k \rangle$ in the above decomposition of A, onto 0. It also maps from a_k onto a_j (respectively, onto $p^{n_j - n_k} a_j$) if $j < k$ (respectively, $j > k$). Then

(1) $\gamma_{jk}\varepsilon_k = \gamma_{jk} = \varepsilon_j \gamma_{jk}$ for all $j \neq k$;
(2) $\gamma_{kj}\gamma_{jk} = p^{|n_j - n_k|}\varepsilon_k$, for all $j \neq k$;
(3) $\gamma_{ij}\gamma_{jk} = \gamma_{ik}$ if $i < j < k$ or $i > j > k$.

The endomorphisms ε_k^* and γ_{jk}^* of the group C also satisfy conditions (1)–(3). The subgroups $\varepsilon_k^* C$ are cyclic direct summands of the group C of orders coinciding with the orders of the groups $\varepsilon_k A$, by the property (d) from Sec. 2. It follows from the condition (2) that the endomorphism $\gamma_{k,k+1}^*$ maps $\varepsilon_{k+1}^* C$ onto $\varepsilon_k^* C$. We set $\varepsilon_k^* C = \langle c_k \rangle$ and show that we can choose the generators c_k to satisfy the relations $\gamma_{k,k+1}^*(c_{k+1}) = c_k$ for all k. Indeed, if the elements $c_1, c_2, \ldots, c_k$ are already chosen and the element c'_{k+1} generates the subgroup $\varepsilon_{k+1}^* C$, then $\gamma_{k,k+1}^*(c'_{k+1}) = tc_k$ for some $t \in \mathbb{Z}$. Further, it follows from (2) that $\gamma_{k+1,k}^*(tc_k) = p^{n_{k+1}-n_k} c'_{k+1}$; by considering the orders of the elements, that $(p, t) = 1$. We take the element $c_{k+1} = sc'_{k+1}$, where $st \equiv 1(\mathrm{mod} p^{n_k})$. Then $\gamma_{k,k+1}^*(c_{k+1}) = c_k$. Furthermore, it follows from (3) that $\gamma_{jk}^*(c_k) = c_j$ for all $j < k$.

For an arbitrary element $a \in A$, we take an endomorphism $\eta \in \operatorname{End} A$ such that $\eta(a_k) = a$ for some $k \in \mathbb{N}$. We define the mapping $\varphi : A \to C$ by setting $\varphi(a) = \eta^*(c_k)$. We verify that the mapping φ is well defined. Let $\eta_1(a_j) = a$, where $\eta_1 \in \operatorname{End} A$ and $j \geqslant k$. Then $(\eta\gamma_{kj} - \eta_1)\varepsilon_j = 0$, whence $(\eta^*\gamma_{kj}^* - \eta_1^*)\varepsilon_j^* = 0$; therefore, $\eta^*(c_k) = \eta_1^*(c_j)$.

Finally, similarly to the proof of Theorem 4.1, we can verify that the constructed mapping φ is an isomorphism which induces the isomorphism ψ. □

Similarly to the case of vector spaces, we obtain a corollary related to automorphisms of torsion groups.

Corollary 5.2. *For a torsion group G, every automorphism of the endomorphism ring of G is inner.*

6. Topological Isomorphisms of Endomorphism Rings

On the endomorphism ring, we have the finite topology, so it is natural to consider isomorphisms of endomorphism rings that are continuous in both directions. They more accurately determine the structure of the source module. We call such isomorphisms *topological* and discuss *theorems of topological isomorphism*.

Thus, by a topological isomorphism $\psi : \operatorname{End} A \to \operatorname{End} C$, we mean a ring isomorphism ψ such that ψ and ψ^{-1} are continuous with respect to the finite topology. It can be directly verified that every group isomorphism $A \to C$ induces a topological ring isomorphism $\operatorname{End} A \to \operatorname{End} C$. We also pay attention to the fact that the ring isomorphisms appearing in Theorems 4.1 and 5.1 are topological. This follows from Proposition 3.2.

In this section, all groups are torsion-free. Most of the concepts related to torsion-free groups are defined in Sec. 1. In addition, we recall that a type is said to be idempotent if it contains a characteristic consisting of the symbols 0 and ∞. The type of torsion-free group A of rank 1 is idempotent if and only if A is isomorphic to the additive group of some subring of the rational number field $\mathbb{Q}$.

Every Abelian group A is a natural left module over its endomorphism ring. Let A be torsion free. In this case, the $\mathbb{Q}$-algebra $\operatorname{End} A \otimes \mathbb{Q}$ is called the *quasi-endomorphism ring* or the *quasi-endomorphism algebra* of the group A. The action of the ring $\operatorname{End} A$ on the group A extends to the action of the ring $\operatorname{End} A \otimes \mathbb{Q}$ on the divisible hull $A \otimes \mathbb{Q}$ of the group A. Thus, we get the left $(\operatorname{End} A \otimes \mathbb{Q})$-module $A \otimes \mathbb{Q}$.

We assume that the group A is embedded in the $\mathbb{Q}$-space $A \otimes \mathbb{Q}$ by identifying an element $a \in A$ with the element $a \otimes 1$. We accept the same agreement with respect to $\operatorname{End} A$ and $\operatorname{End} A \otimes \mathbb{Q}$.

Pure fully invariant subgroups of the group A are briefly called *pfi-subgroups*. It is easy to verify that the correspondences

$$H \mapsto H \otimes \mathbb{Q}, \quad W \mapsto W \cap A$$

are mutually inverse isomorphisms between the lattice of pfi-subgroups of the group A and the submodule lattice of the $(\operatorname{End} A \otimes \mathbb{Q})$-module $A \otimes \mathbb{Q}$.

