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MODEL VARIATIONAL PHASE TRANSITION PROBLEM IN CONTINUUM MECHANICS

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We formulate and study the model variational problem describing phase transitions in two-phase media. Based on this study, we obtain an information about properties of solutions to the problem on equilibrium of a two-phase medium in the traditional statement. Bibliography: 11 *titles.*

1 Introduction

We assume that $\Omega \subset R^m$, $m \geqslant 1$, is a bounded domain, $R_s^{m\times m}$ is the linear space of symmetric $(m \times m)$-matrices equipped with the inner product $\langle \alpha, \beta\rangle = \operatorname{tr} \alpha\beta$, $\alpha, \beta \in R_s^{m\times m}$. In the Hilbert space $L_2(\Omega, R_s^{m\times m})$ equipped with the inner product

$$(\widetilde{e}', \widetilde{e}'')_{L_2(\Omega)} = \int\limits_\Omega \langle \widetilde{e}'(x), \widetilde{e}''(x)\rangle \, dx, \tag{1.1}$$

we consider the closed linear subspace

$$\mathbb{L}(\Omega) = \Big\{\widetilde{e}(\cdot) \in L_2(\Omega, R_s^{m\times m}) : \int\limits_\Omega \widetilde{e}(x)\, dx = 0\Big\}. \tag{1.2}$$

The orthogonal complement is the set of all elements of $L_2(\Omega, R_s^{m\times m})$ independent of $x \in \Omega$. It is obvious that

$$\mathbb{L}_0(\Omega) = \{\widetilde{e}(\cdot) \in \mathbb{L}(\Omega) : \widetilde{e}(x) = e(\nabla u(x)),\, u \in \mathbb{X}(\Omega)\}, \tag{1.3}$$

is a closed linear subspace of $\mathbb{L}(\Omega)$. Here,

$$\mathbb{X}(\Omega) = \overset{\circ}{W}{}^1_2(\Omega, R^m), \quad e_{ij}(\nabla u) = \frac{1}{2}(u^i_{x_j} + u^j_{x_i}), \quad i, j = 1, \ldots, m.$$

We introduce the functional

$$\widetilde{I}[\widetilde{e}, \chi, t, \Omega] = \int\limits_\Omega \{\chi(x)(\widetilde{F}^+(\widetilde{e}(x)) + t) + (1 - \chi(x))\widetilde{F}^-(\widetilde{e}(x))\}\, dx, \tag{1.4}$$

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where $\widetilde{e}(\cdot) \in L_2(\Omega, R_s^{m\times m})$, $\chi(\cdot) \in \mathbb{Z}(\Omega)$, $\widetilde{F}^{\pm}(\widetilde{e}) = \langle A^{\pm}(\widetilde{e} - \zeta^{\pm}), \widetilde{e} - \zeta^{\pm}\rangle$, $\zeta^{\pm} \in R^{m\times m}$, $t \in R^1$, $\mathbb{Z}(\Omega)$ is the set of all measurable characteristic functions, $A^{\pm}$ are linear symmetric positive definite mappings from $R_s^{m\times m}$ to $R_s^{m\times m}$. If we restrict ourselves to elements $\widetilde{e} \in \mathbb{L}_0(\Omega)$, then the functional (1.4) becomes the traditional energy functional of a two-phase medium

$$I[u, \chi, t, \Omega] = \int\limits_{\Omega} \{\chi(F^{+}(\nabla u) + t) + (1 - \chi)F^{-}(\nabla u)\}\, dx, \tag{1.5}$$

where $u \in \mathbb{X}(\Omega)$, $\chi \in \mathbb{Z}(\Omega)$, $F^{\pm}(\nabla u) = \langle A^{\pm}(e(\nabla u) - \zeta^{\pm}), e(\nabla u) - \zeta^{\pm}\rangle$, $u(x)$ is the displacement field, $\chi(x)$ determines the phase distribution, $\zeta^{\pm}$ are the residual strain tensors, $A^{\pm}$ are the tensors of moduli of elasticity, and t denotes the temperature. If the functional (1.4) is considered only for $\widetilde{e} \in \mathbb{L}(\Omega)$, then we obtain the functional, referred to as the *model energy functional* of a two-phase medium, that can be regarded as an extension of the functional (1.5).

We are interested in two variational problems: the model problem on equilibrium of a composite medium

$$\widetilde{I}[\widetilde{e}^{\chi}, \chi, t, \Omega] = \inf_{\widetilde{e} \in \mathbb{L}(\Omega)} \widetilde{I}[\widetilde{e}, \chi, t, \Omega], \quad \widetilde{e}^{\chi} \in \mathbb{L}(\Omega), \tag{1.6}$$

and the model problem on equilibrium of a two-phase medium

$$\widetilde{I}[\widehat{e}_t, \widehat{\chi}_t, t, \Omega] = \inf_{\widetilde{e} \in \mathbb{L}(\Omega), \chi \in \mathbb{Z}(\Omega)} \widetilde{I}[\widetilde{e}, \chi, t, \Omega], \quad \widehat{e}_t \in \mathbb{L}(\Omega), \quad \widehat{\chi}_t \in \mathbb{Z}(\Omega). \tag{1.7}$$

It turns out that the problems (1.6) and (1.7) are considerably simpler than those generated by the functional (1.5), but their solutions inherit many properties of the solutions to the problem with the functional (1.5).

It is convenient to use the following representation of the functional (1.4) for $\widetilde{e} \in \mathbb{L}(\Omega)$:

$$\widetilde{I}[\widetilde{e}, \chi, t, \Omega] = \Xi[\widetilde{e}, \chi, \Omega] + |\Omega|(t - t^*) + |\Omega|\langle A^{-}\zeta^{-}, \zeta^{-}\rangle, \tag{1.8}$$

where

$$\Xi[\widetilde{e}, \chi, \Omega] = \int\limits_{\Omega} \{\langle(\chi(x)A^{+} + (1 - \chi(x))A^{-})\widetilde{e}(x), \widetilde{e}(x)\rangle - 2(\chi(x) - Q)\langle\widetilde{e}(x), \xi\rangle\}\, dx,$$

$$\xi = [A\zeta], \quad Q = \frac{1}{|\Omega|} \int\limits_{\Omega} \chi(x)\, dx, \quad t^* = -[\langle A\zeta, \zeta\rangle].$$

Hereinafter, we use the notation $[\gamma] = \gamma^{+} - \gamma^{-}$.

The paper is organized as follows. We obtain an explicit solution to the problem (1.6) in Section 2 and to the problem (1.7) in Section 3. The solutions to the problem (1.7) are compared with the equilibrium states of a two-phase medium (the minimizers of the functional (1.5)) in Section 4. In Section 5, we consider situations where the functional (1.5) has no minimizers.

2 Model Problem in Composite Media

We begin with the following auxiliary assertion about the solvability of the variational problem for the functional $\Xi[\widetilde{e}, \chi, \Omega]$ in (1.8).

