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TURING COMPUTABILITY: STRUCTURAL THEORY

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***Abstract*.** In this work, we review results of the last years related to the development of the structural theory of n-c.e. Turing degrees for $n > 1$. We also discuss possible approaches to solution of the open problems.

***Keywords and phrases*:** computably enumerable set, Turing degree, Ershov's hierarchy, definability.

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CONTENTS

1. Introduction

Theory of Turing degrees of unsolvability consists of two parts: the local theory, which studies degrees below $\mathbf{0}'$ (the degree of the halting problem) by Turing reducibility, and the global theory, which assumes investigation of the general structure of degrees of unsolvability.

The local degree theory presents more interest and have been developed more deeply. First, it contains all computably enumerable (breifly, c.e.) degrees, which are the core of the computability theory. Moreover, the sets with degrees below $\mathbf{0}'$ possess the characteristic property

$$A \leq_T \emptyset'$$

if and only if there exists a computable function $f(s,x)$ such that $A(x) = \lim_s f(s,x)$. Here $A(x)$ is the characteristic function of the set A:

$$A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin A. \end{cases}$$

Thus, the condition $A \leq_T \emptyset'$ is equivalent to the fact that A can be computably approximated in the following sense: there exists a computable set of uniformly computable sequences $\{f(s,x) \mid s \in \omega\}$ of 0 and 1 such that for each x the limit of the sequence $f(0,x), f(1,x), \ldots$ exists and equals $A(x)$.

In 1965, H. Putnam (see [69]) defined n-c.e. sets as a generalization of c.e. (computably enumerable) sets as follows.

Definition 1.1. For given $n > 0$, a set $A \subseteq \omega$ is called an *n-c.e.* set if there exists a computable sequence of sets $\{A_s\}_{s\in\omega}$ such that for all $x \in \omega$,

$$A_0(x) = 0, \quad A(x) = \lim_s A_s(x), \quad \Big|\big\{s \in \omega \mid A_s(x) \neq A_{s+1}(x)\big\}\Big| \leq n$$

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(thus, c.e. sets are 1-c.e. sets.)

Yu. L. Ershov (see [35–37]) was the first who investigated such sets in detail (also he generalized them to α-c.e. sets for computable ordinals α). The induced hierarchy completely covers all Δ^0_2-sets and is called the *Ershov hierarchy.*

Turing degrees of sets from different levels of the Ershov hierarchy were intensively investigated since 1970s. It was found out that they (considered as partial ordering induced by Turing reducibility) have a rich inner structure and replicate many (but not all as was found out later) properties of its most important representative, namely, the class of degrees containing c.e. sets.

In this work, we observe results obtained during the last 10 years by participants of the seminar on the computability theory in the Institute of Mathematics and Mechanics of the Kazan Federal University. In the first part, we review results related to the investigation of structural theory of degrees of unsolvability. In the second part, we review results related to the model-theoretical properties of the Turing degree structures. In the third part, we emphasize the definability questions of the most important classes of degrees in a wider structures of degrees. Some of these questions are still open despite much effort of research teams from different countries.

We assume that the reader is familiar with a standard university courses of the mathematical logic and the computability theory, in particular, with the first four chapters of R. Soare's book *Recursively Enumerable Sets and Degrees* (its translation from the English original was made in 2000 by the Kazan Mathematical Society). We also follow notation used in this book. In particular, the set of all natural numbers is denoted by ω, the same symbol is used for the first infinite cardinal (these cases can be distinguished by the context). For a set $A \subseteq \omega$, its complement $\omega - A$ is denoted by $\bar{A}$. The cardinality of a set A is denoted by $|A|$. Standard numerations of all c.e. sets and partial computable functions are denoted by $\{W_x\}_{x\in\omega}$ and $\{\Phi_x\}_{x\in\omega}$, respectively.

The binary function $\langle x, y\rangle$ defined as

$$\langle x, y\rangle := \frac{(x+y)^2 + 3x + y}{2}, \quad x, y \in \omega,$$

and performing a bijection from ω^2 onto ω is called the *Cantor numbering function.* For $n > 2$, the n-ary function $\langle x_1, \dots x_n\rangle$ is defined as follows:

$$\langle x_1, \dots x_n\rangle = \Big\langle \Big\langle \dots \langle x_1, x_2\rangle, x_3\Big\rangle, \dots, x_n\Big\rangle.$$

If a function f is defined on an element x, then we denote it by $f(x)\downarrow$, otherwise we write $f(x)\uparrow$. By $A[x]$ we denote the finite set $A \cap \{0, 1, \dots, x-1\}$.

2. Semilattice of n-c.e. Degrees: Structural Theory

2.1. Comparison of elementary theories. The first results about n-c.e. sets (n*-c.e. degrees*) were obtained by Lachlan (the end of 1960s, unpublished) who proved that for any n-c.e. degree $\boldsymbol{d} > \mathbf{0}$, $n > 1$, there exists a c.e. degree $\boldsymbol{a}$ with $\boldsymbol{d} > \boldsymbol{a} > \mathbf{0}$, and also by Cooper (see [19]) who proved that there exists a *properly d-c.e. degree*, i.e., a Turing degree which contains a d-c.e. set, but does not contain a c.e. set (where "d-c.e." is an alternative notation for "2-c.e.").

It is clear that d-c.e. degrees and, in general, n-c.e. Turing degrees form an intermediate structure between the structures of c.e. and Δ^0_2-degrees. It is natural to compare n-c.e. degrees with degrees of these structures, especially since n-c.e. degrees share some properties of degrees from both of these structures.

The result of Lachlan mentioned above and the existence of a minimal Δ^0_2-degree (see [72]) show that the structures of n-c.e. degrees and Δ^0_2-degrees are not elementarily equivalent. The first elementary difference between c.e. and n-c.e. degree for $n > 1$ was obtained by Arslanov (see [5, 6]) who showed that each nonzero n-c.e. degree $\boldsymbol{d}$ *is cuppable to* $\mathbf{0}'$ by some d-c.e. degree, namely, there exists a d-c.e. degree $\boldsymbol{e} < \mathbf{0}'$ such that $\boldsymbol{d} \cup \boldsymbol{e} = \mathbf{0}'$, withal Yates and Cooper (1973, unpublished, see also D. Miller [67]) showed that this does not hold within c.e. degrees. It implies that elementary theories of $\mathcal{R}$ and $\mathcal{D}_n$ are different at Σ_3^0-level. The next elementary differences between these structures were found by Downey (see [29]), who showed that Lachlan's theorem (see [50]) about nonembedding of the diamond into c.e. degrees preserving the greatest and least element does not hold in d-c.e. degrees, and by Cooper, Harrington, Lachlan, Lempp and Soare (see [23, 24]), who established the nondenseness of the ordering of n-c.e. degrees for all $n > 1$ contrary to the Sacks theorem about the denseness of c.e. degrees. It follows that elementary theories of semilattices of $\mathcal{R}$ and $\mathcal{D}_n$ are different already at the Σ_2^0-level.

Since any partially ordered set is embeddable into $\mathcal{R}$ (thus, also into $\mathcal{D}_n$ for $n \geq 1$) respecting $\leq_T$, the elementary theories of $\mathcal{D}_n$ and $\mathcal{D}_m$ for $n \neq m$, $n, m < \omega$, coincide at the Σ_1^0-level. It is known (see, e.g., Soare [83]) that there exist ω-c.e. minimal degrees. Since an n-c.e. degree for $n \geq 1$ cannot be minimal, the elementary theories of $\mathcal{D}_n$, $n \geq 1$, and $\mathcal{D}_\omega$ are different at the Σ_2^0-level.

Elementary differences between structures of n-c.e. degrees for different $n > 1$ for a long time was unknown. Downey [29] even stated a conjecture that structures of m-c.e. and n-c.e. degrees for different $m, n > 1$ must be elementarily equivalent.

This conjecture was disproved in the work of Arslanov, Kalimullin, and Lempp [11], where they constructed a Σ_2^0-formula distinguishing the structures of 2-c.e. and 3-c.e. degrees. Also in this work they provided the first example of an infinite class of c.e. degrees which is definable in the semilattice $\mathcal{D}_2$.

Theorem 2.1 (Arslanov, Kalimullin, and Lempp [11])**.** *There exist a d-c.e. set D and a c.e. set A such that $\emptyset <_T A <_T D \oplus A$ and for any d-c.e. set U, the following assertion holds: if $U \leq_T D \oplus A$, then either $A \leq_T U$ or $U \leq_T A$.* (*In this case, we say that the degrees of sets A and $D \oplus A$ form a* double bubble.)

Corollary 2.2. *There exists d-c.e. degrees $\boldsymbol{d} > \boldsymbol{a} > \boldsymbol{0}$ such that for any d-c.e. degree $\boldsymbol{u}$, the following assertion holds: if $\boldsymbol{u} \leq \boldsymbol{d}$ then either $\boldsymbol{a} \leq \boldsymbol{u}$ or $\boldsymbol{u} \leq \boldsymbol{a}$.*

If d-c.e. degrees $\boldsymbol{a}$ and $\boldsymbol{d}$ possess this property, then the degree $\boldsymbol{a}$ must be computably enumerable. Indeed, let d-c.e. sets D and A be such that $\emptyset <_T A <_T D \oplus A$ and for any d-c.e. set U the following assertion holds: if $U \leq_T D \oplus A$, then either $A \leq_T U$ or $U \leq_T A$.

Denote by $X \leq_T D$ a c.e. set such that the d-c.e. set $D \oplus A$ is c.e. relative to X (the existence of such a set follows from the unpublished result of Lachlan mentioned above; we will consider it in detail in Sec. 4.4). If $X \not\equiv_T A$, then one of the following possibilities holds.

Case 1. $A <_T X \leq_T D \oplus A$. Let $\boldsymbol{x} = \deg(X)$ and $\boldsymbol{a} = \deg(A)$. By the Sacks splitting theorem (see [73]), the degree $\boldsymbol{x}$ is splittable in c.e. degrees avoiding the upper cone of $\boldsymbol{a}$: $\boldsymbol{x} = \boldsymbol{x}_0 \cup \boldsymbol{x}_1$, $\boldsymbol{a} \not\leq \boldsymbol{x}_i$, $i = 0, 1$. It is clear that at least one of the c.e. degrees $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ is incomparable with $\boldsymbol{a}$; a contradiction.

Case 2. $\emptyset <_T X <_T A$. This case is impossible due to the following theorem.

Theorem 2.3 (Arslanov, Kalimullin, and Lempp [11])**.** *Let D and A be d-c.e. sets and let X be a c.e. set such that $A <_T D$, $X \leq_T D$, $A \not\leq_T X$, and A and D be c.e. relative to X. Then there exist a d-c.e. set U such that $U \leq_T D$ and $U \,|_T\, A$.*

Since the cases 1 and 2 are impossible, it follows that $X \equiv_T A$ and, consequently, A has a c.e. degree such that $\deg(D \oplus A)$ is c.e. relative to this degree.

Remark 2.4. In the conditions of Theorem 2.1 for the degree $\deg(D \oplus A)$, there exists a unique c.e. degree $\deg(A)$ with this property. Namely, if for some c.e. set U such that $\emptyset <_T U <_T D \oplus A$ we have $(\forall\ d\text{-c.e. set } V)[V \leq_T D \oplus A \to U \leq_T V \vee V \leq_T U]$, then $U \equiv_T A$.

Proof. To the contrary, assume that this assertion is invalid. Then either $A <_T U$ or $U <_T A$. In both cases we have a contradiction with the Sacks splitting theorem (with avoiding the upper cone of degrees). □

Theorem 2.5 (Arslanov, Kalimullin, and Lempp [11]). *There are no 2-c.e. degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ such that for any 2-c.e. degree $\boldsymbol{u}$ the following assertions hold*:

(i) *if $\boldsymbol{u} \le \boldsymbol{f}$, then either $\boldsymbol{u} \le \boldsymbol{e}$ or $\boldsymbol{e} \le \boldsymbol{d} \cup \boldsymbol{u}$, and*
(ii) *if $\boldsymbol{u} \le \boldsymbol{e}$ then either $\boldsymbol{d} \le \boldsymbol{u}$ or $\boldsymbol{u} \le \boldsymbol{d}$.*

Proof. By the contrary, assume that such degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ exists. Due to the observation after Corollary 2.2, the degree $\boldsymbol{d}$ must be c.e. Let $\boldsymbol{f}' \le \boldsymbol{f}$ and $\boldsymbol{e}' \le \boldsymbol{e}$ be c.e. degrees such that $\boldsymbol{f}$ and $\boldsymbol{e}$ are c.e. relative to $\boldsymbol{f}'$ and $\boldsymbol{e}'$, respectively. Consider the degree $\boldsymbol{x} = \boldsymbol{d} \cup \boldsymbol{e}' \cup \boldsymbol{f}'$. It is clear that $\boldsymbol{d} \le \boldsymbol{x} \le \boldsymbol{f}$.

Note that $\boldsymbol{x}$ is computably enumerable. By item (i), we have either $\boldsymbol{x} \le \boldsymbol{e}$ or $\boldsymbol{e} \le \boldsymbol{d} \cup \boldsymbol{x} = \boldsymbol{x}$. In the first case, a splitting of $\boldsymbol{x}$ with avoiding the upper cone of $\boldsymbol{d}$ provides a c.e. degree incomparable with $\boldsymbol{d}$, which leads to a contradiction. Thus, $\boldsymbol{e} < \boldsymbol{x}$, but then $\boldsymbol{e}$ and $\boldsymbol{x}$ are enumerable relative to $\boldsymbol{d}$, and applying Theorem 2.3, we get a 2-c.e. degree, which is between $\boldsymbol{d}$ and $\boldsymbol{f}$ and incomparable with $\boldsymbol{e}$, which leads to a contradiction again. □

2.2. Generalization to higher levels. Now a natural question arises about the existence of 3-c.e. sets F, E, and D such that $\emptyset <_T D <_T D \oplus E <_T D \oplus E \oplus F$, and for any 3-c.e. set U, the following assertion holds: if $U \le_T D \oplus E \oplus F$, then either $D \oplus E \le_T U$ or $U \le_T D \oplus E$, and if $U \le_T D \oplus E$, then either $D \le_T U$ or $U \le_T D$. (In this case, we say that the degrees of the sets D, $D \oplus E$, and $D \oplus E \oplus F$ form the 3-*bubble*.)

A positive answer to this question allows one to construct a Σ^0_2-formula which differs from the semilattices of 2-c.e. and 3-c.e. degrees since Theorem 2.5 mentioned above implies that this property does not hold in the semilattice of 2-c.e. degrees.

Theorem 2.10 (and Proposition 2.9) imply that the answer to this question is positive. First, we state a necessary definition.

Definition 2.6. Let 3-c.e. degrees $\boldsymbol{f}$, $\boldsymbol{e}$, and $\boldsymbol{d}$ be such that $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ and for any 3-c.e. degree $\boldsymbol{u}$ the following conditions hold:

(i) if $\boldsymbol{u} \le \boldsymbol{f}$, then either $\boldsymbol{u} \le \boldsymbol{e}$ or $\boldsymbol{e} \le \boldsymbol{d} \cup \boldsymbol{u}$, and
(ii) if $\boldsymbol{u} \le \boldsymbol{e}$, then either $\boldsymbol{d} \le \boldsymbol{u}$ or $\boldsymbol{u} \le \boldsymbol{d}$.

In this case, we say that the degrees $\boldsymbol{d}$, $\boldsymbol{e}$, and $\boldsymbol{f}$ form a *weak* 3-*bubble.*

Also we use the following result obtained by M. M. Arslanov jointly with American mathematicians G. LaForte and T. Slaman.

Theorem 2.7 (Arslanov, LaForte, and Slaman [12]). *Let C be an ω-c.e. set and let a set $A \oplus W^A$ be c.e. relative to the c.e. set A. If $C \le_T A \oplus W^A$, then there exists a d-c.e. set D such that $C \le_T D \le_T A \oplus W^A$.*

Corollary 2.8 (Arslanov, LaForte, and Slaman [12]). *Any ω-c.e. degree, which is 2-CEA degree, is a d-c.e. degree.*

Now we prove the following assertion.

Proposition 2.9. *Any weak 3-bubble, which is formed by 3-c.e., 2-c.e., and c.e. degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d}$, is a 3-bubble.*

Proof. Assume that a weak 3-bubble is formed by 3-c.e., 2-c.e., and c.e. degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d}$. To the contrary, assume that there exists a 3-c.e. degree $\boldsymbol{f}_1 < \boldsymbol{f}$ which is incomparable with $\boldsymbol{e}$ and $\boldsymbol{d}$.

The degree $\boldsymbol{f}_1$ cannot be a 2-c.e. degree; otherwise, the degree $\boldsymbol{f}_1 \cup \boldsymbol{d}$ also must be a 2-c.e. degree. Since, clearly, $\boldsymbol{f}_1 \cup \boldsymbol{d} > \boldsymbol{e}$, the degrees $\boldsymbol{f}_1 \cup \boldsymbol{d} > \boldsymbol{e} > \boldsymbol{d}$ form a weak 3-bubble, which cannot be valid due to Theorem 2.5.

Thus, the degree $\boldsymbol{f}_1$ is a properly 3-c.e. degree. However, then the degrees of Lachlan's sets (which will be introduced in Sec. 4.4) for this degree must be less than $\boldsymbol{e}$ (otherwise, we get a 2-c.e. degree incomparable with $\boldsymbol{e}$ and $\boldsymbol{d}$, a contradition). Denote by $\boldsymbol{e}_1$ the degree of some Lachlan's set for a 3-c.e. set from $\boldsymbol{f}_1$. It cannot be $\geq \boldsymbol{d}$ (otherwise, $\boldsymbol{f}_1 \geq \boldsymbol{d}$ too), then $\boldsymbol{e}_1 < \boldsymbol{d}$.