We recall that a group A is said to be *irreducible* if it does not have a proper pfi-subgroup. The irreducibility of the group A is equivalent to the irreducibility of the $(\operatorname{End} A \otimes \mathbb{Q})$-module $A \otimes \mathbb{Q}$.

Definition 6.1. A torsion-free group G is said to be *fully transitive* if for any its elements a, $b \neq 0$ with $\chi(a) \leqslant \chi(b)$, there exists an endomorphism $\alpha \in \operatorname{End} G$ with $\alpha(a) = b$.

Homogeneous separable groups and algebraically compact groups are simplest examples of fully transitive groups.

Lemma 6.2. *A homogeneous fully transitive torsion-free group G is irreducible. In addition, if G is of an idempotent type, then every its pure subgroup contains a generator of the* $\operatorname{End} G$*-module G.*

Proof. We assume that H is a nonzero pfi-subgroup of the group G. Let $a \in H$ and $b \in G$ be some nonzero elements. We take a positive integer n with $\chi(a) \leqslant \chi(nb)$. Then $\alpha(a) = nb$ for some $\alpha \in \operatorname{End} G$. Since the subgroup H is fully invariant, we have $nb \in H$. Since the subgroup H is pure, $b \in H$. Consequently, $H = G$; therefore, G does not contain a proper pfi-subgroup.

If the group G is of idempotent type, then every its nonzero pure subgroup contains element $a \neq 0$ with characteristic $\chi(a)$ consisting of 0 and ∞. Then $\chi(a) \leqslant \chi(b)$ for any nonzero element $b \in G$. Therefore, $b \in (\operatorname{End} G)a$ and $(\operatorname{End} G)a = G$, i.e., the element a generates the $\operatorname{End} G$-module G. □

Theorem 6.3 (see [16])**.** *Let G and H be two homogeneous fully transitive torsion-free groups whose types are idempotent. Then every topological ring isomorphism between* $\operatorname{End} G$ *and* $\operatorname{End} H$ *is induced by some group isomorphism between G and H.*

Proof. Let $\psi : \operatorname{End} G \to \operatorname{End} H$ be some topological ring isomorphism. For convenience, we use the following notation:

$$V = G \otimes \mathbb{Q}, \quad W = H \otimes \mathbb{Q}, \quad R = \operatorname{End} G \otimes \mathbb{Q}, \quad S = \operatorname{End} H \otimes \mathbb{Q}.$$

Then V is a faithful irreducible R-module and W is a faithful irreducible S-module by Lemma 6.2. Further, we set

$$D = \operatorname{End}_R V, \quad F = \operatorname{End}_S W, \quad K = \operatorname{End}_D V, \quad L = \operatorname{End}_F W.$$

Here D and F are division rings by the Schur lemma. By the familiar density theorem of Jacobson–Chevalley for irreducible modules, the ring R is dense in the finite topology of the ring K, and the ring S is dense in the finite topology of the ring L.

We identify the ring $\operatorname{End} G$ (respectively, $\operatorname{End} H$) with its image under the canonical embedding $\operatorname{End} G \to R$ (respectively, $\operatorname{End} H \to S$). Then finite topology of the ring $\operatorname{End} G$ (respectively, $\operatorname{End} H$) coincides with the topology induced by the finite topology of the ring K (respectively, L). Therefore, $\psi \otimes id_{\mathbb{Q}}$ is a topological ring isomorphism of R and S which is also denoted by ψ. Since R (respectively, S) is dense in the complete ring K (respectively, L), we have that ψ can be uniquely extended to the ring isomorphism $K \to L$ which is also denoted by ψ. As above, we write η^* instead of $\psi(\eta)$.

Let g be some fixed generator of the $\operatorname{End} G$-module G which exists by Lemma 6.2. We denote by A the subspace of the D-space V generated by the element g and consider the projection $\pi : V \to A$. Then $\pi \in K$ and $\pi^2 = \pi$. Hence $(\pi^*)^2 = \pi^*$ and $\pi^* : W \to \pi^* W$ is the projection. In addition,

$$D \cong \operatorname{End}_D A \cong \pi K \pi \cong \pi^* L \pi^* \cong \operatorname{End}_F(\pi^* W)$$

Consequently, $\dim_F(\pi^* W) = 1$; see the property (b) from Sec. 2 about the isomorphism $\operatorname{End}_D A \cong \pi K \pi$ and a similar isomorphism for the ring L. In $\pi^* W \cap H$, we take some element h generating the $\operatorname{End} H$-module H.

We define the mapping $\varphi : G \to H$ as follows. For an arbitrary element $a \in G$, we take an endomorphism $\eta \in \operatorname{End} G$ with $a = \eta(g)$. We set $\varphi(a) = \eta^*(h)$. We verify that the action of φ does not depend on the choice of endomorphism η. If $a = \eta_1(g)$, where $\eta_1 \in \operatorname{End} G$, then $(\eta - \eta_1)(g) = 0$. Consequently, $(\eta - \eta_1)\pi = 0$, since the D-space A is one-dimensional. Hence $(\eta^* - \eta_1^*)\pi^* = 0$ and $(\eta^* - \eta_1^*)(h) = 0$, i.e., $\eta^*(h) = \eta_1^*(h)$. Thus, the mapping φ is well defined.

It remains to verify that φ is an isomorphism which induces the isomorphism ψ. In general, it repeats the corresponding places in the proof of Theorem 4.1. □

Of course, in an indirect form, the proof of Theorem 6.3 is based on the Kaplansky method.