Lemma 2.1. *For every $\chi \in \mathbb{Z}(\Omega)$ the problem*

$$\Xi[\widetilde{e}^{\chi}, \chi, \Omega] = \inf_{\widetilde{e} \in \mathbb{L}(\Omega)} \Xi[\widetilde{e}, \chi, \Omega], \quad \widetilde{e}^{\chi} \in \mathbb{L}(\Omega), \tag{2.1}$$

has a unique solution

$$\widetilde{e}^{\chi} = \big(\chi(1-Q)(A^+)^{-1} - (1-\chi)Q(A^-)^{-1}\big)\xi - Q(1-Q)(\chi(A^+)^{-1} + (1-\chi)(A^-)^{-1})(Q(A^+)^{-1} + (1-Q)(A^-)^{-1})^{-1}[A^{-1}]\xi; \tag{2.2}$$

moreover,

$$\Xi[\widetilde{e}^{\chi}, \chi, \Omega] = |\Omega|\widetilde{g}(Q), \tag{2.3}$$

where $g(Q) = -Q(1-Q)\langle\overline{\xi}(Q), \xi\rangle$ *and*

$$\overline{\xi}(Q) = ((1-Q)(A^+)^{-1} + Q(A^-)^{-1})\xi - Q(1-Q)[A^{-1}](Q(A^+)^{-1} + (1-Q)(A^-)^{-1})^{-1}[A^{-1}]\xi.$$

Proof. The problem (2.1) is uniquely solvable because the quadratic functional $\Xi[\cdot, \chi, \Omega]$ is strictly convex. The solution satisfies the identity

$$((\chi A^+ + (1-\chi)A^-)\widetilde{e}^{\chi} - (\chi - Q)\xi, h)_{L_2(\Omega, R_s^{m\times m})} = 0 \quad \forall\, h \in \mathbb{L}(\Omega). \tag{2.4}$$

Therefore, there exists a matrix $\mu \in R_s^{m\times m}$ such that

$$(\chi A^+ + (1-\chi)A^-)\widetilde{e}^{\chi} = \mu + (\chi - Q)\xi. \tag{2.5}$$

Since for the characteristic functions χ we have

$$(\chi A^+ + (1-\chi)A^-)^{-1} = \chi(A^+)^{-1} + (1-\chi)(A^-)^{-1},$$

from (2.5) it follows that

$$\widetilde{e}^{\chi} = \chi(A^+)^{-1}(\mu + (1-Q)\xi) + (1-\chi)(A^-)^{-1}(\mu - Q\xi). \tag{2.6}$$

Since $\widetilde{e}^{\chi} \in \mathbb{L}(\Omega)$, we can integrate the last equality over Ω:

$$\mu = -Q(1-Q)(Q(A^+)^{-1} + (1-Q)(A^-)^{-1})^{-1}[A^{-1}]\xi. \tag{2.7}$$

Combining (2.6) with (2.7), we obtain (2.2). From (2.4) with $h = \widetilde{e}^{\chi}$ we find

$$\Xi[\widetilde{e}^{\chi}, \chi, \Omega] = -\int\limits_{\Omega} (\chi - Q)\langle\xi, \widetilde{e}^{\chi}\rangle\, dx.$$

By (2.6), we have

$$\langle e^{\chi}, \xi\rangle = \langle(\chi(A^+)^{-1} + (1-\chi)(A^-)^{-1})\xi, \mu\rangle + \langle((1-Q)\chi(A^+)^{-1} - Q(1-\chi)(A^-)^{-1})\xi, \xi\rangle.$$

Consequently,

$$(\chi - Q)\langle\widetilde{e}^{\chi}, \xi\rangle = \chi(1-Q)\langle(A^+)^{-1}\xi, \mu\rangle - (1-\chi)Q\langle(A^-)^{-1}\xi, \mu\rangle + \chi(1-Q)^2\langle(A^+)^{-1}\xi, \xi\rangle + (1-\chi)Q^2\langle(A^-)^{-1}\xi, \xi\rangle.$$

Thus,

$$|\Omega|^{-1}\Xi[\widetilde{e}^{\chi}, \chi, \Omega] = -Q(1-Q)\{\langle[A^{-1}]\xi, \mu\rangle + \langle((1-Q)(A^+)^{-1} + Q(A^-)^{-1})\xi, \xi\rangle\}.$$

The last equality and (2.7) imply (2.3). $\square$

In the variational elasticity problems for composite media, the passage from the space $\mathbb{L}_0(\Omega)$ to the space $\mathbb{L}(\Omega)$ is often used to obtain estimates [1].

Lemma 2.2. *For $g(Q)$ the following assertions hold*:

$$
\begin{aligned}
&1)\quad \widetilde{g}(\cdot) \in C^\infty[0,1], \quad \widetilde{g}(0) = \widetilde{g}(1) = 0,\\
&2)\quad \widetilde{g}(Q) = 0, \quad \xi = 0, \quad Q \in [0,1], \quad \widetilde{g}(Q) < 0, \quad \xi \neq 0, \quad Q \in (0,1), \qquad (2.8)\\
&3)\quad \widetilde{g}(\cdot) \ \textit{is strictly convex for} \ \xi \neq 0.
\end{aligned}
$$

Proof. The first assertion and a part of the second one are obvious consequences of formula (2.3) for $\widetilde{g}(Q)$. For $Q \in (0,1)$ the function χ is not a constant identically. Therefore, there is $\varphi \in L_2(\Omega)$ with zero mean value such that

$$\int_\Omega \chi(x)\varphi(x)\,dx > 0.$$

Then for all sufficiently small $\varepsilon > 0$ and $\xi \neq 0$ we have the inequality $\Xi[\widetilde{e}_\varepsilon, \chi, \Omega] < 0$, where $\widetilde{e}_\varepsilon(\cdot) = \varepsilon\varphi(\cdot)\xi \in \mathbb{L}(\Omega)$, which implies that the function $\widetilde{g}(Q)$ is negative for $Q \in (0,1)$, $\xi \neq 0$.

Let us prove that $\widetilde{g}(Q)$ is convex. Assume that $K = (0,1)^m$ is the unlit cube in R^m, $K^\lambda = \{x \in K : x_m \in (0,\lambda)\}$, $K^{1-\lambda} = \{x \in K : x_m \in (1-\lambda, 1)\}$, $\lambda \in (0,1)$. For fixed $Q_0, Q_1 \in [0,1]$, $Q_0 < Q_1$, and any λ we construct a function $\chi_\lambda \in \mathbb{Z}'(\Omega)$ such that

$$Q_0 = \frac{1}{|K^\lambda|}\int_{K^\lambda} \chi_\lambda\,dx, \quad Q_1 = \frac{1}{|K^{1-\lambda}|}\int_{K^{1-\lambda}} \chi_\lambda\,dx.$$

Since $|K^\lambda| = \lambda$, $|K^{1-\lambda}| = 1-\lambda$, $|K| = 1$, for $Q_\lambda = \lambda Q_0 + (1-\lambda)Q_1$ it is obvious that

$$Q_\lambda = \frac{1}{|K|}\int_K \chi_\lambda(x)\,dx.$$

Since for every λ the space

$$\mathbb{L}^\lambda(K) = \left\{\widetilde{e}(\cdot) \in L_2(K, R_s^{m\times m}) : \int_{K^\lambda} \widetilde{e}(x)\,dx = \int_{K^{1-\lambda}} \widetilde{e}(x)\,dx = 0\right\}$$

is narrower than the space $\mathbb{L}(K)$, we have

$$
\begin{aligned}
\widetilde{g}(\lambda Q_0 + (1-\lambda)Q_1) &= \widetilde{g}(Q_\lambda) = \inf_{\widetilde{e}\in\mathbb{L}(K)} \Xi[\widetilde{e}, \chi_\lambda, K] \leqslant \inf_{\widetilde{e}\in\mathbb{L}^\lambda(K)} \Xi[\widetilde{e}, \chi_\lambda, K]\\
&= \inf_{\widetilde{e}\in\mathbb{L}(K^\lambda)} \Xi[\widetilde{e}, \chi_\lambda, K^\lambda] + \inf_{\widetilde{e}\in\mathbb{L}(K^{1-\lambda})} \Xi[\widetilde{e}, \chi_\lambda, K^{1-\lambda}] = \lambda\widetilde{g}(Q_0) + (1-\lambda)\widetilde{g}(Q_1), \qquad (2.9)
\end{aligned}
$$

which implies the convexity of $\widetilde{g}(Q)$.