Consider $\boldsymbol{v} = \boldsymbol{f}_1 \cup \boldsymbol{d}$. The degree $\boldsymbol{v}$ is computable enumerable relative to $\boldsymbol{d}$ and, consequently, is a 2-c.e. degree (see Corollary 2.8 of Theorem 2.7), which cannot be valid (since $\boldsymbol{f}_1 \cup \boldsymbol{d} > \boldsymbol{e} > \boldsymbol{d}$ form a weak 3-bubble). Thus, there is no a 3-c.e. degree, which is incomparable simultaneously with $\boldsymbol{e}$ and $\boldsymbol{d}$. □

Theorem 2.10 (Arslanov, Kalimullin, and Lempp [11])**.** *There exists a* 3*-c.e. degree $\boldsymbol{f}$, a d-c.e. degree $\boldsymbol{e}$, and a c.e. degree $\boldsymbol{d}$ which form a weak* 3*-bubble.*

Corollary 2.11. $\mathcal{D}_2 \not\equiv \mathcal{D}_3$ *at Σ_2^0-level.*

Proof. The following Σ_2^0-formula φ is valid in $\mathcal{D}_3$ and is invalid in $\mathcal{D}_2$:

$$\varphi \doteq (\exists \boldsymbol{d} > \boldsymbol{0})(\exists \boldsymbol{e} > \boldsymbol{d})(\exists \boldsymbol{f} > \boldsymbol{e})\Big\{(\forall \boldsymbol{u} \leq \boldsymbol{f})[\boldsymbol{u} \leq \boldsymbol{e} \ \vee \ \boldsymbol{e} \leq \boldsymbol{u}] \ \& \ (\forall \boldsymbol{u} \leq \boldsymbol{e})[\boldsymbol{u} \leq \boldsymbol{d} \ \vee \ \boldsymbol{d} \leq \boldsymbol{u}]\Big\}. \qquad \square$$

This result disproves the Downey conjecture (see [29]) about the elementarily difference of these structures.

2.3. Splitting into incomparable degrees. In what follows, we will use various methods of splitting of n-c.e. degrees into two incomparable degrees; we start from consideration of these questions.

Definition 2.12. A *splitting* of a degree $\boldsymbol{d}$ is pair of degrees $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ such that $\boldsymbol{d}_0, \boldsymbol{d}_1 < \boldsymbol{d}$ and $\boldsymbol{d} = \boldsymbol{d}_0 \cup \boldsymbol{d}_1$. We say that it is a *splitting above* a degree $\boldsymbol{a} < \boldsymbol{d}$ if $\boldsymbol{a} \leq \boldsymbol{d}_0, \boldsymbol{d}_1$. Eventually, a splitting of $\boldsymbol{d}$ into $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ is a splitting avoiding the upper cone of degrees of $\boldsymbol{b}$ if $\boldsymbol{b} \not\leq \boldsymbol{d}_i$, $i \leq 1$.

In the semilattice of c.e. degrees, the first results about splitting belong to Sacks and Robinson. In [73], Sacks established that each c.e. degree $\boldsymbol{a} > 0$ is splittable into two c.e. degrees avoiding the upper cone of degrees of $\boldsymbol{b}$ for any Δ_2^0-degree $\boldsymbol{b} > 0$. In [70], Robinson proved that each c.e. degree $\boldsymbol{a} > \boldsymbol{0}$ is splittable into two low c.e. degrees above any low c.e. degree. Later, Lachlan [55] obtained the following results which is known as "Lachlan's Monster Theorem" due to highly intricacy of its proof (in this work, Lachlan first used $0'''$-priority method): there exist c.e. degrees $\boldsymbol{a} < \boldsymbol{b}$ such that $\boldsymbol{b}$ is not splittable in c.e. degrees above $\boldsymbol{a}$. Harrington showed (see [43]) that the degree $\boldsymbol{b}$ can be taken as $\boldsymbol{0}'$ here.

Below in Theorem 2.13 we will see that a splitting of any d-c.e. degree $\boldsymbol{b}$ in d-c.e. degrees is possible above any c.e. degree $\boldsymbol{a} < \boldsymbol{b}$.

Let $\boldsymbol{a} > \boldsymbol{0}$ be an n-c.e. degree for some $n > 1$ and $\boldsymbol{b}$ be a c.e. degree such that $\boldsymbol{b} < \boldsymbol{a}$. Since $\boldsymbol{a}$ is c.e. relative to some $(n-1)$-c.e. degree $\boldsymbol{a}_0 < \boldsymbol{a}$, by the Sacks splitting theorem relativized to $\boldsymbol{a}_0 \cup \boldsymbol{b} < \boldsymbol{a}$, the degree $\boldsymbol{a}$ is splittable into two Δ_2^0-degrees above $\boldsymbol{b}$. Items (b) and (c) of Theorem 2.13 state that such splitting is possible above low d-c.e. degrees within d-c.e. degrees.

Theorem 2.13.

(a) *Each d-c.e. degree $\boldsymbol{a}$ is splittable in d-c.e. degrees above any c.e. degrees $\boldsymbol{b} < \boldsymbol{a}$ (Cooper* [22] *for the case $\boldsymbol{b} = \boldsymbol{0}$ and Cooper and Li* [25] *for the general case);*

(b) *any c.e. degree $\boldsymbol{a}$ is splittable in d-c.e. degrees above any low d-c.e. degrees $\boldsymbol{b} < \boldsymbol{a}$ (Arslanov, Cooper, and Li* [9, 10]*); moreover,*

(c) *any d-c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ is splittable in d-c.e. degrees above some low d-c.e. degree $\boldsymbol{b} < \boldsymbol{d}$ (Li* [63]*); however,*

(d) *there exists a d-c.e. degree $\boldsymbol{d}$ such that $\boldsymbol{0}'$ is not splittable even in ω-c.e. degrees above $\boldsymbol{d}$ (Cooper, Harrington, Lachlan, Lempp, and Soare* [23, 24]*).*

Earlier, Cooper [21] proved that density and splitting properties can be combined in low_2 n-c.e. degrees. It is known (Shore and Slaman [78]) that low_2 c.e. degrees also can be split in $\mathcal{R}$ above any c.e. degree, which is below the given one. This and other structural properties, which are similar for the partial orders of low_2 c.e. and low_2 n-c.e. degrees for $n > 1$, allowed Downey and Stob [32] to conjecture that the ordering of low_2 d-c.e. degrees should be equivalent to the ordering of low_2 c.e. degrees. Later, this conjecture was disproved by Yamaleev [88].

For the case of low_3 n-c.e. degrees, Cooper and Li [26] established the following result.

Theorem 2.14. *For any $n > 1$, there exist a low_3 n-c.e. degree $\boldsymbol{a}$ and a c.e. degree $\boldsymbol{b}$, $\mathbf{0} < \boldsymbol{b} < \boldsymbol{a}$, such that for any splitting of $\boldsymbol{a}$ into n-c.e. degrees $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$, at least of the degrees $\boldsymbol{a}_0$ or $\boldsymbol{a}_1$ is above $\boldsymbol{b}$.* (*Thus, $\boldsymbol{a}$ cannot be split avoiding the upper cone of degrees of $\boldsymbol{b}$.*)

Since in $\mathcal{R}$ such splitting of low_3 c.e. degrees always can be done by avoiding a given cone of degrees, it follows that these two orderings are not elementarily equivalent.

Theorem 2.15 (Shore and Slaman [79])**.** *Let $\boldsymbol{d}$, $\boldsymbol{a}$, and $\boldsymbol{b}$ be such that $\boldsymbol{d}$ is an n-c.e. degree for some $n \geq 1$ and $\boldsymbol{a}$ and $\boldsymbol{b}$ be Δ^0_2-degrees such that $\boldsymbol{a} < \boldsymbol{d}$ and $\boldsymbol{a} \ngeq \boldsymbol{b}$. Then $\boldsymbol{d}$ is splittable in Δ^0_2-degrees above $\boldsymbol{a}$ avoiding the upper cone of $\boldsymbol{b}$.*

It is clear that the last result cannot be strengthened considering the splittings in d-c.e. degrees (it was proved in [26] that it cannot be done even for $\boldsymbol{a} = \mathbf{0}$). Yamaleev [87] proved that such a splitting is possible for $\boldsymbol{a} = \mathbf{0}$ if the degree $\boldsymbol{d}$ is a properly d-c.e. degree and the degree $\boldsymbol{b}$ is noncomputable Δ^0_2-degree such that there are no c.e. degrees between $\boldsymbol{d}$ and $\boldsymbol{b}$.

Theorem 2.13 implies that Theorem 2.15 cannot be strengthened considering the splittings of $\boldsymbol{d}$ in d-c.e. degrees; this it cannot be done even for $\boldsymbol{b} = \mathbf{0}'$. Below we put Theorem 4.1 of Yamaleev [87], where it is obtained that such splitting of $\boldsymbol{d}$ is possible if $\boldsymbol{a} = \mathbf{0}$, the degree $\boldsymbol{d}$ is properly d-c.e. degree, and $\boldsymbol{b} < \boldsymbol{d}$ is a noncomputable Δ^0_2-degree such that there are no c.e. degrees between $\boldsymbol{d}$ and $\boldsymbol{b}$.

3. Model-Theoretic Properties

3.1. Bounded theories. We saw that there is a bunch of results about the investigation of the structural theory of (Turing) degrees of undecidability which contain sets from different levels of the Ershov hierarchy. The next natural step should be a systematic investigation of model-theoretic properties of these structures; however, not much work was done in this area. Except for the above-mentioned results of Arslanov, Downey, and other specialists about elementary differences of the semilattices of c.e. degrees and of degrees from the levels n, $n > 1$, of the Ershov hierarchy, there are the following several great achievements obtained in this direction during these years:

- (i) the set of n-c.e. degrees in signature $\{\leq\}$ does not form a Σ_1-substructure of m-c.e. degrees for all n and m, where $1 \leq n < m$ (Cai, Shore and Slaman [16]);
- (ii) for any $m \geq 1$, partial orderings of low_n c.e. and low_m d-c.e. degrees are not elementarily equivalent (Yamaleev [88]. For $m = 1$, this result was obtained independently also by Faizrahmanov [41]).

Despite a lot of effort, there are several natural questions which are still not answered. These include, for example, the following questions:

- (i) (un)decidability of bounded theories of n-c.e. ($n \geq 1$) degrees;
- (ii) the problem of definability (either with parameters or not) of c.e. degrees in the structures of n-c.e. degrees for $n > 1$ (in the general case, it is the problem of definability of m-c.e. degrees in structures of n-c.e. degrees for $1 \leq m < n$).

The problem of definability (either with parameters or not) of n-c.e. degrees for some (all) $n > 1$ in Δ^0_2-degrees is also uninvestigated. It is interesting to find natural classes of n-c.e. degrees that are definable with parameters in Δ^0_2-degrees. Also, there are unknown answers for the following questions: Does a single n-c.e. degree exist, $n \geq 1$, which is definable in $\mathcal{D}_n$ and which is different from $\mathbf{0}$ and $\mathbf{0}'$? Assume that $\mathcal{D}_n$, $n \geq 1$, is fixed. Does a class of m-c.e. degrees ($1 \leq m \leq n$) $\mathcal{C}$ and n-c.e. degrees $\boldsymbol{a}, \boldsymbol{b}$ exist such that $|\mathcal{C} \cap [\boldsymbol{a}, \boldsymbol{b}]| = 1$?

- The problem of elementarily equivalence of semilattices $\mathcal{D}_n$ and $\mathcal{D}_m$, for $m \neq n$ and $n, m \geq 3$, is still open.

We assume that theories of these semilattices are pairwise different; moreover, evidence of the differences can be obtained from a natural generalization of the above-mentioned sentence (which was used in the proof of the difference of semilattices $\mathcal{D}_2$ and $\mathcal{D}_3$)

$$(\exists \boldsymbol{d} > \boldsymbol{0})(\exists \boldsymbol{e} > \boldsymbol{d})(\exists \boldsymbol{f} > \boldsymbol{e})\Big\{(\forall \boldsymbol{u} \leq \boldsymbol{f})\big[\boldsymbol{u} \leq \boldsymbol{e} \ \vee \ \boldsymbol{e} \leq \boldsymbol{u} \cup \boldsymbol{d}\big] \ \& \ (\forall \boldsymbol{u} \leq \boldsymbol{e})\big[\boldsymbol{u} \leq \boldsymbol{d} \ \vee \ \boldsymbol{d} \leq \boldsymbol{u}\big]\Big\}$$

to the corresponding levels of the hierarchy. However, the proof of this claim can be quite intricate if one follows the way of simple generalization of the proof of Arslanov, Kalimullin, and Lempp [11].

We can try to enhance this conjecture if we require that the existential quantifier from the previous formula φ chooses same elements from the structures $\mathcal{D}_n$ and $\mathcal{D}_{n+1}$: if $\mathcal{D}_n \models \forall \bar{x} \psi(\bar{a}, \bar{x})$ for some $\bar{a}$ from $\mathcal{D}_n$ then also $\mathcal{D}_{n+1} \models \forall \bar{x} \psi(\bar{a}, \bar{x})$.

Now we will see that the enhanced version of the conjecture does not hold.

Definition 3.1. A substructure $\mathcal{L}_1$ of a structure $\mathcal{L}_2$ is its Σ_k-substructure for some $k \geq 1$ if for any Σ_k-formula $\varphi(x_1, \ldots, x_r)$ and for any $a_1, \ldots, a_r \in \mathcal{L}_1$,

$$\mathcal{L}_1 \models \varphi(a_1, \ldots, a_r) \iff \mathcal{L}_2 \models \varphi(a_1, \ldots, a_r).$$

We denote it by $\mathcal{L}_1 \preceq_{\Sigma_k} \mathcal{L}_2$.

In 1983, Slaman established that c.e. degrees do not form a Σ_1-substructure of Δ^0_2-degrees. Namely, he proved that the sentence

$$\varphi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \exists x(\boldsymbol{0} < \boldsymbol{x} < \boldsymbol{a} \ \& \ \boldsymbol{c} \not\leq \boldsymbol{b} \cup \boldsymbol{x}),$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are c.e. degrees, is valid in Δ^0_2-degrees and invalid in c.e. degrees.

Since c.e. degrees are in particular n-c.e. degrees for any $n > 1$, it immediately follows that $\mathcal{D}_n \not\preceq_{\Sigma_1} \mathcal{D}(\leq \boldsymbol{0}')$.

Later, Yang and Yu [89] proved the following theorem, which implies that the c.e. degrees do not form a Σ_1-substructure of 2-c.e. degrees.

Theorem 3.2. *There exists c.e. degrees $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{e}$ (parameters) such that*

(i) *there exists a 2-c.e. degree $\boldsymbol{d} < \boldsymbol{a}$ such that $\boldsymbol{d} \not\leq \boldsymbol{e}$ and $\boldsymbol{c} \not\leq \boldsymbol{d} \cup \boldsymbol{b}$;*
(ii) *for any c.e. degree $\boldsymbol{w} < \boldsymbol{a}$, we have either $\boldsymbol{w} \leq \boldsymbol{e}$ or $\boldsymbol{c} \leq \boldsymbol{w} \cup \boldsymbol{b}$.*

Finally, $\mathcal{D}_n$ is not a Σ_1-substructure of $\mathcal{D}_{n+1}$ for any $n > 1$. It was obtained recently by Cai, Shore, and Slaman [16].

Theorem 3.3 (Cai, Shore and Slaman [16])**.** *For any $n \geq 1$, there exist c.e. degrees $\boldsymbol{g}$, $\boldsymbol{p}$, $\boldsymbol{q}$, n-c.e. degree $\boldsymbol{a}$, and $(n+1)$-c.e. degree $\boldsymbol{d}$ such that*

(1) *$\boldsymbol{d} \leq \boldsymbol{a}$, $\boldsymbol{q} \not\leq \boldsymbol{d} \cup \boldsymbol{p}$ and $\boldsymbol{d} \not\leq \boldsymbol{g}$;*
(2) *for any n-c.e. degree $\boldsymbol{w} \leq \boldsymbol{a}$, we have either $\boldsymbol{q} \leq \boldsymbol{w} \cup \boldsymbol{p}$ or $\boldsymbol{w} \leq \boldsymbol{g}$.*

It follows from this theorem that the sentence

$$\varphi(\boldsymbol{a}, \boldsymbol{g}, \boldsymbol{p}, \boldsymbol{q}) = \exists \boldsymbol{w}\Big(\boldsymbol{w} \leq \boldsymbol{a} \ \& \ \boldsymbol{q} \not\leq \boldsymbol{w} \cup \boldsymbol{p} \ \& \ \boldsymbol{w} \not\leq \boldsymbol{g}\Big),$$

where $\boldsymbol{g}, \boldsymbol{p}, \boldsymbol{q}$ are c.e. degrees and $\boldsymbol{a}$ is an n-c.e. degree, is valid in the $(n+1)$-c.e. degrees and invalid in the n-c.e. degrees. Thus, $\mathcal{D}_n \not\preceq_{\Sigma_1} \mathcal{D}_{n+1}$ for any $n \geq 1$. Again, since n-c.e. degrees are m-c.e. degrees for any $m > n$, we have that $\mathcal{D}_n \not\preceq_{\Sigma_1} \mathcal{D}_m$ for any $n < m$.

By Theorem 2.10, there exist a c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ and a d-c.e. degree $\boldsymbol{e} > \boldsymbol{d}$ such that each 3-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ is comparable with $\boldsymbol{d}$. Is it possible to enhance this sentence in the following way: do a c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ and a d-c.e. degree $\boldsymbol{e} > \boldsymbol{d}$ exist such that any n-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ for any $n < \omega$ is comparable with $\boldsymbol{d}$?