If we apply Theorem 6.3 to a homogeneous fully transitive group, then we obtain the result which is similar to Corollaries 4.4 and 5.2.

Corollary 6.4. *For a homogeneous fully transitive torsion-free group of idempotent type, every topological automorphism of its endomorphism ring is inner.*

We apply Theorem 6.3 to homogeneous separable torsion-free groups defined in Sec. 1. It is easy to prove that such a group is fully transitive. In addition, it follows from Proposition 3.2(3) that every isomorphism between endomorphism rings of two separable torsion-free groups is topological. Then we have the following result.

Corollary 6.5. *If G and H are two homogeneous separable torsion-free groups of idempotent types, then every isomorphism* $\operatorname{End} G \to \operatorname{End} H$ *is induced by some isomorphism* $G \to H$.

We also have the following assertion.

Corollary 6.6. *Let G and H be two fully decomposable torsion-free groups such that types of all homogeneous components of these groups are idempotent. Then every isomorphism* $\operatorname{End} G \to \operatorname{End} H$ *is induced by some isomorphism* $G \to H$.

Proof. Let the canonical decompositions of the groups G and H be of the form

$$G = \bigoplus_{\mathbf{t} \in \Omega(G)} G_{\mathbf{t}}, \quad H = \bigoplus_{\mathbf{t} \in \Omega(H)} H_{\mathbf{t}},$$

where $G_{\mathbf{t}}$ and $H_{\mathbf{t}}$ are so-called homogeneous components of the groups G and H, respectively. Let $\psi : \operatorname{End} G \to \operatorname{End} H$ be some ring isomorphism. For every $\mathbf{t} \in \Omega(G)$, we denote by $\varepsilon_{\mathbf{t}}$ the projection $G \to G_{\mathbf{t}}$. There are isomorphisms

$$\operatorname{End} G_{\mathbf{t}} \cong \varepsilon_{\mathbf{t}}(\operatorname{End} G)\varepsilon_{\mathbf{t}} \cong \psi(\varepsilon_{\mathbf{t}})(\operatorname{End} H)\psi(\varepsilon_{\mathbf{t}}) \cong \operatorname{End}\bigl(\psi(\varepsilon_{\mathbf{t}})H\bigr).$$

Since $G_{\mathbf{t}}$ is a homogeneous group of idempotent type, $\psi(\varepsilon_{\mathbf{t}})H$ also is a homogeneous group of idempotent type. By Corollary 6.5, we have the isomorphism $G_{\mathbf{t}} \cong \psi(\varepsilon_{\mathbf{t}})H$, whence $\Omega(G) \subseteq \Omega(H)$. By symmetry, we obtain the converse inclusion. Thus, $\Omega(G) = \Omega(H)$.

Now we can construct an isomorphism from the group G into the group H by the Kaplansky method. For every type $\mathbf{t} \in \Omega(G)$, we fix the direct summand $A_{\mathbf{t}}$ of rank 1 of the group $G_{\mathbf{t}}$. We take a nonzero element $a_{\mathbf{t}} \in A_{\mathbf{t}}$ with characteristic consisting of the symbols 0 and ∞. Let $\pi_{\mathbf{t}} : G \to A_{\mathbf{t}}$ be the projection. Then $\psi(\pi_{\mathbf{t}})H$ is a direct summand of rank 1 of the group $H_{\mathbf{t}}$. In $\psi(\pi_{\mathbf{t}})H$, we take a nonzero element $b_{\mathbf{t}}$ with characteristic consisting of the symbols 0 and ∞. Then $\{a_{\mathbf{t}}\}_{\mathbf{t} \in \Omega(G)}$ is a generator system of the $\operatorname{End} G$-module G and $\{b_{\mathbf{t}}\}_{\mathbf{t} \in \Omega(H)}$ is a generator system of the $\operatorname{End} H$-module H. Any element a of the group G can be represented in the form

$$a = \alpha_{\mathbf{t}_1}(a_{\mathbf{t}_1}) + \alpha_{\mathbf{t}_2}(a_{\mathbf{t}_2}) + \ldots + \alpha_{\mathbf{t}_k}(a_{\mathbf{t}_k}),$$

where $\alpha_{\mathbf{t}_1}, \alpha_{\mathbf{t}_2}, \ldots, \alpha_{\mathbf{t}_k} \in \operatorname{End} G$. Then we set

$$\varphi(a) = \psi(\alpha_{\mathbf{t}_1})(b_{\mathbf{t}_1}) + \psi(\alpha_{\mathbf{t}_2})(b_{\mathbf{t}_2}) + \ldots + \psi(\alpha_{\mathbf{t}_k})(b_{\mathbf{t}_k}).$$

Similarly to the proof of Theorems 4.1, 5.1, and 6.3, we can verify that the mapping φ is well-defined and it is an isomorphism which induces the isomorphism ψ. $\square$

At the end of this section, we use topological isomorphisms to extend the Baer–Kaplansky theorem (Theorem 5.1) to the case of arbitrary groups.

First, we have the following useful fact. Let A be a group and ε be an idempotent of the ring $\operatorname{End} A$. Then the canonical isomorphism $\operatorname{End}(\varepsilon A) \cong \varepsilon(\operatorname{End} A)\varepsilon$, specified in the property (b) of Sec. 2, is topological if we assume that $\operatorname{End}(\varepsilon A)$ is provided by the finite topology which coincides on $\varepsilon(\operatorname{End} A)\varepsilon$ with the topology induced by the finite topology of the ring $\operatorname{End} A$.