We prove the strict convexity of $\widetilde{g}(Q)$ for $\xi \neq 0$. Assume the contrary. Then there is an interval $[Q_0, Q_1] \subset [0,1]$, $Q_0 < Q_1$, and a number $\lambda_0 \in (0,1)$ such that

$$\widetilde{g}(\lambda_0 Q_0 + (1-\lambda_0)Q_1) = \lambda_0\widetilde{g}(Q_0) + (1-\lambda_0)\widetilde{g}(Q_1).$$

From (2.9) and the above identity it follows that for χ_λ

$$\inf_{\widetilde{e}\in\mathbb{L}(K)} \Xi[\widetilde{e},\chi_{\lambda_0},\Omega] = \inf_{\widetilde{e}\in\mathbb{L}^{\lambda_0}(K)} \Xi[\widetilde{e},\chi_{\lambda_0},\Omega]. \tag{2.10}$$

By (2.2), a unique minimizer of the left-hand side of (2.10) has the form

$$\begin{aligned}\widetilde{e}^{\chi_{\lambda_0}} &= (\chi_{\lambda_0}(1-Q_{\lambda_0})(A^+)^{-1} - (1-\chi_{\lambda_0})Q_{\lambda_0}(A^-)^{-1})\xi - Q_{\lambda_0}(1-Q_{\lambda_0})(\chi_{\lambda_0}(A^+)^{-1} \\ &\quad + (1-\chi_{\lambda_0})(A^-)^{-1})(Q_{\lambda_0}(A^+)^{-1} + (1-Q_{\lambda_0})(A^-)^{-1})^{-1}[A^{-1}]\xi, \\ Q_{\lambda_0} &= \lambda_0 Q_0 + (1-\lambda_0)Q_1.\end{aligned} \tag{2.11}$$

To construct a minimizer of the right-hand side of (2.10), we consider two problems

$$\begin{aligned}&\Xi[\widetilde{e}^{\chi_{\lambda_0}}_{\lambda_0},\chi_{\lambda_0},K^{\lambda_0}] = \inf_{\widetilde{e}\in\mathbb{L}(K^{\lambda_0})} \Xi[\widetilde{e},\chi_{\lambda_0},K^{\lambda_0}], \quad \widetilde{e}^{\chi_{\lambda_0}}_{\lambda_0} \in \mathbb{L}(K^{\lambda_0}), \\ &\Xi[\widetilde{e}^{\chi_{\lambda_0}}_{1-\lambda_0},\chi_{\lambda_0},K^{1-\lambda_0}] = \inf_{\widetilde{e}\in\mathbb{L}(K^{1-\lambda_0})} \Xi[\widetilde{e},\chi_{\lambda_0},K^{1-\lambda_0}], \quad \widetilde{e}^{\chi_{\lambda_0}}_{1-\lambda_0} \in \mathbb{L}(K^{1-\lambda_0}).\end{aligned}$$

For $\widetilde{e}^{\chi_{\lambda_0}}_{\lambda_0}$ and $\widetilde{e}^{\chi_{\lambda_0}}_{1-\lambda_0}$ we have a formula similar to (2.11). By the next to last equality in (2.9), the minimizer $\widetilde{e}^{\chi_{\lambda_0}}$ of the right-hand side of (2.10) has the form

$$\widetilde{e}^{\chi_{\lambda_0}}(x) = \begin{cases} \widetilde{e}^{\chi_{\lambda_0}}_{\lambda_0}(x), & x \in K^{\lambda_0}, \\ \widetilde{e}^{\chi_{\lambda_0}}_{1-\lambda_0}(x), & x \in K^{1-\lambda_0}. \end{cases}$$

By (2.10), $\widetilde{e}^{\chi_{\lambda_0}}(x)$ is also a minimizer of the left-hand side of (2.10). By uniqueness, the integrals over K^{λ_0} and $K^{1-\lambda_0}$ on the right-hand side of (2.11) vanish:

$$\begin{aligned}0 &= (Q_0(1-Q_{\lambda_0})(A^+)^{-1} - (1-Q_0)Q_{\lambda_0}(A^-)^{-1})\xi - Q_{\lambda_0}(1-Q_{\lambda_0})(Q_0(A^+)^{-1} \\ &\qquad + (1-Q_0)(A^-)^{-1})(Q_{\lambda_0}(A^+)^{-1} + (1-Q_{\lambda_0})(A^-)^{-1})^{-1}[A^{-1}]\xi, \\ 0 &= (Q_1(1-Q_{\lambda_0})(A^+)^{-1} - (1-Q_1)Q_{\lambda_0}(A^-)^{-1})\xi - Q_{\lambda_0}(1-Q_{\lambda_0})(Q_1(A^+)^{-1} \\ &\qquad + (1-Q_1)(A^-)^{-1})(Q_{\lambda_0}(A^+)^{-1} + (1-Q_{\lambda_0})(A^-)^{-1})^{-1}[A^{-1}]\xi.\end{aligned}$$

Subtracting the first identity from the second one and canceling by $Q_1 - Q_0$, we get $\overline{\xi}(Q_{\lambda_0}) = 0$ which contradicts the second assertion in (2.8). □

Example 2.1. In the one-dimensional case,

$$F^{\pm}(\widetilde{e}) = a_{\pm}(\widetilde{e} - c_{\pm}), \quad a_{\pm}, c_{\pm} \in R^1, \quad a_{\pm} > 0, \quad \widetilde{e}(\cdot) \in L_2(0,l), \quad \int_0^l \widetilde{e}(x)\,dx = 0. \tag{2.12}$$

From (2.3) it follows that

$$\widetilde{g}(Q) = -\frac{[ac]^2 Q(1-Q)}{a_- Q + a_+(1-Q)}, \quad Q \in [0,1]. \tag{2.13}$$

We indicate necessary and sufficient conditions for the function $\widetilde{g}$ defined by (2.3) to be equal to the function

$$\widetilde{g}_0(Q) = -\frac{Q(1-Q)}{\alpha Q + \beta(1-Q)}, \quad \alpha, \beta > 0. \tag{2.14}$$

Since

$$\widetilde{g}_0'(0) = -\frac{1}{\beta}, \quad \widetilde{g}_0'(1) = \frac{1}{\alpha}, \quad \widetilde{g}'(0) = -\langle (A^+)^{-1}\xi, \xi\rangle, \quad \widetilde{g}'(1) = \langle (A^-)^{-1}\xi, \xi\rangle, \tag{2.15}$$

this is possible only if

$$\widetilde{g}_0(Q) = -\frac{\langle (A^+)^{-1}\xi, \xi\rangle \langle (A^-)^{-1}\xi, \xi\rangle Q(1-Q)}{\langle (A^+)^{-1}\xi, \xi\rangle Q + \langle (A^-)^{-1}\xi, \xi\rangle (1-Q)}. \tag{2.16}$$

Since formula (2.16) is simpler than (2.3), we can calculate

$$\widetilde{g}_0''(Q) = \frac{2\langle (A^+)^{-1}\xi, \xi\rangle^2 \langle (A^-)^{-1}\xi, \xi\rangle^2}{(\langle (A^+)^{-1}\xi, \xi\rangle Q + \langle (A^-)^{-1}\xi, \xi\rangle (1-Q))^3}. \tag{2.17}$$

Since the second order derivative is positive for $\xi \neq 0$, the function $\widetilde{g}_0$ is strictly convex.