If the answer is positive, then we get a quite interesting property of degrees from finite levels of the Ershov hierarchy. On the other hand, in the case of a negative answer we obtain the following argument: let $\boldsymbol{d} > \boldsymbol{0}$ and $\boldsymbol{e} > \boldsymbol{d}$ be a c.e. degree and a d-c.e. degree, respectively, and let $n \geq 3$ be the greatest natural number such that each n-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ is comparable with $\boldsymbol{d}$ and there exists an $(n+1)$-c.e. degree $\boldsymbol{v} \leq \boldsymbol{e}$ incomparable with $\boldsymbol{d}$. Consider the following Σ_1-sentence:

$$\varphi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \equiv \exists \boldsymbol{u}\Big(\boldsymbol{x} < \boldsymbol{y} < \boldsymbol{z} \ \& \ \boldsymbol{u} \leq \boldsymbol{z} \ \& \ \boldsymbol{u} \not\leq \boldsymbol{y} \ \& \ \boldsymbol{y} \not\leq \boldsymbol{u}\Big).$$

If $\boldsymbol{d}$ and $\boldsymbol{e}$ are the above-mentioned degrees and n is a natural number which exists according to the negative answer, then we have $\mathcal{D}_{n+1} \models \varphi(\boldsymbol{0}, \boldsymbol{d}, \boldsymbol{e})$ and $\mathcal{D}_n \models \neg\varphi(\boldsymbol{0}, \boldsymbol{d}, \boldsymbol{e})$. Thus, in this case $\varphi(\boldsymbol{0}, \boldsymbol{d}, \boldsymbol{e})$ is another sentence which shows that $\mathcal{D}_n$ is not a Σ_1-substructure of $\mathcal{D}_{n+1}$; moreover, it has a smaller number of parameters.

Thus, we see that either a positive or negative answer for the posed question leads to interesting corollaries.

We assume that the following conjecture holds.

Conjecture 3.4. For any $n \geq 1$, there exists an n-c.e. degree $\boldsymbol{d}$ and an $(n+1)$-c.e. degree $\boldsymbol{e}$ such that $\boldsymbol{0} < \boldsymbol{d} < \boldsymbol{e}$ and for each $(n+1)$-c.e. degree $\boldsymbol{c} \leq \boldsymbol{e}$, either $\boldsymbol{c} \leq \boldsymbol{d}$ or $\boldsymbol{d} \leq \boldsymbol{c}$, but there exists an $(n+2)$-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ such that $\boldsymbol{u}$ is incomparable with $\boldsymbol{d}$.

Moreover, probably this conjecture holds in a stronger version (see the previous discussion of this part of the conjecture):

For any $n \geq 2$, there exist a c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ and a d-c.e. degree $\boldsymbol{e} > \boldsymbol{d}$ such that each n-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ is comparable with $\boldsymbol{d}$. (For $n = 2$, this is Theorem 2.1.)

All known so far sentences (which use the language of partial ordering), which are valid in n-c.e. degrees and invalid in $(n+1)$-c.e. degrees, for some $n \geq 1$, belong to the level $\forall\exists$ or to higher levels. These and other observations allow one to state the following interesting conjecture.

Conjecture 3.5. For any $n \geq 1$ and for all $\exists\forall$-sentences φ, we have

$$\mathcal{D}_n \models \varphi \ \Rightarrow \ \mathcal{D}_{n+1} \models \varphi$$

(i.e., $\exists\forall$-theory of n-c.e. degrees is a subtheory of $\exists\forall$-theory of $(n+1)$-c.e. degrees).

When one passes from the structure $\mathcal{R}(\leq, \cup)$ to the structure $\mathcal{D}_2(\leq)$, Σ_1-formulas become Σ_2-formulas, which evidences the naturalness of this conjecture.

The following conjecture is an enhancement of the previous one.

Conjecture 3.6. For any $n \geq 1$, $\mathcal{D}_n$ is an Σ_1-substructure of $\mathcal{D}_{n+1}$.

Indeed, assume that $\mathcal{D}_n$ is a Σ_1-substructure of $\mathcal{D}_{n+1}$, but also there exist a Σ_2-formula

$$\varphi = \exists\overline{x}\forall\overline{y}\psi(\overline{x}, \overline{y}),$$

which is valid in $\mathcal{D}_n$ and invalid in $\mathcal{D}_{n+1}$. Then

$$\neg\varphi = \forall\overline{x}\exists\overline{y}\psi(\overline{x}, \overline{y})$$

is valid in $\mathcal{D}_{n+1}$. Then as $\overline{x}$ we take a cortege of n-c.e. degrees $\overline{a}$, which provides the validity of φ in $\mathcal{D}_n$, and consider the Σ_1-formula

$$\theta = \exists\overline{y}\psi(\overline{a}, \overline{y}).$$

We obtain that θ is valid in $\mathcal{D}_{n+1}$ on n-c.e. parameters and also $\mathcal{D}_n$ is a Σ_1-substructure of $\mathcal{D}_{n+1}$. Then θ is valid in $\mathcal{D}_n$. However, this means that on the degrees $\overline{a}$ the formula $\exists\overline{y}\psi(\overline{a}, \overline{y})$ is also valid in $\mathcal{D}_{n+1}$; a contradiction.

3.2. The problem of decidability of bounded fragments of theories. The undecidability (and also the nonaxiomatizability) of the elementary theory of $\mathcal{D}$ was established by Lachlan [53]. For the theory of $\mathcal{D}(\leq \mathbf{0}')$, this was done independently by Epstein [34] and Lerman [59]. Harrington and Shelah [44] proved that the theory of $\mathcal{R}$ is also undecidable. Finally, recently Cai, Shore, and Slaman [16] obtained undecidability of elementary theories of $\mathcal{D}_n$ for any $n > 1$.

A simple finite injury priority argument allows one to embed any finite partial ordering in the c.e. degrees preserving the order. It follows the decidability of $\forall$-theories of all these structures in the signature $\leq$.

The undecidability of the $\forall\exists\forall$-theories for $\mathcal{D}$, $\mathcal{D}(\leq \mathbf{0}')$, and $\mathcal{R}$ in the signature $\leq$ was established by Shmerl (for the theories of $\mathcal{D}$ and $\mathcal{D}(\leq \mathbf{0}')$) and by Lempp, Nies, and Slamen [58] (for the theory of $\mathcal{R}$).

The $\forall\exists$-theories of $\mathcal{D}$ and $\mathcal{D}(\leq \mathbf{0}')$ found out to be decidable. These facts were proved by Shore [76] and Lerman and Shore [61], respectively.

3.3. Properties of embeddability of finite lattices. For a given finite lattice $\mathcal{L}$, it is easy to construct a Σ^0_2-sentence, which is valid in $\mathcal{D}_n$ if and only if $\mathcal{L}$ is embeddable into $\mathcal{D}_n$. For instance, the diamond embedding in the structure of d-c.e. degrees is equivalent to the validity of the following Σ^0_2-sentence:

$$\exists a, b, c, d \left\{ (a < b, c < d) \ \& \ \forall x (x \leq b, c \to x \leq a) \ \& \ (x \geq b, c \to x \geq d) \right\}.$$

However, the problem of description of finite lattices embeddable into $\mathcal{D}_n$, $n \geq 1$, is even more difficult. There are a lot of publications devoted to this problem (see the bibliography, e.g., in the paper of Lerman [60]), but it is doubtful that the final result will be obtained soon.

It is also known (Lerman [59]) that the validity of a Σ^0_2-sentence in $\mathcal{D}_n$, $n \geq 1$, is equivalent to the following recognition problem: for a given set of pairs of finite lattices $\{\mathcal{P}, \mathcal{Q}_i\}$, $1 \leq i \leq m$, where $\mathcal{Q}_i$ is an extension of $\mathcal{P}$ as a partial ordering, and for a given embedding of $\mathcal{P}$ into $\mathcal{D}_n$, dicide whether there exists an embedding of some $\mathcal{Q}_i$ into $\mathcal{D}_n$, which extends the embedding of $\mathcal{P}$. For some special cases, the problem has a positive solution (see, e.g., Slaman and Soare [80, 81] for the case $m = 1$, $n = 1$); however, it is still far away from the general solution. Now we consider an example which shows the difficulty of the problem for $\mathcal{D}_2$ even for $m = 1$.

Assume that $\mathcal{P} = \{a, b, c \mid a < b < c\}$, $\mathcal{Q} = \{a, b, c, d \mid a < b < c, \ a < d < c, \ b \not\leq d, \ d \not\leq b\}$. It is clear that $\mathcal{Q}$ extends $\mathcal{P}$; thus, assume that $f : \mathcal{P} \mapsto \mathcal{D}_2$ is an embedding of $\mathcal{P}$ into $\mathcal{D}_2$ such that $f(a) = \mathbf{0}$. However, we saw in Theorem 2.1 that there exist a c.e. degree $\boldsymbol{b} > \mathbf{0}$ and a d-c.e. degree $\boldsymbol{c} > \boldsymbol{b}$ such that for any d-c.e. degree $\boldsymbol{d}$ the following condition holds: if $\boldsymbol{d} \leq \boldsymbol{c}$, then either $\boldsymbol{b} \leq \boldsymbol{d}$ or $\boldsymbol{d} \leq \boldsymbol{b}$. As we can see, an embedding of $\mathcal{Q}$ (which extends $\mathcal{P}$) into $\mathcal{D}_2$ may not exist.

Which finite lattices can be embedded into the structures of n-c.e. degrees, $n \geq 1$? Downey [29] assumed that all finite lattices are embeddable into the d-c.e. degrees preserving 0 and 1.

A positive solution of the problem of embedding of finite lattices can help one to obtain a proof of decidability of the two-quantifier theory of n-c.e. degrees.

Problem of extending of embeddings. Given two finite partial-ordered sets $\mathcal{P} \hookrightarrow \mathcal{Q}$, is it valid that any embedding of $\mathcal{P}$ into the n-c.e. degrees can be extended to an embedding of $\mathcal{Q}$?

- The problem has a positive solution for $|\mathcal{P}| = 2$ and $|\mathcal{Q}| < 5$.
- There exist examples of $|\mathcal{P}| = 2$ and $|\mathcal{Q}| = 5$ with a negative solution.
- There exists an example of $|\mathcal{P}| = 3$ and $|\mathcal{Q}| = 4$ with a negative solution.

Despite a lot of effort, there are is still open several natural questions about the structure of semilattices of $\mathcal{D}_n$ for $n > 1$ and about their algorithmic complexity. The most important problems are the following.

A. Problem of description of embeddable finite p.o. into the degree structures $\mathcal{R}$ and $\mathcal{D}_n, n > 1$ (with preserving the corresponding relations). The most interesting is the finding of an "effective" description of finite lattices embeddable into these structures preserving the lattice operations. It looks like a solution of this problem is one of the main steps in the proof of decidability of $\forall\exists$-theories of these degree structures, which is another important open problem in the computability theory. The point is that for a given finite lattice $\mathcal{L}$, it is possible to construct a $\exists\forall$-sentence φ, which is valid in $\mathcal{D}_n$ if and only if $\mathcal{L}$ is embeddable into it. For instance, an embedding of the 4-element lattice $\{0, a, b, 1,\ a \not\leq b,\ b \not\leq a,\ a \vee b = 1,\ a \wedge b = 0\}$ into $\mathcal{D}_n$ with preserving of the least and greatest element is equivalent to the validity of the following Σ^0_2-sentence in $\mathcal{D}_n$:

$$\exists a, b, c, d\ \forall x\ \Big\{(a < b, c < d)\ \&\ (x \leq b, c \rightarrow x \leq a)\ \&\ (x \geq b, c \rightarrow x \geq d)\Big\}.$$

As was pointed out in [56], if it is possible to get such a description of lattices embeddable into $\mathcal{R}$, then with a technique developed in the works of Soare and Slaman [81] and Ambos-Spies, Jockusch, Shore, and Soare [1], it would be possible to obtain a proof of decidability of $\forall\exists$-theory of $\mathcal{R}$.

B. Recognition problem of possibility of the embedding extensions: Does an algorithm exist which for a given finite partially ordered sets $\mathcal{P} \hookrightarrow \mathcal{Q}_0, \ldots \mathcal{Q}_n$, $n \geq 0$, allows one to decide whether any embedding $f : \mathcal{P} \rightarrow \mathcal{D}_n$ can be extended to an embedding $g : \mathcal{Q}_i \rightarrow \mathcal{D}_n$ for some $i \leq n$? (It is clear that here i depends on the choice of an embedding of $\mathcal{P}$ into $\mathcal{D}_n$.)

Lerman [59] noted that this problem has a positive solution for $\mathcal{R}$ (and for $\mathcal{D}_n, n \geq 1$) if and only if the $\forall\exists$-theory of $\mathcal{R}$ (of $\mathcal{D}_n$, respectively) is decidable.

Moreover, almost all significant results about the structure of semilattices $\mathcal{R}$ and $\mathcal{D}_n$, $n > 1$, can be considered as theorems about embeddings or extensions of embeddings of some lattices. For instance, the result of above mentioned Theorem 2.1 about the existence of double bubble can be posed as follows: any embedding of the 3-element p.o. set $\mathcal{L} = \{0 < a < b\}$ into $\mathcal{R}$ can be extended to an embedding of the p.o. set

$$\mathcal{L}' = \big\{0, a, b, d \mid 0 < a < b,\ 0 < d < b,\ a \not\leq d,\ d \not\leq a\big\},$$

however, there exist an embedding of $\mathcal{L}$ into $\mathcal{D}_2$ such that it is impossible to extend it to an embedding of $\mathcal{L}'$.

Nowadays there are a lot of results on the solution of these problem for the semilattice $\mathcal{R}$. The most interesting works among them (except for the above mentioned paper) are the works of Ambos-Shies and Lerman [2, 3], where conditions were found that are sufficient for the impossibility of embedding (as a Π_2-sentence), and conditions that are sufficient for the possibility of embedding (as a more complicated Π_3-sentence) of finite lattices into $\mathcal{R}$, and also the work of Slaman and Soare [81], where they provided a full solution of the problem of extendability of an embedding into $\mathcal{R}$ for the case where the extending family of sets consists of one set.

The sufficient condition of Ambos-Spies and Lerman about the impossibility of embedding is as follows: $\mathcal{L}$ satisfied this condition if it contains the critical Slaman triple $a, b, c \in L$ and also two other incomparable elements $p, q \in L$ such that $a \leq p \wedge q \leq a \vee c \leq q$. (Elements $a, b, c \in L$ form a *critical Slaman triple* if they are pairwise incomparable and $a \vee c = b \vee c$ and $a \wedge b \leq c$.)

In particular, it follows that the 8-element lattice $\mathcal{S}_8$, which consists of 0 and 1 and also of pairwise incomparable element a, b, c and p, q such that

$$a \vee b < p, \quad a \vee b < q, \quad p \wedge q = a \vee b = a \vee c = b \vee c, \quad p \vee q = 1, \quad a \wedge b = a \wedge c = b \wedge c = 0,$$

is not embeddable into $\mathcal{R}$ (this results was obtained earlier by Slaman and Soare [81]). Here a, b, and c form a critical Slaman triple, and p and q are those two auxiliary elements, which appears in the theorem of Ambos-Spies and Lerman.

Respectively, the theorem (more precisely, criterium) of Slaman and Soare [81] can be presented as follows: Assume that $\mathcal{P}$ and $\mathcal{Q}$ are finite p.o. sets with 0 and 1 such that $\mathcal{Q}$ extends $\mathcal{P}$, if $\mathcal{P}$ and $\mathcal{Q}$

satisfy at least one of the following conditions (1) or (2); then there exists an embedding of $\mathcal{P}$ into $\mathcal{R}$ such that it cannot be to the embedding of $\mathcal{Q}$ into $\mathcal{R}$:

$$\big(\exists x, y \in \mathcal{Q}\big) \Big[x \not\geq y \ \&\ \mathfrak{B}(\mathfrak{A}(y)) \subseteq \mathfrak{B}\big(\mathfrak{A}(\mathfrak{B}(x))\big)\Big], \tag{1}$$

$$\big(\exists x \in \mathcal{Q} - \mathcal{P}\big) \Big[\mathfrak{T}(x) \neq \emptyset \ \&\ \mathfrak{B}\big(\mathfrak{A}(\mathfrak{T}(x) \cup \mathfrak{B}(x))\big) \not\subseteq \mathfrak{B}(x)\Big], \tag{2}$$

where for $S \subseteq \mathcal{Q}$,

$$\mathfrak{A}(S) = \big\{a \in \mathcal{P} \mid (\forall x \in S)(a \geq x)\big\},$$
$$\mathfrak{B}(S) = \big\{b \in \mathcal{P} \mid (\forall x \in S)(x \geq b)\big\},$$
$$\mathfrak{T}(S) = \big\{z \in \mathcal{Q} - \mathcal{P} \mid x > z \ \&\ \mathfrak{B}(x) \not\subseteq \mathfrak{B}(\mathfrak{A}(z))\big\}.$$

Otherwise, any embedding of $\mathcal{P}$ into $\mathcal{R}$ can be extended to the embedding of $\mathcal{Q}$ into $\mathcal{R}$.

For the semilattices $\mathcal{D}_n$, $n \geq 1$, these problems were almost untouched (if one does not take into account the structural theorems, which as can be considered as results about embeddings and the embedding extensions as we mentioned above). There is still the open Downey's conjecture about embedding of any finite lattice into the semilattice of degrees of $\mathcal{D}_2$ with preserving 0 and 1. This theorem implies that it is valid at least for the lattice $\mathcal{S}_8$ (this was proved also by G. Wu, a private communication).

Theorem 3.7. *The lattice S_8 is embeddable into $\mathcal{D}_2$ with preserving* 0 *and* 1*.*

The main part of the proof of the sufficient conditions of Slaman and Soare [81] about the existence of nonextendable embeddings is the following theorem.

Theorem 3.8. *There exist incomparable c.e. degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ such that for any c.e. degree $\boldsymbol{z} < \boldsymbol{a}$ either $\boldsymbol{z} < \boldsymbol{b}$ or $\boldsymbol{z} \cup \boldsymbol{b} = \mathbf{0}'$.*

For transferring of the Slaman–Soare conditions about the existence of nonextendable embeddings to the case of n-c.e. degrees, $n > 1$, the following fact is important: in the previous theorem the degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ should be c.e. degrees, and should go through the n-c.e. degrees.