We recall that the torsion part of the mixed group G, i.e., the largest torsion subgroup of G, is denoted by $t(G)$. In addition, if G is a torsion group, then $t(G) = G$ and $t(G) = 0$ for a torsion-free group G.

Theorem 6.7 (May [23]). *The following assertions hold.*

1. *Let G and H be two groups and $\psi : \operatorname{End} G \to \operatorname{End} H$ be a topological isomorphism. Then there exists an isomorphism $\varphi : t(G) \to t(H)$ such that $\psi(\eta)$ and $\varphi\eta\varphi^{-1}$ coincide on $t(H)$ for every $\eta \in \operatorname{End} G$.*
2. *Let T be a torsion group and H be an arbitrary group. Then every topological isomorphism $\operatorname{End} T \to \operatorname{End} H$ is induced by an isomorphism $T \to H$.*

Proof. 1. The required isomorphism φ can be constructed by the Kaplansky method which is restricted here to the torsion parts of the groups G and H. It is only necessary to clarify the following point. If ε is an idempotent of the ring $\operatorname{End} G$ and $\varepsilon G \cong \mathbb{Z}_{p^k}$, $k \geqslant 1$, then it is clear that $\psi(\varepsilon)H \cong \mathbb{Z}_{p^k}$. Let $\varepsilon G \cong \mathbb{Z}_{p^\infty}$. Then $\operatorname{End}(\varepsilon G) \cong \widehat{\mathbb{Z}}_p$ ($\widehat{\mathbb{Z}}_p$ is the ring or group of p-adic integers) and, consequently, $\operatorname{End}\big(\psi(\varepsilon)H\big) \cong \widehat{\mathbb{Z}}_p$. It follows from this property that $\psi(\varepsilon)H \cong \mathbb{Z}_{p^\infty}$ or $\psi(\varepsilon)H \cong \widehat{\mathbb{Z}}_p$. The theorem that $\operatorname{End}(\varepsilon G)$ and $\operatorname{End}\big(\psi(\varepsilon)H\big)$ are topologically isomorphic rings was mentioned before. However, the finite topologies on the rings $\operatorname{End} \mathbb{Z}_{p^\infty}$ and $\operatorname{End} \widehat{\mathbb{Z}}_p$ are distinct, since the first topology is the p-adic topology and the second topology is the discrete topology. Therefore, $\psi(\varepsilon)H \cong \mathbb{Z}_{p^\infty}$ is only possible. Now the way to the application of the Kaplansky method is open.

2. Let $\psi : \operatorname{End} T \to \operatorname{End} H$ be some topological isomorphism. Since ψ is continuous, we have that for an arbitrary element $y \in H$, there exist elements $x_1, x_2, \dots, x_n \in T$ such that if $\alpha \in \operatorname{End} T$ and $\alpha(x_i) = 0$, $i = 1, 2, \dots, n$; then we have $\psi(\alpha)(y) = 0$. There exists a positive integer m such that $mx_i = 0$ for all i. Hence $\psi(m \cdot id_T)(y) = (m \cdot id_H)(y) = my = 0$. Consequently, H is a torsion group and we can use Theorem 5.1. $\square$

7. Definability of p-groups by Radical of Endomorphism Rings

One can raise the question of the determinability of a group not by the whole ring of endomorphisms but by some part of it. It follows from Theorem 7.1 that a p-group with an unbounded basic subgroup is determined by the Jacobson radical of its endomorphism ring (as a ring without unity). Other similar results are given in the remarks. The section is based on the paper [14]. Various facts about radicals of endomorphism rings are given in the fourth part of the monograph [17].

Let us make one terminological remark. Group terms applied to a ring, ideal, or module refer to their additive groups. The same applies to the individual elements of these objects. Thus, for example, the order of an element of a ring (an ideal or a module) means its order as an element of the corresponding additive group.

Let G be some group. Then $\mathrm{J}(\operatorname{End} G)$ is the Jacobson radical of its endomorphism ring and $\mathrm{K}(G)$ is the torsion subgroup of the ideal $\mathrm{J}(\operatorname{End} G)$. It is clear that $\mathrm{K}(G)$ is an ideal in $\operatorname{End} G$ and the ring $\mathrm{K}(G)$ is not unital . We often write K instead of $\mathrm{K}(G)$.

Theorem 7.1. *Let G be a p-group whose basis subgroup is an unbounded group, and G' an arbitrary p-group. Then every ring isomorphism $\psi : \mathrm{K}(G) \to \mathrm{K}(G')$ is induced by some group isomorphism $\varphi : G \to G'$, i.e., $\psi(\eta) = \varphi\eta\varphi^{-1}$ for any $\eta \in \mathrm{K}(G)$.*

First, we prove a series of auxiliary assertions. But first we note that any ring isomorphism $\operatorname{End} G \to \operatorname{End} G'$, of course, induces the ring isomorphism $\mathrm{K}(G) \to \mathrm{K}(G')$.