Lemma 2.3. *The function $\widetilde{g}(Q)$ defined by* (2.3) *coincides with the function $\widetilde{g}_0(Q)$ defined by* (2.16) *if and only if for some λ*

$$(A^+)^{-1}\xi = \lambda (A^-)^{-1}\xi. \tag{2.18}$$

Proof. By (2.3), the equality $\widetilde{g}(Q) = \widetilde{g}_0(Q)$, $Q \in [0,1]$, is equivalent to the equality

$$\langle ((1-Q)(A^+)^{-1} + Q(A^-)^{-1})\xi, \xi\rangle \langle (Q(A^+)^{-1} + (1-Q)(A^-)^{-1})\xi, \xi\rangle - \langle (A^+)^{-1}\xi, \xi\rangle \langle (A^-)^{-1}\xi, \xi\rangle$$
$$= Q(1-Q)\langle (Q(A^+)^{-1} + (1-Q)(A^-)^{-1})^{-1}[A^{-1}]\xi, [A^{-1}]\xi\rangle \langle (Q(A^+)^{-1} + (1-Q)(A^-)^{-1})\xi, \xi\rangle.$$

Collecting like terms, we obtain the equality

$$\langle [A^{-1}]\xi, \xi\rangle^2 = \langle B^{-1}[A^{-1}]\xi, [A^{-1}]\xi\rangle \langle B\xi, \xi\rangle,$$

where $B = B(Q) = Q(A^+)^{-1} + (1-Q)(A^-)^{-1}$, which is equivalent to the identity

$$\langle B^{-1/2}[A^{-1}]\xi, B^{1/2}\xi\rangle^2 = |B^{-1/2}[A^{-1}]\xi|^2 \, |B^{1/2}\xi|^2.$$

Since the Schwarz inequality becomes equality only on proportional vectors, the assertion of the lemma is equivalent to the existence of $\gamma(Q)$ such that $B^{-1/2}[A^{-1}]\xi = \gamma(Q)B^{1/2}\xi$, $Q \in (0,1)$. Thus, we have

$$[A^{-1}]\xi = \gamma(Q)(Q(A^+)^{-1} + (1-Q)(A^-)^{-1})\xi.$$

Hence

$$\gamma(Q) = \frac{\langle [A^{-1}]\xi, \xi\rangle}{Q\langle (A^+)^{-1}\xi, \xi\rangle + (1-Q)\langle (A^-)^{-1}\xi, \xi\rangle}.$$

Substituting $\gamma(Q)$ into the above equality, we find

$$\langle (A^-)^{-1}\xi, \xi\rangle (A^+)^{-1}\xi = \langle (A^+)^{-1}\xi, \xi\rangle (A^-)^{-1}\xi,$$

which means that $(A^+)^{-1}\xi$ and $(A^-)^{-1}\xi$ are proportional. □

From the above lemmas and formula (1.8) we obtain the following assertion.

Theorem 2.1. *For each function $\chi \in \mathbb{Z}(\Omega)$ the problem* (1.6) *is uniquely solvable. The solution $\widetilde{e}^{\chi}$ is given by formula* (2.3), *and the equilibrium energy is given by*

$$\widetilde{I}[\widetilde{e}^{\chi}, \chi, t, \Omega] = |\Omega| \widetilde{G}(Q, t), \tag{2.19}$$

where $\widetilde{G}(Q,t) = (t - t^*)Q + \widetilde{g}(Q) + \langle A^- \zeta^-, \zeta^- \rangle$, $t \in R^1$, $Q \in [0, 1]$.

We note that the equilibrium energy is independent of a given function χ of the phase distribution, but depends on the volume fraction of the domain occupied by the phase labeled by " + ".

3 Model Problem in Two-Phase Media

We begin by considering the minimization problem

$$\widetilde{G}(\widetilde{Q}(t), t) = \min_{Q \in [0,1]} \widetilde{G}(Q, t), \quad \widetilde{Q}(t) \in [0, 1], \quad t \in R^1. \tag{3.1}$$

Since the function $\widetilde{G}(\cdot, t)$ is strictly convex and

$$\widetilde{G}_Q(0, 1) = t - \widetilde{t}_+, \quad \widetilde{G}_Q(1, t) = t - \widetilde{t}_-, \quad \widetilde{t}_\pm = t^* \pm \langle (A^\pm)^{-1} \xi, \xi \rangle, \tag{3.2}$$

for the problem (3.1) the following assertions hold [2, Lemma 2.2]:

$$\begin{aligned} &\text{1) in the case } \xi \neq 0, \\ &\widetilde{t}_- < \widetilde{t}_+,\ \widetilde{Q}(t) = 1 \text{ for } t \leqslant \widetilde{t}_-,\ \widetilde{Q}(t) = 0 \text{ for } t \geqslant \widetilde{t}_+,\ \widetilde{Q}(t) \text{ is a unique solution to} \\ &\text{the equation } \widetilde{G}_Q(Q, t) = 0 \text{ for } t \in (\widetilde{t}_-, \widetilde{t}_+), \\ &\text{2) in the case } \xi = 0, \\ &\widetilde{t}_- = \widetilde{t}_+ = t^*,\ \widetilde{Q}(t) = 1 \text{ for } t < t^*,\ \widetilde{Q}(1) = 0 \text{ for } t > t^*, \text{ the values of } \widetilde{Q}(t^*) \\ &\text{occupy the segment } [0, 1]. \end{aligned} \tag{3.3}$$

We assume that $\xi \neq 0$, $t_1, t_2 \in [\widetilde{t}_-, \widetilde{t}_+]$, $t_2 > t_1$. From (3.2) and (3.3) it follows that $\widetilde{G}_Q(\widetilde{Q}(t_2), t_2) = \widetilde{G}_Q(\widetilde{Q}(t_1), t_1)$. Then

$$\widetilde{g}'(\widetilde{Q}(t_2)) - \widetilde{g}'(\widetilde{Q}(t_1)) = -(t_2 - t_1). \tag{3.4}$$

Since the function $\widetilde{g}'(Q)$ is strictly monotonically increasing on $[0, 1]$, from (3.4) we find

$$\widetilde{Q}(t_2) < \widetilde{Q}(t_1). \tag{3.5}$$

Since the bounded function $\widetilde{Q}(t)$, $t \in [\widetilde{t}_-, \widetilde{t}_+]$, is monotone, $\widetilde{Q}(t_1) \to Q$ as $t_1 \to t_2$. By (3.4), we have $Q = \widetilde{Q}(t_2)$ which implies the continuity of $\widetilde{Q}(\cdot)$ on $[\widetilde{t}_-, \widetilde{t}_+]$ and, consequently, on R^1.