In the semilattices $\mathcal{D}_n$, $n > 1$, this theorem of Slaman and Soare holds in more general form. Namely, Li and Yi in [64] constructed d-c.e. degrees which form so called "two-sided" strong minimal pair in the n-c.e. degrees, $n > 1$, i.e., d-c.e. degrees $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ such that for each n-c.e. degrees $\boldsymbol{z} > \mathbf{0}$, if $\boldsymbol{z} < \boldsymbol{a}_i$, then $\boldsymbol{z} \cup \boldsymbol{a}_{1-i} = \mathbf{0}'$. It easily follows that the degree $\boldsymbol{a}_0$ can be constructed as a c.e. degree; however, it requires changing the condition "two-sided" of the strong minimal pair into "one-sided."

Theorem 3.9 (Li and Yi [64])**.** *There exist a c.e. degree $\boldsymbol{a}$ and a d-c.e. degree $\boldsymbol{b}$ such that $\boldsymbol{a} \cap \boldsymbol{b} = \mathbf{0}$ and for each nonzero n-c.e. degree $\boldsymbol{z} > \mathbf{0}$ either $\boldsymbol{z} \leq \boldsymbol{b}$ or $\boldsymbol{z} \cup \boldsymbol{b} = \mathbf{0}'$.*

We obtained that for such $\boldsymbol{a}$ and $\boldsymbol{b}$, there also exist two-sided minimal pairs.

Theorem 3.10. *There exist incomparable a c.e. degree $\boldsymbol{a}_0$ and a d-c.e. degree $\boldsymbol{a}_1$ such that $\boldsymbol{a}_0 \cap \boldsymbol{a}_1 = \mathbf{0}$ and for each n-c.e. degree $\boldsymbol{x} > \mathbf{0}$, if $\boldsymbol{x} < \boldsymbol{a}_i$, then $\boldsymbol{x} \cup \boldsymbol{a}_{1-i} \geq \boldsymbol{a}_i$.*

In [15], the authors left open the question about the existence of strong two-sided minimal pairs of c.e. degrees in the semilattice $\mathcal{R}$. We can add to it the following question.

Question 3.11. Do c.e. degrees exist that form a strong two-sided minimal pair in the n-c.e. degrees for each (or at least for some) $n > 1$? In the other words, is it possible in the previous theorem to make the d-c.e. degree $\boldsymbol{a}_1$ computably enumerable?

The study of one-element extensions of embeddings of finite p.o. sets presents fundamental importance. The proof of the following theorem can be obtained by applying the above-mentioned criterion of Slaman and Soare to such extensions.

Theorem 3.12 (Arslanov [8]). *Assume that $\mathcal{P}$ is a finite lattice and $\mathcal{Q} = \mathcal{P} \cup \{y\}$ is its one-element extension as a partial order. Then any embedding of $\mathcal{P}$ as a lattice into $\mathcal{R}$ can be extended to an embedding of $\mathcal{Q}$ as a partial order if and only if for each $x \in \mathcal{P}$ the following condition holds:*

$$y \not< x \to \exists z \in \mathcal{P} \left(z \not\leq x \; \& \; \forall v \in \mathcal{P} \; (v > y \to v \geq z) \right),$$

$$x \not< y \to \exists z, u \in \mathcal{P} \left(z \leq x \; \& \; z \not\leq u \; \& \; \forall v \in \mathcal{P} \; (v < y \to v \leq u) \right).$$

Remark 3.13. This assertion is also valid if $\mathcal{P}$ is a p.o. set with 0 and 1 (not necessary lattice).

Since there exists a lattice $\mathcal{P}$ and its extension $\mathcal{Q}$ such that any embedding of $\mathcal{P}$ into $\mathcal{R}$ can be extended to the embedding of $\mathcal{Q}$ (as partial order), but there exists an embedding of $\mathcal{P}$ into $\mathcal{D}_2$ such that $\mathcal{Q}$ cannot extend it (the minimal example is $\mathcal{P} = \{a, b \mid a < b\}$, $\mathcal{Q} = \{a, b, c \mid a < c < b\}$), then the conditions of the previous theorem are not sufficient for the existence of one-element extensions in $\mathcal{D}_2$.

On the other hand, there is no evidence that these conditions are not necessary within $\mathcal{D}_2$. Moreover, all known examples confirm the following conjecture.

Conjecture 3.14. Assume that $\mathcal{P}$ is a finite lattice and $\mathcal{Q}$ is its one-element extension. If any embedding of $\mathcal{P}$ into $\mathcal{D}_2$ can be extended to the embedding of $\mathcal{Q}$, then it holds also for the embedding of $\mathcal{P}$ into $\mathcal{R}$.

Remark 3.15. We saw above that there exists a c.e. degree $\boldsymbol{a}$, $\boldsymbol{0} < \boldsymbol{a} < \boldsymbol{0}'$, such that $\boldsymbol{a} \cup \boldsymbol{b} < \boldsymbol{0}'$ for any c.e. degree $\boldsymbol{b} < \boldsymbol{0}'$ (Cooper and Yates, unpublished), but $\boldsymbol{a} \cup \boldsymbol{d} = \boldsymbol{0}'$ for some d-c.e. degree $\boldsymbol{d} < \boldsymbol{0}'$ (Arslanov [5, 6]). It follows that the previous conjecture does not hold if $\mathcal{Q}$ is the semilattice extension of $\mathcal{P}$ (with preserving operation $\vee$).

3.4. Isomorphic copies. Some model-theoretic properties of the semilattices $\mathcal{D}_n$, $n \geq 1$, as well as of the lattices of the n-c.e. sets $\mathcal{C}_n$, $n \geq 1$ ($\mathcal{C}_1 = \mathcal{C}$ denotes the lattice of all c.e. sets), can be deduced from the corresponding results for the semilattices $\mathcal{R}$ and $\mathcal{C}$, which have been studied much better. For instance, similarly to the computably enumerable case, the following result can be obtained by combining the well-known Lachlans theorems [51] about hyperhypersimple sets (see, e.g., Rodgers [71, Theorem 12-XIX]) and about boolean algebras.

Theorem 3.16. *For each $n \geq 1$, the lattice $\mathcal{C}_n$ of all n-c.e. sets does not have a computable presentation, i.e., in particular, it is not isomorphic to any computable linear ordering.*

Proof. For $n = 1$, the results follows immediately from the Lachlan theorem [51], which asserts that any Σ^0_3-boolean algebra is isomorphic to the boolean algebra of c.e. supersets of some hypersimple set, and from the Feiner theorem [42] about the existence of a Σ^0_3-boolean algebra which does not have a computable presentation.

Now assume that $n \geq 2$ and A is a hyperhypersimple set. If $A \subseteq B$, B is an n-c.e. set, and the difference $B - A$ is infinite, then it is co-c.e. (i.e., its complement is c.e.). Indeed, $B - A$ must be a $2k$-c.e. set for some even $2k$ (otherwise, $\omega - A$ is not immune) and

$$B - A = (A_1 - A_2) \cup (A_3 - A_4) \cup \ldots (A_{2k-1} - A_{2k})$$

for some c.e. sets $A_1 \supset A_2 \supset \ldots \supset A_{2k}$. The Lachlan theorem about the hyperhyperimmune difference of c.e. sets asserts that if c.e. sets X and Y are such that $X \supset Y$ and $X - Y$ hyperhyperimmune, then there exists a computable set R such that $X - Y \subseteq R \subseteq X$. It follows that there exist computable sets $R_1, R_2, \ldots R_k$ such that

$$A_{2i-1} - A_{2i} \subseteq R_i \subseteq A_{2i-1}, \quad A \cup \{\omega - B\} = \overline{B - A} = \overline{R_1} \cup \left\{ \bigcup_{i=1}^{k} \left(A_{2i} \cap \{\overline{\cup_{m>2i} R_m}\} \right) \right\}.$$

Now the above-mentioned Lachlan theorem about hyperhypersimple sets implies that the set B is also computably enumerable. Thus, the boolean algebras $\mathcal{C}(A)$ and $\mathcal{C}_n(A)$ of c.e. and n-c.e., respectively, sets (by modulo of finite differences) of the hyperhypersimple set A coincide. □

In the study of isomorphic copies of $\mathcal{D}_n$ (briefly, degrees of presentations of $\mathcal{D}_n$), the following result of Shore [77] is useful (the result was obtained by him for c.e. degrees and in more general form).

Theorem 3.17. *Let A be a Π^0_2-set. There exists a partial lattice (i.e., an upper semilattice where not all pairs have the greatest lower bound) $\mathcal{L}$ such that*

(1) *$\mathcal{L}$ is isomorphically embeddable into any semilattice $\mathcal{D}_n$, $n \geq 1$;*
(2) *if $\mathcal{L}$ is isomorphically embeddable into some semilattice $\mathcal{S}$, then the set A is computable relative to the (Turing) jump of any presentation of $\mathcal{S}$.*

In particular, A is computable relative to the jump of any presentation of each semilattice $\mathcal{D}_n$, $n \geq 1$. Thus, the following assertion holds.

Corollary 3.18. *There is no semilattice $\mathcal{D}_n$, $n \geq 1$, that has a computable presentation. Moreover, the degree of any presentation of each of them is greater than or equal to $\mathbf{0}'$.*

The proof of the following theorem can be obtained with help of the proof of the similar result obtained by Nies, Shore and Slaman [68] for the c.e. degrees.

Theorem 3.19. *Assume that $n > 1$. Then for any n-c.e. degree $\boldsymbol{a} > \mathbf{0}$, the semilattice $\mathcal{D}_n(\leq \boldsymbol{a})$ does not have a computable presentation.*

Sketch of the proof. The proof of the fact (see [68]) that semilattices of c.e. degrees below a given c.e. degree $\boldsymbol{c} > \mathbf{0}$ do not have computable presentation can be adapted to the case of n-c.e. degrees below given n-c.e. degree for an arbitrary $n > 1$: the operations $\cup$ and $\cap$ applied to c.e. degrees give c.e. degrees, even if they are defined for all degrees $\leq \mathbf{0}'$ (in particular, it holds for the n-c.e. degrees for any $n > 1$). Then, the technique from Shore's paper [77], where the results about embeddings of so-called TRR-lattices into the c.e. degrees were obtained (they allow one to show that there is no a computable presentation of the c.e. degrees), applied to the proof of Nies, Shore, and Slaman allows one to do the same for the structures of n-c.e. degrees below given n-c.e. degree $>\mathbf{0}$. □

The degree spectra of presentation of semilattices $\mathcal{D}_n$, $n \geq 1$, as well as of their fragments $\mathcal{D}_n(\leq \boldsymbol{a})$, was not investigated at all.

The following theorem for the case $n = 1$ can be found in the work of Lerman, Shore, and Soare [62] for the case $n = 1$; however, this proof is also valid in the general case.

Theorem 3.20. *For any $n \geq 1$, $\mathcal{D}_n$ is not countably categorical.*

Proof. Lerman, Shore, and Soare [62] defined a countable set of pairwise nonisomorphic, finite partial-lattices (where the greatest lower bound does not exist for some pairs) that are defined by three elements and each of these lattices is embeddable into $\mathcal{R}$, thus, into $\mathcal{D}_n$ for each $n > 1$.

Each such partial lattice produces a special 3-type, which can be realized within $\mathcal{D}_n$. Now the desired proposition follows from the characterization of ω-categorical theories (also known as the Ryl-Nardzevsky theorem; see Ershov and Palutin [38]). □

4. Definability of the Degree Classes

4.1. Definability of m-c.e. degrees in semilattices of n-c.e. degrees for $1 \leq m < n$. Consider in detail the problems of definability in semilattices of n-c.e. degrees. One of the possible methods of the proof of definability of m-c.e. degrees in the n-c.e. degree structures for $m < n$ is as follows:

(a) find a set $\mathcal{S}$ of m-c.e. degrees, which is infinite definable in $\mathcal{D}_n$, and
(b) prove that the degrees from $\mathcal{S}$ generate the degrees $\mathcal{D}_m$ by using the operations $\cup$ and $\cap$.

Theorem 4.1 (Yamaleev [87]). *Let Turing degrees $\boldsymbol{d}$ and $\boldsymbol{b}$ be such that $\boldsymbol{d}$ is a proper d-c.e. degree and let $\boldsymbol{b}$ be a Δ^0_2-degree such that it is not c.e., $\boldsymbol{b} < \boldsymbol{d}$, and let the interval $(\boldsymbol{d}, \boldsymbol{b})$ not contain c.e. degrees. Then $\boldsymbol{d}$ is splittable within d-c.e. degrees avoiding the upper cone of degrees for $\boldsymbol{b}$.*

Corollary 4.2. *If d-c.e. degrees $\boldsymbol{d} > \boldsymbol{b}$ are such that $\boldsymbol{b}$ is not c.e., then there exists a d-c.e. degree $\boldsymbol{a}$ such that $\boldsymbol{b} < \boldsymbol{a} \leq \boldsymbol{d}$ and $\boldsymbol{a}$ is splittable within d-c.e. degrees avoiding the upper cone of degrees for $\boldsymbol{b}$.*

Proof. If there exists a c.e. degree $\boldsymbol{a}$ such that $\boldsymbol{b} < \boldsymbol{a} \leq \boldsymbol{d}$, then the desired splitting of $\boldsymbol{a}$ exists by the Sacks splitting theorem. If such a degree $\boldsymbol{a}$ does not exist, then $\boldsymbol{d}$ is splittable by Theorem 4.1. □

Let

$$\varphi(\boldsymbol{x}) \doteq (\exists \boldsymbol{b} > \boldsymbol{x})(\forall \boldsymbol{d})\Big[\boldsymbol{x} < \boldsymbol{d} \leq \boldsymbol{b} \to (\forall \boldsymbol{d}_0, \boldsymbol{d}_1)\big[\boldsymbol{d} = \boldsymbol{d}_0 \cup \boldsymbol{d}_1 \to \boldsymbol{x} \leq \boldsymbol{d}_0 \ \vee \ \boldsymbol{x} \leq \boldsymbol{d}_1\big]\Big].$$

From the above it follows that $\mathcal{D}_2 \models \varphi(\boldsymbol{a}) \Rightarrow \boldsymbol{a}$ is c.e. Therefore, the formula φ describes a definable in d-c.e. degrees class $\mathcal{S}_2$, which consists only of c.e. degrees. An analysis of the proof of theorem of Arslanov, Kalimullin, and Lempp allows one to see that this class contains infinitely many c.e. degrees. Note also that $\mathcal{S}_2$ does not coincide with the class of all c.e. degrees. It follows from the existence of nonisolating c.e. degree, i.e., a degree $\boldsymbol{a}$ such that for any d-c.e. degree $\boldsymbol{d} > \boldsymbol{a}$, there exists a c.e. degree $\boldsymbol{b}$ such that $\boldsymbol{d} > \boldsymbol{b} > \boldsymbol{a}$ (Arslanov, Lempp, and Shore [13]).

Indeed, if $\mathcal{D}_2 \models \varphi(\boldsymbol{a})$ for some c.e. degree $\boldsymbol{a}$, then let $\boldsymbol{b} > \boldsymbol{a}$ be a d-c.e. degree such that it is not splittable avoiding the upper cone of degrees of $\boldsymbol{a}$. Then between $\boldsymbol{a}$ and $\boldsymbol{b}$ there are no c.e. degrees, except for $\boldsymbol{a}$, i.e., $\boldsymbol{a}$ isolates $\boldsymbol{b}$.

Thus, the set

$$\mathcal{S} = \Big\{\boldsymbol{x} \geq \boldsymbol{0} \ \Big| \ (\exists \boldsymbol{y} > \boldsymbol{x} \forall \boldsymbol{z})\Big(\boldsymbol{x} < \boldsymbol{z} \leq \boldsymbol{y} \to (\forall \boldsymbol{z}_0, \boldsymbol{z}_1)\big(\boldsymbol{z}_0 \cup \boldsymbol{z}_1 = \boldsymbol{z} \ \& \ \boldsymbol{z}_0 \mid \boldsymbol{z}_1 \to \boldsymbol{x} \leq \boldsymbol{z}_0 \ \vee \ \boldsymbol{x} \leq \boldsymbol{z}_1\big)\Big)\Big\}$$

consists only of c.e. degrees and is infinite and definable in $\mathcal{D}_2$ by the formula $\varphi(x)$.

Now we assume that $\boldsymbol{d}$ is a properly d-c.e. degree. Then for any splitting of $\boldsymbol{d}$ into two d-c.e. degrees $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$, at least one of the degrees $\boldsymbol{d}_i$, $i \leq 1$, must possess the following property: *For any d-c.e. degree $\boldsymbol{u}$, $\boldsymbol{d}_i \leq \boldsymbol{u} \leq \boldsymbol{d}$, $\boldsymbol{u}$ is splittable into d-c.e. degrees avoiding the upper cone of degrees above $\boldsymbol{d}_i$.*

Otherwise, this means (by Theorem 4.1) that for each degree $\boldsymbol{d}_i$, $i \leq 1$, there is a c.e. degree between $\boldsymbol{d}_i$ and $\boldsymbol{d}$ and, therefore, the degree $\boldsymbol{d}$ itself is computably enumerable (as a least upper bound of such degrees).

Answers for the following question are still unknown. The above reasonings imply that a positive answer for any of these questions means the definability of c.e. degrees in $\mathcal{D}_2$.

1. Is it true that each c.e. degree $\boldsymbol{a} > \boldsymbol{0}$ is the least upper bound of two incomparable degrees from $\mathcal{S}$?

This question has a connection with the following one: Is it true that degrees from $\mathcal{S}$ are dense in the c.e. degrees, i.e., is it true that for any two c.e. $\boldsymbol{a} < \boldsymbol{b}$, there exists some degree from $\mathcal{S}$? A positive answer for this question immediately gives a positive answer for question 1: in order to get it, we split the c.e. degree $\boldsymbol{a} > \boldsymbol{0}$ into two incomparable c.e. degrees $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ (this can be done by the Sacks splitting theorem), and find the desired degrees $\boldsymbol{b}_i \in \mathcal{S}$ between the degrees $\boldsymbol{a}$ and $\boldsymbol{a}_i$, $i \leq 1$. Thus, we have $\boldsymbol{a} = \boldsymbol{b}_0 \cup \boldsymbol{b}_1$.