Up to the end of the section, G is some p-group and $\mathrm{N}(\operatorname{End} G)$ is the nil-radical of the ring $\operatorname{End} G$, i.e., the sum of all nil-ideals of $\operatorname{End} G$. We also define the *Pierce ideal* $\mathrm{P}(G)$ of the group G; namely, we set

$$\mathrm{P}(G) = \big\{\alpha \in \operatorname{End} G \mid x \in G[p],\ h(x) < \infty\ \Rightarrow\ h(x) < h\big(\alpha(x)\big)\big\}.$$

It is easy to verify that $\mathrm{P}(G)$ is an ideal of the ring $\operatorname{End} G$. We also have the relation

$$\mathrm{P}(G) = \{\alpha \in \operatorname{End} G \mid \alpha\big((p^n G)[p]\big) \subseteq \big(p^{n+1} G\big)[p] \text{ for all } n \geqslant 0\}.$$

Various facts about the Pierce ideal can be found in [17, Sec. 20]. The most important of these is the inclusion $\mathrm{J}(\operatorname{End} G) \subseteq \mathrm{P}(G)$. Consequently, we have the inclusions

$$\mathrm{N}(\operatorname{End} G) \subseteq \mathrm{J}(\operatorname{End} G) \subseteq \mathrm{P}(G). \tag{$*$}$$

Further, we prove that the torsion parts of these ideals coincide. First, we note that if I is an ideal of some ring, then $t(I)$ is also an ideal. In addition, $t(\operatorname{End} G)$ is a p-group in our case.

Lemma 7.2. *There exist the relations*

$$t\big(\mathrm{N}(\operatorname{End} G)\big) = t\big(\mathrm{J}(\operatorname{End} G)\big) = t\big(\mathrm{P}(G)\big).$$

Proof. It follows from $(*)$ that it suffices to show that every element of $t\big(\mathrm{P}(G)\big)$ is nilpotent. Let $\alpha \in t\big(\mathrm{P}(G)\big)$. Then $p^n\alpha = 0$ for some $n \in \mathbb{N}$ and $\alpha G \subseteq G[p^n]$. Let $\alpha' = \alpha\big|_{G[p^{n+1}]}$. Since $\alpha \in \mathrm{P}(G)$, it is clear that $(\alpha')^m = 0$ for some $m \in \mathbb{N}$. Then $\alpha^{m+1} = 0$. $\square$

We recall that the ideal $\mathrm{K}(G)$ is denoted by the single letter K.

Lemma 7.3. *The ideal* K *is not bounded if and only if the group G has an unbounded basis subgroup.*

Proof. We assume that there is a decomposition $G = \langle a\rangle \oplus \langle b\rangle \oplus H$, $o(a) < o(b)$. We define an endomorphism α of the group G by setting $\alpha(b) = a$ and $\alpha\big(\langle a\rangle \oplus H\big) = 0$. Then $\alpha \in \mathrm{K}$ and $o(\alpha) = o(a)$. This proves that the ideal K is not bounded if G has an unbounded basis subgroup. To prove the converse, we assume that $G = A \oplus D$, where $p^n A = 0$ for some n and D is a divisible group. Let $\beta \in \mathrm{K}$. Then $p^n\beta = 0$, since a divisible group does not have nonzero endomorphisms of finite order. Consequently, $p^n\,\mathrm{K} = 0$. $\square$

We use the following notation. For an endomorphism $\eta \in \mathrm{K}$ and a subset $L \subseteq \mathrm{K}$, we denote by $L \cdot \eta$ the set $\{\lambda\eta \mid \lambda \in L\}$; the set $\eta \cdot L$ is similarly defined.

Lemma 7.4. *Let G be a group with unbounded basis subgroup, t be a positive integer, and let $\eta \in \mathrm{K}$. The relation $\mathrm{K}[p^t] \cdot \eta = 0$ holds if and only if $\eta G \subseteq p^t G$.*

Proof. Let $\alpha \in \mathrm{K}$. Clearly, $p^t\alpha = 0 \Leftrightarrow (p^t\alpha)G = 0$. Therefore, if $\eta G \subseteq p^t G$, then $\mathrm{K}[p^t] \cdot \eta = 0$. To prove the converse, we assume that $\eta G \nsubseteq p^t G$. We construct an endomorphism $\lambda \in \mathrm{K}[p^t]$ with $\lambda\eta \neq 0$. We take an element $x \in \eta G$ such that $x \notin p^t G$. Then $h(x) < t$. Consequently, there exists a decomposition $G = \langle y\rangle \oplus Y$, and $x = p^s y + u$, where $p^s y \neq 0$, $s < t$, and $u \in Y$ (see Sec. 1). Let $o(y) = p^n$ and $m = \min\{n, t\}$; then $m > s$. Since a basis subgroup of the group G is not bounded, Y has a direct summand $\langle z\rangle$ of order p^k with $k > m$. We define an endomorphism λ by setting $\lambda(y) = p^{k-m}z$ and $\lambda Y = 0$. Then $\lambda \in \mathrm{K}[p^t]$ and $\lambda(x) = p^{k-(m-s)}z \neq 0$. Consequently, $\lambda\eta \neq 0$, which is required. $\square$

Proposition 7.5. *If the conditions of Lemma 7.4 hold and $\eta \in \mathrm{K}[p^t]$, then $\mathrm{K}[p^t] \cdot p^{t-1}\eta \neq 0$ if and only if the group G has decompositions $G = \langle y\rangle \oplus Y = \langle \eta(y)\rangle \oplus X$ such that $o(\eta(y)) = p^t$.*

Proof. First, we assume that $\mathrm{K}[p^t] \cdot p^{t-1}\eta \neq 0$. Then $(p^{t-1}\eta)G \nsubseteq p^t G$ by Lemma 7.4. Consequently, there exists a direct summand $\langle y\rangle$ of G with $\eta(p^{t-1}y) \notin p^t G$. This implies that the element $\eta(p^{t-1}y)$ is of order p and of height $t-1$. Therefore, $\langle \eta(y)\rangle$ is a direct summand in G (see Sec. 1). It is also clear that $o(\eta(y)) = p^t$.