Using the positivity of $\widetilde{g}''(Q)$, $Q \in [0, 1]$, and applying the implicit function theorem to the equation $\widetilde{G}_Q(Q, t) = 0$, we prove that $\widetilde{Q}(t)$, $t \in [\widetilde{t}_-, \widetilde{t}_+]$, is infinitely differentiable and the following equality holds:

$$\widetilde{Q}'(t) = -\frac{1}{\widetilde{g}''(\widetilde{Q}(t))}, \quad t \in [\widetilde{t}_-, \widetilde{t}_+], \tag{3.6}$$

which can be regarded as an ordinary first order differential equation for $\widetilde{Q}(t)$ with separated. variables.

Now, we study the solvability of the problem (1.7). In what follows,

$$i_{\min}(t) = \begin{cases} t + \langle A^+\zeta^+, \zeta^+\rangle, & t \leqslant t^*, \\ \langle A^-\zeta^-, \zeta^-\rangle, & t \geqslant t^*, \end{cases} \tag{3.7}$$

and the *equilibrium energy* is defined by

$$\widetilde{i}(t) = \inf_{\widetilde{e}\in\mathbb{L}(\Omega),\chi\in\mathbb{Z}(\Omega)} |\Omega|^{-1}\widetilde{I}[\widetilde{e},\chi,t,\Omega] = \inf_{\chi\in\mathbb{Z}(\Omega)}\inf_{\widetilde{e}\in\mathbb{L}(\Omega)} |\Omega|^{-1}\widetilde{I}[\widetilde{e},\chi,t,\Omega]$$
$$= \inf_{\chi\in\mathbb{Z}(\Omega)} |\Omega|^{-1}\widetilde{I}[\widetilde{e}^{\chi},\chi,t,\Omega] = \min_{Q\in[0,1]} \widetilde{G}(Q,t) = \widetilde{G}(\widetilde{Q}(t),t). \tag{3.8}$$

Theorem 3.1. *For every $t \in R^1$ the problem* (1.7) *is solvable, and the set of solutions is exhausted by the pairs $(\widehat{e}_t, \widehat{\chi}_t)$, where $\widehat{e}_t = \widetilde{e}^{\widehat{\chi}_t}$ is defined by* (2.2) *and $\widehat{\chi}_t$ is an arbitrary function such that*

$$\frac{1}{|\Omega|}\int\limits_{\Omega} \widehat{\chi}_t(x)\,dx = \widetilde{Q}(t),$$

where $\widetilde{Q}(t)$ is a solution to the problem (3.1). *The equilibrium energy $\widetilde{i}(t)$ possesses the following properties*:

1) *in the case $\xi \neq 0$, $\widetilde{i}(\cdot) \in C^1(R^1)$ is strictly concave on $[\widetilde{t}_-, \widetilde{t}_+]$, $\widetilde{i}'(t) = \widetilde{Q}(t)$, $\widetilde{i}(t) = i_{\min}(t)$ for $t \notin (\widetilde{t}_-, \widetilde{t}_+)$,* (3.9)

2) *in the case $\xi = 0$, $\widetilde{i}(t) = i_{\min}(t)$.*

Proof. By (3.8), the problem (1.7) is solvable and the set of solutions possesses the required properties.

By (3.8) and (2.19), the equilibrium energy $\widetilde{i}(\cdot)$ is the infimum with respect to $Q \in [0,1]$ of the family of linear functions $\widetilde{G}(Q,\cdot)$. Therefore, it is concave. From (3.3) and (2.19) for $\widetilde{G}(Q,t)$ we obtain the identities in (3.9) for $\widetilde{i}(t)$, $t \notin (\widetilde{t}_-, \widetilde{t}_+)$.

By properties of the equilibrium energy, the following assertions hold [3].

1) There exists a set $\mathscr{L} \subset R^1$ with at most countable complement, where the function $\widetilde{i}(t)$ has the monotonically decreasing continuous derivative $\widetilde{i}'(t) \in [0,1]$ on $\mathscr{L}$.

2) At each point $t \in R^1 \setminus \mathscr{L}$, the function $\widetilde{i}(t)$ has one-sided derivatives $1 \geqslant \widetilde{i}'(t-0) > \widetilde{i}'(t+0) \geqslant 0$; moreover, $\widetilde{i}'(t-0) = \lim\limits_{\tau\in\mathscr{L},\tau<t,\tau\to t} \widetilde{i}'(\tau)$ and $\widetilde{i}'(t+0) = \lim\limits_{\tau\in\mathscr{L},\tau>t,\tau\to t} \widetilde{i}'(\tau)$.

Using these assertions, we can complete the proof of (3.9).

For any t and t' it is obvious that

$$\widetilde{i}(t') \leqslant |\Omega|^{-1}\widetilde{I}[\widehat{e}_t, \widehat{\chi}_t, t'] = |\Omega|^{-1}\widetilde{I}[\widehat{e}_t, \widehat{\chi}_t, t] + (t'-t)\widetilde{Q}(t) = \widetilde{i}(t) + (t'-t)\widetilde{Q}(t).$$

Therefore,

$$\frac{\widetilde{i}(t') - \widetilde{i}(t)}{t'-t} \leqslant \widetilde{Q}(t), \quad t' > t,$$

$$\frac{\widetilde{i}(t) - \widetilde{i}(t')}{t-t'} \geqslant \widetilde{Q}(t), \quad t > t'.$$

For $t \in \mathscr{L}$, passing to the limit in the above inequalities as $t' \to t$, we get

$$\widetilde{i}'(t) = \widetilde{Q}(t) \quad \forall\, t \in \mathscr{L}. \tag{3.10}$$

By assertions 1), 2) and the continuity of $\widetilde{Q}(t)$, we have $\mathscr{L} = R^1$ and formula (3.10) holds for all t. This means that the equilibrium energy $\widetilde{i}(\cdot)$ is continuously differentiable and $\widetilde{i}'(t)$ is strictly monotonically decreasing for $t \in [\widetilde{t}_-, \widetilde{t}_+]$, which implies the strict concavity of $\widetilde{i}(t)$ on this interval. □

Example 3.1. We consider the particular case

$$A^+ = A^- = A, \quad [\zeta] \neq 0. \tag{3.11}$$

Taking into account the identity $\xi = A[\zeta]$, from (2.2), (2.3), (2.17), (3.6), (3.10) we find

$$\begin{gathered} \widetilde{e}^\chi = (\chi - Q)[\zeta], \quad \widetilde{g}(Q) = -Q(1-Q)\langle A[\zeta], [\zeta]\rangle, \quad \widetilde{Q}(t) = \frac{\widetilde{t}_+ - t}{\widetilde{t}_+ - \widetilde{t}_-}, \\ \widetilde{i}(t) = -\frac{1}{2}\frac{(\widetilde{t}_+ - t)^2}{\widetilde{t}_+ - \widetilde{t}_-} + \langle A\zeta^-, \zeta^-\rangle, \quad t \in [\widetilde{t}_-, \widetilde{t}_+], \quad \widetilde{t}_\pm = -2\langle A[\zeta], \zeta_\mp\rangle. \end{gathered} \tag{3.12}$$