2. Is it true that for each c.e. degree $\boldsymbol{a}$, there exists a splitting (within d-c.e. degrees) $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ such that for some d-c.e. degrees $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ such that $\boldsymbol{a}_i < \boldsymbol{b}_i < \boldsymbol{a}$, the degree $\boldsymbol{b}_i$ is not splittable avoiding the upper cone of degrees of $\boldsymbol{a}_i$ for each $i \in \{0, 1\}$?

4.2. New approach to the definability of m-c.e. degrees in semilattices of n-c.e. degrees for $1 \leq m < n$. An investigation of the problem of definability of c.e. degrees in semilattices $\mathcal{D}_n$, $n > 2$, can be done also according to the following outline.

1. Let $1 < n$ and let properly $(n+1)$-c.e. degrees $\boldsymbol{d}$ and $\boldsymbol{b}$ be such that $\boldsymbol{d} > \boldsymbol{b}$ and the interval $(\boldsymbol{b}, \boldsymbol{d})$ does not contain n-c.e. degrees. Is it true that $\boldsymbol{d}$ is splittable within $(n+1)$-c.e. degrees avoiding the cone of degrees above $\boldsymbol{b}$?
2. Is it true that each n-c.e. degree $\boldsymbol{a}$ is splittable into two $(n+1)$-c.e. degrees $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ such that there exist $(n+1)$-c.e. degrees $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$, $\boldsymbol{a}_i < \boldsymbol{b}_i < \boldsymbol{a}$, such that $\boldsymbol{b}_i$ is not splittable in $(n+1)$-c.e. degrees avoiding the upper cone of degrees above $\boldsymbol{a}_i$ for each $i \in \{0, 1\}$?

We assume that for each $n \geq 1$, the following definable in $\mathcal{D}_{n+1}$ set of $(n+1)$-c.e. degrees is infinite and consists of n-c.e. degrees:

$$\mathcal{S}_n = \Big\{ \boldsymbol{x} \geq \boldsymbol{0} \;\Big|\; (\exists \boldsymbol{y} > \boldsymbol{x})(\forall \boldsymbol{z})\Big(\boldsymbol{x} < \boldsymbol{z} \leq \boldsymbol{y} \to (\forall \boldsymbol{z}_0, \boldsymbol{z}_1)\big(\boldsymbol{z}_0 \cup \boldsymbol{z}_1 = \boldsymbol{z} \;\&\; \boldsymbol{z}_0 \mid \boldsymbol{z}_1 \to \boldsymbol{x} \leq \boldsymbol{z}_0 \;\vee\; \boldsymbol{x} \leq \boldsymbol{z}_1\big)\Big)\Big\}.$$

We also assume that this set of n-c.e. degrees in $\mathcal{D}_{n+1}$ generates the whole class of n-c.e. degrees, and thus *for any $n > 1$, the n-c.e. degrees are uniformly definable in $\mathcal{D}_{n+1}$.*

Conjecture 4.3. We assume that the finite families of c.e. degrees are not definable without parameters in $\mathcal{D}_2$.

In order to prove this sentence for a given finite family M of c.e. degrees, one can try to construct an automorphism of $\mathcal{D}_2$, which would leave M stable.

4.3. Definability and isolation properties. Now we consider a property of isolation from side and discuss how it can be used in the solution of the definability problem.

Explicitly, this property was first distinguished in [86]; however, it was actively used in other degree structures but did not receive enough attention in the Turing degrees. Implicitly this property was used in [16, 89], where the authors established that $\mathcal{D}_n$ are not Σ_1-substructure $\mathcal{D}_m$ for each $n < m$. Restricting ourselves by 2-c.e. degrees, we recall that in Theorem 3.2, the following assertion was proved: there exist c.e. degrees $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{e}$ (parameters) such that

(i) there exists a 2-c.e. degrees $\boldsymbol{d} < \boldsymbol{a}$ such that $\boldsymbol{d} \not\leq \boldsymbol{e}$ and $\boldsymbol{c} \not\leq \boldsymbol{d} \cup \boldsymbol{b}$,
(ii) for each c.e. degree $\boldsymbol{w} < \boldsymbol{a}$, either $\boldsymbol{w} \leq \boldsymbol{e}$ or $\boldsymbol{c} \leq \boldsymbol{w} \cup \boldsymbol{b}$.

Here all c.e. degrees below $\boldsymbol{d}$ are also below $\boldsymbol{e}$. This suggested (see [86]) the following definition of isolation of a degree $\boldsymbol{d}$ from side by a degree $\boldsymbol{e}$.

Definition 4.4. The degree $\boldsymbol{d}$ is isolated in the class of degrees $\mathcal{C}$ from side by a degree $\boldsymbol{e}$ if $\boldsymbol{d} \not\leq \boldsymbol{e}$ and for any degree $\boldsymbol{c} \in \mathcal{C}$, we have $\boldsymbol{c} < \boldsymbol{d} \to \boldsymbol{c} < \boldsymbol{e}$.

In what follows, we assume that the class $\mathcal{C}$ consists of c.e. degrees. Also, the isolated degree $\boldsymbol{d}$ usually will a 2-c.e. degree, and the isolating degree $\boldsymbol{e}$ will be either c.e. or 2-c.e. degree. The above definition is a generalization of the well-studied conception of isolation where the c.e. degree $\boldsymbol{e}$ must be strictly below $\boldsymbol{d}$.

Now we show how this notion of isolation from side can be used for the solution of the definability problem. For the case of usual isolation, it is known that there exist isolated 2-c.e. degrees and not isolated ones. So far, it is unknown whether it is true for the isolation from side.

Question 4.5. Is it true that any properly 2-c.e. degree $\boldsymbol{d}$ is isolated from side by some c.e. degree $\boldsymbol{e}$?

Assume that the answer is positive. Let be $\boldsymbol{d}$ be a properly 2-c.e. degree. Then there exists a c.e. degree $\boldsymbol{e}$ such that $\boldsymbol{d} \not\leq \boldsymbol{e}$ and all c.e. degrees below $\boldsymbol{d}$ are also below $\boldsymbol{e}$. It is clear that this is valid for all definable subclasses of c.e. degrees.

This property does not hold for c.e. degrees by the Sacks splitting theorem. However, in order to obtain a formula that would distinguish all c.e. degrees among the 2-c.e. degrees, it is necessary that all degrees that are involved in this use of the Sacks splitting theorem also be definable. As we saw in the previous sections, potential candidates as such degrees could be, for instance, the centers of double bubbles. For this reason, it is necessary that the centers of double bubbles be downward dense and could avoid the lower cones of given 2-c.e. degrees. Thus, we pose it as the following conjecture.

Conjecture 4.6. Let a formula $\psi(x)$ be valid in $\mathcal{D}_2(\leq 0')$ if and only if x is the center of a double bubble.

1. If $\boldsymbol{d}$ is a properly 2-c.e. degree, then

$$\exists \boldsymbol{e}\Big[\boldsymbol{d} \not\leq \boldsymbol{e} \wedge \forall \boldsymbol{w}\big(\psi(\boldsymbol{w}) \wedge \boldsymbol{w} < \boldsymbol{d} \longrightarrow \boldsymbol{w} < \boldsymbol{e}\big)\Big].$$

2. If $\boldsymbol{a}$ is a c.e. degree, then (the quantifiers go though the 2-c.e. degrees)

$$\forall \boldsymbol{e}\Big[\boldsymbol{a} \not\leq \boldsymbol{e} \longrightarrow \exists \boldsymbol{w}\big(\psi(\boldsymbol{w}) \wedge \boldsymbol{w} < \boldsymbol{a} \wedge \boldsymbol{w} \not\leq \boldsymbol{e}\big)\Big]$$

Thus, here we also deal with double bubbles and their distribution in the structure of 2-c.e. degrees. Although the density problem was substituted by an easier density problem and by the isolation from side, the first part of the question (about isolation from side of properly 2-c.e. degrees) can become crucial.

4.4. Definability and relative enumberability. The properties of relative enumerability were already used in Secs. 2.1 and 2.2. In this section, we consider them in detail, and also we consider their possible applications for the definability problem. First, we recall the corresponding definition of CEA-hierarchy.

Definition 4.7. A set A is a B-CEA set if $B \leq_T A$ and A is c.e. relative to B. Moreover, a set A is an n-CEA set for $n > 1$ if A is a B-CEA set for some $(n-1)$-CEA set B, and also A is a 0-CEA set if it is computable. A degree $\boldsymbol{a}$ is an n-CEA degree if it contains an n-CEA set A. A degree $\boldsymbol{a}$ is a $\boldsymbol{b}$-CEA degree if some set $A \in \boldsymbol{a}$ is a B-CEA set for some $B \in \boldsymbol{b}$.

Thus, we can see that the 1-CEA degrees are exactly the c.e. degrees. For the higher levels of the hierarchy, the situation is more complicated and currently we know the following relations between n-c.e. Turing degrees and n-CEA Turing degrees.

Theorem 4.8 (Lachlan, unpublished)**.** *For a given* 2*-c.e. degree* $\mathbf{d}$*, there exist a c.e. degree* $\mathbf{e}$ *such that* $\mathbf{d}$ *is an* $\mathbf{e}$*-CEA degree.*

In order to prove this theorem, we *associate* with each 2-c.e. set $D = B_1 - B_2$ the c.e. set $As(D) = \{\langle s, x\rangle : x \in D_s - D\}$. Obviously, $As(D) \leq_T D$ and D is c.e. in $As(D)$; therefore, D is an $As(D)$-CEA set. It is clear that the definition of $As(D)$ depends on enumerations of the c.e. sets B_1 and B_2, and we can uniformly find the c.e. index for $As(D)$ from those of B_1 and B_2. However, the Turing degree of $As(D)$ is defined uniquely by the set D and does not depend on an enumeration of D. It follows from the following proposition.

Proposition 4.9 (Ishmukhametov [46])**.** *Let* $D = B_1 - B_2$ *be a* 2*-c.e. set and* $As(D)$ *be the corresponding associated set. If* D *is c.e. in a set* B*, then* $As(D) \leq_T B$*.*

Proof. Let $D = \mathrm{dom}(\Phi_e(B))$ for some e. For each $\langle s, x\rangle$, if $x \notin D_s$, then $\langle s, x\rangle \notin As(D)$. If $x \in D_s$, then let $t > s$ be a stage in an enumeration of D such that

— either $x \notin D_t$ (in this case, $\langle s, x\rangle \in As(D)$);
— or $\Phi_e(B, x)\downarrow [t]$ (then $\langle s, x\rangle \notin As(D)$).

Note that for an n-c.e. set D, we can similarly define an $(n-1)$-c.e. set $As(D) \leq_T D$ such that D is c.e. in $As(D)$. Sometimes, the alternative definition for $As(D)$

$$L(D) = \{s \mid \exists x \in D_s - D\}$$

is more convenient; here we assume that an enumeration of D satisfies the condition $|D_{s+1} - D_s| \leq 1$. It is easy to see that $L(D)$ has the same Turing degree as $As(D)$. The sets $As(D)$ and $L(D)$ are called *associated Lachlan sets* for the set D.

Thus, 2-c.e. degrees form a subset of 2-CEA degrees, and n-c.e. degrees, for $n > 2$, form a subset of n-CEA degrees. The reverse inclusion does not hold. □

Theorem 4.10 (Jockusch, Shore [49]). *There exists a 2-CEA degree that is not an n-c.e. degree for all $n < \omega$.*

Therefore, levels of CEA-hierarchy provides wider classes of degrees comparing with the same levels of the Ershov hierarchy.

On the other hand, by the following result, there is no a properly n-c.e. degree which is a 2-CEA degree.

Theorem 4.11 (Arslanov, LaForte, and Slaman [12]). *If an ω-c.e. degree $\boldsymbol{d}$ is also an A-CEA degree for some c.e. set A, then there exists a 2-c.e. set $D \in \boldsymbol{d}$, which is A-CEA.*

For completeness, we also mention the following results.

Theorem 4.12 (Jockusch and Shore [49]). *There exists a 3-c.e. degree, which is not a 2-CEA degree.*

Theorem 4.13 (Arslanov, LaForte, and Slaman [12]). *There exists a 2-c.e. set D such that for each $n \geq 3$, there exists a set A_n, which is D-CEA and has a properly n-c.e. degree.*

Theorem 4.14 (Arslanov, Lempp, and Shore [14]). *There exists a c.e. degree $\boldsymbol{a}$ such that if some degree $\boldsymbol{d} \leq \boldsymbol{0}'$ is $\boldsymbol{a}$-CEA, then $\boldsymbol{d}$ is c.e.*

We can consider a similar definition if we proceed from sets to degrees; however, the situation becomes more complicated since in the general case the degree of $L(D)$ depends on the choice of the set 2-c.e. set D from a 2-c.e. degree $\boldsymbol{d}$. This leads us to c.e. predecessors of degree $\boldsymbol{d}$ in which $\boldsymbol{d}$ is c.e.; namely, Ishmukhametov [46] considered the class

$$R[\boldsymbol{d}] = \{\boldsymbol{a} \mid \boldsymbol{d} \text{ is } \boldsymbol{a}\text{"=CEA}\}.$$

However, it was found that this class is the same as

$$L[\boldsymbol{d}] = \{\deg(L(D)) \mid \exists \text{ 2-c.e. } D \in \boldsymbol{d}\},$$

which was considered in [40]. It is easy to see that $L[\boldsymbol{d}] \subset R[\boldsymbol{d}]$, thus we cite the proof for the reverse inclusion.

Proposition 4.15 (Fang, Wu, and Yamaleev [40]). *Let $\boldsymbol{d}$ be a 2-c.e. degree and $\boldsymbol{a} < \boldsymbol{d}$ be a c.e. degree such that $\boldsymbol{a} \in R[\boldsymbol{d}]$. Then $\boldsymbol{a} \in L[\boldsymbol{d}]$.*

Proof. Let A be a c.e. set from $\boldsymbol{a}$. By Theorem 4.11, we find a 2-c.e. set $U \in \boldsymbol{d}$ such that U c.e. relative to A. From Proposition 4.9 it follows that $\deg(L(U)) \leq \boldsymbol{a}$. Now consider $D = U \oplus (\omega - A)$; it is clear that D is 2-c.e., $D \equiv_T U$ and $L(D) \in \ \boldsymbol{a}$. □

Therefore, the class $L[\boldsymbol{d}]$ is upward closed relative to c.e. degrees below $\boldsymbol{d}$, similarly to the class $R[\boldsymbol{d}]$. In [46], Ishmukhametov started the detailed investigation of $R[\boldsymbol{d}]$ for various $\boldsymbol{d}$, and obtained the following series of results (further we consider $L[\boldsymbol{d}]$ for the sake of convenience).

Theorem 4.16 (Ishmukhametov [46]). *There is a 2-c.e. degree $\boldsymbol{d}$ such that $L[\boldsymbol{d}] = [\boldsymbol{a}, \boldsymbol{b}]$, where $\boldsymbol{a} \neq \boldsymbol{b}$ are noncomputable c.e. degrees.*

By Proposition 4.15 and by properties of c.e. degrees, we obtain the following result.

Corollary 4.17. *There exists a 2-c.e. degree $\boldsymbol{d}$ such that $L[\boldsymbol{d}]$ contains incomparable elements.*

Note that in Theorem 4.16, the degree $\boldsymbol{a}$ is the least element of $L[\boldsymbol{d}]$, and $\boldsymbol{b}$ is the greatest elements of $L[\boldsymbol{d}]$. Note that properly 2-c.e. degrees $\boldsymbol{d}$ with greatest elements in $L[\boldsymbol{d}]$ coincide with well-studied class of isolated 2-c.e. degrees introduced by Cooper and Yi [28]. Then the isolated degrees were investigated by Wu; in particular, one of the applications is the use of isolated degrees for the diamond embedding into the 2-c.e. degrees, which simplified the proof of Downey's result [29], and, moreover, allowed making one-half of the diamond as c.e. In contrast to the isolated 2-c.e. degrees, the properly 2-c.e. degrees $\boldsymbol{d}$ whose $L[\boldsymbol{d}]$ has the least element, were not investigated systematically. Complementing Theorem 4.15, Ishmukhametov also proved that the degrees $\boldsymbol{a}$ and $\boldsymbol{b}$ from Theorem 4.16 can coincide.

Theorem 4.18 (Ishmukhametov [46]). *There is a* 2*-c.e. degree* $\boldsymbol{d}$ *such that* $|L[\boldsymbol{d}]| = 1$.

He called such degrees *exact* 2-c.e. degrees. It was shown in Sec. 2.1 that the top of a double bubble must be an exact 2-c.e. degree. Thus, the investigation of distribution of exact 2-c.e. degrees is a subproblem of distribution of double bubbles, whose importance was shown in Secs. 2.1 and 4.1. However, there is no such results about combining of this property with other properties; probably this is an effect of 2-c.e. degrees $\boldsymbol{d}$ with the least element in $L[\boldsymbol{d}]$, which have problems when combining them with other properties. In [65], it was proved that exact 2-c.e. degrees are downward dense.

Theorem 4.19 (Liu, Wu, and Yamaleev [65]). *Given a noncomputable c.e. degree* $\boldsymbol{a}$*, there exists an exact* 2*-c.e. degree* $\boldsymbol{d} \leq \boldsymbol{a}$.

The next theorem establishes that this property does not hold for the tops of double bubbles. Thus, the tops of double bubbles and the exact 2-c.e. degrees form different classes.

Theorem 4.20 (Andrews, Kuyper, Lempp, Soskova, and Yamaleev [4]). *There exists a noncomputable c.e. degree* $\boldsymbol{a}$ *such that there are no double bubbles below it* (*and, in particular, there are no the tops of double bubbles*).

The next question in investigation of Lachlan degrees is the following question of Ishmukhametov posed him in [46]: Does a 2-c.e. degree $\boldsymbol{d}$ have a minimum element in $L[\boldsymbol{d}]$? Unfortunately, the answer has been negative contrary to the set case. Ishmukhametov [47] and later Fang, Wu, and Yamaleev [40] obtained the following results.