To prove the converse, we assume that the group G has the mentioned decompositions. The height of the element $p^{t-1}\eta(y)$ is less than t; this implies that $(p^{t-1}\eta)G \nsubseteq p^t G$. It follows from Lemma 7.4 that $\mathrm{K}[p^t] \cdot p^{t-1}\eta \neq 0$. $\square$

We introduce one new notion.

Definition 7.6. A right K-module T is said to be *strongly homogeneous* if T is a torsion group and for any nonzero $\sigma, \tau \in T$, the intersection $\sigma K \cap \tau K$ contains an element α with $o(\alpha) = \min\{o(\sigma), o(\tau)\}$.

Proposition 7.7. *Let the conditions of Lemma* 7.4 *hold and let* $\eta \in \mathrm{K}[p^t]$ *such that* $\mathrm{K}[p^t] \cdot p^{t-1}\eta \neq 0$. *The subgroup* ηG *is a cyclic group if and only if* $\eta\,\mathrm{K}$ *is a strongly homogeneous* K*-module.*

Proof. By Proposition 7.5, there exist decompositions

$$G = \langle y \rangle \oplus Y = \langle x \rangle \oplus X$$

such that $\eta(y) = x$ and $o(x) = o(\eta) = p^t$.

First, we assume that ηG is a cyclic group. Then we have $\eta G = \langle x \rangle$. Let endomorphisms $\tau_i \in \mathrm{K}$ satisfy $\eta\tau_i \neq 0$, $i = 1, 2$. Let $o(\eta\tau_i) = p^{n_i}$. Then $n_i \leqslant t$, $i = 1, 2$; without loss of generality, we assume that $n_1 \leqslant n_2$. There exist elements $g_i \in G$, $i = 1, 2$ such that $\eta\tau_i(g_i) = p^{t-n_i}x$. We have the decomposition $G = \langle z \rangle \oplus Z$ such that the order of the element z exceeds the orders of the elements g_1 and g_2. We take endomorphisms $\sigma_i \in \mathrm{K}$ such that $\sigma_i(z) = g_i$ and $\sigma_i Z = 0$, $i = 1, 2$. We set $\lambda_i = \eta\tau_i\sigma_i$, $i = 1, 2$. Then $\lambda_i Z = 0$ and $\lambda_i(z) = p^{t-n_i}x$; therefore, $o(\lambda_i) = p^{n_i} = o(\eta\tau_i)$, $i = 1, 2$. Further, we have that $p^{n_2-n_1}\lambda_2 = \lambda_1$ and $\lambda_1 \in (\eta\tau_1)\,\mathrm{K} \cap (\eta\tau_2)\,\mathrm{K}$. In addition, the order of the element λ_1 is equal to the minimal order of the elements $\eta\tau_1$ and $\eta\tau_2$.

To prove the converse, we assume that $\eta G \neq \langle x \rangle$. Then there exists an element g such that $\eta(g) = w$ for some $w \notin \langle x \rangle$. If $o(w) = p^s$, then it is clear that $s \leqslant t$. Again, we take a decomposition $G = \langle z \rangle \oplus Z$ with the property that the order of the element z strictly exceeds the orders of the elements y and g. In such the situation, there exist endomorphisms $\sigma, \tau \in \mathrm{K}$ such that $\sigma(z) = y$, $\tau(z) = g$ and $\sigma Z = 0 = \tau Z$. Then we have $\eta\sigma(z) = x$, $\eta\tau(z) = w$, $o(\eta\sigma) = p^t$ and $o(\eta\tau) = p^s$. We assert that the intersection $(\eta\sigma)\,\mathrm{K} \cap (\eta\tau)\,\mathrm{K}$ does not contain elements of order p^s. We assume the contrary: there exist $\alpha, \beta \in \mathrm{K}$ such that the element $\eta\sigma\alpha = \eta\tau\beta$ is of order p^s. Let a be an arbitrary element of the group G. We have $\alpha(a) = mz + z'$ and $\beta(z) = nz + z''$ for some integers m and n and elements $z', z'' \in Z$. Now we have $\eta\sigma\alpha(a) = mx$, $\eta\tau\beta(a) = nw$ and $mx = nw$. Since $w \notin \langle x \rangle$, we have that p divides n. Then $p^{s-1}\eta\tau\beta(a) = 0$ and $p^{s-1}\eta\tau\beta = 0$. The obtained contradiction completes the proof. ☐

We now state the main auxiliary result.

Proposition 7.8. *A* p*-group* G *has an unbounded basis subgroup if and only if there exists a sequence* $1 < n_1 < n_2 < \ldots$ *of positive integers, and there exist elements* $\eta_1, \eta_2, \ldots$ *of* K *such that the following conditions hold for every* $i \geqslant 1$.