4 Equilibrium States of Two-Phase Media for the Model and Original Problems

By the *original problem* in a two-phase medium we mean the variational problem of minimizing the functional (1.5)

$$I[\overline{u}_t, \overline{\chi}_t, t] = \inf_{u \in \mathbb{X}(\Omega), \chi \in \mathbb{Z}(\Omega)} I[u, \chi, t], \quad \overline{u}_t \in \mathbb{X}(\Omega), \quad \overline{\chi}_t \in \mathbb{Z}(\Omega). \tag{4.1}$$

It is obvious that

$$\inf_{u \in \mathbb{X}(\Omega), \chi \in \mathbb{Z}(\Omega)} I[u, \chi, t, \Omega] = \inf_{\widetilde{e} \in \mathbb{L}_0(\Omega), \chi \in \mathbb{Z}(\Omega)} \widetilde{I}[\widetilde{e}, \chi, t, \Omega]. \tag{4.2}$$

An analog of the representation (1.8) of the functional (1.5) has the form

$$I[u, \chi, t, \Omega] = \Lambda[u, \chi, \Omega] + |\Omega|(t - t^*)Q + |\Omega|\langle A^-\zeta^-, \zeta^-\rangle, \tag{4.3}$$

where

$$Q = \frac{1}{|\Omega|}\int\limits_\Omega \chi\, dx,$$

$$\Lambda[u, \chi, \Omega] = \int\limits_\Omega \{\langle(\chi A^+ + (1-\chi)A^-)e(\nabla u), e(\nabla u)\rangle - 2(\chi - Q)\langle e(\nabla u), \xi\rangle\}\, dx.$$

For all Ω there exists a function $g(Q)$, $Q \in [0,1]$, such that [4]

$$\begin{gathered} \inf_{u \in \mathbb{X}(\Omega), \chi \in \mathbb{Z}_Q(\Omega)} I[u, \chi, t, \Omega] = |\Omega| g(Q), \\ \mathbb{Z}_Q(\Omega) = \left\{\chi \in \mathbb{Z}(\Omega) : \frac{1}{|\Omega|}\int\limits_\Omega \chi\, dx = Q,\ Q \in [0,1]\right\}. \end{gathered} \tag{4.4}$$

The function $g(Q)$ is convex, $g(0) = g(1) = 0$, $g(Q) \equiv 0$ for $\xi = 0$, $g(Q) < 0$ for $\xi \neq 0$, $Q \in (0,1)$.

By the *equilibrium energy* of a two-phase medium in the case of the original problem we mean

$$
\begin{aligned}
i(t) &= |\Omega|^{-1} \inf_{u\in\mathbb{X}(\Omega),\chi\in\mathbb{Z}(\Omega)} I[u,\chi,t,\Omega] \\
&= |\Omega|^{-1} \inf_{Q\in[0,1]} \inf_{u\in\mathbb{X}(\Omega),\chi\in\mathbb{Z}_Q(\Omega)} \{\Lambda[u,\chi,\Omega] + |\Omega|(t-t^*)Q + |\Omega|\langle A^-\zeta^-,\zeta^-\rangle\} = \min_{Q\in[0,1]} G(Q,t), \\
G(Q,t) &= (t-t^*)Q + g(Q) + \langle A^-\zeta^-,\zeta^-\rangle.
\end{aligned}
\tag{4.5}
$$

The function $i(t)$ has been studied less than its model counterpart (3.9) (cf. [5]) We know that $i(t)$ is concave and the following assertions hold:

1) in the case $\xi \neq 0$, there exist $t_- < t^* < t_+$ such that $i(t) = i_{\min}(t)$ for $t \notin (t_-,t_+)$ and $i(t) < t_{\min}(t)$ for $t \in (t_-,t_+)$, (4.6)

2) in the case $\xi = 0$, $i(t) = i_{\min}(t)$.

In addition, assertions 1), 2) in Section 3 hold for $i(t)$.

By (2.3), the function $\widetilde{g}(Q)$ is the same for all $\chi \in \mathbb{Z}_Q(\Omega)$. Then

$$
\begin{aligned}
|\Omega|\widetilde{g}(Q) &= \inf_{\widetilde{e}\in\mathbb{L}(\Omega)} \Xi[\widetilde{e},\chi,\Omega] = \inf_{\widetilde{e}\in\mathbb{L}(\Omega),\chi\in\mathbb{Z}_Q(\Omega)} \Xi[\widetilde{e},\chi,\Omega] \\
&\leqslant \inf_{\widetilde{e}\in\mathbb{L}_0(\Omega),\chi\in\mathbb{Z}_Q(\Omega)} \Xi[\widetilde{e},\chi,\Omega] = \inf_{u\in\mathbb{X}(\Omega),\chi\in\mathbb{Z}_Q(\Omega)} \Lambda[u,\chi,\Omega] = |\Omega|g(Q).
\end{aligned}
$$

Therefore,

$$
\widetilde{g}(Q) \leqslant g(Q) \quad \forall\, Q \in [0,1]. \tag{4.7}
$$

The equality in (4.7) is attained at $Q = 0$ and $Q = 1$.

Lemma 4.1. *We assume that $\widetilde{g}(Q_0) = g(Q_0)$ for some $A^\pm$, $\zeta^\pm$, $Q_0 \in [0,1]$. Then the set of solutions (u^Q,χ^Q) to the problem*

$$
\Lambda[u^Q,\chi^Q,\Omega] = \inf_{u\in\mathbb{X}(\Omega),\chi\in\mathbb{Z}_Q(\Omega)} \Lambda[u,\chi,\Omega], \quad u^Q \in \mathbb{X}(\Omega), \quad \chi^Q \in \mathbb{Z}_Q(\Omega) \tag{4.8}
$$

coincides with the set of solutions to the equation

$$
\begin{aligned}
e(\nabla u^Q) &= (\chi^Q(1-Q)(A^+)^{-1} - (1-\chi^Q)Q(A^-)^{-1})\xi - Q(1-Q)(\chi^Q(A^+)^{-1} \\
&\quad + (1-\chi^Q)(A^-)^{-1})(Q(A^+)^{-1} + (1-Q)(A^-)^{-1})^{-1}[A^{-1}]\xi, \\
u^Q &\in \mathbb{X}(\Omega), \quad \chi^Q \in \mathbb{Z}_Q(\Omega),
\end{aligned}
\tag{4.9}
$$

with $Q = Q_0$.