Theorem 4.21 (Ishmukhametov [47]). *There exists a* 2*-c.e. degree* $\boldsymbol{d}$ *such that* $L[\boldsymbol{d}]$ *does not have the least element.*

Theorem 4.22 (Fang, Wu, and Yamaleev [40]). *There exists a* 2*-c.e. degree* $\boldsymbol{d}$ *such that* $L[\boldsymbol{d}]$ *does not have a minimal element.*

Fang, Liu, Wu, and Yamaleev [65] showed that the notions "minimal" and "least" are equivalent for $L[\boldsymbol{d}]$. Thus, Theorems 4.21 and 4.22 are equivalent from this point of view.

Theorem 4.23 (Fang, Liu, Wu, and Yamaleev [39]). *Given a* 2*-c.e. degree* $\boldsymbol{d}$*, if* $\boldsymbol{a}, \boldsymbol{b} \in L[\boldsymbol{d}]$*, then there exists* $\boldsymbol{c} \in L[\boldsymbol{d}]$ *such that* $\boldsymbol{c} \leq \boldsymbol{a}, \boldsymbol{b}$.

Since for any properly 2-c.e. degree $\boldsymbol{d}$ we have $\boldsymbol{0} \notin L[\boldsymbol{d}]$, as a corollary we obtain the following assertion.

Corollary 4.24 (Fang, Liu, Wu, and Yamaleev [39]). *If* $\boldsymbol{d}$ *is a properly* 2*-c.e. degree, then* $L[\boldsymbol{d}]$ *does not contain a minimal pair.*

It is easy to see that if a degree $\boldsymbol{e}$ is c.e., then $L[\boldsymbol{e}] = [\boldsymbol{0}, \boldsymbol{e}]$; in particular, the least and greatest degrees always exist. If $\boldsymbol{d}$ is a properly 2-c.e. degree, then (see Ishmukhametov [46]) there exist exact degrees, namely, degrees that, in some sense, are "far" from c.e. degrees and are "more proper" 2-c.e. degrees. The following question about the existence of properly 2-c.e. degrees, which are "closer" to c.e. degrees, is open.

Question 4.25. Does a 2-c.e. degree $\boldsymbol{d}$ exist such that $L[\boldsymbol{d}] = (\boldsymbol{0}, \boldsymbol{d})$ (an interval without endpoints)?

A particular motivation for an investigation of Lachlan degrees is a finding of structural properties which would link $\boldsymbol{d}$ and $L[\boldsymbol{d}]$ in cases where $\boldsymbol{d}$ is a proper 2-c.e. degree. Yamaleev tried the following approach for a solution of the question of the definability of c.e. degrees in the 2-c.e. degrees. Consider the following question.

Question 4.26. Is it true that for any properly 2-c.e. degree $\boldsymbol{d}$, there exists a c.e. degree $\boldsymbol{a}$ such that $\boldsymbol{a} \leq \boldsymbol{x}$ for any $\boldsymbol{x} \in L[\boldsymbol{d}]$?

A positive answer to this question could be used in the following way. If a degree $\boldsymbol{d}$ is properly 2-c.e. degree, then let $\boldsymbol{a} < \boldsymbol{d}$ be the corresponding c.e. degree. Since $\boldsymbol{a}$ is a lower bound for the spectra of Lachlan degrees, then both parts of any splitting of $\boldsymbol{d}$, which avoid the upper cone of $\boldsymbol{a}$, must be properly 2-c.e. degrees. On the contrary, assume that $\boldsymbol{d} = \boldsymbol{b} \cup \boldsymbol{c}$ and assume that $\boldsymbol{c}$ is c.e. Then let $C \in \boldsymbol{c}$ and $B \in \boldsymbol{b}$; consequently, $\deg(C \oplus B) = \boldsymbol{d}$, but $L(C) \oplus L(B) \equiv_T \emptyset \oplus L(B) \equiv_T L(B)$, and thus $\boldsymbol{a} \not\leq \deg(L(C) \oplus L(B))$, a contradiction. (Also we note that such splittings could not exist. However, then the degree $\boldsymbol{d}$ is immediately recognized as properly 2-c.e.) We assume that this does not hold for the c.e. degrees; namely, any c.e. degree is splittable avoiding any upper cone in such a way that one part of the splitting is definable (the part with definability can use the centers of double bubbles). This statement is one of anticipated properties of c.e. degrees which we pose as a conjecture.

Conjecture 4.27. For any noncomputable c.e. degree $\boldsymbol{c}$, for any noncomputable 2-c.e. degree $\boldsymbol{f}$, there exists c.e. degrees $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\boldsymbol{u}$ is the center of a double bubble, $\boldsymbol{c} = \boldsymbol{u} \cup \boldsymbol{v}$, and $\boldsymbol{f} \not\leq \boldsymbol{u}$, $\boldsymbol{v}$.

Thus, a potential candidate for recognition of c.e. degrees among all 2-c.e. degrees is the following formula:

$$\varphi(\boldsymbol{x}) = \forall \boldsymbol{a}\ \exists \boldsymbol{u}\ \exists \boldsymbol{v}\ \Big[\boldsymbol{a} \leq \boldsymbol{x} \to \big(\psi(\boldsymbol{u}) \,.\wedge.\, \boldsymbol{x} = \boldsymbol{u} \cup \boldsymbol{v} \,.\wedge.\, \boldsymbol{a} \not\leq \boldsymbol{u} \,.\wedge.\, \boldsymbol{a} \not\leq \boldsymbol{v}\big)\Big],$$

where the formula $\psi(\boldsymbol{u})$ is valid if and only if $\boldsymbol{u}$ is the center of the double bubble $D(\leq \mathbf{0}')$. The above conjecture states that the formula $\varphi(\boldsymbol{x})$ is valid if and only if $\boldsymbol{x}$ is a c.e. degree.

4.5. Example of working with definable classes. The next part of this section is devoted to main ideas of the proof of Theorem 4.20, which are examples of working with double bubbles and the use of Lachlan sets. Our notation and terminology follows Soare [83] and Downey and Hirschfeldt [30]; we also use the technique for presenting priority constructions, which is standard for the last years. Below we present main ideas of the existence of a noncomputable c.e. degree below which there are no double bubbles.

Requirements. Recall that the top of a double bubble is always an exact degree. Now assume that 2-c.e. degrees $\boldsymbol{d}_1 > \boldsymbol{d}_2 > \mathbf{0}$ form a double bubble. If $D \in \boldsymbol{d}_1$ is a 2-c.e. degree, then by Sec. 4.4 we have $L(D) \in \boldsymbol{d}_2$.

So in order to prove the theorem, we must construct a noncomputable c.e. set A such that for any noncomputable 2-c.e. set $D \leq_T A$, if $\mathbf{0} < \deg(L(D)) < \deg(D)$, then there is a 2-c.e. set $E \leq_T D$ that is Turing incomparable with $L(D)$. Fix a computable listing of all tuples $\langle \Phi, \Psi, \Theta, \Omega, D\rangle$ of partial computable functionals Φ, Ψ, Θ, and Ω and 2-c.e. sets D. It suffices to construct a c.e. set A satisfying the following requirements:

$$\mathcal{P}_\Theta : A \neq \Theta;$$

$$\mathcal{R}_{\Phi,D} : D = \Phi^A \ \Rightarrow\ \exists E\, \exists \Lambda_{\Phi,D}\, (E = \Lambda^D_{\Phi,D} \wedge E \mid_T L(D)) \vee D \leq_T L(D) \vee L(D) \leq_T \emptyset,$$

where each $\mathcal{R}$-requirement has its own infinite list of subrequirements:

$$\mathcal{T}_\Psi : E = \Psi^{L(D)} \ \Rightarrow\ \exists \Gamma_\Psi\ (D = \Gamma_\Psi^{L(D)});$$

$$\mathcal{S}_\Omega : L(D) = \Omega^E \ \Rightarrow\ \exists \Delta_\Omega\ (L(D) = \Delta_\Omega) \vee \exists \Gamma_\Omega\ (D = \Gamma_\Omega^{L(D)}).$$

We will usually suppress the subscripts on the functionals above when they are clear from the context. We will construct A using a tree of strategies and the gap/co-gap method. The proof uses a $\mathbf{0}'''$-priority argument and its full version can be found in [4]. In the proof sketch below, we describe the work of each strategy in isolation and their interaction on the strategy tree.

Further, for the sake of convenience we assume that at each stage of an algorithm which enumerates our set, the set changes at most at one element. The notation $s^D(x)$ means a stage when x enters D. If there is no such stage, then we assume $s^D(x)\uparrow$. Thus, if x leaves D, then $s^D(x)$ enters $L(D)$. We consider first main ideas of satisfaction of each strategy in isolation.

Basic $\mathcal{P}$-strategy. The basic $\mathcal{P}$-strategy is a variant of the standard Friedberg–Muchnik strategy. We choose a fresh witness a, wait for a stage s such that $\Theta(a)[s] \downarrow= 0$, and enumerate a into A.

Basic $\mathcal{R}$-strategy. An $\mathcal{R}$-strategy ρ serves as the mother strategy for all of its substrategies. It monitors the length of agreement between D and Φ^A. At nonexpansionary stages, it takes the finitary outcome fin. At expansionary stages, it makes progress towards constructing the functional Λ so that $\Lambda^D = E$ and takes its infinite outcome ∞, allowing its $\mathcal{S}$- and $\mathcal{T}$-substrategies to act (we will also call them child strategies of the strategy $\mathcal{R}$).

Basic $\mathcal{T}$-strategy. A $\mathcal{T}$-strategy τ is a child strategy of some $\mathcal{R}$-strategy. In isolation, it checks the length of agreement between E and $\Psi^{L(D)}$. At expansionary stages, τ constructs Γ_τ so that $\Gamma_\tau^{L(D)} = D$. The strategy has two possible outcomes, Γ and fin.

Basic $\mathcal{S}$-strategy. An $\mathcal{S}$-strategy σ, say, is a child strategy of some $\mathcal{R}$-strategy. It checks the length of agreement between $L(D)$ and Ω^E, and if the stage is expansionary, then σ first tries to construct Δ_σ such that $\Delta_\sigma = L(D)$. However, when interacting with other strategies (we will describe it in the following), this functional can be spoiled; thus it causes σ to construct a backup functional Γ_σ such that $\Gamma_\sigma^{L(D)} = D$. Thus, the strategy has three possible outcomes Γ, Δ, and fin.

Interactions between strategies. Now we consider nontrivial interactions between strategies and describe how to overcome the corresponding problems. Since all problems begin when a $\mathcal{P}$-strategy enumerates an element into A, we will always assume that there is a $\mathcal{P}$-strategy below the other strategies we consider.

$\mathcal{T}$-Strategy τ below its mother $\mathcal{R}$-strategy ρ. Let us consider the behavior of a strategy τ in more detail, and assume that there is $\mathcal{P}$-strategy π below it Γ-outcome. For every x, we need to correctly define $\Gamma^{L(D)}(x) = D(x)$. We pick a big $y = y_x$ first and wait until the length of agreement between $\Psi^{L(D)}$ and E is larger than y. At the first expansionary stage s at which this happens, we define $\Gamma^{L(D)}(x)[s] = D(x)[s]$ with use-function $\gamma(x)[s] = s > \psi(y)[s]$. From now on (assuming τ is along the true path), the equality between $\Gamma^{L(D)}(x)$ and $D(x)[s]$ can be broken only if a witness a of $\mathcal{P}$-strategy $\pi \supseteq \tau^\frown\Gamma$ is enumerated into A. It is worth noting that a must have been chosen before stage s, and so this can happen at most finitely many times (since all new witnesses of $\mathcal{P}$-strategies after initialization will be chosen big enough and there are only finitely many old witnesses).

The change in A allows a change in D on any x with Φ-use $\varphi(x)[s] \geq a$. We have the following possible cases:

Case 1: x enters D but there is no change in $L(D) \upharpoonright (\gamma(x) + 1)$: then we enumerate $y = y_x$ into E (since a permission from D is obtained) and we initialize all strategies below τ. So we have

$$1 = E(y) \neq \Psi^{L(D)}(y) = \Psi^{L(D)}(y)[s] = 0,$$

and τ wins. Initialized strategies must pick fresh witnesses; thus, from this moment on only strategies of higher priority than τ can enumerate numbers into A that allow changes of $\Psi^{L(D)}(y)[s]$. Indeed, if $\Psi^{L(D)}(y)[s]$ changes at a stage $s_1 > s$, then a number x_1 leaves D where $s^D(x_1) \leq \psi(y) < s$. It follows that some $a_1 \leq \varphi(x_1)[s] < s$ entered A after stage s, so a_1 must have been chosen before stage s.

Case 2: x enters or leaves D and there is a change in $L(D) \upharpoonright (\gamma(x) + 1)$: then we update $D(x) = \Gamma^{L(D)}(x)$ with new big use $\gamma(x)$. Note that a new update of $\Gamma^{L(D)}(x)$ can only be caused by a number $a_1 < a$ entering A. It is easy to see that when a is enumerated into A by a $\mathcal{P}$-strategy, we initialize all lower-priority strategies, and hence all strategies with witnesses greater than a. New witnesses will be greater than the current use $\varphi(x)$ and will not be able to change computations related to x. Thus, an increase in $\varphi(x)$ can only be caused by the enumeration of some $a_1 < a$, and as we noted above, this can happen at most finitely often.

Note that if x leaves D, then a change in $L(D) \upharpoonright (\gamma(x)+1)$ must occur because we defined $\Gamma^{L(D)}(x)$ correctly at stage s, when we have that x is already in D, and so $s^D(x) < s = \gamma(x)$. It follows that the two cases above exhaust all possibilities.

Intuitively, when we construct a functional Γ, we can think of it as *opening a gap* and allowing for some number a to enter A. Thus, during this time intervals of opened gaps, the restraint of strategy τ is dropped in the following sense: if there is an undesired change for τ because of some a enters A during this time, then τ can be satisfied forever. Therefore, these changes are undesired only at the first glance; in fact, they allow a *successful closing* of a gap (respectively, if there are no such changes, then the gap is *closed unsuccessfully*). Hence, either a gap will be closed successfully, namely, at some point we have case 1 and a diagonalization at τ, or all gaps will be closed unsuccessfully, namely, we always have case 2, in which case we will correctly reduce D to $L(D)$.

In the construction, this method also involves links which allow us to coordinate actions of a mother strategy with its child strategies during intervals of opened gaps; in particular, they allow one to jump from a mother strategy to its child strategy skipping all intermediate strategies. For instance, working with the above-mentioned ρ and τ using the corresponding link, we jump directly from ρ to τ and decide whether we want to enumerate y into E while keeping $E = \Lambda^D_\rho$ correct. So we will enumerate a number into or extract a number from E only when we come to a substrategy of ρ using a link (if there is no link at ρ, then we change E at ρ); otherwise, we will not need to change E at ρ, since at ρ we will not be in a position in which we must change E back due to D returning to an old initial segment (except for the situation when some $\mathcal{P}$-strategy between ρ and τ enumerates a small number into A, which allows a D-change which can force us to change E back at ρ but also causes τ to be initialized).

$\mathcal{T}$-Strategy τ below an $\mathcal{S}$-strategy σ below their mother $\mathcal{R}$-strategy ρ. The real conflict, which also causes this priority argument to be a $\mathbf{0}'''$-argument rather than just an infinite-injury argument, first arises in the following scenario. Suppose we have an $\mathcal{R}$-strategy ρ with an $\mathcal{S}$- substrategy σ and a $\mathcal{T}$-substrategy τ below such that τ is below the finite outcome of σ. Furthermore, assume that we have three $\mathcal{P}$-strategies π_2, π_1, and π_0 below the Γ-outcome of τ, the Δ-outcome of σ and the Γ-outcome of σ, respectively. Suppose now the following sequence of events.

First, the $\mathcal{P}$-strategy π_2 enumerates a witness a_2 into A, allowing a number x to enter D and causing τ to enumerate a number $y = y_x$ into E in order to diagonalize τ. Next, the $\mathcal{P}$-strategy π_1 enumerates a witness $a_1 < a_2$ into A, allowing x to leave D, which would normally force y to be extracted from E in order to keep Λ correct. However, for the stage $s^D(x)$ at which x entered D, $s^D(x)$ will enter $L(D)$ when x leaves D, while σ has possibly already defined $\Delta(s^D(x)) = 0$, which cannot be corrected. We resolve this conflict by threatening to let σ construct a Turing functional $\Gamma^{L(D)} = D$ to permanently satisfy ρ.

However, letting σ construct Γ (and taking an infinite Γ-outcome to the left of the infinite Δ-outcome) creates a new problem: suppose our $\mathcal{P}$-strategy π_0 below the Γ-outcome of σ next enumerates a number a_0 into A, allowing D to change at a number on which $\Gamma^{L(D)}$ is already defined and now possibly wrong. The strategy for σ can use the following procedure to force an $L(D)$-change and correct $\Gamma^{L(D)}$: before letting π_0 choose its witness a_0, we have a number x from some $\mathcal{P}$-strategy π_1 ready that just left D and caused the function Δ of σ to be incorrect. Similarly to the coordination work of ρ and τ, we use a link from ρ to σ, so that we can visit σ directly before ρ has a chance to extract y from E, allowing Λ^D to be temporarily incorrect. Therefore, if the functional Δ is now wrong on $s^D(x)$, then we create a second link from ρ to σ and move to outcome Γ, only then allowing a_0 to be enumerated in A. Suppose that this causes a change in $D(x')$.

(1) If x' enters D, then there need not be any $L(D)$-change and thus $\Gamma^{L(D)}(x')$ may now be incorrect. If $\Gamma^{L(D)}(x')$ is defined, then this means that x' is sufficiently small to allow us to preserve y in E while still keeping Λ^D correct. This causes a permanent disagreement between Ω^E and $L(D)$ at $s^D(x)$, since the old definition of $\Omega^E(s^D(x)) = 0$ is still valid while $s^D(x) \in L(D)$; this disagreement can only be

undone by an action of a strategy of higher priority than σ, since σ can now switch to a permanent finitary diagonalization outcome unless initialized later.