(1) $o(\eta_i) = p^{n_i}$;
(2) $\mathrm{K}[p^{n_i}] \cdot p^{n_i-1}\eta_i \neq 0$;
(3) $\eta_i\,\mathrm{K}$ *is a strongly homogeneous* K*-module*;
(4) *the mapping* $f_i : \mathrm{K}\,\eta_i \to \mathrm{K}\,\eta_{i+1}$, $\alpha\eta_i \mapsto \alpha\eta_i\eta_{i+1}$, $\alpha \in \mathrm{K}$, *is a monomorphism.*

In addition, for a given sequence $1 < n_1 < n_2 < \ldots$ *and given endomorphisms* $\eta_i \in \mathrm{K}$ *satisfying properties* (1)–(4), *there exist elements* $d_1, d_2, \ldots$ *in* G *such that for every* $i \geqslant 1$, *we have*

$$o(d_i) = p^{n_i}, \quad \eta_i(d_{i+1}) = d_i, \quad \eta_i G = \langle d_i \rangle, \quad \langle d_i \rangle \text{ ”— is a direct summand of the group } G.$$

Proof. Let G have a unbounded basis subgroup. Then for any $i \geqslant 1$, there exists an element $a_i \in G$ of order p^{n_i}, where $1 < n_1 < n_2 < \ldots$, and the relations

$$G = \langle a_i \rangle \oplus \langle a_2 \rangle \oplus \ldots \oplus \langle a_t \rangle \oplus H_t, \quad H_t = \langle a_{t+1} \rangle \oplus H_{t+1}.$$

hold. We define endomorphisms η_i by setting $\eta_i(a_{i+1}) = a_i$ and assuming that η_i annihilates all summands complement to $\langle a_{i+1} \rangle$. It is clear that η_i is of order p^{n_i}. It follows from Propositions 7.5 and 7.7 that the conditions (2) and (3) hold. The condition (4) is directly verified.

To prove the converse, it suffices to note that by Lemma 7.3, it follows from the condition (1) that G has an unbounded basis subgroup.

Now we assume that the conditions (1)–(4) hold and verify the existence of the elements d_i with required properties. By Proposition 7.5, for every $i \geqslant 1$, there exist decompositions $G = \langle y_i \rangle \oplus Y_i = \langle x_i \rangle \oplus X_i$ such that $\eta_i(y_i) = x_i$ and $o(x_i) = p^{n_i}$. It follows from Proposition 7.7 that $\eta_i G = \langle \eta_i(y_i) \rangle$. This implies the relation $G = \langle y_i \rangle \oplus \ker \eta_i$. Consequently, we have $x_{i+1} = k_i y_i + z_i$, where k_i is an integer and $\eta_i(z_i) = 0$.

We assume that p divides k_i. Then

$$p^{n_i-1}\eta_i\eta_{i+1}G = p^{n_i-1}\langle \eta_i(x_{i+1}) \rangle = p^{n_i-1}\langle k_i x_i \rangle = 0.$$

On the other hand, we have $p^{n_i-1}\eta_i\eta_{i+1} = f_i(p^{n_i-1}\eta_i)$ and f_i is an injection. This is a contradiction. Thus, k_i is relatively prime to p.

Let $d_1 = x_1$ and let t be a positive integer such that for all $i = 1, 2, \ldots, t$, the element d_i is already defined in such a way that $d_i = m_i x_i$, where the integer m_i is relatively prime to p and $\eta_i(d_{i+1}) = d_i$ for every $i < t$. It follows from the relations $x_{t+1} = k_t y_t + z_t$ (see the above) that $\eta_t(x_{t+1}) = k_t x_t$. Since k_t and m_t are relatively prime to p, there exists an integer m_{t+1} that is also relatively prime to p such that the relations

$$\eta_t(m_{t+1}x_{t+1}) = m_{t+1}k_t x_t = m_t x_t = d_t$$

hold. We set $d_{t+1} = m_{t+1}x_{t+1}$. Thus, for every i, the group $\langle d_i \rangle = \langle x_i \rangle$ is a direct summand of the group G of order p^{n_i}, which is required. The remaining properties of the elements d_i are valid by the choice of d_i. □

Proof of Theorem 7.1 We assume that the symbol K' denotes the ideal $\mathrm{K}(G')$, and K denotes $\mathrm{K}(G)$, as above. Let $\psi : \mathrm{K} \to \mathrm{K}'$ be some ring isomorphism of nonunital rings. For an endomorphism $\alpha \in \mathrm{K}$, we set $\alpha' = \psi(\alpha) \in \mathrm{K}'$. By Proposition 7.8, there exist positive integers $n_1 < n_2 < \ldots$ and endomorphisms $\eta_i \in \mathrm{K}$ satisfying the conditions(1)–(4) of Proposition 7.8. The corresponding properties are preserved under ring isomorphisms, whence the endomorphisms $\eta_i' \in \mathrm{K}'$ satisfy similar conditions $(1')$–$(4')$. We apply Proposition 7.8 again and obtain that there exist elements $d_i \in G$ and $d_i' \in G'$ with the following properties. These elements generate direct summands of order p^{n_i} and $\eta_i(d_{i+1}) = d_i$, $\eta_i G = \langle d_i \rangle$, $\eta_i'(d_{i+1}') = d_i'$, $\eta_i' G' = \langle d_i' \rangle$.

We define the mapping $\varphi : G \to G'$ as follows. For an element $x \in G$, we take a positive integer k with $o(x) < p^{n_k}$. Let ε be an endomorphism of the group G such that $\varepsilon(d_k) = x$ and ε annihilates the complement to $\langle d_k \rangle$ summand. Then $\varepsilon \in t\big(\mathrm{P}(G)\big) = \mathrm{K}$ by Lemma 7.2. We set $\varphi(x) = \varepsilon'(d_k')$. The mapping φ is well defined. Indeed, we assume that $x = \omega(d_j)$ for some $j \geqslant k$ and $\omega \in \mathrm{K}$. Then $d_k = (\eta_k\eta_{k+1} \cdot \ldots \cdot \eta_{j-2}\eta_{j-1})d_j$, which implies $(\varepsilon\eta_k \cdot \ldots \cdot \eta_{j-2}\eta_{j-1} - \omega)d_j = 0$. Therefore,

$$(\varepsilon\eta_k \cdot \ldots \cdot \eta_{j-2}\eta_{j-1} - \omega)\eta_j\eta_{j+1}G = \langle(\varepsilon\eta_k \cdot \ldots \cdot \eta_{j-2}\eta_{j-1} - \omega)d_j\rangle = 0.$$

It follows from (4) that $(\varepsilon\eta_k \cdot \ldots \cdot \eta_{j-2}\eta_{j-1} - \omega)\eta_j = 0$, whence we have $\varepsilon'\eta_k' \cdot \ldots \cdot \eta_{j-1}'\eta_j' = \omega'\eta_j'$. As a result, we obtain the relation $\omega'(d_j') = \varepsilon'(d_k')$ which means that the action of φ does not depend on the choice of the element d_k and endomorphism ε.