Proof. Since $|\Omega|\widetilde{g}(Q_0) = \Xi[\widetilde{e}^\chi,\chi,\Omega]$ for all $\chi \in \mathbb{Z}_{Q_0}(\Omega)$ and

$$
|\Omega|g(Q_0) = \inf_{u\in\mathbb{X}(\Omega),\chi\in\mathbb{Z}_{Q_0}(\Omega)} \Lambda[u,\chi,\Omega] = \inf_{\widetilde{e}\in\mathbb{L}_0(\Omega),\chi\in\mathbb{Z}_{Q_0}(\Omega)} \Xi[\widetilde{e},\chi,\Omega],
$$

the equality $\widetilde{g}(Q_0) = g(Q_0)$ implies

$$
\Xi[\widetilde{e}^\chi,\chi,\Omega] = \inf_{\widetilde{e}\in\mathbb{L}_0(\Omega),\chi\in\mathbb{Z}_{Q_0}(\Omega)} \Xi[\widetilde{e},\chi,\Omega] \quad \forall\, \chi \in \mathbb{Z}_{Q_0}(\Omega), \tag{4.10}
$$

and the solvability of the problem (4.8) is equivalent to the solvability of the problem

$$\Xi[\widetilde{e}_0, \chi_0, \Omega] = \inf_{\widetilde{e}\in\mathbb{L}_0(\Omega), \chi\in\mathbb{Z}_{Q_0}(\Omega)} \Xi[\widetilde{e}, \chi, \Omega], \quad \widetilde{e}_0 \in \mathbb{L}_0(\Omega), \quad \chi_0 \in \mathbb{Z}_{Q_0}(\Omega). \tag{4.11}$$

If there exists a function $\chi_0 \in \mathbb{Z}_{Q_0}(\Omega)$ such that $\widetilde{e}_0 = \widetilde{e}^{\chi_0}$, then the solvability of the problem (4.10) implies the solvability of the problem (4.11), which implies

$$\Xi[\widetilde{e}^{\chi}, \chi, \Omega] = \Xi[\widetilde{e}_0, \chi_0, \Omega] \quad \forall\, \chi \in \mathbb{Z}_{Q_0}(\Omega).$$

Setting $\chi = \chi_0$ and taking into account the uniqueness of a solution to the problem (2.1), we have $\widetilde{e}^{\chi_0} \in \mathbb{L}_0(\Omega)$. □

By the relation

$$|\Omega|\widetilde{i}(t) = \inf_{\widetilde{e}\in\mathbb{L}(\Omega), \chi\in\mathbb{Z}(\Omega)} \widetilde{I}[\widetilde{e}, \chi, t, \Omega] \leqslant \inf_{\widetilde{e}\in\mathbb{L}_0(\Omega), \chi\in\mathbb{Z}(\Omega)} \widetilde{I}[\widetilde{e}, \chi, t, \Omega] = |\Omega| i(t),$$

we conclude that

$$\widetilde{i}(t) \leqslant i(t) \quad \forall\, t \in R^1. \tag{4.12}$$

From (4.12) we obtain the relations connecting the phase transition temperatures $t_\pm$ and $\widetilde{t}_\pm$ for the original and model problems

$$\widetilde{t}_- \leqslant t_- \leqslant t^* \leqslant t_+ \leqslant \widetilde{t}_+. \tag{4.13}$$

If $\xi = 0$, then all the inequalities in (4.13) becomes equalities and

$$\widetilde{i}(t) = i(t) = i_{\min}(t), \quad t \in R^1. \tag{4.14}$$

Lemma 4.2. *Let $\xi \neq 0$. Assume that for some t_0*

$$\widetilde{i}(t_0) = i(t_0). \tag{4.15}$$

Then the problem

$$G(\overline{Q}(t), t) = \inf_{Q\in[0,1]} G(Q, t), \quad \overline{Q}(t) \in [0, 1], \tag{4.16}$$

at $t = t_0$ has a unique solution $\overline{Q}(t_0)$ and

$$\overline{Q}(t_0) = \widetilde{Q}(t_0), \quad g(\overline{Q}(t_0)) = \widetilde{g}(\widetilde{Q}(t_0)). \tag{4.17}$$

Proof. By (3.9), for $\xi \neq 0$ the function $\widetilde{i}(t)$ is continuously differentiable. We prove that, under the assumption (4.15), the function $i(t)$ has the derivative $i'(t_0) = \widetilde{i}'(t_0)$. By (4.12) and (4.15), $i(t) - i(t_0) \geqslant \widetilde{i}(t) - \widetilde{i}(t_0)$ for all t. Consequently,

$$\frac{i(t) - i(t_0)}{t - t_0} \geqslant \frac{\widetilde{i}(t) - \widetilde{i}(t_0)}{t - t_0}, \quad t > t_0, \quad \frac{i(t) - i(t_0)}{t - t_0} \leqslant \frac{\widetilde{i}(t) - \widetilde{i}(t_0)}{t - t_0}, \quad t < t_0.$$

At each point $t = t_0$, the function $i(t)$ has the one-sided derivative $i'(t_0 \pm 0)$ satisfying the inequality $i'(t_0 - 0) \geqslant i'(t_0 + 0)$. Passing to the limit in the above relations as $t \to t_0$, we conclude that $i'(t_0 + 0) \geqslant \widetilde{i}'(t_0) \geqslant i'(t_0 - 0)$, the derivative $i'(t_0)$ exists and coincides with $\widetilde{i}'(t_0)$.

Let $\{u_n \in \mathbb{X}(\Omega),\, \chi_n \in \mathbb{Z}(\Omega)\}$ be the set of all sequences minimizing the functional (1.5) with $t = t_0$. Denote by $\check{\mathbb{Q}}(t_0)$ the union of sets of all concentration points for the number sequences

$$Q_n = \frac{1}{|\Omega|} \int\limits_{\Omega} \chi_n \, dx.$$

As known [6], if the derivative $i'(t_0)$ exists, this set consists of the single point $i'(t_0)$.

Let $\overline{Q}(t_0)$ be a solution to the problem (4.16), and let $\{u_n \in \mathbb{X}(\Omega), \chi_n \in \mathbb{Z}_{\overline{Q}(t_0)}(\Omega)\}$ be a minimizing sequence for the problem (4.4). By (4.5), this sequence also minimizes the functional (1.5) with $t = t_0$. Therefore, $\overline{Q}(t_0) \in \check{\mathbb{Q}}(t_0)$ and, consequently, $\overline{Q}(t_0) = i'(t_0)$.

Thus, if the derivative $i'(t_0)$ exists, then the problem (4.16) has a unique solution $\widehat{Q}(t_0) = i'(t_0)$. In this case, formula (3.10) implies the first identity in (4.17). Since

$$|\Omega| i(t_0) = g(\overline{Q}(t_0)) + (t_0 - t^*)\overline{Q}(t_0) + \langle A^- \zeta^-, \zeta^- \rangle,$$

$$|\Omega| \widetilde{i}(t_0) = \widetilde{g}(\widetilde{Q}(t_0)) + (t_0 - t^*)\widetilde{Q}(t_0) + \langle A^- \zeta^-, \zeta^- \rangle,$$

we obtain the second identity in (4.17). □

Theorem 4.1. *If the condition* (4.15) *holds, then the sets of solutions to the problem* (4.1) *with* $t = t_0$ *and Equation* (4.9) *with* $Q_0 = \widetilde{Q}(t_0)$ *coincide.*

It suffices to use (4.5) and Lemmas 4.2 and 4.1.