(2) If x' leaves D (and had previously entered D at a stage $s^D(x')$), then it will follow from the way we construct Γ that $\gamma(x') \geq s^D(x')$. So x' leaving D causes $s^D(x')$ to enter $L(D)$ and allow $\Gamma^{L(D)}(x')$ to be corrected.

Similarly to the previous case, we *open a second gap* when we allow the number a_0 to enter A. Thus, either one of these gaps will be *closed successfully* (i.e., at some point, we have case (1)) and we have a permanent win at σ, or all gaps will be *closed unsuccessfully* (i.e., we always have case (2)), then we correctly reduce D to $L(D)$. Again, in the construction, we create a link from σ to ρ since we jump from ρ to σ when we need to decide whether to enumerate y into E or not, and the link allows us to keep $E = \Lambda_\rho^D$ correct.

Interaction of several $\mathcal{R}$-strategies. In our intuitive analysis, we restrict ourselves to two $\mathcal{R}$-strategies ρ_0 and ρ_1. Assume that we have $\rho_0 \subset \rho_1$, and that they have substrategies σ_0 and σ_1, respectively (also assume that σ_0 and σ_1 have an Γ-outcome). The conceivable relative priorities for these strategies are as follows:

(1) $\rho_0 \subset \sigma_0 \subset \rho_1 \subset \sigma_1$;
(2) $\rho_0 \subset \rho_1 \subset \sigma_1 \subset \sigma_0$, and
(3) $\rho_0 \subset \rho_1 \subset \sigma_0 \subset \sigma_1$.

The third case could produce nonnested links; so we disallow it as follows: When σ_0 changes the global outcome of ρ_0 along the true path, we introduce another version of ρ_1, say, an $\mathcal{R}$-strategy ρ_1', first, and only allow substrategies of ρ_1' but not of ρ_1 below ρ_1'. This reduces the third case above to the first, in the usual manner of $\mathbf{0}'''$-arguments.

In the first case, there is no real conflict, since ρ_1 already knows that σ_0 will construct its Γ, which permanently satisfies ρ_0. In the second case, there may be links from ρ_0 directly to σ_0, over ρ_1 and σ_1; but if σ_0 truly has Γ-outcome, then we again introduce another version of ρ_1, say, an $\mathcal{R}$-strategy ρ_1', below σ_0 and only allow substrategies of ρ_1' but not of ρ_1 below σ_0.

Thus, in such a way the work of two $\mathcal{R}$-strategies and their substrategies is coordinated. In case of several $\mathcal{R}$-strategies, they can be coordinated in the same manner; the details and the full construction can be found in [4].

4.6. Definability with parameters. Now we proceed to the questions of definability with parameters.

Theorem 4.28. *There exists an infinite family of c.e. degrees such that each of them is definable in $\mathcal{D}_2$ with one parameter from $\mathcal{D}_2 - \mathcal{R}$.*

Proof. For a given $\boldsymbol{a} \in \mathcal{S}$ let $\boldsymbol{d}$ be a degree such that for any degree $\boldsymbol{z}$ if $\boldsymbol{a} < \boldsymbol{z} \leq \boldsymbol{d}$ and $\boldsymbol{z}_0 \cup \boldsymbol{z}_1 = \boldsymbol{z}$ for some d-c.e. degrees $\boldsymbol{z}_0$ and $\boldsymbol{z}_1$, and $\boldsymbol{z}_0 \mid \boldsymbol{z}_1$. Then either $\boldsymbol{a} \leq \boldsymbol{z}_0$ or $\boldsymbol{a} \leq \boldsymbol{z}_1$. It is clear that $\boldsymbol{d}$ is not computable, computably enumerable, and uniquely define the c.e. degree $\boldsymbol{a}$. In particular, $S \cap [\mathbf{0}, \boldsymbol{d}] = \{\boldsymbol{a}\}$. □

Slaman and Woodin [82] proved that the class $\mathcal{R}$ of c.e. degrees is definable with parameters in $\mathcal{D}(\leq \mathbf{0}')$. Their proof sufficiently uses Theorem 4.30.

Definition 4.29. A family $\mathcal{A}$ of low degrees is called uniformly low family if there exist a sequence of sets $\langle X(n) \mid n \in \omega \rangle$ and a $\emptyset'$-computable function f such that $\{deg(X(n)) \mid n \in \omega\} = \mathcal{A}$ and $\Phi_{f(n)}^{\emptyset'} = (X(n))'$.

Theorem 4.30. *Let $\mathcal{A}$ be a uniformly low family of Δ_2^0-degrees such that it is bounded by some low degree $\boldsymbol{a}$. Then $\mathcal{A}$ is definable with parameters in $\mathcal{D}(\leq \mathbf{0}')$.*

It follows from this theorem that the definability with parameters of $\mathcal{R}$ in $\mathcal{D}(\leq \mathbf{0}')$ can be obtained with help of the following theorem.

Theorem 4.31 (Welch [84])**.** *There exist two uniformly low families of c.e. degrees $\mathcal{A}_0$ and $\mathcal{A}_1$ such that each of them if bounded by some low degree, and each c.e. degree $\boldsymbol{a}$ is the least upper bound of some $\boldsymbol{a}_0 \in \mathcal{A}_0$ and $\boldsymbol{a}_1 \in \mathcal{A}_1$.*

Proof. By the Sacks splitting theorem, the creative set $K = \{\langle e, n\rangle \mid n \in W_e\}$ can be split into two low T-incomparable sets sets A_0 and A_1. Thus we define the families $\mathcal{A}_i, i \leq 1$ as the following families of degrees of sets

$$\mathcal{B}_i = \big\{e \mid \langle e, n\rangle \in A_i\big\}_{n\in\omega}.$$

Hence by the previous theorem the class of c.e. degrees $\mathcal{R}$ is definable with parameters in $\mathcal{D}(\leq \mathbf{0}')$. □

Theorem 4.32. *Let $\boldsymbol{a}$ be a low c.e. degree $> \mathbf{0}$. The family of n-c.e. degrees $\{\boldsymbol{b} \mid \boldsymbol{b} \leq \boldsymbol{a}\}$ is definable with parameters in $\mathcal{D}(\leq \mathbf{0}')$ for each $n \geq 1$.*

Proof. We need to define a computable function g such that family of degrees $\{\deg(V_{n,g(e)}) \mid e \in \omega\}$ coincides with the family of all n-c.e. degrees $\leq\boldsymbol{a}$. Here $\{V_{n,e}\}_{e\in\omega}$ is an effective numbering of all n-c.e. sets, $n \geq 2$.

For $n = 1$, such a function g can be obtained by the Yates theorem about characterization of Σ^0_3-index sets of classes of c.e. sets. For the case $n > 1$, such a function g can be obtained with the help of the following proposition: for any $n > 1$ and any total function $f \leq_T \emptyset''$, there exists a computable function g such that for any $e \in \omega$, $V_{n,f(e)} \equiv_T V_{n,g(e)}$. It was proved in [6] for the case when instead of numbering of n-c.e. sets $\{V_n\}$, the author considered the numbering of c.e. sets $\{W_n\}$. However, the proof for general case is similar with obvious and necessary changes.

Let A be a c.e. set from the degree $\boldsymbol{a}$. Since $\{e : V_{n,e} \leq_T A\}$ is a Σ^A_3-set (e.g., see Ishukhametov [46], where it is also proved that this index set is Σ^A_3-complete) and since $A' \leq_T \emptyset'$, we conclude that $\{e : V_{n,e} \leq_T A\}$ is a Σ^0_3-set. If follows that there exists a function $f \leq_T \emptyset''$ such that

$$\big\{V_{n,e} : V_{n,e} \leq_T A,\ e \in \omega\big\} = \big\{V_{n,f(e)} : e \in \omega\big\}.$$

Now, let g be a computable function such that for each $e \in \omega$, $V_{n,f(e)} \equiv_T V_{n,g(e)}$ we have

$$\big\{\boldsymbol{b} \in \mathcal{D}_2 \mid \boldsymbol{b} \leq \boldsymbol{a}\big\} = \big\{deg(V_{n,g(e)}) \mid e \in \omega\big\}.$$

The following function c is computable relative to $\emptyset'$:

$$c(e, x) = \begin{cases} 1, & \text{if } x \in V_{n,g(e)}, \\ 0, & \text{if } x \notin V_{n,g(e)}. \end{cases}$$

Therefore, there exists a computable relative to $\emptyset'$ function α such that for each $e \in \omega$, $V_{n,g(e)} = \Phi^{\emptyset'}_{\alpha(e)}$. Now the theorem follows from Theorem 4.30. □

Theorem 2.1 states that there exist a c.e. degree $\boldsymbol{b} > \mathbf{0}$ and a d-c.e. degree $\boldsymbol{c} > \boldsymbol{b}$ such that for any c.e. degree $\boldsymbol{d}$, if $\boldsymbol{d} \leq \boldsymbol{c}$, then either $\boldsymbol{b} \leq \boldsymbol{d}$ or $\boldsymbol{d} \leq \boldsymbol{b}$. It easy to check that satisfaction of an additional requirement

$$(\forall e)(\exists^\infty s)\big(\Phi^B_e(e)[s]\downarrow \rightarrow \Phi^B_e(e)\downarrow\big),$$

which allows one to make in this theorem the degree of set $B \in \boldsymbol{b}$ as low, can easily be combined with satisfaction of other requirements. Thus, there exist c.e. sets of *low degree*, which satisfies the conditions of the above theorem. We do not know whether there exist sets of *nonlow degrees* with these properties, i.e., whether there exist d-c.e. degrees $\boldsymbol{d} > \mathbf{0}$ such that their splittings into d-c.e. degrees cannot be low:

$$(\boldsymbol{d} = \boldsymbol{a} \cup \boldsymbol{b}) \rightarrow \boldsymbol{a}' > \mathbf{0}' \ \&\ \boldsymbol{b}' > \mathbf{0}'.$$

If the above theorem could be enhanced in such a way, then it would mean that the above method of Slaman and Woodin cannot be applied in principle for the proof of definability with parameters of n-c.e. degrees in $\mathcal{D}(\leq \mathbf{0}')$. Namely, in the general case an n-c.e. degree for $n > 1$ cannot be split into two low n-c.e. degrees, contrary to the case of c.e. degrees.

4.7. Definability of m-c.e. degrees on the lattice language of n-c.e. sets.

Definition 4.33. The set of Turing degrees $\mathcal{C}$ is definable in $\mathcal{E}_n$ for some $n \geq 1$, if there exist a class of sets $S \subset \mathcal{E}_n$ definable in $\mathcal{E}_n$ such that $\mathcal{C} = \{\deg(B) \mid B \in S\}$.

Example 4.34. The degree $\mathbf{0}$ is definable in each $\mathcal{E}_n$, $1 \leq n < \omega$, since $\mathbf{0} = \deg(\emptyset)$.

For each $n \geq 0$, let

$$H_n = \{\boldsymbol{a} \text{ c.e.} \mid \boldsymbol{a}^{(n)} = \mathbf{0}^{(n+1)}\}, \quad L_n = \{\boldsymbol{a} \text{ c.e.} \mid \boldsymbol{a}^{(n)} = \mathbf{0}^{(n)}\}$$

be the classes of high$_n$ and low$_n$ sets, respectively. By definition, $\boldsymbol{a}^0 = \boldsymbol{a}$, thus $H_0 = \{\mathbf{0}'\}$ and $L_0 = \{\mathbf{0}\}$.

Starting from 1960s, various authors investigated the questions of definability in the mentioned above sense for the classes H_n and L_n, $n \geq 0$ in the lattice $\mathcal{E}_1$ of the c.e. sets. The results about nondefinability of jumps for these classes follows from the following theorem of Cholak [17] and Harrington and Soare [45].

Theorem 4.35 (Cholak [17]; Harrington and Soare [45])**.** *Each noncomputable c.e. set A by some automorphism of $\mathcal{E}$ can be mapped to some high c.e. set H (in this case, we say that A is automorphic H).*

Since automorphisms preserve all lattice-theoretic properties, it follows from this theorem that the classesL_n, $n > 0$, and $\overline{H_n}$, $n > 0$, being downward dense relative to the Turing reduction, are not definable. (Downward density of a class of degree $\mathcal{A}$ means that $\boldsymbol{a} \in \mathcal{A}$ and $\boldsymbol{b} \leq \boldsymbol{a} \to \boldsymbol{b} \in \mathcal{A}$.)

Nondefinability for the class $\overline{H}_0$ of all not T-complete c.e. degrees was established by Harrington and Soare [45].

The results about definability for the classes of jumps H_n and $\bar{L}_n$ for $n \geq 2$ were obtained by Cholak and Harrington [18].

Theorem 4.36 (Cholak and Harrington [18])**.** *For $n \geq 2$, H_n and $\bar{L}_n$ are definable.*

Note that the definability of the class $\overline{L_2}$ follows from earlier works of Lachlan [52] and Shoenfield [75].

Since the class of high degrees H_1 coincides with the class of degrees which contain a maximal set (Martin [66]), the high c.e. degrees are also definable in $\mathcal{E}$.

For the last class $\overline{L}_1$, Epstein showed that it is nondefinable.

Theorem 4.37 (Epstein [33])**.** *The family of non-low c.e. degrees are not definable in $\mathcal{E}$.*

The definability of c.e. sets (and, therefore, of c.e. degrees) in $\mathcal{E}_2$ was established by Lempp and Nies [57]. They proved that an element from $\mathcal{E}_2$ is c.e. if and only if it is the supremum of two elements with unique complements. Their proof is based on the following observation.

Lemma 4.38 (Lempp and Nies [57])**.** *A c.e. set A has a d-c.e. complement $C - D$ in $\mathcal{E}_2^*$ if and only if for some c.e. set B, $C - D =^* \overline{B}$, and also either $A =^* B$ or A is a major subset of B (denoted as $A \subset_m B$).*

(Recall that a c.e. set A is a major subset of a c.e. set B if $A \subset_\infty B$ and for each c.e. set W, $\overline{B} \subseteq^* W \to \overline{A} \subseteq^* W$.)

It follows that each c.e. set A is a union of two disjoint c.e. sets with unique complements: A is a union of two disjoint c.e. sets A_0 and A_1 of low degrees by the well-known Sacks theorem [73] (also see Soare [83, Theorem VII.3.2]), and, therefore, A_0, A_1 cannot be major subsets of any sets, because major subsets have high degrees (Jockusch [48], also see Soare [83, Ex. X1.1.19]).

Below we will generalize this result of Lempp and Nies about the definability to higher levels of the Ershov hierarchy; our arguing is based on the same observations.

Lemma 4.39. *Let $n > 2$ and A be a c.e. set. If X is an n-c.e. complement of A in $\mathcal{E}_n^*$, then there exists a c.e. set B such that $X =^* \overline{B}$ and either $A =^* \overline{B}$ or $A \subset_m B$.*

Proof. Assume that $X = (B_1 - B_2) \cup \ldots B_n$ for some c.e. sets $B_1 \supseteq B_2 \supseteq \ldots B_n$. It is clear that $X \cap A = \emptyset$, and we have $A \cup B_1 =^* \omega$ since $A, X \subseteq A \cup B_1$. By the reduction principle there exists a computable set $R \subset B_1$ such that $A \cup R = \omega$. It is clear that we can assume that X can be presented as $(R - B_2) \cup (R \cap B_3 - B_4) \cup \ldots \cup (R \cap B_{2m-1} - B_{2m})$ (if n is odd, then $B_{2m} = \emptyset$).

If $\overline{A} \cap B_{2m} \neq^* \emptyset$ then for a 3-c.e. set $M = (\omega - B_{2m}) \cup A$ we have $A, X \subseteq M$ and $M \subset_\infty \omega$, a contradiction. Therefore, $\overline{A} \cap B_{2m} =^* \emptyset$. On the other hand,

$$B_1 \supseteq^* \overline{A}, \quad \overline{A} \cap (B_{2i} - B_{2i+1}) =^* \emptyset$$

for each $i, 1 \leq i < m$ (otherwise, if for some $i, 1 \leq i < m$,

$$\overline{A} \cap (B_{2i} - B_{2i+1}) = \infty,$$

then $A, X \subseteq M$, where $M = (R - B_2) \cup \ldots \cup (R \cap B_{2i-1} - B_{2i}) \cup B_{2i+1} \cup A$, a contradiction). Therefore, $X =^* B_1 - A \cap B_1 = B_1 - A$, and the statement follows from Lemma 4.38. □

Definition 4.40. Let $A \subset_\infty B$ be c.e. sets. A is called *small* in B (notation $A \subset_s B$) if for each pair of c.e. sets X, Y, from $X \cap (B - A) \subseteq Y$ follows that $Y \cup (X - B)$ is a c.e. set.

Lemma 4.41. *Assume that $n \geq 1$, and A and B are c.e. sets, $A \subset_s B$. Then for each n-c.e. set $S \subset A$ if $B - S$ is c.e. then the set $\omega - S$ is also c.e.*

Proof. Assume that $X = \omega$ and $Y = B - S$. Then $X \cap (B - A) = B - A \subset Y$. Therefore $Y \cup (X - B) = Y \cup (\omega - B) = \omega - S$ is c.e. □

The following proposition was proved in Lempp and Nies (see [57, proof of Theorem 2.4]).

Lemma 4.42. *If a d-c.e. set $A - B$ has a unique complement in $\mathcal{E}_2$, then it is c.e.*

We generalize it as follows.

Lemma 4.43. *Assume that $0 < n < \omega$. If an n-c.e. set X has a unique complement in $\mathcal{E}_n$, then it is d-c.e.*

Proof. For simplicity we consider the case where $n = 3$: $X = (A - B) \cup C$ for some c.e. sets $A \supseteq B \supseteq C$. In the general case the proposition is proved by induction using the similar arguments.

Let a 3-c.e. set $Y = (P - Q) \cup L$ be a complement for $(A - B) \cup C$. Consider two cases. (For a c.e. set S_1 and an n-c.e. set S_2, $S_1 \subset S_2$, we say that S_1 is a major subset of S_2 ($S_1 \subset_m S_2$) if for each c.e. set W, $\overline{S_2} \subseteq^* W \to \overline{S_1} \subseteq^* W$.)