Finally, similarly to the proof of Theorem 4.1, it is directly verified that φ is an isomorphism which induces the isomorphism ψ. □

Remarks 1. It is interesting that a group with a sufficiently rich divisible subgroup is determined by its topological endomorphism ring in the class of all groups. Namely, May proved (see [22]) the following assertion: *Let a group G contain copies of the groups $\mathbb{Q}$ and $\mathbb{Z}_{p^\infty}$ for every prime p. Then for any group H, every topological isomorphism* $\operatorname{End} G \to \operatorname{End} H$ *is induced by some isomorphism* $G \to H$ (cf. Theorem 6.7).

There are not many papers devoted to the definability of torsion-free groups by endomorphism rings. The paper [3] of Bazzoni and Metelli is very important. They proved that *a separable torsion-free group G is determined by its endomorphism ring in the class of all such groups if and only if every direct summand of rank* 1 *of the group G is divided by almost all prime integers.* It should be noted that torsion-free groups often have few endomorphisms, and, generally speaking, the endomorphism ring has little effect on the original group. For torsion-free groups, the isomorphism theorem is a very rare phenomenon.

For mixed groups, on the contrary, there is a rather rich literature on isomorphism theorems. But we immediately point out that even for mixed groups G with $G/t(G) \cong \mathbb{Q}$, the following two central questions have negative answers.

(i) *Will the isomorphism* $G \cong H$ *follow from the isomorphism* $\operatorname{End} G \cong \operatorname{End} H$*?*
(ii) *Is every automorphism of the ring* $\operatorname{End} G$ *inner?*

The corresponding examples can be found in the paper of May and Toubassi (see [31]).

If the isomorphism theorem is invalid for some groups, we can extend the problem and try to find conditions for isomorphism of endomorphism rings of two given groups G and H. May and Toubassi (see [35]) did this for mixed groups of rank 1 with totally projective torsion parts.

Various other results connected with the determination of mixed groups by their endomorphism rings are contained in the papers May and Toubassi [23, 29, 30].

At the end of Sec. 1, we pointed out the close connections of Abelian groups with modules over domains of discrete valuation. The ring $\mathbb{Q}_p$ of rational numbers with denominators relatively prime to p gives an example of such a domain. In fact, the $\mathbb{Q}_p$-modules coincide with the groups G such that $nG = G$ for all integers n with $(n, p) = 1$.

$\widehat{\mathbb{Z}}_p$ is a complete discrete valuation domain. It is the completion of the ring $\mathbb{Q}_p$ in the p-adic topology. The $\widehat{\mathbb{Z}}_p$-modules are also called *p-adic modules*.

Many papers are devoted to isomorphism theorems for endomorphism rings (or endomorphism algebras) of mixed modules over discrete valuation domains. It is clear that all the results obtained in this case are applicable, in particular, to $\mathbb{Q}_p$-modules and $\widehat{\mathbb{Z}}_p$-modules. Unfortunately, even if we accept the strongest assumptions, i.e., if we consider a complete discrete valuation domain R and topological isomorphisms of endomorphism R-algebras, even for mixed modules of rank 1, the two central questions formulated above have a negative solution. This follows, for example, from the paper of May [28].

Most results on the isomorphism problem for mixed modules refer to modules with totally projective torsion submodules or to Warfield modules. A typical result here is the following theorem of May and Tubassi [32]: *Let M be a mixed module of rank* 1 *over a discrete valuation domain R with totally projective torsion submodule. If N is a module of rank* 1*, then every isomorphism* $\operatorname{End}_R M \to \operatorname{End}_R N$ *is induced by some isomorphism $M \to N$*. It follows from the paper of Göbel and May [12] that this result cannot be carried over to mixed modules of other finite ranks, even if one assumes the divisibility of their torsion-free factor modules. However, this is possible if the domain R is complete (see [24]).

Other results on this topic can be found in [4–6, 25–27]. Note that very unexpected examples have been constructed in these and other articles. In particular, they show that there are various serious obstacles to finding isomorphism theorems for endomorphism rings of mixed modules over discrete valuation domains, including if they are complete.

As for Theorem 7.1, the papers [13] of Hausen and Johnson and [34] Schulz completely clarify when for two p-groups G and H, every ring isomorphism $\mathrm{J}(\operatorname{End} G) \to \mathrm{J}(\operatorname{End} H)$ is induced by some isomorphism $G \to H$.

In the papers of Flagg [7–10], the determinability of the module over a discrete valuation domain by the radical of its endomorphism ring is studied. In [9], the author succeeded in replacing the endomorphism ring by the radical of the endomorphism ring in well-known theorems on the definability of mixed modules with totally projective torsion submodules and Warfield modules.

Using some results on isomorphisms of endomorphism rings of modules, one can get acquainted with the survey [33].

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