5 Equilibrium States of Original Problem

Based on the results obtained in the above sections, we show that the problem (4.1) has no solutions under certain restrictions on the parameters of the problem. If the condition (3.11) in [7] holds, then we can find an explicit formula for $g(Q)$ (cf. [8, formula (3.2)]). Analyzing this formula for $g(Q)$, we conclude that the equality $g(Q_0) = \widetilde{g}(Q_0)$ is valid for some $Q_0 \in (0,1)$ if and only if

$$[\zeta] = e(k \otimes c) \text{ for some } k, c \in R^m, \quad m \geqslant 2, \quad |k| = 1, \quad c \neq 0; \tag{5.1}$$

here, $(k \otimes c)_{ij} = k_i c_j$, $e(k \otimes c)_{ij} = \frac{1}{2}(k_i c_j + k_j c_i)$, $i, j = 1, \dots, m$, and the validity of the equality $g(Q_0) = \widetilde{g}(Q_0)$ at some point $Q_0 \in (0,1)$ implies that this equality $g(Q) = \widetilde{g}(Q)$ holds at all points $Q \in (0,1)$. Thus, the condition (5.1) can be regarded as a criterion for the functions $g(Q)$ and $\widetilde{g}(Q)$ to be equal.

Since $\widetilde{g}(Q)$ and $g(Q)$ coincide in view of (3.8) and (4.5), the equilibrium energies $\widetilde{i}(t)$ and $i(t)$ also coincide. By Lemma 4.2, this means that $\widetilde{g}(Q)$ and $g(Q)$ coincide. Therefore, the condition (5.1) is also a criterion for the coincidence of the equilibrium energies in the original and model problems. We note that, in this case, $\widetilde{t}_\pm = t_\pm$.

If (3.11) and (5.1) hold, then Equation (4.9) takes the form

$$e(\nabla u^Q) = (\chi^Q - Q) e(k \otimes c), \quad u^Q \in \mathbb{X}(\Omega), \quad \chi^Q \in \mathbb{Z}_Q(\Omega), \quad Q \in [0,1]. \tag{5.2}$$

Lemma 5.1. *For* $Q \in (0,1)$ *the problem* (5.2) *has no solutions in any domain* $\Omega \subset R^m$, $m \geqslant 2$.

Proof. For the sake of brevity we omit the superscript Q in the notation of u and χ and write Equation (5.2) in the coordinate form

$$u^i_{x_j} + u^j_{x_i} = (\chi - Q)(k_i c_j + k_j c_i). \tag{5.3}$$

Let $k^\perp$ be a unit vector in R^m, orthogonal to k. Multiplying both sides of (5.3) by $k_i k_j$, $k_i^\perp k_j^\perp$, $k_i^\perp k_j$ and summarizing with respect to i, j from 1 to m, we find

$$\begin{aligned}
&k \cdot \nabla(u \cdot k) = (\chi - Q)(c \cdot k),\\
&k^\perp \cdot \nabla(u \cdot k^\perp) = 0,\\
&k \cdot \nabla(u \cdot k^\perp) + k^\perp \cdot \nabla(u \cdot k) = (\chi - Q)(c \cdot k^\perp).
\end{aligned} \tag{5.4}$$

From the second identity in (5.4) it follows that $u \cdot k^\perp$ is constant along any direction $k^\perp$, which implies $u \cdot k^\perp = 0$ for any $k^\perp$. Therefore, by the first and third identities in (5.4),

$$\begin{aligned}
&k \cdot \nabla(u \cdot k) = (\chi - Q)(c \cdot k),\\
&k^\perp \cdot \nabla(u \cdot k) = (\chi - Q)(c \cdot k^\perp)
\end{aligned} \tag{5.5}$$

for all $k^\perp$. We multiply the first identity by $c \cdot k^\perp$ and the second one by $c \cdot k$. Subtracting, we find

$$\tau \cdot \nabla(u \cdot k) = 0, \quad \tau = (c \cdot k^\perp)k - (c \cdot k)k^\perp. \tag{5.6}$$

Since $|\tau|^2 = (c \cdot k)^2 + (c \cdot k^\perp)^2$, $c \neq 0$, there exists a vector $k^\perp$ such that $\tau \neq 0$. Then from (5.5) it follows that the function $u \cdot k$ is constant along τ. By the boundary conditions, we have $u \cdot k = 0$. Thus, the function satisfying (5.3) vanishes. Then from (5.5) we find

$$(\chi - Q)^2((c \cdot k)^2 + (c \cdot k^\perp)^2) = 0 \quad \forall k^\perp.$$

Integrating over Ω, we get $Q(1 - Q)((c \cdot k)^2 + (c \cdot k^\perp)^2) = 0$, which is impossible for $Q \in (0, 1)$ and $c \neq 0$. □

We assume that $\xi = e(k \otimes c)$, $k, c \in R^2$, $|k| = 1$, $c \neq 0$. A simple calculation leads to the following assertions.

(1) If k and c are linearly independent, then the eigenvalues of the matrix ξ have different signs: $\lambda_+ > 0$ and $\lambda_- < 0$ and these eigenvalues and the corresponding normed eigenvectors are expressed by

$$\begin{aligned}
\lambda_+ &= \frac{1}{2}(k \cdot c + |c|), \quad z_+ = \frac{1}{\sqrt{2}|c|}\frac{1}{(1 + |c|^{-1}c \cdot k)^{1/2}}(|c|k + c),\\
\lambda_- &= \frac{1}{2}(k \cdot c - |c|), \quad z_- = \frac{1}{\sqrt{2}|c|}\frac{1}{(1 - |c|^{-1}c \cdot k)^{1/2}}(|c|k - c).
\end{aligned} \tag{5.7}$$

(2) if k and c are linearly dependent and $c = \pm|c|k$, then the eigenvalues of the matrix ξ and the corresponding eigenvectors take the form

$$\lambda_1 = \pm|c|, \quad z_1 = k, \quad \lambda_2 = 0, \quad z_2 = k^\perp. \tag{5.8}$$

Assume that λ_1, λ_2 are eigenvalues and z_1, z_2 are the corresponding orthonormal eigenvectors of a matrix $\xi \in R_s^{2\times 2}$. Then the following assertions hold.

(3) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, then $\xi = e(k \otimes c)$ with $k = z_1$ and $c = \lambda_1 z_1$.

(4) if $\lambda_1 > 0$ and $\lambda_2 < 0$, then $\xi = e(k \otimes c)$ with

$$k = \frac{1}{\sqrt{\lambda_1 - \lambda_2}}(z_1\sqrt{\lambda_1} + z_2\sqrt{|\lambda_2|}), \quad c = \sqrt{\lambda_1 - \lambda_2}(z_1\sqrt{\lambda_1} - z_2\sqrt{|\lambda_2|}).$$

By the above assertions (1)–(4), Lemma 5.1, and Theorem 4.1, we derive the following assertion.

Theorem 5.1. *Assume that* $A^{\pm} = A$, $[\zeta] \neq 0$ *and* Ω *is a domain in* R^m, $m \geqslant 2$. *Then the following assertions hold*:

(1) *if* (5.1) *holds, then the problem* (4.1) *has no solution for any* $t \in (t_-, t_+)$,

(2) *if* $m = 2$, *then the problem* (4.1) *has no solution for any* $[\zeta]$ *such that* $\det[\zeta] \leqslant 0$ *and* $t \in (t_-, t_+)$.

The solvability of the problem (4.1) was studied, for example, in [4] and [8]–[10]. The results obtained in the cited works concern the case of isotropic moduli of elasticity. Theorem 5.1 is an attempt to study the problems without the isotropy condition. We note that it is reasonable to study the solvability of the problem (4.1) only for $t \in (t_-, t_+)$ (cf. [11]).

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