Case 1. P is computable. Exactly as in [57], we can prove that

$$A - (B - C) =^* (A - P) \cup (A \cap (Q - L)) =^* A_1 \cup (A \cap Q) - A_1 \cup (A \cap L),$$

(since $(A - P) \cap (Q - L) =^* \emptyset$), where $A_1 = A - P$ is a c.e. set. Therefore, $A - (B - C)$ is a d-c.e. set.

Case 2. P is not computable. As in [57], we prove that $\tilde{P} - (Q - L)$ is also a complement of $A - (B - C)$, where $\tilde{P} = U \cup (Q - L)$ and U is a small major subset of P ($U \subset_{sm} P$).

Assuming that $(A - B) \cup C$ is not a d-c.e. set, we prove that $\tilde{P} - (Q - L) \subset_\infty P - (Q - L)$; thus, $A - (B - C)$ has two different complements.

For this we repeat the arguments from Lempp and Nies [57] (case 2 of the proof of the similar theorem) with minor changes: let R be a computable set such that $R \subseteq A$ and $\overline{R} \subseteq P$. The inclusion $U \subset_{sm} P$ implies that $\overline{R} \subseteq U$. Now assume that

$$\tilde{P} - (Q - L) =^* P - (Q - L).$$

Let $S = R \cap (B - (C \cup L)) \cap P$. S is a d-c.e. set, which splits P with other c.e. part $\overline{R} \cup Q$. Lemma 4.41 implies that S has a c.e. complement. Let Z be a c.e. set such that $\overline{S} = Z$. It is clear that $X, Y \subseteq Z$ and, therefore, the set S in finite.

Now let $S_1 = R \cap (B - C) \cap P$. Since $S =^* \emptyset$ we have $S_1 =^* B \cap L \cap R$. Therefore, S_1 is c.e. Thus, it follows that

$$Z = \overline{R} \cup S_1 \cup C \cup \overline{(B - C)}$$

is a co-d-c.e. set and, clearly, $A - (B - C), P - (Q - L) \subseteq Z$.

Now we prove that $\tilde{P} - (Q - L) =^* P - (Q - L)$ implies that the set $\overline{Z}$ is infinite (it will contradict the condition $A - (B - C) \vee P - (Q - L) =^* 1$). For contrary, assume that $\overline{Z}$ is finite, then

$$A - (B - C) =^* R \cap \overline{S_1} \cup (Q - L) \cap A.$$

However, $(Q - L) \cap A = Q \cap C$; thus,

$$A - (B - C) =^* (R \cap \overline{S_1}) \cup (Q \cap C)$$

and, therefore, $A - (B - C)$ is a d-c.e. set, a contradiction. □

Lemma 4.44.

(a) *If a d-c.e. set $A - B$ has a unique 3-c.e. complement $X = (P - Q) \cup L$ in $\mathcal{E}_3$, then $A - B$ also is a unique d-c.e. complement of $(P - Q) \cup L$ in $\mathcal{E}_3$.*

(b) *If a 3-c.e. set X has a unique d-c.e. complement $A - B$ in $\mathcal{E}_3$, then X is a d-c.e. set.*

Proof. Assume that $A_1 - B_1$ is another complement for X. If, for instance,

$$(A_1 - B_1) - (A - B) = \infty,$$

then

(i) if $\overline{A} \cap (A_1 - A_2) = \infty$, then the 3-c.e. set $(P \cup A_1 - (A \cup B_1 \cup Q)) \cup L$ is another complement for $A - B$);

(ii) if $B \cap (A_1 - B_1) = \infty$, then the 3-c.e. set

$$(P - (Q \cup (B \cap A_1 - B_1 \cap B))) \cup L = (P - Q \cup B \cap A_1) \cup B_1 \cap B \cup L$$

is another complement for $A - B$.

(b) The proof is similar to the proof of Lemma 4.43. We consider two cases: when the set A is computable, and when it is not. In the first case, the proof is exactly the same as [57, proof of Theorem 2.4] with necessary, but obvious, changes (in this case, we will obtain that $A - B$ is a c.e. set). The second case repeats the same case from Lemma 4.43. □

Remark 4.45. Lemma 4.44(b) is a stronger version of Lemma 4.43 for the case $n = 3$: in the conditions of Lemma 4.44(b) it states that in order for a 3-c.e. set X be a d-c.e. set, it suffices that this set does not have two different d-c.e. complements in $\mathcal{E}_3$ (therefore, X can have such 3-c.e. sets).

Theorem 4.46. *An element of $\mathcal{E}_n$ is c.e. if and only if it is the supremum of two elements from $\mathcal{E}_n$ with unique complements.*

Proof for the case $n = 3$. The part $(\Rightarrow)$ is the same as in [57]: if some 3-c.e. set A is c.e., then we split A into two disjoint c.e. sets A_0 and A_1 of low degrees. Each A_i has a unique complement $\overline{A_i}$; otherwise, if A_i has another d-c.e. complement, then by Lemma 4.39 it would have to be a major subset of a c.e. set, but the degrees of major subsets must be high.

The part ($\Leftarrow$) of the theorem follows from Lemmas 4.43 and 4.44: if a 3-c.e. set A is the supremum of two 3-c.e. sets A_0 and A_1 with unique 3-c.e. complements $\tilde{A}_0$ and $\tilde{A}_1$, respectively, then by Lemma 4.43, A_0 and A_1 are d-c.e. sets. By Lemma 4.44(a), A_0 and A_1 are unique complements for $\tilde{A}_0$ and $\tilde{A}_1$, respectively. Therefore, by Lemma 4.44(b), both $\tilde{A}_0$ and $\tilde{A}_1$ are d-c.e. sets. Thus, d-c.e. sets A_0 and A_1 have a unique d-c.e. complement, and by Lemma 4.42 A_0 and A_1 are c.e. sets. Since A is a union of A_0 and A_1, it is also a c.e. set. □

Corollary 4.47. *The class of all c.e. degrees and the class of all noncomputable c.e. degrees are definable without parameters in each $\mathcal{E}_n$, $n \geq 2$.*

Corollary 4.48. *Let $1 \leq m < n < \omega$. Then $\mathcal{E}_m$ is definable without parameters in $\mathcal{E}_n$.*

Theorem 4.49. *The class of all high c.e. sets H_1 is definable in each $\mathcal{E}_n$, $1 \leq n < \omega$.*

Proof. Fix $n \geq 1$ and let $\mathcal{A}$ be the following class of n-c.e. sets which is definable in $\mathcal{E}_n$:

$$\mathcal{A} = \Big\{ X : \ X \text{ is } n\text{-c.e. and } (\forall e) \ \big(V_{n,e} \cap \overline{X} \text{ is finite} \vee \overline{V}_{n,e} \cap \overline{X} \text{ is finite} \big) \Big\}.$$

It is well known that for each set U of high degree, there exists a maximal set $M \equiv_T U$ (Martin [66]; also see Soare [83, Chap. 11, Sec. 1]). Thus, for each maximal set M, we have $M \in \mathcal{A}$: if some n-c.e. set

$$B = \bigcup_{i=0}^{\left[\frac{n-1}{2}\right]} \{(R_{2i+1} - R_{2i}) \cup (R_{2i} - R_{2i+1})\}$$

divides $\bar{M}$ into infinitely many parts, then, clearly, a c.e. set R_i for some $0 \leq i \leq n-1$ also divides $\bar{M}$ into infinitely many parts, which is impossible. Thus, $\mathcal{A}$ contains all high c.e. degrees: $\mathcal{H} \subseteq \mathcal{A}$.

Now we prove the reverse inclusion $\mathcal{A} \subseteq \mathcal{H}$. Let V be an n-c.e. set such that

$$(\forall e) \ \ (V_{n,e} \cap V \text{ is finite} \vee \bar{V}_{n,e} \cap V \text{ is finite}).$$

It is clear that n must be an even number,

$$V = \bigcup_{i=1}^{\left[\frac{n+1}{2}\right]} \{(A_{2i-1} - A_{2i})\}$$

for some c.e. sets $A_1 \supseteq \ldots \supseteq A_n$, and V is a hyperhyperimmune set. The Lachlan theorem about the hyperhyperimmune difference of c.e. sets (Lachlan [54]) states that if for some c.e. sets X and Y such that $X \supset Y$ the set $X - Y$ is hyperhyperimmune, there exists a computable set R such that $X - Y \subseteq R \subseteq X$. Therefore, there exist computable sets $R_1, R_2, \ldots R_k$ such that

$$A_{2i} - A_{2i-1} \subseteq R_i \subseteq A_{2i-1}, \quad \{\omega - V\} = \overline{R_1} \cup \left\{ \bigcup_{i=1}^{n/2} \left(A_{2i} \cap \left\{ \overline{\bigcup_{m>2i} R_m} \right\} \right) \right\}.$$

This implies that V is c.e. and the degree of V is high. Thus, $\mathcal{A} \subseteq \mathcal{H}$. □

So far we did not have examples of definable subclasses of $\mathcal{E}_n$, $1 < n < \omega$, which contain non-c.e. sets and are different from $\mathcal{E}_m$, $m < n$. Assume that

$$\mathcal{D}_n := \mathcal{E}_{n+1} \cap \overline{\mathcal{E}_{n+1}}.$$

We have

$$\mathcal{E}_n \subset \mathcal{D}_n \subset \mathcal{E}_{n+1}$$

for each n, $1 \leq n < \omega$. For each $n \geq 0$, the class of sets $\mathcal{D}_n$ is definable in $\mathcal{E}_{n+1}$ by the formula

$$(\forall X \in \mathcal{E}_{n+1})(X \in D_n \leftrightarrow \omega - X \in \mathcal{E}_{n+1}).$$

Therefore, if for some $n > 0$ we have

$$\{\deg(A) \mid A \in \mathcal{E}_n\} \subset \{\deg(A) \mid A \in \mathcal{E}_{n+1} \cap \overline{\mathcal{E}_{n+1}}\} \subset \{\deg(A) \mid A \in \mathcal{E}_{n+1}\},$$

then we obtain a definable in $\mathcal{E}_{n+1}$ class of $(n+1)$-c.e. degrees different from $\mathcal{E}_n$. However, as follows from Theorem 4.50, it is invalid.

Theorem 4.50. *For each $n > 0$, we have*

$$\{\deg(A) \mid A \in \mathcal{E}_n\} = \{\deg(A) : A \in \mathcal{E}_{n+1} \cap \overline{\mathcal{E}_{n+1}}\}.$$

Proof. Let X and Y be $(n+1)$-c.e. sets such that $X \cup Y = \omega$ and $X \cap Y = \emptyset$. Let $X = A - B$ and $Y = C - D$ for some c.e. sets A, C and n-c.e. sets B and D, $B \subset A$ and $D \subset C$. Let f and g be $1-1$-computable functions such that $A = \mathrm{rng}(f)$ and $C = \mathrm{rng}(g)$, and let $U = f^{-1}(B) \oplus g^{-1}(D)$. Then U is an n-c.e. set, and since $Y = \omega - X$ we have $U \leq_T A - B$. Now we prove that $A - B \leq_T U$ too. For some x, we enumerate sets A and C until x enters A or C. If, for instance, $x \in A$, then let for some $y, f(y) = x$. We have $x \in A - B$ if and only if $2y \notin U$. □

Generalizing the definition of a c.e. major subset, we call an n-c.e. set A ($n > 1$) as major subset of a c.e. B if $A \subset_\infty B$ and for each c.e. W,

$$\overline{B} \subseteq^* W \to \overline{A} \subseteq^* W$$

In a similar way, as was done for the case of c.e. major subsets (see Soare [83, X1.1.19]) it can be proved that n-c.e. major subsets of c.e. sets (if they exist) have high degrees.

Theorem 4.51 (Arslanov [7])**.** *Let $n > 0$. If n is an even number then there are no n-c.e. major subsets. If n is an odd number, then each noncomputable c.e. set B has a major n-c.e. major subset A of a properly n-c.e. degree.*

Proof. Assume that $n = 2k$, $R_1 \supset R_2 \supset \ldots \supset R_{2k}$ are c.e. sets and

$$A = \bigcup_{i=1}^{k} R_{2i} - R_{2i-1}$$

is an n-c.e. subset of a c.e. set B such that $B - A$ is infinite. Clearly, we can assume that R_{2k} is infinite, and, since B is c.e., each R_i is a subset of B. Let C be infinite computable subset of R_{2k} such that $R_{2k} - C$ infinite. Then the computable set $\overline{C}$ witnesses that A is not a major subset of B.

Now let $n = 2k + 1$ and B be a noncomputable c.e. set. We construct an n-c.e. set A of a properly n-c.e. degree such that $A \subset_m B$.

We construct the desired n-c.e. set A by combining two strategies: the Lachlan strategy for construction of a major subset of a noncomputable c.e. set (see Soare [83, Theorem X.4.6]) and the Cooper strategy for construction of an n-c.e. set of a properly n-c.e. degree.

We construct A so that for each e, $A \not\equiv_T V_e$, where V_e is the eth $(n-1)$-c.e. set (for some fixed computable numbering of all $(n-1)$-c.e. sets $\{V_e\}_{e \in \omega}$). We do this with the help of the fact that the maximum number of changes of $A(x)$ is greater by 1 than the number of changes of $V_e(x)$.

Again, for convenience we assume that $n = 3$. The desired 3-c.e. set A is constructed as $(D_1 - D_2) \cup D_3$, where $D_1 \supseteq D_2 \supseteq D_3$ are c.e. sets and $A \subset B$. In order to satisfy the condition that "the degree of A is not a d-c.e. degree" for each d-c.e. set V and for all partial-computable functionals Φ and Ψ, we satisfy the following requirements:

$$\mathcal{R}_{V,\Phi,\Psi} : A \neq \Phi^V \vee V \neq \Psi^A,$$

where $\{(W_e, \Phi_e, \Psi_e)\}_{e \in \omega}$ is an effective numbering of all triples of c.e. sets W and partial-computable functionals Φ and Ψ.

To satisfy the requirement $R_{V,\Phi,\Psi}$, we choose a fresh witness $x \in B$ and wait for a stage s such that

$$A_s(x) = \Phi^V(x)[s] \wedge V\lceil\varphi(x)[s] = \Psi_s^A\lceil\varphi(x)[s]$$

(if it never happens, then the requirement $R_{V,\Phi,\Psi}$ is satisfied), set up the restraint for all other strategies on the interval $A\lceil\psi_s\varphi_s(x)$, enumerate x into D_1. and wait for a stage s' such that

$$A_{s'}(x) = \Phi^{V\lceil\varphi(x)}(x)[s'] \wedge V\lceil\varphi(x)[s'] = \Psi^{A\lceil\psi\varphi(x)}\lceil\varphi(x)[s']$$

(again, if it never happens, then the requirement is satisfied). Enumerate x into D_2 (i.e., remove x from A) and restrain the interval $A\lceil\psi_{s'}\varphi_{s'}(x)$ for other requirements. Wait for a stage s'' such that

$$A_{s''}(x) = \Phi^{V\lceil\varphi(x)}(x)[s''] \wedge V\lceil\varphi(x)[s''] = \Psi^{A\lceil\psi\varphi(x)}\lceil\varphi(x)[s'']$$

(again, if it never happens, then the requirement is satisfied). Now we put x into D_3 and restrain interval $A\lceil\psi_{s''}\varphi_{s''}(x)$ for other requirements.

Changes of $\Phi^V(x)$ between the stages s' and s'' could happen only because of changes in $V\lceil\varphi(x)$; they are irreversible since V is a 2-c.e. set.

In order to make the set A a major subset of B, we use Soare's modification of the Lachlan construction of major subset of c.e. set (see Soare [83, Theorem X.4.6]). Thus, we choose a sequence of movable markers $\{\Gamma_n\}_{n\in\omega}$, having in the end $B - A = \{d_0 < d_1 < \ldots\}$, where d_n^s is a position of Γ_n at the end of stage s, and $d_n = \lim_s d_n^s$. We need to satisfy the following requirements:

(i) $\mathcal{N}_e$: the marker Γ_e is moved at most finitely many times, and

(ii) $\mathcal{P}_e$: $\overline{B} \subseteq W_e \to \overline{A} \subseteq^* W_e$.

We modify the construction of Theorem X.4.6 from [83] as follows. At the stage $s+1$ satisfying the requirement $\mathcal{P}_e$, we work with elements $y \in B$ and $d_e^s < y$ only if they are not restrained by $\mathcal{R}$-requirements of higher priority. (If such y or e does not exist, then we just proceed to the next stage.)

Since each requirement $R_{V,\Phi,\Psi}$ is satisfied after finite number of injuries, we can satisfy all requirements similar to [83] with obvious changes. □

Corollary 4.52. *For each $n > 1$ and for each odd number $m < n$, there exists a definable in $\mathcal{E}_n$ subclass of properly m-c.e. degrees.*

The proof follows immediately from Theorems 4.51 and 4.46 (namely, from Corollary 4.48).

It is known (e.g., see Arslanov [6]) that for each n, $1 < n < \omega$, there exists a high properly n-c.e. degree. The following question naturally arises: Is it true that the class of high n-c.e. degrees is definable in $\mathcal{E}_n$ for each n, $1 < n < \omega$? The next theorem gives a partial answer to this question.

Theorem 4.53. *For each odd number n, $1 \leq n < \omega$, for each high n-c.e. degree $\boldsymbol{d}$ and for each noncomputable c.e. set A, there exists an n-c.e. major subset M of the set A such that* $\deg(M) = \boldsymbol{d}$.

The proof of this theorem can be obtained by a straightforward generalization (to the case of n-c.e. sets) of Lerman's proof (see Soare [83, XI.2.14]) of the following result: For each high c.e. degree $\boldsymbol{d}$ and for each noncomputable c.e. set A, there exists its major subset M such that $\deg(M) = \boldsymbol{d}$.

Corollary 4.54. *For each odd number n, $1 \leq n < \omega$, the class of all high n-c.e. degrees is definable in $\mathcal{E}_n$.*

Question 4.55. Is it true that the class of all high n-c.e. degrees is definable in $\mathcal{E}_n$ for even numbers n?

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